Introduction to Percolation Theory

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Original Problem

- Suppose a large porous rock is submerged under water for a long time, will the water reach the center of the stone.
- Posed by Broadbent and Hammersley in 1957

Related Problems

- How far from each other should trees in an orchard be planted in order to minimize the spread of fire?
- How infectious does a strain of flu have to be to create a pandemic? What is the expected size of an outbreak?

Mathematical Setup

- Stone: a large two dimensional grid of channels (edges). Edges in the grid are open or present with probability p ($0 \leq p \leq 1$) and closed or absent with probability $1 - p$.
- Pores: open edges and p determines the porosity of the stone
- A connected component of the graph of open edges is called a cluster. The water will reach the center of the stone if there is a cluster joining its center with the outside.

The Orchard Example

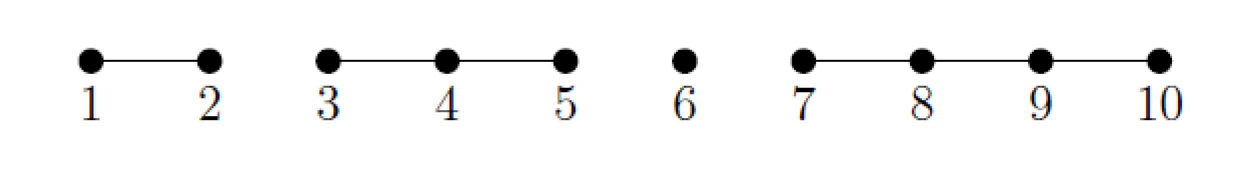
- p is the probability that fire will spread to an adjacent tree and minimizing the spread corresponds to minimizing the size of the largest open cluster

General Setup

- The space of the model is Z^n or any infinite graph.
- The edges are open or closed with probability p, which may depend on the properties of the edge (e.g. degree).
- Open cluster is a connected component of the open edge graph.
- The network is said to percolate if there is an infinite open cluster containing the origin.

Example 1

- Consider the graph of a path of vertexes 1 to N
- Ex.



- What is the probability that vertex 1 is connected to vertex N?

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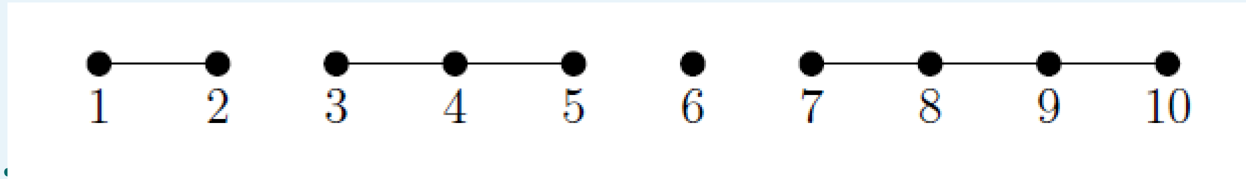

- What is the probability that vertex 1 is connected to vertex N?
- p^{N-1}
- Which as N grows, this goes toward 0.

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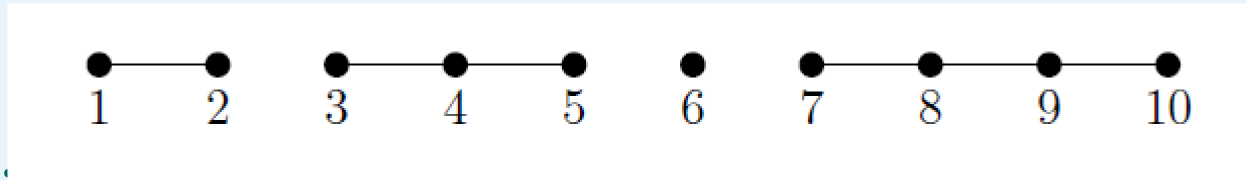
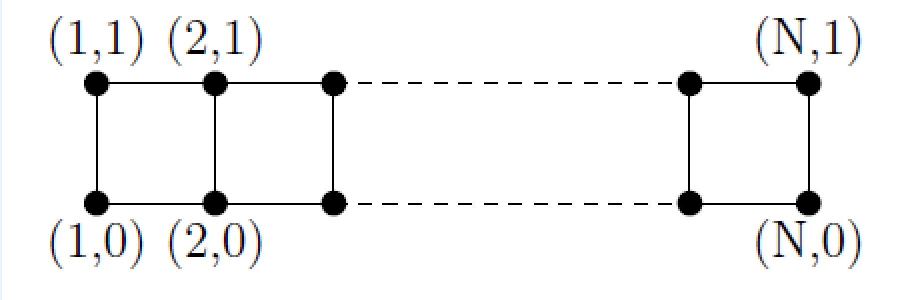


- What is the probability that vertex 1 is connected to vertex N?
- p^{N-1}
- Which as N grows, this goes toward 0.

