Graph Optimization
Lagrangian Relaxation

GO - Lagrangian relaxation

Lagrangian relaxation

Consider an ILP problem

$$\begin{cases} z(P) = \max \ c^T x \\ A\,x \leq b \\ B\,x \leq d \\ x \in \mathbb{Z}^n \end{cases} \tag{P}$$

where constraints $A\,x \leq b$ are 'difficult', i.e the problem would be 'easier' to solve without such constraints.
Instead of removing such constraints, their violation is penalized, by adding their combination to the objective function.

Lagrangian relaxation

Definition

The Lagrangian relaxation of problem (P), with respect to constraints $A\,x \leq b$, is the following ILP problem:

$$\begin{cases} \max \; z(RL) = c^T x + \lambda^T (b - A\,x) \\ B\,x \leq d \\ x \in \mathbb{Z}^n \end{cases} \qquad (\mathrm{RL}_\lambda)$$

where parameters $\lambda \geq 0$ are the *Lagrangian multipliers*.

Lagrangian relaxation

The Lagrangian relaxation provides a bound of the optimal solution of the original problem.

Property

For each set of multipliers $\lambda \geq 0$

$$z(P) \leq z(RL_\lambda).$$

Proof

$$z(P) = \max_{\substack{A\,x \leq b \\ B\,x \leq d \\ x \in \mathbb{Z}^n}} c^\mathsf{T} x \leq \max_{\substack{A\,x \leq b \\ B\,x \leq d \\ x \in \mathbb{Z}^n}} c^\mathsf{T} x + \lambda^\mathsf{T}(b - A\,x)$$

$$\leq \max_{\substack{B\,x \leq d \\ x \in \mathbb{Z}^n}} c^\mathsf{T} x + \lambda^\mathsf{T}(b - A\,x) = z(RL_\lambda)$$

Example

Given the knapsack problem:

$$\begin{aligned} \max\ & 10\,x_1 + 12\,x_2 + 3\,x_3 & \\ & 2\,x_1 + 3\,x_2 + x_3 & \leq 4 \\ & x_j \in \{0,1\}, \forall j = 1,\ldots,3 & \end{aligned}$$

the Lagrangian relaxation is:

$$\begin{aligned} \max\ & 10\,x_1 + 12\,x_2 + 3\,x_3 + \lambda\,(4 - 2\,x_1 - 3\,x_2 - x_3) \\ & x_j \in \{0,1\}, \forall j = 1,\ldots,3 \end{aligned}$$

Example

The Lagrangian relaxation can be easily solved for any $\lambda \geq 0$.

$$\begin{aligned} &\max\ 10\,x_1 + 12\,x_2 + 3\,x_3 + \lambda\,(4 - 2\,x_1 - 3\,x_2 - x_3) \\ &= (10 - 2\,\lambda)\,x_1 + (12 - 3\,\lambda)\,x_2 + (3 - \lambda)\,x_3 + 4\,\lambda = \\ &= \overline{p}_1 x_1 + \overline{p}_2 x_2 + \overline{p}_3 x_3 + 4\,\lambda \\ &x_j \in \{0,1\}, \forall j = 1, \dots, 3 \end{aligned}$$

If $\overline{p}_i > 0$ then $x_i = 1$, otherwise $x_i = 0$.
In case of $\lambda = 4$ the optimal solution of the Lagrangian relaxation is $(1,0,0)$ with objective function $z(RL_4) = 18$.

If the optimal solution of (RL_λ) is feasible for (P), then is it optimal also for (P)?
In general NO, as the objective function of (P) and (RL_λ) are different.
In the above example $(1, 0, 0)$ is optimal for (RL_4) and it is feasible for (P), but it is not optimal for (P).

Property

If $\bar{x}$ is the optimal solution of (RL_λ), $\bar{x}$ is feasible for (P) and $\lambda^\mathsf{T}(b - A\,\bar{x}) = 0$ (complementary slackness condition), then $\bar{x}$ is optimal also for (P).

