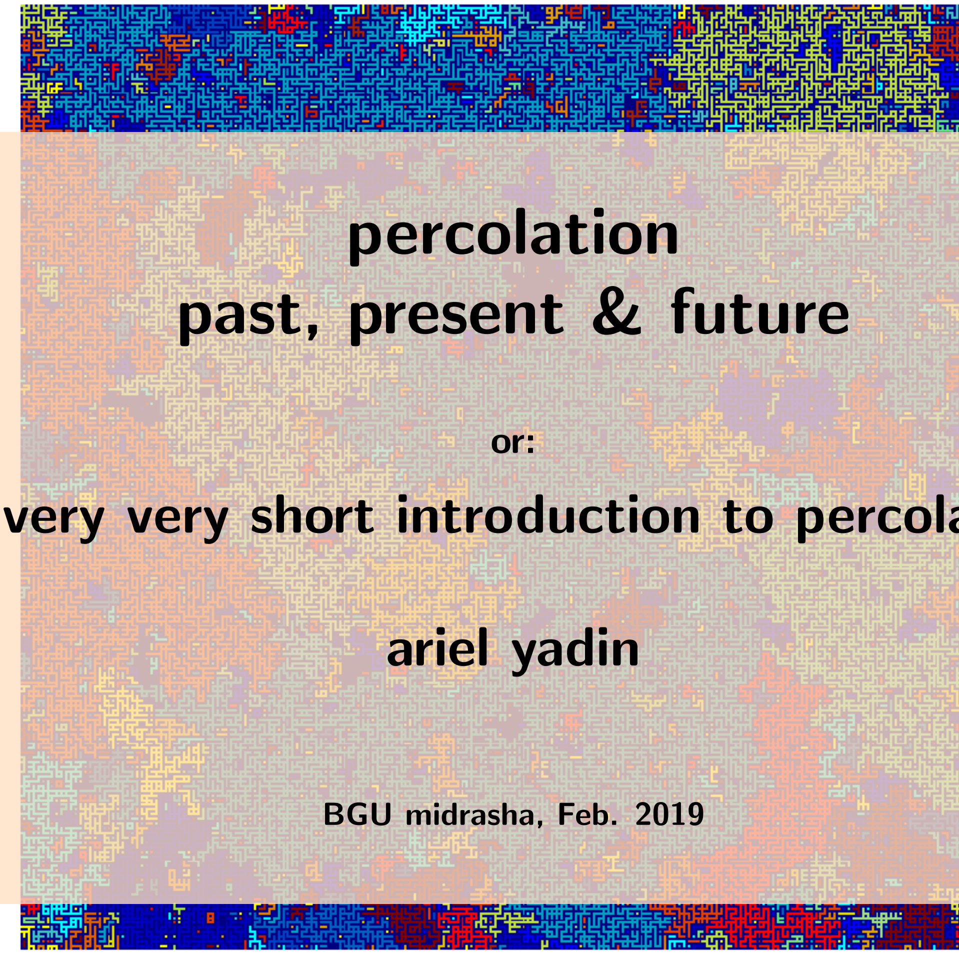


percolation
past, present & future

or:

a very very short introduction to percolation

ariel yadin

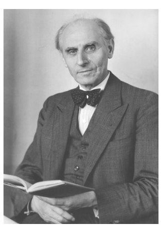
BGU midrasha, Feb. 2019

Contents

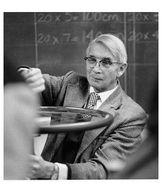
Chapter 1

past

Pierre Curie



Wilhelm Lenz



Ernst Ising

Rudolf Peierls

In the second half of the 19th century, Pierre Curie experimented with magnetism of iron. When placing an iron block in a magnetic field, sometimes the iron will stay magnetized after removing it from the field, and sometimes the iron will lose the magnetic properties after removal of the field. Curie discovered that this has to do with the temperature; if the temperature is below some critical temperature T_c, then the iron will retain magnetism. If the temperature is higher than T_c then the iron will lose its magnetic properties.

In the 1920's, Lenz gave his student Ising a mathematical model for the above phenomena, and Ising proceeded to study it, solving the one dimensional case in his thesis in 1924. Ising proved that in dimension one, the magnetization is lost in all temperatures, and conjectured that this is also the case in higher dimensions.

This was widely accepted until 1936 (!) when Peierls showed that in dimension 2 and up, there is a phase transition at some finite temperature.

What is percolation?

Let us think of glaciers and ice sheets. As the temperature in the ice rises, electromagnetic bonds between the molecules are becoming less prevalent than the kinetic energy of the molecules, so bounds between the molecules are starting to break. At some point, a lot of the molecules are no longer bound to each other, and they start to move more as individuals than as a structured lattice. The glacier is melting.

This is not exactly what is going on, but it will supply some "real world" motivation for the mathematical model we are about to introduce.

Figure 1.1: Glacier du Trient near the French-Swiss border on the Mont-Blanc massif.

In this mini-course we will deal with a cousin of the so-called Ising model: a model known as *percolation*, first defined by Broadbent and Hammersley in 1957. We will mainly be interested in such phase transition phenomena when some parameter varies (as the temperature in the Ising model).

Simon Broadbent

John Hammersley

1.1 percolation

Let $G = (V, E)$ be a graph. (So V is a set of vertices, and $E \subset \binom{V}{2}$ is a set of edges.) Throughout this mini-course, we always assume G is connected, infinite, and has bounded degree, unless stated explicitly otherwise.

There are two types of percolation models we may consider on G: One is a model percolating on the edges, called **bond percolation** and one percolates on the vertices, call **site percolation**.

Definition 1.1.1 (bond percolation) Fix a graph $G = (V, E)$ and a parameter $p \in [0, 1]$. Let $(\omega(e))_{e \in E}$ be independent Bernoulli-p random variables.

We say an edge $e \in E$ is **open** if $\omega(e) = 1$ and **closed** if $\omega(e) = 0$.

With slight abuse of notation, we think of ω as a subgraph of G (not necessarily connected), by considering the subgraph induced on the open edges of ω. ω is call **bond percolation** on G.

- We write $\mathcal{C}(x) = \mathcal{C}(x; \omega)$ to denote the connected component of a vertex x in the graph ω.
- We say that vertices $x, y \in V$ are **connected** (in ω), denoted $x \leftrightarrow y$, if $y \in \mathcal{C}(x)$.
- We say that a subset $A \subset V$ is connected to a subset $B \subset V$, denoted $A \leftrightarrow B$, if there exist $x \in A, y \in B$ such that $x \leftrightarrow y$.
- We say that x is connected to infinity, denoted $x \leftrightarrow \infty$ if $|\mathcal{C}(x)| = \infty$; that is, x belongs to an infinite connected component.

An analogous process may be defined on the vertices instead of the edges.

Definition 1.1.2 (site percolation) Fix a graph $G = (V, E)$ and a parameter $p \in [0, 1]$. Let $(\sigma(x))_{x \in V}$ be independent Bernoulli-p random variables. Vertices $x \in V$ are called **sites**. We say a site $x \in V$ is **open** if $\sigma(x) = 1$ and **closed** if $\sigma(x) = 0$.

With slight abuse of notation, we think of σ as a subgraph of G (not necessarily connected), by considering the subgraph induced on the open sites of σ. σ is call **site percolation** on G. The definitions of $\mathcal{C}(x) = \mathcal{C}(x; \sigma), x \leftrightarrow y, A \leftrightarrow V, x \leftrightarrow \infty$ are then defined analogously to those of bond percolation.

For this mini-course we will focus for simplicity on bond percolation. Exercise 1.6 shows that site percolation is actually the general theory.

We use the notation $\mathbb{P}_p, \mathbb{E}_p$ to denote the measure and expectation of the probability space of percolation with parameter p. Thus, for example, $\mathbb{P}_p[x \leftrightarrow \infty]$ is the probability that in percolation with parameter p, x is in an infinite component. (σ-algebra details are studied in the exercises.)

1.2 phases


Andrey Kolmogorov

Arguably the most basic question one may ask about percolation, is whether there exist infinite components or not. Kolmogorov's $0-1$ law tells us that this event must either have probability 0 or probability 1. See Exercise 1.3 Thus, we may separate the interval $[0,1]$ into two phases: the *sub-critical* phase where no infinite components exist a.s., and the *super-critical* phase where there a.s. exists an infinite component. Precisely, the sub-critical phase is the set of all p for which $\mathbb{P}_p[\exists\ x\ :\ x \leftrightarrow \infty] = 0$. The $0-1$ law guaranties that the complement is the super-critical phase, which is the set of all p for which $\mathbb{P}_p[\exists\ x\ :\ x \leftrightarrow \infty] = 1$. Of course, 0 is in the sub-critical phase, and 1 is in the super-critical phase. However, the question arises what do these phases look like? We will now see that these are just two complementary intervals.

1.3 monotonicity

Let $(U_e)_{e\in E}$ be independent random variables, each distributed uniformly on $[0,1]$. For $p \in [0,1]$ define

$$\Omega_p : E \to \{0,1\} \qquad \Omega_p(e) = \mathbf{1}_{\{U_e \le p\}}.$$

The important observation here is:

Proposition 1.3.1 **(monotone coupling)** Ω_p has the same distribution as percolation with parameter p.

Moreover, if $p \le q$ then $\Omega_p \le \Omega_q$.

The proof is an exercise.

When writing an inequality between functions above, *e.g.* $\omega \le \eta$, we mean that the inequality holds pointwise. This partial order leads to the notion of *increasing* and *decreasing* events and random variables.

Definition 1.3.2 A random variable $X : \{0,1\}^E \to [-\infty, \infty]$ is called **increasing** (resp. **decreasing**) if for any $\omega \le \eta$ we have $X(\omega) \le X(\eta)$ (resp. $X(\omega) \ge X(\eta)$).

An event $A \in \mathcal{F}$ is called increasing (resp. decreasing) if whenever $\omega \leq \eta$, then $\omega \in A$ implies $\eta \in A$ (resp. $\omega \notin A$ implies $\eta \notin A$).

That is, any increasing event already satisfied by some configuration ω, must also be satisfied by a new configuration η obtained from ω by opening bonds.

We now arrive at:

Theorem 1.3.3 Let G be a connected graph. There exists $p_c = p_c(G) \in [0, 1]$ such that for all $p < p_c$ there are $\mathbb{P}_p$-a.s. no infinite components and for any $p > p_c$ there exists an infinite component $\mathbb{P}_p$-a.s. (*i.e.* $p < p_c$ is sub-critical and $p > p_c$ is super-critical).