Not that interesting!!!

Example 2



- Notation: x~y means the point at x is connected to y

$$
\begin{aligned}
\mathbb{P}(\{L \leftrightarrow R\}) &< \mathbb{P}\left(\bigcap_{i=1}^{N-1} \{(i,1) \sim (i+1,1)\} \cup \{(i,2) \sim (i+1,2)\}\right) \\
&= \prod_{i=1}^{N-1} \mathbb{P}(\{(i,1) \sim (i+1,1)\} \cup \{(i,2) \sim (i+1,2)\}) \\
&= \prod_{i=1}^{N-1} [\mathbb{P}((i,1) \sim (i+1,1)) + \mathbb{P}((i,2) \sim (i+1,2)) - \mathbb{P}(\text{both bonds are open})] \\
&= (2p - p^2)^{N-1}
\end{aligned}
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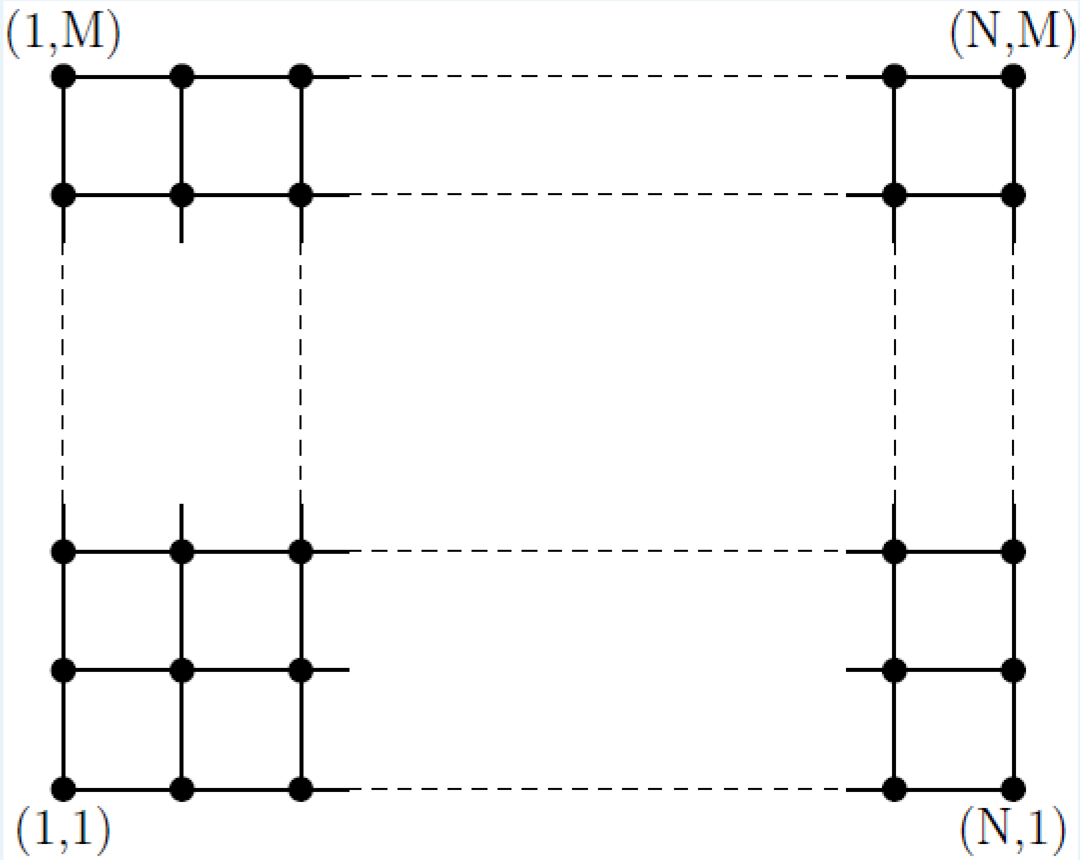
Extension to a MxN grid