Proof

If x is feasible for (P), then

$$c^\mathsf{T}x \leq c^\mathsf{T}x + \lambda^\mathsf{T}\underbrace{(b - A\,x)}_{\geq 0} \overset{\text{hypothesis 1}}{\leq} c^\mathsf{T}\bar{x} + \lambda^\mathsf{T}(b - A\,\bar{x}) \overset{\text{hypothesis 3}}{=} c^\mathsf{T}\bar{x}.$$

$\square$

Lagrangian relaxation: example

The Lagrangian relaxation of (P) is:

$$\begin{aligned} \max\ (10-2\,\lambda)\,x_1 + (12-3\,\lambda)\,x_2 + (3-\lambda)\,x_3 + 4\,\lambda \\ x_j \in \{0,1\}, \qquad \forall\, j \in 1,\dots,3 \end{aligned}$$

The optimal solution is given by $\bar{x}$:

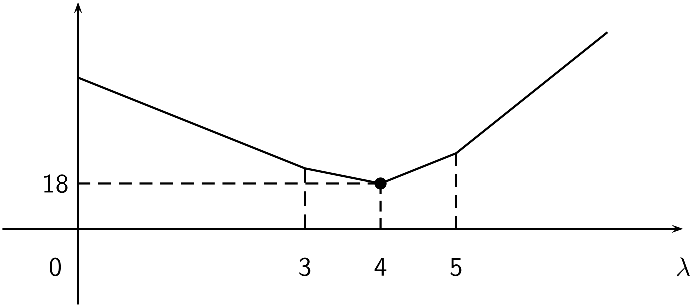
$$\bar{x}_1 = \begin{cases} 1 & \text{se } \lambda < 5 \\ 0 & \text{otherwise} \end{cases} \qquad \bar{x}_2 = \begin{cases} 1 & \text{se } \lambda < 4 \\ 0 & \text{otherwise} \end{cases} \qquad \bar{x}_3 = \begin{cases} 1 & \text{se } \lambda < 3 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the optimal value of (RL_λ) is

$$z(RL_\lambda) = \begin{cases} 4\,\lambda & \text{if } \lambda > 5 \\ 2\,\lambda + 10 & \text{if } \lambda \in [4,5] \\ 22-\lambda & \text{if } \lambda \in [3,4] \\ 25-2\,\lambda & \text{if } \lambda \in [0,3] \end{cases}$$

Lagrangian relaxation: example

Function $z(RL_\lambda)$ is convex and piecewise linear:



The value of $z(RL_\lambda)$ depends on the value of λ. As we want to provide the best approximation of the optimum of (P), we look for the lowest value of $z(RL_\lambda)$. The best bound which can be obtained through Lagrangian relaxation is $\min_\lambda z(RL_\lambda)$. In this case the minimum is 18 and it is obtained for $\bar{\lambda} = 4$.

Lagrangian dual: definition

The problem

$$\begin{cases} \min \ z(RL_\lambda) \\ \lambda \geq 0 \end{cases} \qquad (D)$$

is the *Lagrangian dual* of (P).

We have to solve (D) to obtain the best possible bound.

Property

- $z(D) \geq z(P)$.
- Function $z(RL_\lambda)$ is convex with respect to λ, but, in general, it not differentiable.

Subgradient

To minimize a convex non differentiable function, *subgradient methods* are applied, which are based on a generalization of gradient.

Definition

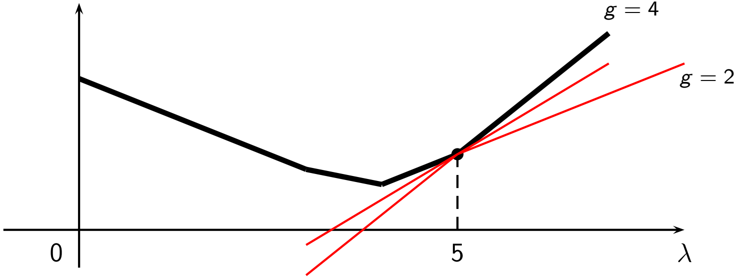
Given a convex function $v : \mathbb{R}^m \to \mathbb{R}$.
A subgradient of v in $\bar{\lambda}$ is a vector $g \in \mathbb{R}^m$ such that

$$v(\lambda) \geq g^{\mathsf{T}}(\lambda - \bar{\lambda}) + v(\bar{\lambda}) \qquad \forall\ \lambda \geq 0,$$

i.e. the plane orthogonal to g in $(\bar{\lambda}, v(\bar{\lambda}))$ "lays below" the graph of v.

