Chapter 9

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9.A

(i)

To show that the function $\varphi : R \to R/I$ given by $r \mapsto r + I$ is in general a ring homomorphism we must show that $\varphi(n+m) = \varphi(n) + \varphi(m)$ and $\varphi(nm) = \varphi(n) \cdot \varphi(m)$, indeed for $n, m \in R$

$$\begin{aligned} \varphi(n+m) &= (n+m) + I && \text{By defintion of } \varphi \\ &= (n+I) + (m+I) && \text{By defintion of addition in } R/I \\ &= \varphi(n) + \varphi(m) && \text{By defintion of } \varphi \end{aligned}$$

and

$$\begin{aligned} \varphi(nm) &= (nm) + I && \text{By defintion of } \varphi \\ &= (n+I) \cdot (m+I) && \text{By defintion of multiplication in } R/I \\ &= \varphi(n) \cdot \varphi(m) && \text{By defintion of } \varphi. \end{aligned}$$

Thus, we have shown that quotient rings are generally ring homomorphisms. It was also shown in Example 9.10 that the ring homomorphism is surjective.

(ii)

To show that the natural inclusion map $\iota : R \to S$ given by $s \mapsto s$ is a ring homomorphism we must show that $\iota(n+m) = \iota(n) + \iota(m)$ for $n, m \in S$. Indeed,

$$\begin{aligned} \iota(n+m) &= (n+m) && \text{By defintion of } \iota \\ &= (n) + (m) && \text{By defintion of subring addition in } R \\ &= \iota(n) + \iota(m) && \text{By defintion of } \iota, \end{aligned}$$

and

$$
\begin{aligned}
\iota(nm) &= (nm) && \text{By defintion of } \iota \\
&= (n) \cdot (m) && \text{By defintion of subring multiplication in } R \\
&= \iota(n) \cdot \iota(m) && \text{By defintion of } \iota.
\end{aligned}
$$

Thus, we have shown that the natural inclusion map of a subring is in general a ring homomorphism. It was also shown in Example 9.11 that this homomorphism is injective.

(iii)

The function $\varphi : \mathbb{Z} \to \mathbb{Z}$ given by $n \mapsto 3n$ is not a ring homomorphism because it violates the property that $\varphi(nm) = \varphi(n)\varphi(m)$ with $n, m \in \mathbb{Z}$. Specifically,

$$
\begin{aligned}
\varphi(2 \cdot 3) &= 3 \cdot (2 \cdot 3) \\
&= 3 \cdot (6) \\
&= 18
\end{aligned}
$$

and with the other definition of φ,

$$
\begin{aligned}
\varphi(2) \cdot \varphi(3) &= 3(2) \cdot 3(3) \\
&= 6 \cdot 9 \\
&= 54.
\end{aligned}
$$

This violates the multiplication property of ring homomorphisms thus, it cannot be a ring homomorphism.

(9.B)

(i)

The function $\varphi : \mathbb{Z} \to 3\mathbb{Z}$ given by $n \mapsto 3n$ is not a homomorphism because the condition that $\varphi(nm) = \varphi(n)\varphi(m)$ does not hold for the case $n = 2, m = 3$, and indeed

$$
\begin{aligned}
\varphi(2 \cdot 3) &= 3 \cdot (2 \cdot 3) \\
&= 3 \cdot (6) \\
&= 18
\end{aligned}
$$

and with the other definition of φ,

$$
\begin{aligned}
\varphi(2) \cdot \varphi(3) &= 3(2) \cdot 3(3) \\
&= 6 \cdot 9 \\
&= 54.
\end{aligned}
$$

Since the property does not hold it cannot be a ring homomorphism.

(ii)

The function $\varphi : 2\mathbb{Z} \to 3\mathbb{Z}$ given by $2n \mapsto 3n$ is not a homomorphism because the condition that $\varphi(nm) = \varphi(n)\varphi(m)$ does not hold for the case $n = 2, m = 4$, and indeed

$$\begin{aligned}\varphi(2 \cdot 4) &= \varphi(8) \\ &= 3 \cdot 4 \\ &= 12\end{aligned}$$

and with the other definition of φ,

$$\begin{aligned}\varphi(2) \cdot \varphi(4) &= 3(1) \cdot 3(2) \\ &= 3 \cdot 6 \\ &= 18.\end{aligned}$$

Since the property does not hold it cannot be a ring homomorphism.