Extension to a MxN grid

$$\begin{aligned}
\mathbb{P}\left(H(N,M)\right) &\leq \mathbb{P}\left(\bigcap_{i=1}^{N-1}\{(i,j)\sim(i+1,j)\ \text{ for some } j=1,\ldots,M\}\right)\\
&= \prod_{i=1}^{N-1}\mathbb{P}\left((i,j)\sim(i+1,j) \text{ for some } j=1,\ldots,M\right)\\
&= \prod_{i=1}^{N-1}\left[1-\mathbb{P}\left((i,j)\not\sim(i+1,j) \text{ for all } j=1,\ldots,M\right)\right]\\
&= \left(1-(1-p)^M\right)^{N-1}
\end{aligned}$$

Percolation on $\mathbb{Z}^2$

- First, some helpful numbers to consider.
- Cluster: $C_x=\{\in \mathbb{Z}^2 : y\sim x\}$
- Having an infinite cluster is called percolation.
- Next i $\mathbb{P}_p$ ne percolation density.
- $\Theta(p):= \quad (|C_x|$ is infinite)
- Note this is actual independent of x
- When $\Theta(p)=0$ this means that there is no percolation.

Transition point

- Due to this we define a transition point:
- $p_H = \inf\{p : \Theta(p) > 0\}$.
- This are transition point for the following reason
- $\Theta(p) > 0$ for all $p > p_H$

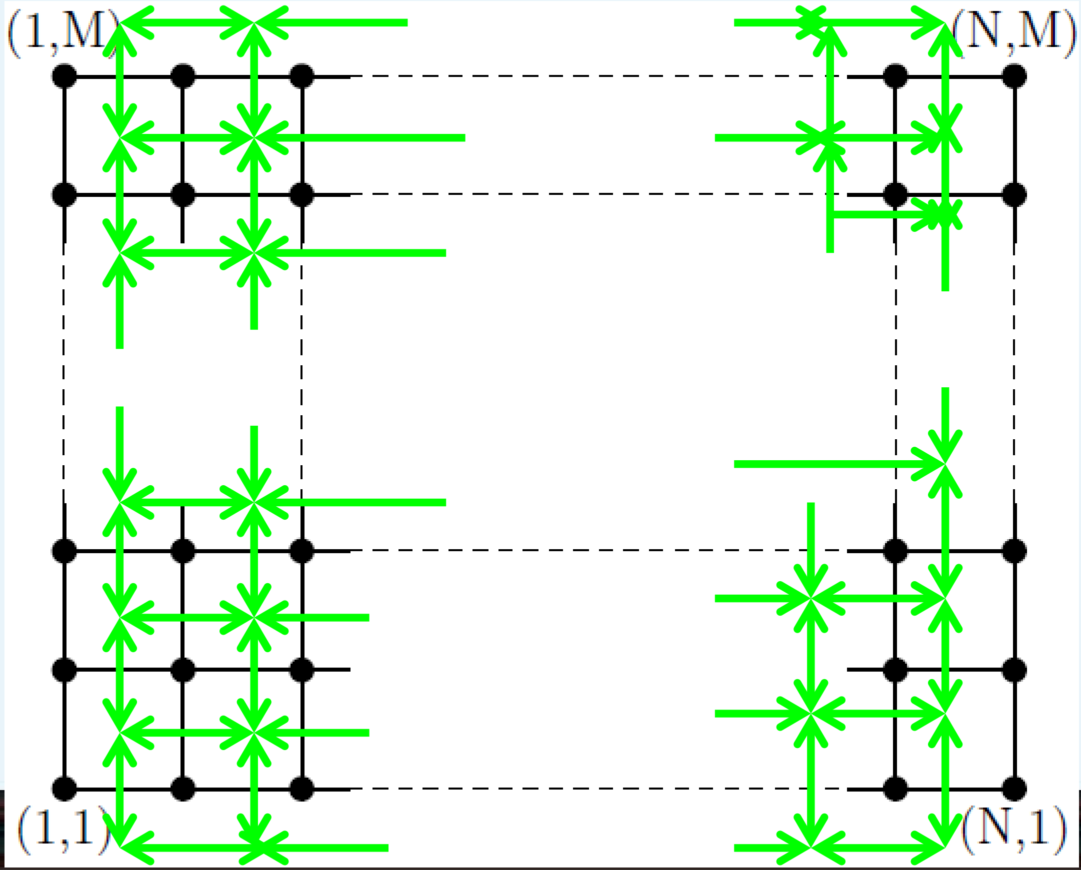
The main idea for $\mathbb{Z}^2$

$$p_H = \frac{1}{2}$$

- T.E. Harris in 1960 proved that the percolation probability of $\mathbb{Z}^2$ is indeed no less than $\frac{1}{2}$
- The upper bound is proven by Kesten in 1980
- Together show that the critical probability for the percolation in $\mathbb{Z}^2$ is indeed ½