Subgradient: example

In the example:



the subgradients of $z(RL_\lambda)$ in $\bar{\lambda} = 5$ are the values in the interval $[2, 4]$.

Subgradient method

To solve the Lagrangian dual

$$\min_{\lambda \geq 0} z(RL_\lambda) \tag{D}$$

Subgradient methods are applied, which update the Lagrangian multipliers according to the subgradient in the current point.

Subgradient procedure

- At each iteration k the subgradient g^k of $z(RL_\lambda)$ in λ^k is computed.
- Lagrangian multipliers are updated:

$$\lambda^{k+1} = \max \left\{ 0, \lambda^k - t_k \frac{g^k}{\|g^k\|} \right\}$$

.

Subgradient method

Property

If steps t_k are such that

$$\lim_{k\to\infty} t_k = 0, \qquad \sum_{k=1}^{\infty} t_k = +\infty,$$

then

$$\lim_{k\to\infty} z(RL_{\lambda^k}) = z(D).$$

Remarks

In general, the sequence $z(RL_{\lambda^1}),\ z(RL_{\lambda^2}),\ \ldots$ is not increasing.

Computing a subgradient

How to compute a subgradient of $z(RL_\lambda)$?

Property

If $\bar{x}$ is the optimal solution of $(RL_{\bar{\lambda}})$, then the vector

$$g = b - A\bar{x}$$

is a subgradient of $z(RL_\lambda)$ in $\bar{\lambda}$.

Computing a subgradient

Proof

$z(RL_\lambda) = \max\limits_{\substack{Bx \le d \\ x \in \mathbb{Z}^n}} c^\mathsf{T} x + \lambda^\mathsf{T}(b - Ax)$, thus $z(RL_{\bar\lambda}) = c^\mathsf{T}\bar{x} + \bar\lambda^\mathsf{T}(b - A\bar{x})$.

Further

$$\begin{aligned} z(RL_\lambda) &\ge c^\mathsf{T}\bar{x} + \lambda^\mathsf{T}(b - A\bar{x}) \\ &= c^\mathsf{T}\bar{x} + \bar\lambda^\mathsf{T}(b - A\bar{x}) + (\lambda - \bar\lambda)^\mathsf{T}(b - A\bar{x}) \\ &= z(RL_{\bar\lambda}) + (\lambda - \bar\lambda)^\mathsf{T}(b - A\bar{x}). \end{aligned}$$

$\square$

Lagrangian dual and continuous relaxation

Given an ILP problem:

$$\begin{cases} \max \ c^{\mathsf{T}} x \\ A\,x \leq b \\ B\,x \leq d \\ x \in \mathbb{Z}^n \end{cases} \qquad \text{(P)}$$

and its Lagrangian relaxation (D) and continuous relaxation (CR):

$$(D) \begin{cases} \min \ v(RL_\lambda) \\ \lambda \geq 0 \end{cases} \qquad\qquad (RC) \begin{cases} \max \ c^{\mathsf{T}} x \\ A\,x \leq b \\ B\,x \leq d \end{cases}$$

Which is the relationship between the optimal solution of (D) and (CR)?