Proof. Define $p_c = \sup\{p : \mathbb{P}_p[\exists\ x : x \leftrightarrow \infty] = 0\}$. Note that for any $p > p_c$ it must be that $\mathbb{P}_p[\exists\ x : x \leftrightarrow \infty] > 0$ so the Kolmogorov $0 - 1$ law implies that this probability is actually 1.

In order to show that $p < p_c$ is sub-critical, it suffices to show that the function $p \mapsto \mathbb{P}_p[\exists\ x : x \leftrightarrow \infty]$ is montone. This follows from the fact that $\{x \leftrightarrow \infty\}$ is an increasing event, so also $\{\exists\ x : x \leftrightarrow \infty\} = \bigcup_x \{x \leftrightarrow \infty\}$ is increasing. $\square$

For percolation on G define

$$\theta_x(p) = \mathbb{P}_p[x \leftrightarrow \infty].$$

In the exercises we see that θ_x is a non-decreasing function. Of course, it is immediate to see that $\theta_x(p) = 0$ for any x if $p < p_c$.

Slightly more involved is to show that if $p > p_c$ then for any x we have $\theta_x(p) > 0$.

1.4 FKG (positive correlation)

In this section we introduce one of the most useful tools in percolation: the so called *FKG inequality* (named after Fortuin, Kasteleyn & Ginibre, however it was first proved by Harris in 1960 in his proof that $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$).

Theorem 1.4.1 **(FKG, Harris' Lemma)** Let $X, Y : \{0,1\}^E \to \mathbb{R}$ be square integrable random variables (*i.e.* $\mathbb{E}_p[X^2], \mathbb{E}_p[Y^2] < \infty$). If X, Y are both increasing then $\mathrm{Cov}_p(X, Y) \geq 0$.

For example, since $\{x \leftrightarrow z\}$ and $\{x \leftrightarrow \infty\}$ are both increasing events,

$$\mathbb{P}_p[x \leftrightarrow z \ , \ x \leftrightarrow \infty] \geq \mathbb{P}_p[x \leftrightarrow z] \cdot \mathbb{P}_p[x \leftrightarrow \infty].$$

It is immediate that $\mathbb{P}_p[x \leftrightarrow z] > 0$ in a connected graph. So we have proved:

Theorem 1.4.2 For percolation on an infinite connected graph G the following are equivalent:

- $\mathbb{P}_p[\exists \ x \ : \ x \leftrightarrow \infty] = 1$.
- There exists x for which $\theta_x(p) > 0$.
- For all x we have $\theta_x(p) > 0$.

As a consequence, although that precise value of $\theta_x(p)$ may depend on x, the property $\theta_x(p) > 0$ or not does not depend on the specific choice of x. Hence,

$$p_c = \sup\{p \ : \ \theta_x(p) = 0\} = \inf\{p \ : \ \theta_x(p) > 0\}$$

which does not depend on the choice of which vertex x we are looking at.

Note that easily $\mathbb{E}_p |\mathcal{C}(x)| = \infty$ for any $p > p_c$. It is, however, much more difficult to prove that $\mathbb{E}_p |\mathcal{C}(x)| < \infty$ when $p < p_c$. Also, exactly at the critical point, $\mathbb{E}_{p_c} |\mathcal{C}(x)| = \infty$, but this also requires more tools.

This discussion leads us to perhaps one of the most important open problems in the theory:

Conjecture 1.4.3 Let G be a transitive infinite connected graph. If $p_c < 1$ then $\theta(p_c) = 0$.

(Recall that a graph G is transitive if its automorphism group acts transitively on the vertices; that is, if for any $x, y \in V(G)$ there exists $\varphi_{xy} \in \mathrm{Aut}(G)$ such that $\varphi_{xy}(x) = y$.)

Harry Kesten

This conjecture is wide open even for the "simple" graphs $\mathbb{Z}^d$ when $3 \leq d \leq 6$. For large d is has been solved, but the methods used are completely non-elementary.

As for the two dimensional case, in this mini-course we will attempt to provide a partial proof of the Harris-Kesten Theorem:

Theorem 1.4.4 **(Harris, Kesten)** For bond percolation on $\mathbb{Z}^2$, one has $p_c = \frac{1}{2}$ and $\theta(1/2) = 0$.

In the 1960's shortly after the model was introduced, Harris showed that $\theta(1/2) = 0$ (which implies that $p_c \geq \frac{1}{2}$). It took about 20 more years for Kesten to prove in the 1980's that $p_c(\mathbb{Z}^2) = \frac{1}{2}$.

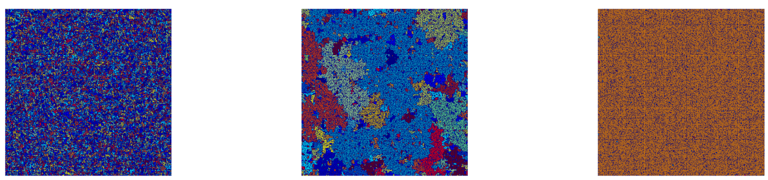


Figure 1.2: Bond percolation on $\mathbb{Z}^2$ with parameters $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ respectively. Colors are the different connected components.

1.5 one dimension

We conclude the first chapter with the one dimensional case, where G is basically $\mathbb{Z}$.

Let $G = (V, E)$ be an infinite connected graph. Let $o \in V$ be some fixed vertex. A subset $A \subset E$ is called a **cutset** (for o) if any infinite simple path in G starting at o must cross some edge in A. A is called a **minimal cutset** if A is a cutset and any $B \subsetneq A$ is not a cutset. Any finite cutset contains a minimal cutset.

Proposition 1.5.1 For bond percolation on G, the component $\mathcal{C}(o)$ is finite if and only if there exists a closed finite cutset A for o (*i.e.* all edges in A are closed).

Proof. If $\mathcal{C}(o)$ is finite then $\{\{x,y\} \in E \ : \ x \leftrightarrow o \ , \ y \not\leftrightarrow o\}$ is a cutset, and all edges in this cutset must be closed by definition.

If $\mathcal{C}(o)$ is infinite, then there exists an infinite simple path γ in G, starting at $\gamma_0 = o$, such that all edges $\{\gamma_n, \gamma_{n+1}\}$ are open. However, if A is any finite cutset, then γ must cross A, so A contains an open edge. $\square$

We say that a (connected, infinite, bounded degree) graph G is **one dimensional** if there exists a sequence $(A_n)_n$ of mutually disjoint cutsets in G such that $\sup_n |A_n| < \infty$.

Example 1.5.2 $\mathbb{Z}$ is one dimensional since $\{\{-2z, -2z-1\}, \{2z, 2z+1\}\}$ forms a sequence of mutually disjoint cutsets all of size 2. $\triangle\triangledown\triangle$

Theorem 1.5.3 If G is one dimensional then $p_c(G) = 1$.

Proof. Let $(A_n)_n$ be a sequence of mutually disjoint cutsets such that $|A_n| \leq M$ for all n. Let $p < 1$. Let $\mathcal{E}_n = \{A_n \text{ is closed}\}$. Then $\mathbb{P}_p[\mathcal{E}_n] \geq (1-p)^M$. Also, since $(A_n)_n$ are mutually disjoint, the events $(\mathcal{E}_n)_n$ are mutually independent.

By Borel-Cantelli this implies that $\mathbb{P}_p$-a.s. infinitely many of the cutsets A_n are closed. $\square$

1.6 exercises I

Exercise 1.1 Let S be an infinite countable set. Consider the space $\Omega = \{0,1\}^S$. For $\omega \in \Omega, s \in S$ define $X_s(\omega) = \omega(s)$, the natural projection to the s coordinate.

Let $\mathcal{F} = \sigma(X_s \ : \ s \in S)$ be the minimal σ-algebra for which these projections are measurable.

For $\omega \in \Omega$ and $J \subset S$ define

$$\mathcal{C}(J,\omega) = \{\eta \in \Omega \ : \ \eta\big|_J = \omega\big|_J\}$$

When J is finite $\mathcal{C}(J,\omega)$ is called a **cylinder set**.

- What is $\mathcal{C}(\emptyset,\omega)$?
- Show that $\eta \in \mathcal{C}(J,\omega)$ if and only if $\mathcal{C}(J,\eta) = \mathcal{C}(J,\omega)$.
- Show that

$$\begin{aligned} \mathcal{F} &= \sigma(\mathcal{C}(J,\omega) \ : \ \omega \in \Omega \ , \ |J| = 1) \\ &= \sigma(\mathcal{C}(J,\omega) \ : \ \omega \in \Omega \ , \ |J| < \infty). \end{aligned}$$

$\mathcal{F}$ is called the **cylinder σ-algebra**. ◇◇◇

Exercise 1.2 If $\sim$ is an equivalence relation on Ω, we say that a subset $A \subset \Omega$ respects $\sim$ if for any $\omega \sim \eta \in \Omega$ we have $\omega \in A \iff \eta \in A$.

- Show that the collection of subsets A that respect the equivalence relation $\sim$ forms a σ-algebra on Ω.
- Let S be an infinite countable set. Consider the space $\Omega = \{0,1\}^S$. Consider the cylinder σ-algebra $\mathcal{F}$ from the previous exercise. Find an equivalence relation $\sim$ on Ω, so that $A \in \mathcal{F}$ if and only if A respects $\sim$.

◇◇◇

Exercise 1.3 (Kolmogorov $0-1$ law) Let S be an infinite countable set. Consider the space $\Omega = \{0,1\}^S$. Let $\mathcal{F}$ be the cylinder σ-algebra. Let $\mathbb{P}_p$ be the product probability measure on Ω, for which $\mathbb{P}_p[\omega(s) = 1] = p$ and all $\omega(s)$ are independent.

- For a subset $I \subset S$ consider the σ-algebra

  $$\mathcal{F}_I := \sigma(\mathcal{C}(J, \omega) \ : \ J \subset I \ , \ \omega \in \Omega).$$

  Show that $\mathcal{F}_I, \mathcal{F}_J$ are independent if $I \cap J = \emptyset$.

- Prove that for any event $A \in \mathcal{F}$ and any $\varepsilon > 0$, there exist a finite subset $J \subset S, |J| < \infty$ and an event $A' \in \mathcal{F}_J$ such that $\mathbb{P}_p[A \triangle A'] < \varepsilon$.

