Lagrange relaxation

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Lagrange relaxation — overview

Let $\mathcal{F} := \{x \in \mathbb{Z}^n \mid Dx \geq d\}$, where D, d integral. Consider

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Theorem

$Z_{IP} \geq Z_D = \min\{cx \mid Ax \geq b, x \in conv.hull(\mathcal{F})\}$.

We assumed $\mathcal{F} := \{x \in \mathbb{Z}^n \mid Dx \geq d\}$, where D, d have integer entries. Then $\text{conv.hull}(\mathcal{F})$ is a polyhedron, and hence

$$\text{conv.hull}(F) = \{x \mid Ux \geq v\}$$

for some U, v.

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Proof.

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- by LP duality, $Z(\lambda) = \max\{zv + \lambda b \mid zU = c - \lambda A, z \geq 0\}$

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$\square$

Lagrange relaxation as a lower bound

$\mathcal{F} := \{x \in \mathbb{Z}^n \mid Dx \geq d\}$, where D, d have integer entries.

$$Z_{IP} := \min\{cx \mid Ax \geq b, x \in \mathcal{F}\}.$$

$$Z_{LP} := \min\{cx \mid Ax \geq b, Dx \geq d\}$$

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$Z_D = \min\{cx \mid Ax \geq b, x \in conv.hull(\mathcal{F})\}.$

In general $Z_{IP} \geq Z_D \geq Z_{LP}$.

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Corollary

1. $Z_D = Z_{IP}$ *for all* $c \iff$ $conv.hull\{x \in \mathcal{F} \mid Ax \geq b\} = conv.hull(\mathcal{F}) \cap \{x \mid Ax \geq b\}$
2. $Z_D = Z_{LP}$ *for all* $c \iff conv(\mathcal{F}) = \{x \mid Dx \geq d\}$

Efficient lower bounds for the Travelling Salesman Problem

Given a graph $G = (V, E)$ and costs $c \in \mathbb{R}^E$ the TSP problem is:

$$\begin{array}{ll} \min & \sum_{e \in E} c_e x_e \\ & \sum_{e \in \delta(v)} x_e = 2 \qquad \text{for all } v \in V, \\ & \sum_{e \in E(S)} x_e \leq |S| - 1 \quad \text{for all } S \subseteq V, 1 < |S| < V - 1 \\ & x \in \{0, 1\}^E \end{array}$$

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Fix $v_0 \in V$, omit the redundant constraints $\sum_{e \in E(S)} x_e \leq |S| - 1$ where $v_0 \in S$, and consider the Lagrange dual that arises by relaxing

$$\sum_{e \in \delta(v)} x_e = 2, \text{for all } v \in V, v \neq v_0$$

Then computing $Z(\lambda)$ amounts to finding an optimal spanning tree.

Computing the Lagrange dual

Definition

We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *concave* if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

for all x, y and all $\alpha \in [0,1]$.

Recall $Z(\lambda) := \min\{cx + \lambda(b - Ax) \mid x \in \mathcal{F}\}$.

Lemma

$Z(\lambda)$ is a concave function of λ.

So $\max\{Z(\lambda) \mid \lambda \geq 0\}$ is a concave optimization problem, to which we can apply the *subgradient optimization algorithm*.

The subgradient optimization algorithm

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function. Then $s \in \mathbb{R}^n$ is a *subgradient of f at x* if

$$f(x) + s(y - x) \geq f(y)$$

for all $y \in \mathbb{R}^n$.

If f is concave, and $(\Theta_t)_{t=1}^{\infty}$ is s.t. $\sum_t \Theta_t = \infty$ and $\lim_{t\to\infty} \Theta_t = 0$, then the following algorithm will converge to an optimal solution x of

$$\max\{f(x) \mid x \geq 0\}$$

The subgradient optimization algorithm

Initialize $x \geq 0$, $t = 1$. Repeat:

- Compute $f(x)$ and a subgradient s of f at x
- Put $x \leftarrow \max\{0, x + \theta_t s\}$, and $t \leftarrow t + 1$

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function. Then $s \in \mathbb{R}^n$ is a *subgradient of* f *at* λ if

$$f(\lambda) + s(\lambda' - \lambda) \geq f(\lambda')$$

for all $\lambda' \in \mathbb{R}^n$.

If $Z(\lambda) = \min\{cx + \lambda(b - Ax) \mid x \in \mathcal{F}\} = cx^* + \lambda(b - Ax^*)$ for some $x^* \in \mathcal{F}$, then

$$Z(\lambda) + (\lambda' - \lambda)(b - Ax^*) = cx^* + \lambda'(b - Ax^*) \geq Z(\lambda')$$

for all λ', so that then, $b - Ax^*$ is a subgradient of Z at λ.

Hence, we may apply the subgradient optimization algorithm to find

$$Z_D = \max\{Z(\lambda) \mid \lambda \geq 0\}$$

Homework

- read 4.3 and 4.4 from the book.
- make exercise 4.6, 4.7 and 4.9.
- install Sage (www.sagemath.org) on your laptop before next friday.