Harris(1960)

- First, consider the dual of Z^2



Here we have a N by M r

The dual is N-1 by M+1 r

Dual Lattice

- Let the dual lattice edge be open only if the edge that is crosses in the lattice is closed.
- Therefore the dual lattice has a probability of it's edges being open of 1-p, where p is the probability the the lattice edge is open.
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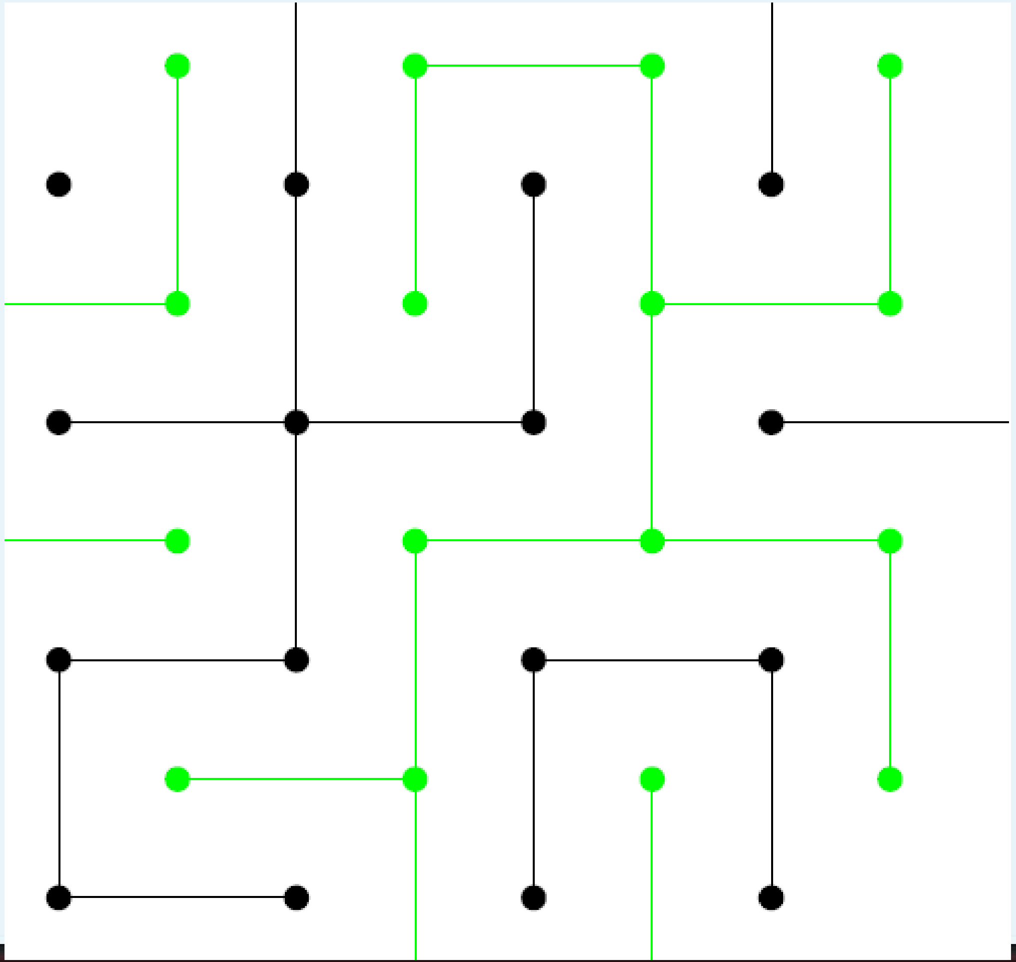
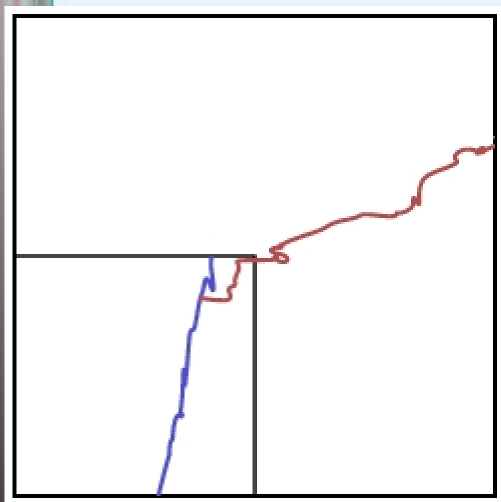
Example of a dual lattice