- Let $\mathcal{T}_J = \mathcal{F}_{S \setminus J}$ and define the **tail σ-algebra** by

  $$\mathcal{T} = \bigcap_{|J| < \infty} \mathcal{T}_J.$$

  Use the above to show that for any $A \in \mathcal{T}$ we have $\mathbb{P}_p[A] \leq \mathbb{P}_p[A]^2$. Conclude that $\mathbb{P}_p[A] \in \{0,1\}$.

◇ ◇ ◇

Exercise 1.4 Let S be an infinite countable set. Consider the space $\Omega = \{0,1\}^S$. Find an equivalence relation $\sim$ on Ω for which $A \in \mathcal{T}$ if and only if A respects $\sim$. ◇ ◇ ◇

Exercise 1.5 Show that the event that there exists an infinite component in (site / bond) percolation is a tail event. ◇ ◇ ◇

Exercise 1.6 Show that any bond percolation on G can be defined as site percolation on some other graph G'. (Hint: consider the *line graph* of G; this is the graph L whose vertices are $V(L) = E(G)$ and edges are defined by the relation $e \sim e'$ if $e \cap e' \neq \emptyset$.) ◇ ◇ ◇

Exercise 1.7 (monotone coupling) Let $(U_e)_{e\in E}$ be independent random variables, each distributed uniformly on $[0,1]$. For $p \in [0,1]$ define

$$\Omega_p : E \to \{0,1\} \qquad \Omega_p(e) = \mathbf{1}_{\{U_e \leq p\}}.$$

Show that Ω_p has the same distribution as percolation with parameter p.

Moreover, show that if $p \leq q$ then $\Omega_p \leq \Omega_q$. ◇◇◇

Exercise 1.8 Show that A is an increasing event if and only if A^c is a decreasing event.

Show that the union of increasing events is increasing. Show that the intersection of increasing events is increasing.

Show that A is an increasing event if and only if $\mathbf{1}_A$ is an increasing random variable. ◇◇◇

Exercise 1.9 For each of the following events, decide if they are increasing, decreasing or neither.

- $\{x \leftrightarrow y\}$
- $\{x \leftrightarrow \infty\}$
- $\{x \leftrightarrow y \ , \ y \not\leftrightarrow z\}$
- $\{A \not\leftrightarrow B\}$

◇◇◇

Exercise 1.10 Show that if X is an increasing random variable then $p \mapsto \mathbb{E}_p[X]$ is a monotone non-decreasing function. ◇◇◇

Exercise 1.11 Show that if $p < p_c$ then $\theta_x(p) = 0$ for any x. ◇◇◇

Exercise 1.12 Let X, Y, Z be square integrable random variables. Assume that X, Y are decreasing and Z is increasing. Show that $\mathrm{Cov}_p(X, Z) \leq 0$ and that $\mathrm{Cov}_p(X, Y) \geq 0$. ◇◇◇

Exercise 1.13 Show that

$$\mathbb{P}_p[x \leftrightarrow z \ , \ x \leftrightarrow \infty] \geq \mathbb{P}_p[x \leftrightarrow z] \cdot \mathbb{P}_p[x \leftrightarrow \infty].$$

◇◇◇

Exercise 1.14 Show that if H is a subgraph of G then $p_c(H) \geq p_c(G)$. ◇◇◇

Exercise 1.15 Show that if $\mathcal{C} \subset V$ is a finite connected subgraph of G, then the edge boundary $\partial_e \mathcal{C} = \{\{x, y\} \in E \ : \ x \in \mathcal{C} \ , \ y \notin \mathcal{C}\}$ is a cutset for any vertex $o \in \mathcal{C}$. ◇◇◇

Exercise 1.16 In this exercise, we will develop the proof of the FKG inequality for random variables that only depend on finitely many coordinates. (The proof of the full FKG inequality for square integrable random variables is then attained using standard L^2 methods.)

The setting is as follows. Let S be a finite set, and consider the space $\Omega = \{0, 1\}^S$ with the product measure $\mathbb{P}[\{\omega\}] = \prod_{s \in S} p^{\omega}(s)(1 - p)^{1 - \omega(s)}$.

Let $X, Y : \Omega \to \mathbb{R}$ be such that $\mathbb{E}[X^2], \mathbb{E}[Y^2] < \infty$, and X, Y are increasing. Prove by induction on $|S|$ that $\mathrm{Cov}(X, Y) \geq 0$. ◇◇◇

Exercise 1.17 (Peierl's argument) Consider the graph $\mathbb{Z}^2$.

Let

$$\mathcal{C}_n = \{A \subset E(\mathbb{Z}^2) \ : \ |A| = n \ , \ A \text{ is a cutset for } 0\}.$$

Show that there exists $C > 1$ such that $|\mathcal{C}_n| \leq C^n$ for all n.

Use this to prove that $p_c(\mathbb{Z}^2) < 1$. (In fact, one may prove that $p_c(\mathbb{Z}^2) \leq \frac{C-1}{C}$). ◇◇◇

Exercise 1.18 Show that this implies that $p_c(\mathbb{Z}^d) < 1$ for all $d \geq 2$. ◇◇◇

Exercise 1.19 Show that if G is an infinite connected graph with maximal degree D then $p_c(G) \geq \frac{1}{D-1}$. ◇◇◇

Exercise 1.20 Let G be an infinite connected bounded degree graph. Fix a vertex o. A **self-avoiding path** started at o is a path $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ such that $\gamma_i \neq \gamma_j$ for all $i \neq j$, and $\gamma_0 = o$ and $\gamma_i \sim \gamma_{i+1}$ for all $0 \leq i < n$. In this case $|\gamma| = n$ is the length of γ.

Let $\mathsf{SAW}_n(o)$ be the set of all self-avoiding paths started at o of length n. Define

$$\mu = \mu(G, o) = \limsup_{n \to \infty} |\mathsf{SAW}_n(o)|^{1/n}.$$

Prove that $p_c(G) \geq \frac{1}{\mu}$. ◇◇◇

Chapter 2

present

In this chapter, we will introduce the more modern tools of percolation theory. The goal is to give a taste of the recently developed methods. Although we do not go into the proofs in a detailed manner, everything introduced may be proven with elementary means.

The main theorem we will focus on is:

> **Harris Kesten Theorem, Theorem 1.4.4.**
> For bond percolation on $\mathbb{Z}^2$ it holds that $p_c = \frac{1}{2}$ and $\theta(p_c) = 0$.

There are a few main components that go into this theorem. The first is the fact that when $p < p_c$, not only are there no infinite components, but the size of a component is very small. In fact, its diameter has an exponential tail. This theorem was originally proved by Menshikov (for subexponentially growing graphs) and Aizenman & Barsky (in a more general context). However, their proofs are highly non-elementary. We shall discuss a more elementary and short argument given in 2015 by Duminil-Copin & Tassion. Indeed, the only tool used for their argument is Russo's formula, which we will describe shortly.

The next component is a phenomenon that is an artifact of the *amenability* of $\mathbb{Z}^2$. The Burton & Keane Theorem that we shall discuss below tells us that in a transitive amenable graph either there are no infinite components, or there is a *unique* infinite component.

The last component that goes into proving the Harris-Kesten Theorem, is specialized to $\mathbb{Z}^2$. It is basically the fact that planar graphs have a dual, and percolation on the primal and dual graphs may be coupled together, so that the geometry of the plane plays a crucial role.

Lucio Russo

Grigory Margulis

2.1 Russo's formula

In this section we introduce one of the most classical tools in the theory of percolation, commonly known as *Russo's formula* (independently discovered by Margulis). This formula computes the derivative of $\mathbb{E}_p[X]$ as a function of p. For example, if A is an increasing event, then the derivative of $\mathbb{P}_p[A]$ controls how fast A is going from a small probability to a large one. In a sense, a large derivative means that there is a "sharp threshold" phenomenon.

Before stating the formula, we require some more notation. For a configuration $\omega \in \{0,1\}^E$ and an edge $e \in E$, define ω^e, ω_e to be

$$\omega^e(\varepsilon) = \begin{cases} \omega(\varepsilon) & \varepsilon \neq e \\ 1 & \varepsilon = e \end{cases} \qquad \omega_e(\varepsilon) = \begin{cases} \omega(\varepsilon) & \varepsilon \neq e \\ 0 & \varepsilon = e \end{cases}$$

That is, ω^e (resp. ω_e) is the configuration ω with the edge e forced to be open (resp. closed).

For a random variable $X : \{0,1\}^E \to \mathbb{R}$ define $X^e(\omega) = X(\omega^e)$ and $X_e(\omega) = X(\omega_e)$. Define also the derivative of X (in direction e) by $\frac{\partial}{\partial e} X = X^e - X_e$.

Note: the expectation $\mathbb{E}_p[\frac{\partial}{\partial e} X]$ is known as the **influence** of the edge e on X.

We say that a random variable $X : \{0,1\}^E \to \mathbb{R}$ **depends on finitely many edges** if there exists a finite set $S \subset E, |S| < \infty$ such that for any $\omega, \eta \in \{0,1\}^E$ with $\omega\big|_S = \eta\big|_S$, we have that $X(\omega) = X(\eta)$. That is, in order to know the value of X, it suffices to consider only the state of those edges in S.

Theorem 2.1.1 **(Russo's formula)** Let $X : \{0,1\}^E \to \mathbb{R}$ be a random variable that depends on finitely many edges. Then the function $p \mapsto$

$\mathbb{E}_p[X]$ is differentiable and

$$\frac{\partial}{\partial p}\,\mathbb{E}_p[X] = \sum_e \mathbb{E}_p[\tfrac{\partial}{\partial e} X].$$

Let us remark that by the exercises the sum in this theorem is actually finite. In fact, if X is measurable with respect to $\mathcal{F}_S$ the sum may be restricted to edges in S.

Proof of Russo's formula. Since X depends only on finitely many edges, there exists a finite set $S \subset E$ such that $\frac{\partial}{\partial e} X = 0$ for $e \notin S$ (see the exercises). Thus, $\sum_e \frac{\partial}{\partial e} X = \sum_{e\in S} \frac{\partial}{\partial e} X$. Assume that $S = \{s_1, \ldots, s_n\}$. Let $S_j = \{s_1, \ldots, s_j\}$ for $1 \leq j \leq n$ and let $S_0 = \emptyset$. (So $S_n = S$.)