Property

$z(D) \leq z(RC)$

Lagrangian dual and continuous relaxation

Proof

$$
\begin{aligned}
z(D) &= \min_{\lambda \geq 0} \; z(RL_\lambda) \\
&= \min_{\lambda \geq 0} \; \lambda^\mathsf{T} b + \max_{\substack{B\,x \leq d \\ x \in \mathbb{Z}^n}} (c^\mathsf{T} - \lambda^\mathsf{T} A)\, x \\
\text{[continuous relaxation of } (RL_\lambda)\text{]} \quad &\leq \min_{\lambda \geq 0} \lambda^\mathsf{T} b + \max_{B\,x \leq d} (c^\mathsf{T} - \lambda^\mathsf{T} A)\, x \\
\text{[duality]} \quad &= \min_{\lambda \geq 0} \lambda^\mathsf{T} b + \min_{\substack{y^\mathsf{T} B = c^\mathsf{T} - \lambda^\mathsf{T} A \\ y \geq 0}} y^\mathsf{T} d \\
&= \min_{\substack{\lambda^\mathsf{T} A + y^\mathsf{T} B = c^\mathsf{T} \\ \lambda, y \geq 0}} \lambda^\mathsf{T} b + y^\mathsf{T} d \\
\text{[duality]} \quad &= \max_{\substack{A\,x \leq b \\ B\,x \leq d}} c^\mathsf{T} x \\
&= z(RC)
\end{aligned}
$$

Lagrangian dual and continuous relaxation

Property

If (RL_λ) and its continuous relaxation have the same optimal value for any $\lambda \geq 0$, then $z(D) = z(RC)$.

The optimum of the Lagrangian relaxation of the knapsack problem

$$\begin{cases} \max \sum_{i \in I} p_i x_i + \lambda \left(B - \sum_{i \in I} w_i \, x_i\right) \\ x_i \in \{0,1\}, \forall i \in I \end{cases} \qquad (RL_\lambda)$$

is the same as the optimum of its continuous relaxation.
Thus the optimum of the Lagrangian dual is the same as the optimum of the continuous relaxation.

Lagrangian dual and continuous relaxation

If the *integrality property* holds for the Lagrangian relaxation, the bound of the Lagrangian relaxation is equivalent to the bound of the continuous relaxation.

Fixed charge capacitated facility location problem

Given

- Set of *client* nodes $\mathcal{T}$ to be served
  - w_j weight of client j
- A set $\mathcal{C}$ of candidate sites to host *facility*
  - A cost f_i of installing one facility in site i
  - Capacity Γ_i of facility
- $g_{ij}, \forall i \in \mathcal{C}, j \in \mathcal{T}$ assignment cost of j to i

Capacitated fixed charge facility location problem

$$\min \sum_{i\in\mathcal{C}} f_i y_i + \sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{T}} g_{ij}x_{ij}$$

$$\begin{aligned}
\sum_{i\in\mathcal{C}} x_{ij} &\geq 1, && \forall j \in \mathcal{T}\\
\sum_{j\in\mathcal{T}} w_j x_{ij} &\leq \Gamma_i y_i && \forall i \in \mathcal{C}\\
x_{ij} &\in \{0,1\}, && \forall i \in \mathcal{C}, j \in \mathcal{T}\\
y_i &\in \{0,1\}, && \forall i \in \mathcal{C}
\end{aligned}$$

Lagrangian relaxation for location problems

Capacitated fixed charge facility location problem

$$\min \sum_{i \in \mathcal{C}} f_i y_i + \sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{T}} g_{ij} x_{ij}$$

$$\sum_{i \in \mathcal{C}} x_{ij} \geq 1, \qquad \forall j \in \mathcal{T}$$

$$\sum_{j \in \mathcal{T}} w_j x_{ij} \leq \Gamma_i y_i \qquad \forall i \in \mathcal{C} \rightarrow \mu_i \geq 0$$

$$x_{ij} \in \{0, 1\}, \qquad \forall i \in \mathcal{C}, j \in \mathcal{T}$$

$$y_i \in \{0, 1\}, \qquad \forall i \in \mathcal{C}$$

Capacitated fixed charge facility location problem

$$\min \sum_{i \in \mathcal{C}} f_i y_i + \sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{T}} g_{ij} x_{ij} - \sum_{i \in \mathcal{C}} \mu_i \left(\Gamma_i y_i - \sum_{j \in \mathcal{T}} w_j x_{ij} \right)$$

$$\begin{aligned} \sum_{i \in \mathcal{C}} x_{ij} &\geq 1, && \forall j \in \mathcal{T} \\ x_{ij} &\in \{0,1\}, && \forall i \in \mathcal{C}, j \in \mathcal{T} \\ y_i &\in \{0,1\}, && \forall i \in \mathcal{C} \end{aligned}$$