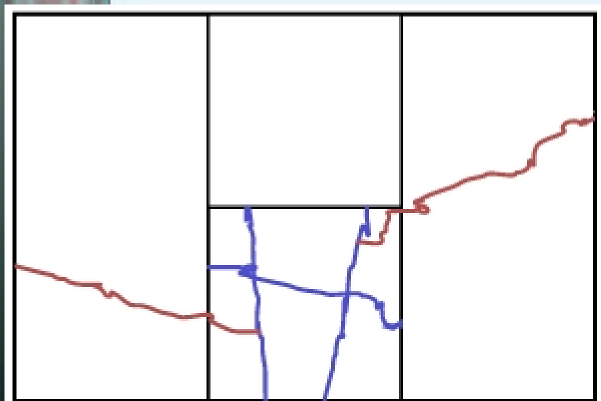
Relationship

- A horizontal crossing of the lattice and a vertical crossing of the dual lattice are two mutually exclusive events, but one must occur.
- $P_p(\mathrm{Hort}(N \text{ by } M)) + P_{1-p}(\mathrm{Vert}(N-1 \text{ by } M+1) = 1$
- $P_p(\mathrm{Hort}(N \text{ by } M)) + P_{1-p}(\mathrm{Hort}(M+1 \text{ by } N-1) = 1$
- If $M=N-1$ then
- $P_p(\mathrm{Hort}(N \text{ by } N-1)) + P_{1-p}(\mathrm{Hort}(N \text{ by } N-1) = 1$
- If $p=½$ then $P_{½}(\mathrm{Hort}(N \text{ by } N-1)) = ½$

- If we extend this to a N-1 by N-1 square then we would have less to travel. Therefore,
- If $p=½$ then $P_{½}(Hort(N-1 \text{ by } N-1)) \leq ½$
- Since this is independent of N, let S be any square then:
- $P_{½}(Hort(S)) \geq ½$



- Consider the event F_1 described by a configuration of open paths, like the blue and red ones shown to the left. We know that the probability of an open vertical path in the smaller square is greater than 1/2. In the same way, the probability of having a horizontal path, like the red one, open is greater than 1/2 as well. The fact that we want the red path to enter the smaller square introduces another factor of 1/2:
- $P_{1/2}(F_1) \geq ½ P_{1/2}(\mathrm{Vert}(S_{small}))P_{1/2}(\mathrm{Hort}(S_{big}))$
- $P_{1/2}(F_1) \geq 2^{-3}$



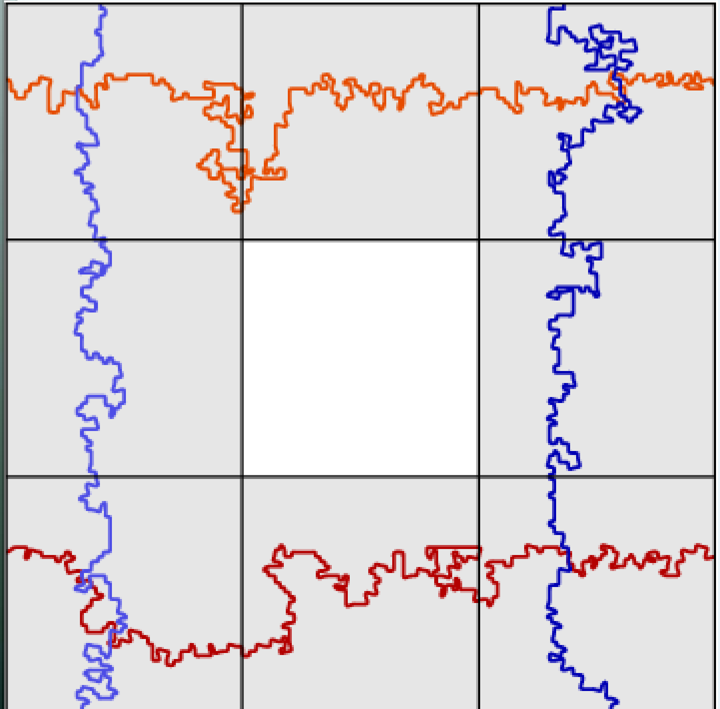
- Now let F_2 be the event that there is an open path across a 3n by 2n rectangle. We break this into smaller pieces by considering the rectangle to consist of two squares of dimension 2n by 2n overlapping in an n by 2n rectangle. Here we have two F_1 events and one square crossing.
- $P_{1/2}(F_2) \geq P_{1/2}(F_1) P_{1/2}(F_1) P_{1/2}(\mathrm{Hort}(S))$
- $P_{1/2}(F_1) \geq 2^{-7}$