We now use a variant of the monotone coupling. Recall that $(U_e)_e$ are i.i.d. uniform on $[0,1]$ and $\Omega_p(e) = \mathbf{1}_{\{U_e \leq p\}}$ has the distribution of bond percolation with parameter p. To compute the derivative, fix p and let $\varepsilon > 0$ be very small.

For $0 \leq j \leq n$, let

$$\Lambda_j(e) = \begin{cases} \mathbf{1}_{\{U_e \leq p\}} & e \notin S_j \\ \mathbf{1}_{\{U_e \leq p+\varepsilon\}} & e \in S_j \end{cases}$$

So $\Lambda_n\big|_S = \Omega_{p+\varepsilon}\big|_S$ and $\Lambda_0 = \Omega_p$. Thus, $X(\Omega_{p+\varepsilon}) = X(\Lambda_n)$ and $X(\Omega_p) = X(\Lambda_0)$. Note that for any $1 \leq j \leq n$,

- If $U_{s_j} \in (p, p+\varepsilon]$ then $X(\Lambda_j) = X^{s_j}(\Lambda_j)$ and $X(\Lambda_{j-1}) = X_{s_j}(\Lambda_{j-1})$.
- If $U_{s_j} \notin (p, p+\varepsilon]$ then then $\Lambda_j\big|_S = \Lambda_{j-1}\big|_S$, so $X(\Lambda_j) = X(\Lambda_{j-1})$.

Thus we may compute for $1 \leq j \leq n$,

$$\begin{aligned} \mathbb{E}[X(\Lambda_j) - X(\Lambda_{j-1})] &= \mathbb{E}\left[(X^{s_j}(\Lambda_j) - X_{s_j}(\Lambda_{j-1}))\mathbf{1}_{\{U_{s_j}\in(p,p+\varepsilon]\}}\right] \\ &= \mathbb{E}\left[X^{s_j}(\Lambda_j) - X_{s_j}(\Lambda_{j-1})\right]\cdot \mathbb{P}[U_{s_j} \in (p, p+\varepsilon]] = \mathbb{E}[\tfrac{\partial}{\partial s_j} X(\Lambda_j)]\cdot\varepsilon, \end{aligned}$$

where we have used the fact that X^e, X_e are both independent of U_e.

It is quite simple to see that

$$\lim_{\varepsilon\to 0} \mathbb{E}[\tfrac{\partial}{\partial s_j} X(\Lambda_j)] = \mathbb{E}[\tfrac{\partial}{\partial s_j} X(\Omega_p)] = \mathbb{E}_p[\tfrac{\partial}{\partial s_j} X].$$

Thus,

$$\mathbb{E}_{p+\varepsilon}[X] - \mathbb{E}_p[X] = \mathbb{E}[X(\Omega_{p+\varepsilon}) - X(\Omega_p)] = \sum_{j=1}^{n} \mathbb{E}[X(\Lambda_j) - X(\Lambda_{j-1})],$$

so

$$\begin{aligned} \tfrac{\partial}{\partial p}\, \mathbb{E}_p[X] &= \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon} \big(\mathbb{E}_{p+\varepsilon}[X] - \mathbb{E}_p[X] \big) \\ &= \sum_{j=1}^{n} \mathbb{E}_p[\tfrac{\partial}{\partial s_j} X] = \sum_{s \in S} \mathbb{E}_p[\tfrac{\partial}{\partial s} X]. \end{aligned}$$

□

Let $A \in \mathcal{F}$ be an event. An edge $e \in E$ is pivotal in ω for A if $\omega^e \in A, \omega_e \not\in A$ or $\omega_e \in A, \omega^e \not\in A$. That is changing the state of the edge e in ω changes whether ω belongs to A or not.

It is not difficult to verify that

$$\{e \text{ is pivotal for } A\} = \{\omega \in \{0,1\}^E \ : \ e \text{ is pivotal in } \omega \text{ for } A\} = A^e \triangle A_e,$$

where

$$A^e = \{\omega^e \ : \ \omega \in A\} \qquad A_e = \{\omega_e \ : \ \omega \in A\}.$$

Also, if A is increasing then

$$\tfrac{\partial}{\partial e} \mathbf{1}_A = \mathbf{1}_{\{e \text{ is pivotal for } A\}} = \mathbf{1}_{A^e \setminus A_e}.$$

2.2 the Duminil-Copin & Tassion argument

Some notation: Fix a vertex o. Let $B_r = B_r(o)$ denote the ball of radius r around o in the graph G. Let $B_r(x)$ denote the ball of radius r around a vertex x. For a subset $A \subset V$, define the **boundary** and **edge boundary** of A by:

$$\partial A = \{y \not\in A \, : \, \exists\, x \in A\, ,\, y \sim x\} \qquad \partial_e A = \{(x,y) \ : \ \{x,y\} \in E\, ,\, x \in A\, ,\, y \not\in A\}.$$

We also require the definition of *connections inside a given set*: For a subset $S \subset V$ we write $x \overset{S}{\leftrightarrow} y$ (resp. $A \overset{S}{\leftrightarrow} B$) if there exists an open path, only using edges contained in S, that connects x, y (resp. some $x \in A, y \in B$).

Theorem 2.2.1 Consider percolation on an infinite connected transitive graph G. Fix a vertex $o \in V$. For any $p < p_c$ there exists a constant $c = c_p > 0$ such that for all $r > 0$,

$$\mathbb{P}_p[o \leftrightarrow \partial B_r] \le e^{-cr}.$$

Hugo Duminil-Copin

Vincent Tassion

Let $S \subset V$ be a finite subset containing o. Define

$$\varphi_p(S) = p \sum_{(x,y)\in\partial_e S} \mathbb{P}_p[o \overset{S}{\leftrightarrow} x].$$

So $\varphi_p(S)$ is p times the expected number of edges in the edge boundary $\partial_e S$ of S, that are connected to o inside S.

Let $r > 0$ be large enough so that $S \cup \partial S \subset B_r$. Thus, if $y \sim S$ then $y \in B_r$.

Lemma 2.2.2 For any $k > 0$ we have that

$$\mathbb{P}_p[o \leftrightarrow \partial B_{kr}] \le \varphi_p(S) \cdot \mathbb{P}_p[o \leftrightarrow \partial B_{(k-1)r}].$$

Proof. Define $\mathcal{C} = \{x \in S \ : \ x \overset{S}{\leftrightarrow} o\}$. Since $S \subset B_r$, the event $\{o \leftrightarrow \partial B_{kr}\}$ implies that there must exist an edge $(x,y) \in \partial_e S$ in the edge boundary of S, such that $o \overset{S}{\leftrightarrow} x$, the edge $\{x,y\}$ is open, and y is connected to ∂B_{kr} by an open path that does not intersect $\mathcal{C}$. (This may be obtained by considering the open paths leading from ∂B_{kr} until first hitting ∂S.) This last event is exactly $y \overset{\mathcal{C}^c}{\leftrightarrow} \partial B_{kr}$.

By summing over all possible options for $\mathcal{C}$, we arrive at the bound

$$\begin{aligned}\mathbb{P}_p[o \leftrightarrow \partial B_{kr}] &\le \sum_{(x,y)\in\partial_e S} \sum_{C\subset S} \mathbb{P}_p[\mathcal{C} = C \ , \ o \overset{S}{\leftrightarrow} x \ , \ \{x,y\} \text{ is open } \ , \ y \overset{C^c}{\leftrightarrow} \partial B_{kr}\}] \\ &= \sum_{(x,y)\in\partial_e S} \sum_{C\subset S} \mathbb{P}_p[\mathcal{C} = C \ , \ o \overset{C}{\leftrightarrow} x] \cdot p \cdot \mathbb{P}_p[y \overset{C^c}{\leftrightarrow} \partial B_{kr}],\end{aligned}$$

because edges in C, in C^c and in $\partial_e C$ are all independent. Now, since $S \cup \partial S \subset B_r$, we have that any y in the sum above must be in $y \in B_r$. Hence, by transitivity of the graph G, we get that $\mathbb{P}_p[y \overset{C^c}{\leftrightarrow} \partial B_{kr}] \le \mathbb{P}_p[o \leftrightarrow \partial B_{(k-1)r}]$ for any such y. Returning to the sum over possibilities for $\mathcal{C}$, we arrive at the conclusion. $\square$

Lemma 2.2.3 For any $r > 0$ and $p \in (0,1)$,

$$\frac{\partial}{\partial p}\mathbb{P}_p[o \leftrightarrow \partial B_r] \geq \frac{1}{p(1-p)} \cdot \inf_{o \in S \subset B_r} \varphi_p(S) \cdot \Big(1 - \mathbb{P}_p[o \leftrightarrow \partial B_r]\Big).$$

Proof. Consider the event $\{o \leftrightarrow \partial B_r\}$. Let P be the (random) set of pivotal edges for this event. Also, set $\mathcal{C} = \{x \in B_r \ : \ x \not\leftrightarrow \partial B_r\}$. One notes that if $\mathcal{C} = S$ then $P = \{\{x,y\} \in E \ : \ (x,y) \in \partial_e S \ : \ x \overset{S}{\leftrightarrow} o\}$. Also, $o \not\leftrightarrow \partial B_r$ if and only if $o \in \mathcal{C} \subset B_r$. Thus,

$$\begin{aligned}\sum_{e \subset B_r} \mathbb{P}_p[e \in P \ , \ o \not\leftrightarrow \partial B_r] &= \sum_{e \subset B_r} \sum_{o \in S \subset B_r} \mathbb{P}_p[e \in P \ , \ \mathcal{C} = S] \\ &= \sum_{o \in S \subset B_r} \sum_{(x,y) \in \partial_e S} \mathbb{P}_p[o \overset{S}{\leftrightarrow} x \ , \ \mathcal{C} = S].\end{aligned}$$

When $S \subset B_r$, the event $\mathcal{C} = S$ depends only on edges not in S. So the events $\{o \overset{S}{\leftrightarrow} x\}, \{\mathcal{C} = S\}$ are independent. Thus,

$$\begin{aligned}\sum_{e \subset B_r} \mathbb{P}_p[e \in P \ , \ o \not\leftrightarrow \partial B_r] &= \sum_{o \in S \subset B_r} \sum_{(x,y) \in \partial_e S} \mathbb{P}_p[o \overset{S}{\leftrightarrow} x] \cdot \mathbb{P}_p[\mathcal{C} = S] \\ &= \sum_{o \in S \subset B_r} \mathbb{P}_p[\mathcal{C} = S] \cdot \tfrac{1}{p}\varphi_p(S) \\ &\geq \mathbb{P}_p[o \not\leftrightarrow \partial B_r] \cdot \tfrac{1}{p} \inf_{o \in S \subset B_r} \varphi_p(S).\end{aligned}$$

The proof is now complete since Russo's formula tells us that

$$\sum_{e \subset B_r} \mathbb{P}_p[e \in P \ , \ o \not\leftrightarrow \partial B_r] = (1-p)\frac{\partial}{\partial p}\mathbb{P}_p[o \leftrightarrow \partial B_r],$$

where we have used that $o \leftrightarrow \partial B_r$ is increasing. □

Proof of theorem 2.2.1. Define

$$q_c = \sup\{p \ : \ \exists \ S \subset V \ , \ |S| < \infty \ , \ \varphi_p(S) < 1\}.$$

Lemma 2.2.2 shows that if there exists a finite set $S \subset V$ containing o such that $\varphi_p(S) < 1$, then for $r > 0$ such that $S \cup \partial S \subset B_r$, we have $\mathbb{P}_p[o \leftrightarrow \partial B_{kr}] \leq \varphi_p(S)^k$ which provides the exponential decay required. So we have exponential decay for all $p < q_c$. This implies that $q_c \leq p_c$.