Lagrangian relaxation for location problems

First subproblem

$$\min \sum_{i \in \mathcal{C}} (f_i - \mu_i \Gamma_i) y_i$$

$$y_i \in \{0, 1\}, \qquad \forall i \in \mathcal{C}$$

Solution

- if $(f_i - \mu_i \Gamma_i) \leq 0$ then $y_i = 1$
- if $(f_i - \mu_i \Gamma_i) > 0$ then $y_i = 0$

Lagrangian relaxation for location problems

Second subproblem

$$\min \sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{T}} (g_{ij} + \mu_i w_j) x_{ij}$$

$$\sum_{i \in \mathcal{C}} x_{ij} \geq 1, \qquad \forall j \in \mathcal{T}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i \in \mathcal{C}, j \in \mathcal{T}$$

Solution

For each *client* j the best *facility* k

$$k \in \mathcal{C} : k = \arg\min_{i \in \mathcal{C}} \{(g_{ij} + \mu_i w_j)\}$$

is computed and $x_{kj} = 1$

Second Lagrangian relaxation

$$
\begin{aligned}
\min \sum_{i \in \mathcal{C}} f_i y_i + \sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{T}} g_{ij} x_{ij} & \\
\sum_{i \in \mathcal{C}} x_{ij} \geq 1, & \qquad \forall j \in \mathcal{T} \to \lambda_j \geq 0 \\
\sum_{j \in \mathcal{T}} w_j x_{ij} \leq \Gamma_i y_i & \qquad \forall i \in \mathcal{C} \\
x_{ij} \in \{0, 1\}, & \qquad \forall i \in \mathcal{C}, j \in \mathcal{T} \\
y_i \in \{0, 1\}, & \qquad \forall i \in \mathcal{C}
\end{aligned}
$$

Lagrangian relaxation for location problems

Second Lagrangian relaxation

$$
\begin{aligned}
\min \sum_{i \in \mathcal{C}} f_i y_i + \quad & \textstyle\sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{T}} g_{ij} x_{ij} \\
& \color{red}{- \textstyle\sum_{j \in \mathcal{T}} \lambda_j \left(\sum_{i \in \mathcal{C}} x_{ij} - 1 \right)} \\
\sum_{j \in \mathcal{T}} w_j x_{ij} \leq \Gamma_i y_i \quad & \forall i \in \mathcal{C} \\
x_{ij} \in \{0, 1\}, \quad & \forall i \in \mathcal{C}, j \in \mathcal{T} \\
y_i \in \{0, 1\}, \quad & \forall i \in \mathcal{C}
\end{aligned}
$$

Lagrangian relaxation for location problems

Relaxed problem

$$\min \sum_{i \in \mathcal{C}} f_i y_i + \sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{T}} (g_{ij} - \lambda_j) x_{ij} + \qquad \textstyle\sum_{j \in \mathcal{T}} \lambda_j$$

$$\sum_{j \in \mathcal{T}} w_j x_{ij} \le \Gamma_i y_i \qquad \forall i \in \mathcal{C}$$

$$x_{ij} \in \{0,1\}, \qquad \forall i \in \mathcal{C}, j \in \mathcal{T}$$

$$y_i \in \{0,1\}, \qquad \forall i \in \mathcal{C}$$

Lagrangian relaxation for location problems

Each client may be assigned more than once or it may be not assigned. Open a facility may be profitable if a set of clients are assigned to it such that $g_{ij} - \lambda_j \leq 0$