Continued results

- $P_{1/2}(\text{Hort}(N \text{ by } 2N)) \leq 2^{-15}$
- $P_{1/2}(\text{Hort}(N \text{ by } 3N)) \leq 2^{-25}$

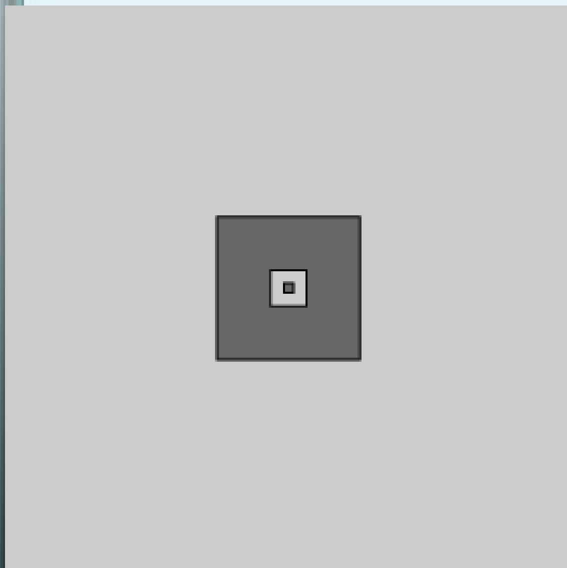
- The main one we need is the N by 3N case

Combining results



- Notice here there are 4 cases of the N by 3N path crossing.
- $P_{½}$(4 crossings)$\geq 2^{-100}$
- If this occurs in the dual lattice then the cluster size must be finite.
- So for an infinite cluster we have probability $1-2^{-100}$

Multiple rings



- Since the size, N, did not matter. We can put loops in side loops.
- In order to have infinite cluster each loop in the dual must not exist.
- If there are k, loops we are looking at in the dual then the probability is: $(1-2^{-100})^k$

- We can have an infinite number of these so the probability i
- So the probability of having an infinite cluster is 0 at a bond

Upper bound

- Menshikov's theorem:
  - For a given $p < p_H$, there is a constant a such that $P_p(\text{path of length } n) \leq e^{-an}$
- Now try to cross a square(N by N). There are N starting points and there has to be at least N-1 length path to cross the square. By Menshikov:
- $P_{1/2}(\text{Hort}(N \text{ by } N)) \leq Ne^{-aN}$; Assuming $1/2 < p_H$
- Choosing N large enough this can be less than ½ which contradicts previous

What is happening?

- This means that if p is less than ½ then the probability of infinite cluster (percolation) is 0
- If p is greater than ½ the probability of an infinite cluster exist.
- What about when p=½?
- It is been shown that no infinite cluster exist if p=½ for the square lattice of points

Application of Site Percolation

- Application: Network Robustness and Fragility
  - Problem: How many random nodes can be removed before a network looses connectivity?
  - How many of the highly connected nodes can be removed before the network looses connectivity?

Continued research

- Conformal Invariant
- Power laws to describe Θ

Resources

- Dr. Jeffery Schenker (MSU)- lecture notes on percolation
  - https://sites.google.com/a/msu.edu/mth496-01-ss10/
- Dr. Kim Christensen (Imperial College London)- Percolation Theory
  - http://www.mit.edu/~levitov/8.334/notes/percol_notes.pdf
- David Austin (GVSU)-Percolation: Slipping through the Cracks
  - http://www.ams.org/samplings/feature-column/fcarc-percolation
- Daniel Gopin (NIST) – Percolation: Theory and