We now show that for any $p > q_c$ we have $\theta(p) > 0$, so $q_c = p_c$.

Let $f(p) = \mathbb{P}_p[o \not\leftrightarrow \partial B_r]$. By Lemma 2.2.3, if $p > q_c$ then

$$\tfrac{\partial}{\partial p} \log f(p) \le -\tfrac{1}{p(1-p)}.$$

Thus, integrating over (q_c, p) we have

$$\log \tfrac{f(q_c)}{f(p)} \ge \int_{q_c}^{p} \tfrac{1}{\xi(1-\xi)} d\xi = \log \tfrac{p(1-q_c)}{q_c(1-p)},$$

which implies that $f(p) \le \frac{q_c}{1-q_c} \cdot \frac{1-p}{p}$. Hence,

$$\mathbb{P}_p[o \leftrightarrow \partial B_r] \ge 1 - \tfrac{q_c}{1-q_c} \cdot \tfrac{1-p}{p} = \frac{p(1-q_c) - q_c(1-p)}{p(1-q_c)} = \frac{p - q_c}{p(1-q_c)}.$$

For any $p > q_c$, the left hand side is positive independent of r, so taking $r \to \infty$, $\theta(p) \ge \frac{p-q_c}{p(1-q_c)} > 0$ for any $p > q_c$. $\square$

2.3 exercises II

Exercise 2.1 Show that the events $\{x \overset{S}{\leftrightarrow} y\}$ and $\{A \overset{S}{\leftrightarrow} B\}$ are in the σ-algebra $\mathcal{F}_S$. ◇◇◇

Exercise 2.2 Show that a random variable X is increasing if and only if $\frac{\partial}{\partial e} X$ is non-negative for every e. ◇◇◇

Exercise 2.3 Show that $X_e, X^e, \frac{\partial}{\partial e} X$ are independent of $\omega(e)$. That is show that these are measurable with respect to the σ-algebra $\mathcal{F}_{E \setminus \{e\}}$. ◇◇◇

Exercise 2.4 Show that X depends on finitely many edges if and only if X is measurable with respect to the σ-algebra $\mathcal{F}_S$ for some finite S. ◇◇◇

Exercise 2.5 Show that X is measurable with respect to the σ-algebra $\mathcal{F}_S$ if and only if $\frac{\partial}{\partial e} X = 0$ for any $e \notin S$. ◇◇◇

Exercise 2.6 Show that if X depends on finitely many edges then also $\frac{\partial}{\partial e} X$ depends on finitely many edges. ◇◇◇

Exercise 2.7 Let $S \subset E$ be a finite subset. For $\alpha \in \{0,1\}^S$ define $\omega_\alpha \in \{0,1\}^E$ by

$$\omega_\alpha(e) = \begin{cases} \alpha(e) & e \in S \\ 0 & e \notin S \end{cases}$$

For $X : \{0,1\}^E \to \mathbb{R}$ define $X' : \{0,1\}^S \to \mathbb{R}$ by $X'(\alpha) = X(\omega_\alpha)$.

Let $\mathbb{P}, \mathbb{E}$ be any probability measure and expectation on $\{0,1\}^E$ with the cylinder σ-algebra $\mathcal{F}$.

Show that if X is measurable with respect to the σ-algebra $\mathcal{F}_S$, then

$$\mathbb{E}[X] = \sum_{\alpha \in \{0,1\}^S} \mathbb{P}[\omega\big|_S = \alpha] X'(\alpha).$$

◇◇◇

Exercise 2.8 In this exercise we use the notation from the proof of Russo's formula. Show that in the proof of Russo's formula,

$$\lim_{\varepsilon \to 0} \mathbb{E}[\tfrac{\partial}{\partial s_j} X(\Lambda_j)] = \mathbb{E}_p[\tfrac{\partial}{\partial s_j} X].$$

(Hint: what is the limit of $\mathbb{P}[\Lambda_j\big|_S = \alpha]$ for $\alpha \in \{0,1\}^S$?) ◇◇◇

Exercise 2.9 Show that

$$\{e \text{ is pivotal for } A\} = \{\omega \in \{0,1\}^E \ : \ e \text{ is pivotal in } \omega \text{ for } A\} = A^e \triangle A_e,$$

where

$$A^e = \{\omega^e \ : \ \omega \in A\} \qquad A_e = \{\omega_e \ : \ \omega \in A\}.$$

Show that if A is increasing then

$$\tfrac{\partial}{\partial e} \mathbf{1}_A = \mathbf{1}_{\{e \text{ is pivotal for } A\}} = \mathbf{1}_{A^e \setminus A_e}.$$

◇◇◇

Exercise 2.10 Show that the event $\{e \text{ is pivotal for } A\}$ is independent of the state of the edge e. (That is, show that $\{e \text{ is pivotal for } A\} \in \mathcal{F}_{E\setminus\{e\}}$.) Specifically, if A is an increasing event, show that

$$\begin{aligned}\mathbb{P}_p[e \text{ is pivotal for } A] &= \tfrac{1}{p}\, \mathbb{P}_p[e \text{ is pivotal for } A \ , \ A] \\ &= \tfrac{1}{1-p}\, \mathbb{P}_p[e \text{ is pivotal for } A \ , \ A^c].\end{aligned}$$

◇◇◇

Robert Burton

Michael Keane

2.4 the Burton Keane theorem

For percolation on G let N be the random variable which counts the number of infinite components. Using elementary arguments it is not difficult to prove the $0 - 1 - \infty$ law:

> **Theorem 2.4.1** If G is a transitive infinite connected graph, then for any $p \in [0,1]$ there exists $k_p \in \{0, 1, \infty\}$ such that $\mathbb{P}_p[N = k_p] = 1$.

A graph $G = (V, E)$ is called **amenable** if

$$\Phi(G) := \inf_{\substack{S \subset V \\ 0 < |S| < \infty}} \frac{|\partial S|}{|S|} = 0.$$

That is, we may find large finite sets with small boundary. This is a very deep an important geometric concept, going back to von-Neumann, but for another course...

Burton and Keane gave a remarkably simple argument showing that in transitive amenable graphs there cannot be infinitely many infinite components.

> **Theorem 2.4.2** **(Burton & Keane)** If G is a transitive amenable graph, for any $p \in [0,1]$ there exists $k_p \in \{0, 1\}$ such that $\mathbb{P}_p[N = k_p] = 1$ (where N is the number of infinite components).

The idea of the proof is to use the notion of **trifurcation points**. Let $\Psi_r(x)$ be the following event:

- $B_r(x)$ intersects some infinite component; *i.e.* there exists $y \in B_r(x)$ such that $y \leftrightarrow \infty$.
- If we close all edges in $B_r(x)$, the boundary $\partial B_r(x)$ is connected to infinity via at least 3 *disjoint* infinite components.

If $\omega \in \Psi_r(x)$ then x is said to be an r**-trifurcation point** of ω.

A cute combinatorial argument shows that one cannot have too many trifurcation points in a set.

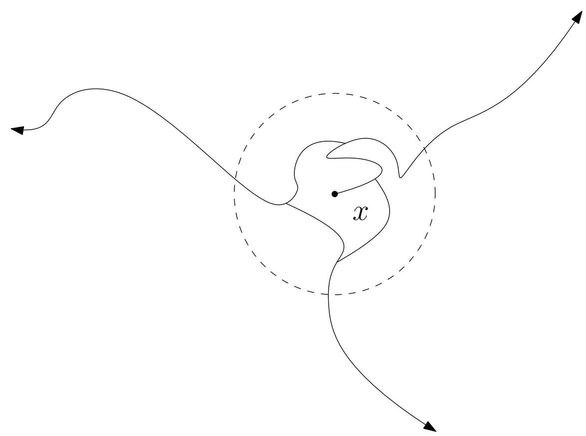


Figure 2.1: The event $\Psi_r(x)$.

Lemma 2.4.3 Let G be an infinite connected bounded degree graph, and $S \subset V$ a finite subset. For any configuration ω, the number of r-trifurcation points in S is at most $D^r|\partial S|$ where D is the maximal degree in G.

Remark: this lemma bounds the number of trifurcation points deterministically.

The relation between trifurcation points and the number of infinite components is:

Lemma 2.4.4 Let G be a transitive infinite connected graph. Let N be the number of infinite components in percolation on G. Suppose that $p \in (0,1)$ is such that $\mathbb{P}_p[N = \infty] = 1$. Then, there exists $r > 0$ such that $\mathbb{P}_p[\Psi_r(x)] > 0$ (for any x).

Together these two lemmas provide us with a proof of the Burton Keane Theorem. Indeed, if $\mathbb{P}_p[N = \infty] = 1$ then choose $r > 0$ such that $\mathbb{P}_p[\Psi_r(x)] > 0$ for any x. For a finite set S let τ_S be the number of r-trifurcation points in S. Then by transitivity, if the degree in G is D,

$$D^r|\partial S| \geq \mathbb{E}_p[\tau_S] = \sum_{x \in S} \mathbb{P}_p[\Psi_r(x)] = |S| \cdot \mathbb{P}_p[\Psi_r(o)].$$

By taking a Folner sequence $(S_n)_n$ for which $\frac{|\partial S_n|}{|S_n|} \to 0$, we obtain that $\Phi_r(o) = 0$, a contradiction.