Solution

- $\forall i \in \mathcal{C}$:

$$z^* = \max \sum_{j \in \mathcal{T}} \left(-(g_{ij} - \lambda_j)\right) u_j$$

$$\sum_{j \in \mathcal{T}} w_j u_j \leq \Gamma_i$$

$$u_j \in \{0, 1\}$$

- if $z^* > f_i$ then $y_i = 1, x_{ij} = u_i$, $\forall j \in \mathcal{T}$
- else $y_i = 0$

Lagrangian relaxation for symmetric TSP

Problem (P):

$$\min \sum_{i} \sum_{j>i} c_{ij}\, x_{ij}$$

$$\sum_{j>i} x_{ij} + \sum_{j<i} x_{ji} = 2 \qquad \forall\ i \neq r$$

$$\sum_{j>r} x_{rj} + \sum_{j<r} x_{jr} = 2$$

$$\sum_{i \in S} \sum_{\substack{j \notin S \\ j>i}} x_{ij} + \sum_{i \notin S} \sum_{\substack{j \in S \\ j>i}} x_{ij} \geq 1 \qquad \forall\ S \subset N, \quad S \neq \emptyset, N$$

$$x_{ij} \in \{0,1\} \quad i < j$$

Lagrangian relaxation for symmetric TSP

Select a node r and relax the constraints related to nodes different from r: a multipler $\lambda_i \in \mathbb{R}$ is assigned to each node $i \neq r$ and $\lambda_r = 0$.

$$\min \sum_{i} \sum_{j>i} c_{ij}\, x_{ij}$$

$$\sum_{j>i} x_{ij} + \sum_{j<i} x_{ji} = 2 \qquad \forall\ i \neq r$$

$$\sum_{j>r} x_{rj} + \sum_{j<r} x_{jr} = 2$$

$$\sum_{i \in S} \sum_{\substack{j \notin S \\ j>i}} x_{ij} + \sum_{i \notin S} \sum_{\substack{j \in S \\ j>i}} x_{ij} \geq 1 \qquad \forall\ S \subset N, \quad S \neq \emptyset, N$$

$$x_{ij} \in \{0,1\} \quad i<j$$

Lagrangian relaxation for symmetric TSP

Problem RL_λ

$$\min \sum_i \sum_{j>i} c_{ij}\, x_{ij} + \sum_i \lambda_i \left(\sum_{j>i} x_{ij} + \sum_{j<i} x_{ji} - 2 \right) =$$

$$\sum_i \sum_{j>i} (c_{ij} + \lambda_i + \lambda_j)\, x_{ij} - 2 \sum_i \lambda_i$$

$$\sum_{j>r} x_{rj} + \sum_{j<r} x_{jr} = 2$$

$$\sum_{i \in S} \sum_{\substack{j \notin S \\ j>i}} x_{ij} + \sum_{i \notin S} \sum_{\substack{j \in S \\ j>i}} x_{ij} \geq 1 \qquad \forall\ S \subset N, \quad S \neq \emptyset, N$$

$$x_{ij} \in \{0,1\} \quad i < j$$

Lagrangian relaxation for symmetric TSP

Definition r-tree

An r–tree is a set of n edges such that

- 2 edges are incident in node r
- $n - 2$ edges provide a spanning tree on the nodes different from r

How to build a minimum cost r–tree?

- 2 minimum cost edges incident in r
- a spanning tree on the nodes different from r (Kruskal)

Lagrangian relaxation for symmetric TSP

Lagrangian relaxation corresponds to find the minimum cost r-tree on a graph, such that the cost of edge $\{i, j\}$ is $c_{ij} + \lambda_i + \lambda_j$, where c_{ij} is the original edge cost.

Remark

- $\lambda_i \in \mathbb{R}$ as the relaxed constraints are equalities
- If the optimal solution of (RL_λ) is feasible for (P), then it is the optimal solution for (P) also.
- The Lagrangian dual is $\max\limits_{\lambda \in \mathbb{R}^n} z(RL_\lambda)$
- Lagrangian multipliers are updated as follows: $\lambda^{k+1} = \lambda^k + t_k \dfrac{g^k}{\|g^k\|}$