2.5 planarity

$\mathbb{Z}^2$ has a very special structure. First of all, it is a planar graph. A graph G is **planar** if it can be embedded in the plane so that different edges do not intersect. Planar graph are special since they have a dual graph. The dual graph $\hat{G}$ of a planar graph G is the graph $\hat{G} = (\hat{V}, \hat{E})$, whose vertices $\hat{V}$ are the *faces* of the planar graph G, and two faces form an edge in $\hat{E}$ if they share an edge of G. If two faces $f, f' \in \hat{V}$ share an edge $e \in E$, we denote the edge $\hat{e} = \{f, f'\}$ and call it the dual edge of e.

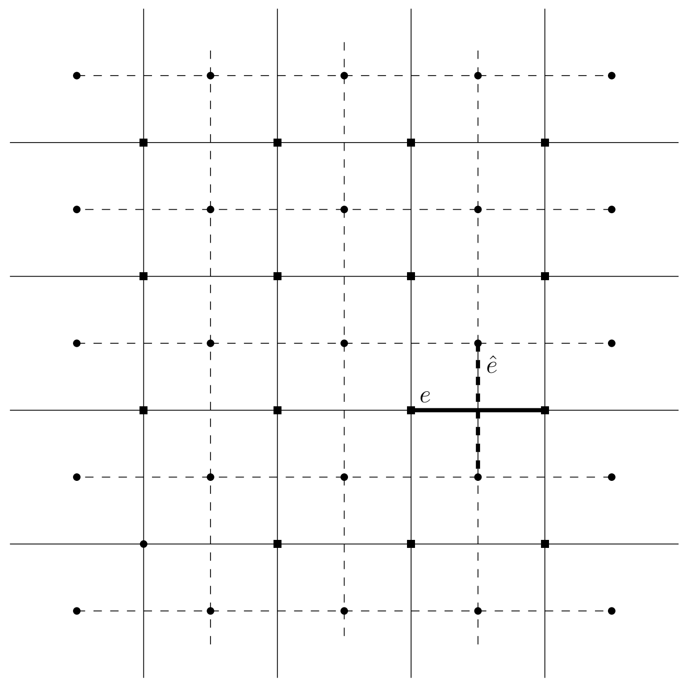


Figure 2.2: $\mathbb{Z}^2$ and its dual graph.

$\mathbb{Z}^2$ is even more special, since it happens to be its own dual. The dual of a rectangle in $\mathbb{Z}$ is again a rectangle in $\hat{\mathbb{Z}}^2$. If the primal rectangle has dimensions $n \times n+1$ then the dual rectangle is a $n+1 \times n$ rectangle in $\hat{\mathbb{Z}}^2$.

Now, if G is a planar graph with dual $\hat{G}$, then we may couple the bond percolation configurations on G and $\hat{G}$ as follows: For $\omega \in \{0,1\}^E$, let $\hat{\omega}(\hat{e}) = 1 - \omega(e)$. That is, the dual edge $\hat{e}$ is open if and only if the primal edge e is closed.

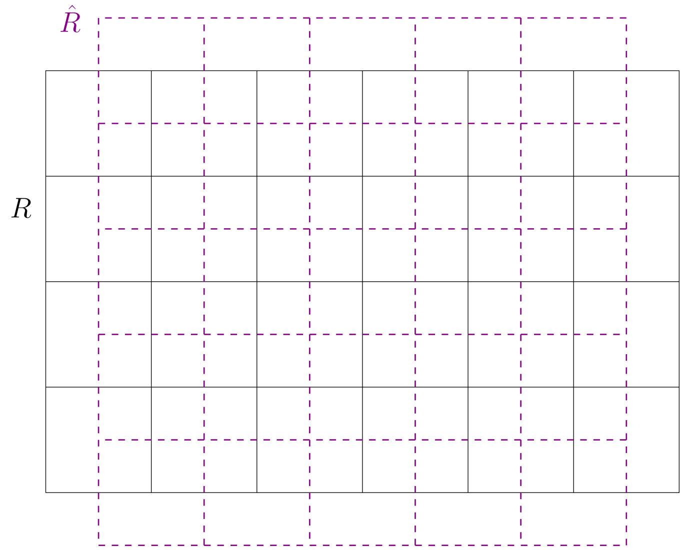


Figure 2.3: A rectangle R and its (vertical) dual $\hat{R}$

For a rectangle R in $\mathbb{Z}^2$ let $\partial_W R, \partial_E R, \partial_N R, \partial_S R$ be the west, east, north, south borders of the rectangle.

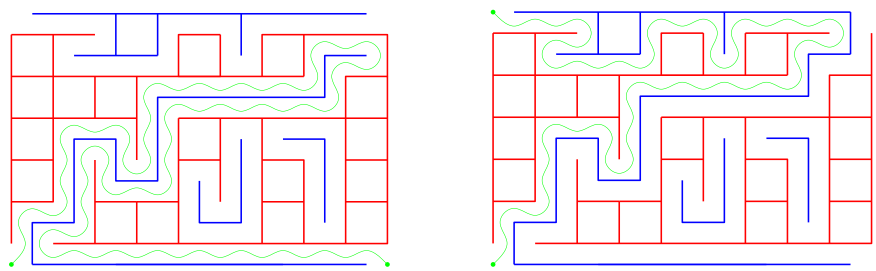
One should convince themself of the following fact.

Lemma 2.5.1 Let R be a rectangle in $\mathbb{Z}^2$. Consider the event that R is crossed by an open path from left to right. Let $\hat{R}$ be the dual rectangle in $\hat{\mathbb{Z}}^2$ and consider the event that $\hat{R}$ is crossed from top to bottom in the dual configuration. Then, these events are mutually disjoint and complementary.

More precisely, for any $\omega \in \{0,1\}^E$ we have that either $\omega \in \{\partial_W R \overset{R}{\leftrightarrow} \partial_E R\}$ or $\hat{\omega} \in \{\partial_N \hat{R} \overset{\hat{R}}{\leftrightarrow} \partial_S \hat{R}\}$, but never both.

Proof sketch. If $\omega \in \{0,1\}^E$, then we may "explore" the interface between the primal and dual components emanating from the south-west corner, be always keeping the primal component (red) to our left and the dual component (blue) to our right, with the convention that $\partial_W R, \partial_E R$ are open in the primal, and $\partial_N \hat{R}, \partial_S \hat{R}$ are open in the dual.

One may convince themself that if $\omega \in \{\partial_W R \overset{R}{\leftrightarrow} \partial_E R\}$, then this exploration must exit through the SE corner, and if $\hat\omega \in \{\partial_N \hat R \overset{\hat R}{\leftrightarrow} \partial_S \hat R\}$ then the exploration must exit from the NW corner. (Exiting from the NE corner is impossible because of our choice of boundary conditions.)



This shows that these two options are complementary events, and at least one of them must hold. □

Now, consider a $n \times n+1$ rectangle R in $\mathbb{Z}^2$. $\hat R$, the dual of R, is a $n+1 \times n$ rectangle in $\hat{\mathbb{Z}}^2$. Thus, a left-right crossing of R is the same as a top-bottom crossing of $\hat R$. That is,

$$\mathbb{P}_p[\partial_W R \overset{R}{\leftrightarrow} \partial_E R] = \mathbb{P}_{1-p}[\partial_N \hat R \overset{\hat R}{\leftrightarrow} \partial_S \hat R].$$

However, by Lemma 2.5.1,

$$\mathbb{P}_p[\partial_W R \overset{R}{\leftrightarrow} \partial_E R] + \mathbb{P}_p[\partial_N \hat R \overset{\hat R}{\leftrightarrow} \partial_S \hat R] = 1.$$

This implies that

$$\mathbb{P}_{1/2}[\partial_W R \overset{R}{\leftrightarrow} \partial_E R] = \mathbb{P}_{1/2}[\partial_N \hat R \overset{\hat R}{\leftrightarrow} \partial_S \hat R] = \tfrac{1}{2}.$$

This probability is the same regardless of the size of the rectangle R.

We thus have basically demonstrated:

Theorem 2.5.2 $p_c^{\text{bond}}(\mathbb{Z}^2) \le \frac{1}{2}$.

Proof. If $\frac{1}{2} < p_c$ then using the exponential decay, there exists $c > 0$ such that for any $r > 0$ and any $x \in \mathbb{Z}^2$, we have $\mathbb{P}_{1/2}[x \leftrightarrow \partial B_r(x)] \le e^{-2cr}$.

Now, let R be a $n \times n+1$ rectangle. Note that the event $\{\partial_W R \overset{R}{\leftrightarrow} \partial_E R\}$ implies that there exists $x \in \partial_W R$ such that $\{x \leftrightarrow \partial B_r(x)\}$ where $r = n/2$. Since $|\partial_W R| = n$, we have that

$$\tfrac{1}{2} = \mathbb{P}_{1/2}[\partial_W R \overset{R}{\leftrightarrow} \partial_E R] \leq \sum_{x \in \partial_W R} \mathbb{P}_{1/2}[x \leftrightarrow \partial B_r(x)] \leq n \cdot e^{-cn} \to 0,$$

contradiction! □

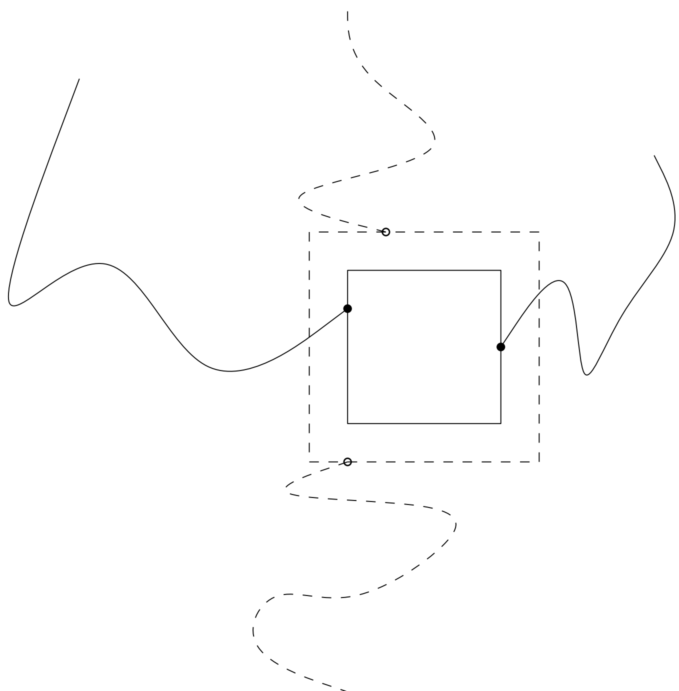
2.6 Zhang's argument

We complement Theorem 2.5.2 with an argument of Zhang proving:

Theorem 2.6.1 For bond percolation on $\mathbb{Z}^2$ we have $\theta(1/2) = 0$.

Proof. Assume for a contradiction that $\theta(1/2) > 0$. That is with some positive probability there exists an infinite component in $\mathbb{Z}^2$.

Let R be an $n \times n$ square in $\mathbb{Z}^2$. Let $\hat{R}$ be the $n+1 \times n+1$ square dual to R in $\hat{\mathbb{Z}}^2$. Consider the events $W = \{\partial_W R \overset{R^c}{\leftrightarrow} \infty\}, E = \{\partial_E R \overset{R^c}{\leftrightarrow} \infty\}$ that the west and east boundaries of R are connected to infinity only using edges *not* in R. Similarly, consider the events N (resp. S) to be the events that there is an open dual path outside $\hat{R}$ connecting $\partial_N \hat{R}$ (resp. $\partial_S \hat{R}$) to infinity in the dual graph $\hat{\mathbb{Z}}^2$.



Note that as $n \to \infty$, since $\theta(1/2) > 0$ we have $\mathbb{P}_{1/2}[A] \to 1$ for $A \in \{W, E, N, S\}$. (Note that this is where $1/2$ is special, since we are using $\theta(1/2) > 0$ for both the primal configuration, in which edges are open with probability p, and the dual configuration, in which edges are open with probability $1 - p$. $1/2$ is special because it is the only $p \in (0, 1)$ for which $p = 1 - p$.) Thus, we may choose n large enough so that

$$\mathbb{P}_{1/2}[E \cap W \cap N \cap S] \geq \tfrac{1}{2}.$$

Now, the event $E \cap W \cap N \cap S$ depends only on the edges not contained in R. Thus, it is independent of $\mathcal{F}_{E(R)}$. Let $\mathcal{E}$ be the event that all edges contained in R are closed. Note that $\mathcal{E}$ and $E \cap W \cap N \cap S$ are independent.

Because of planarity, we get that $\mathcal{E} \cap E \cap W \cap N \cap S$ implies that $\partial_E R, \partial_W R$ are connected to infinity by open paths, but these paths cannot be in the same component. Indeed, any path connecting these two paths must cross dual edges which are either in $\hat{R}$ or in one of the paths connecting the north or south boundaries of $\hat{R}$ to infinity.

Thus,

$$\begin{aligned}\tfrac{1}{2}\cdot(1-\tfrac{1}{2})^{|E(R)|} &\leq \tfrac{1}{2}\cdot\mathbb{P}_{1/2}[\mathcal{E}] = \mathbb{P}_{1/2}[\mathcal{E}\cap E\cap W\cap N\cap S] \\ &\leq \mathbb{P}_{1/2}[\text{there are at least 2 infinite open components}]\end{aligned}$$

But this contradicts the Burton & Keane Theorem. □

2.7 conclusion

Together, theorems 2.5.2 and 2.6.1 prove

> **Harris Kesten Theorem, Theorem 1.4.4.**
> For bond percolation on $\mathbb{Z}^2$, we have $p_c(\mathbb{Z}^2) = \frac{1}{2}$ and $\theta(1/2) = 0$.

Theorem 2.6.1 was initially proved by Harris, where he introduced his correlation inequality, usually known as the FKG inequality. Kesten proved in the 1980's theorem 2.5.2 using a completely different approach, and not the exponential decay. As the theory developed the exponential decay of the sub-critical regime was proved by Aizenman & Barsky, and Menshikov.

Planarity plays a very important role in all proofs of the Harris-Kesten theorem, as well as in other deep results pertaining to percolation theory in the plane, which are not in the scope of this mini-course.

Even simple extensions to non-planar situations are much more difficult if not completely open at this stage.

2.8 exercises III

Exercise 2.11 Consider percolation on an infinite connected transitive graph G.

An event $\mathcal{E} \in \mathcal{F}$ is called translation invariant if for any automorphism φ of the graph G $\varphi\mathcal{E} \subset \mathcal{E}$. (Here, φ acts on $\omega \in \{0,1\}^E$ by $\varphi\omega(e) = \omega(\varphi^{-1}(e))$, and thus $\varphi\mathcal{E} = \{\varphi\omega \ : \ \omega \in \mathcal{E}\}$.)

Show that any translation invariant event $\mathcal{E}$ admits a $0-1$ law: *i.e.* $\mathbb{P}_p[\mathcal{E}] \in \{0,1\}$ for all p. ◇◇◇

Exercise 2.12 Show that if N is the random variable counting the number of infinite components in percolation on a graph G, and φ is an automorphism of G, then $N \circ \varphi^{-1} = N$.

Show that this implies that if G is transitive then N is a.s. constant. ◇◇◇

Exercise 2.13 Give an example of a (non-transitive) infinite connected bounded degree graph G such that the number of infinite components in percolation on G is not a.s. a constant. ◇◇◇

Exercise 2.14 Show that $\mathbb{Z}^d$ is an amenable graph.

Show that regular trees of degree at least 3 are not amenable. ◇◇◇

Exercise 2.15 Prove Lemma 2.4.4. ◇◇◇

Exercise 2.16 Prove Lemma 2.5.1. ◇◇◇

Exercise 2.17 Consider percolation on a transitive graph G. Let $(A_n)_n$ be an increasing sequence of subsets $A_n \subset A_{n+1}$ such that $\bigcup_n A_n = G$. Show that if $\theta(p) > 0$, then for any ε there exists n such that $\mathbb{P}_p[A_n \leftrightarrow \infty] > 1 - \varepsilon$. ◇◇◇

Exercise 2.18 For this exercise we consider site percolation on the faces of the hexagonal lattice. That is, the vertices are the hexagons of the tiling of $\mathbb{R}^2$ by regular hexagons of the same size. Two hexagons are adjacent if they share an edge. See Figure 2.4

Let R be a rhombus of side length n in the hexagonal lattice. See Figure 2.8.

Let $\{\partial_{NW} R \overset{R}{\leftrightarrow} \partial_{SE} R\}$ be the event that the north-west boundary of R is connected by an open path to the south-east boundary. Similarly let $\{\partial_{NE} R \overset{\hat{R}}{\leftrightarrow} \partial_{SW} R\}$ denote the event that the north-west boundary is connected by a *closed* path to the south-west boundary.

Show that $\{\partial_{NW} R \overset{R}{\leftrightarrow} \partial_{SE} R\} = \{\partial_{NE} R \overset{\hat{R}}{\leftrightarrow} \partial_{SW} R\}^c$.

Conclude that $\mathbb{P}_p[\partial_{NW} R \overset{R}{\leftrightarrow} \partial_{SE} R] + \mathbb{P}_{1-p}[\partial_{NE} R \overset{R}{\leftrightarrow} \partial_{SW} R] = 1$.

Use this to prove that

$$\mathbb{P}_{1/2}[\partial_{NW} R \overset{R}{\leftrightarrow} \partial_{SE} R] = \mathbb{P}_{1/2}[\{\partial_{NE} R \overset{R}{\leftrightarrow} \partial_{SW} R] = \tfrac{1}{2}.$$

◇ ◇ ◇

Exercise 2.19 Show that for site percolation on the hexagonal lattice we have $p_c = \frac{1}{2}$. ◇ ◇ ◇

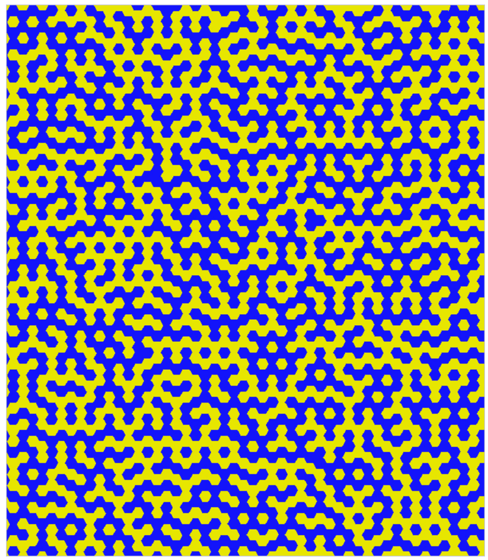


Figure 2.4: Percolation on the hexagonal lattice.

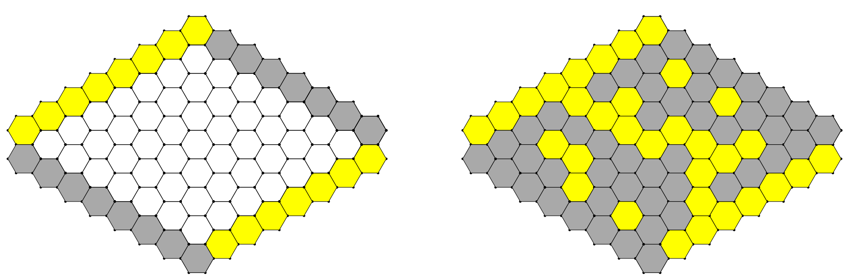


Figure 2.5: A rhombus of side length 8, with percolation and boundary conditions.

Chapter 3

future

We now review some of the fundamental open problems in percolation theory. This is naturally a subjective list.

3.1 no percolation at criticality

Conjecture 3.1.1 Let G be a transitive infinite connected graph. If $p_c < 1$ then $\theta(p_c) = 0$.

The assumption of transitivity (or at least quasi-transitivity) is necessary, there are examples of trees for which critical percolation survives. This conjecture has been verified for many cases. Non amenable Cayley graphs were done by Benjamini, Lyons, Peres & Schramm. $\mathbb{Z}^d$ for large enough d by Hara & Slade using a method known as *lace-expansion*. For Cayley graphs of exponential growth this has been recently proven by Hutchcroft, and also for Cayley graphs of large growth by Hutchcroft and Hermon.

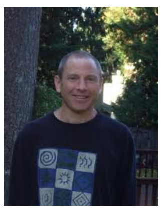
However, the critical percolation on $\mathbb{Z}^d$ for $3 \leq d \leq 6$ is (perhaps surprisingly) still not understood. Proving that $\theta(p_c) = 0$ in these cases has baffled some of the great minds in mathematics of the past decades.

Even slight extensions to other 2-dimensional cases are quite difficult to analyze. For example, it has only recently been shown that $\theta(p_c) = 0$ for $\mathbb{Z}^2 \times \{0, 1, \ldots, n\}$ (by Duminil-Copin, Sidoravicious, Tassion).

3.2 non-trivial phase transition

In their highly influential paper "*percolation beyond* $\mathbb{Z}^d$" from 1996, Benjamini & Schramm conjectured that basically any graph with $p_c = 1$ should be one dimensional in a certain sense. We have already seen that the converse always holds (where in this direction one-dimensional means having an infinite sequence of bounded size cutsets).

Itai Benjamini



Oded Schramm

It has only very recently been shown that this conjecture holds for transitive graphs.

Theorem 3.2.1 If G is an infinite transitive graph with $p_c = 1$ then G is quasi-isometric to $\mathbb{Z}$.

The proof of the above theorem actually shows that if G has *isoperimetric dimesion* at least 4 the $p_c(G) < 1$. The lower dimensional cases are then classified using the known theory of polynomial growth groups, which in this case reduce the problem to understanding $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3$.

3.3 uniqueness

Recall that by the $0, 1, \infty$ law, in any transitive graph G, for any p, the number of infinite components is a.s. in $\{0, 1, \infty\}$. It makes sense to define:

$$p_u(G) = \inf\{p \in [0,1] \ : \ \text{there is a unique infinite component } \}.$$

By definition $p_u \geq p_c$. Also, if $p_c < p_u$ then for any $p \in (p_c, p_u)$ there are infinitely many infinite components a.s. The content of the Burton & Keane Theorem is exactly $p_c = p_u$ for transitive amenable graphs.

It is not very difficult to show that bond percolation on a d-regula tree admits that $p_c = \frac{1}{d-1}$, that $\theta(p_c) = 0$, and that $p_u = 1$. That is, there are infinitely many infinite components for any $p > p_c$, unless $p = 1$.

One may ask what happens for $p > p_u$? Can there be cases where there are infinitely many components for some $p > p_u$ and a unique infinite component for some $1 > q > p > p_u$? This was answered in the negative (by Hagstromm & Peres and Schonmann) where they showed a monotonicity property for $p > p_u$.

The next question that naturally arises is which graphs admit $p_c < p_u$? This is still open even in the transitive case.

Conjecture 3.3.1 Let G be an infinite transitive graph. Then $p_c < p_u$ if and only if G is non-amenable.

We have already seen that the Burton & Keane Theorem shows that $p_c < p_u$ implies non-amenable. The other direction is basically still open. Perhaps the only known result is by Pak & Smirnova which states that for any non-amenable finitely generated group G, there exists a Cayley graph for which $p_c < p_u$.

3.4 conformal invariance

The subject of conformal invariance of different models of statistical mechanics in the plane is deep and spans an immense amount of literature. In the percolation context perhaps the most basic idea is Cardy's formula.

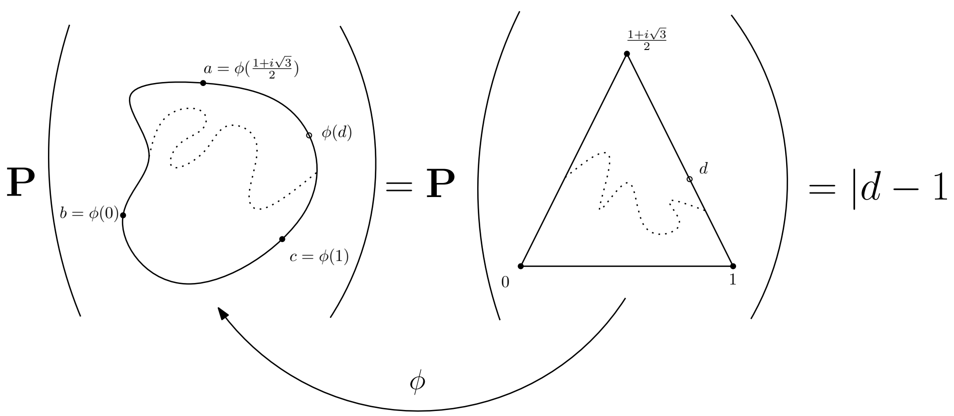


Figure 3.1: Cardy's formula.

If we fix a planar graph G (*i.e.* a graph embedded in the complex plane $\mathbb{C}$ with edges non-crossing), then we may consider δG, scaling that graph by a factor of $\delta > 0$ (where we think of $\delta \to 0$). Fix some pre-compact simply connected domain D in the plane. Let a, b, c, d be four points on the boundary of D, in (say) counter-clockwise order.

For a percolation configuration on δG, we may consider the event that the segment (a, b) on the boundary of D is connected by an open path that stays inside D to the segment (c, d) on the boundary of D. Denote this by $(a, b) \overset{D,\delta}{\leftrightarrow} (c, d)$. We then take the limit as $\delta \to 0$ to get

$$P(D; a, b, c, d) = \lim_{\delta \to 0} \mathbb{P}_{p_c}[(a, b) \overset{D,\delta}{\leftrightarrow} (c, d)].$$

Cardy's formula states that this limit exists and is invariant under conformal transformations. Thus (by the Reimann mapping theorem) this number is given by $P(T; a, b, c, d)$ where T is an equilateral triangle of side length 1, a, b, c are its vertices, and d is some point on the line between c and a. Cardy predicted that this number should be $|d - c|$.

Cardy's formula was proved by Smirnov on the hexagonal lattice (for which he received a Fields medal). The theory is deeply related to Schramm Leowner Evolution, which is one of the most fascinating theories of late, and again a topic for a different course. Although there are good heuristic reasons to believe that Cardy's formula should hold for other planar lattices as well, this is still open even for $\mathbb{Z}^2$.

Theorem 3.4.1 **(Smirnov)** Consider site percolation on the hexagonal lattice. Then, the limit $P(D; a, b, c, d)$ defined above exists. Also, if $\phi : D \to \phi(D)$ is a conformal map then $P(\phi(D); \phi(a), \phi(b), \phi(c), \phi(d)) = P(D; a, b, c, d)$.

Moreover, if T is the triangle whose vertices are $a = \frac{1}{2} + i\frac{\sqrt{3}}{2}, b = 0, c = 1$ and d is on the segment between c and a, then $P(T; a, b, c, d) = |d - 1|$.

3.5 the value of p_c

This problem may be less interesting from the perspective of statistical mechanics, although many newcomers to the field are surprised at the amount of theorems that can be proven regarding critical percolation without knowing the precise value of p_c.

We have seen that $p_c^{\text{bond}}(\mathbb{Z}^2) = \frac{1}{2}$. Also, p_c^{site} of the hexagonal lattice is $\frac{1}{2}$. Both these proofs use planarity and duality.

Numerically, there are good algorithms to approximate p_c as close as we wish. However, precise computation of p_c for different graphs are rare. It is known that $p_c(\mathbb{Z}^d) \approx \frac{1}{2d-1}$ for large d, but this is not so easy to prove, and is related to *lace expansion* methods. Also, for the d-regular tree $p_c = \frac{1}{d-1}$, and this is not too difficult to prove.

A curious question is to compute p_c for certain "simple" graphs, *e.g.* $p_c^{\text{site}}(\mathbb{Z}^2)$, $p_c(\mathbb{Z}^3)$. Of course it may be that these numbers do not have any aesthetic way of representing them.

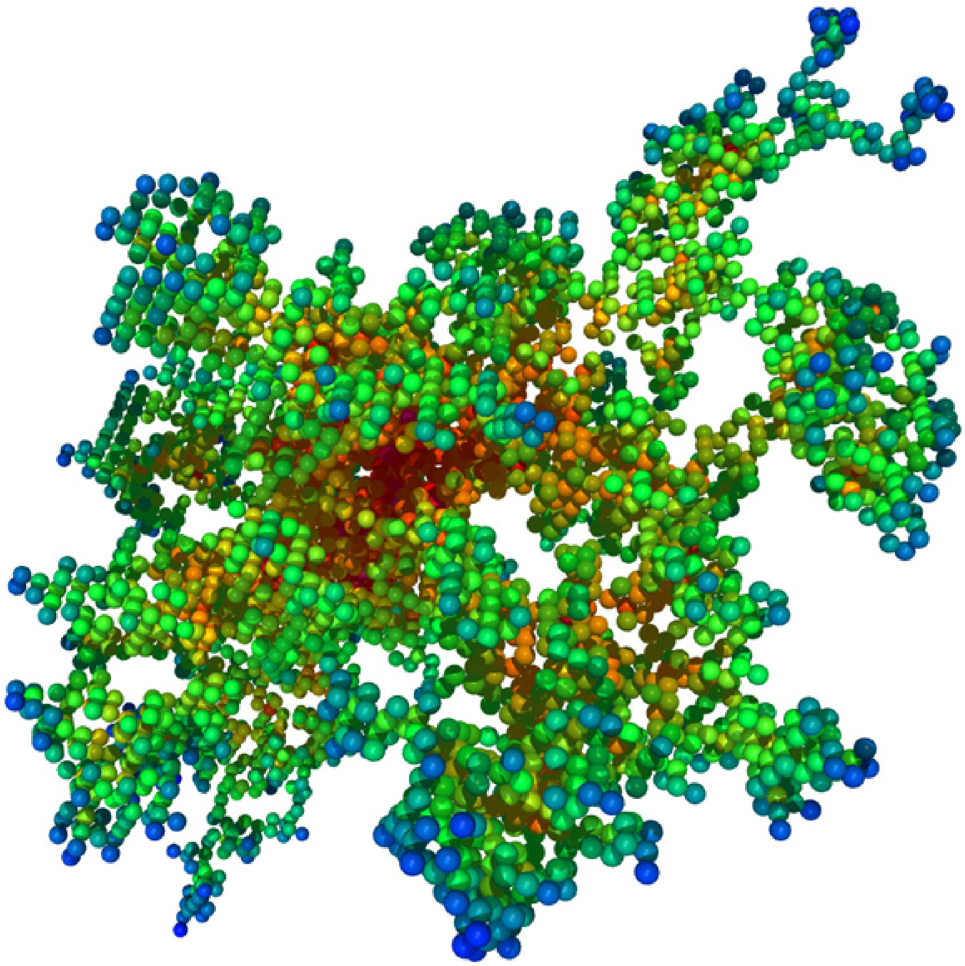


Figure 3.2: Component of the origin, site percolation on $\mathbb{Z}^3$.