Integer Programming
IE418

Lecture 11

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Reading for This Lecture

- Wolsey Chapter 2
- Nemhauser and Wolsey Sections II.3.1, II.3.6, II.4.1, II.4.2, II.5.4

Relaxation

For simplicity, we now consider a pure integer program IP defined by

$$
\begin{aligned}
z_{IP} &= \max\{cx \mid x \in S\}, \\
S &= \{x \in \mathbb{Z}^n_+ \mid Ax \leq b\}.
\end{aligned}
$$

Definition 1. *A relaxation of IP is a maximization problem defined as*

$$z_R = \max\{z_R(x) \mid x \in S_R\}$$

with the following two properties:

$$
\begin{aligned}
S &\subseteq S_R && (1) \\
cx &\leq z_R(x), \ \forall x \in S. && (2)
\end{aligned}
$$

Importance of Relaxations

- The main purpose of a relaxation is to obtain an upper bound on z_{IP}.
- Relaxation is used as a method of bounding in branch and bound.
- The idea is to choose a relaxation that is much easier to solve than the original problem.
- Note that the relaxation must be solved to optimality to yield a valid bound.
- We will consider three basic types of relaxations.
  - LP relaxation
  - Combinatorial relaxation
  - Lagrangian relaxation
- Relaxations are also used in some other bounding schemes we will look at.

Aside: How Do You Spell “Lagrangian?”

- Some spell it “Lagrangean.”
- Some spell it “Lagrangian.”
- We ask Google.
- In 2002:
  - “Lagrangean” returned 5620 hits.
  - “Lagrangian” returned 143000 hits.
- In 2007:
  - “Lagrangean” returns 208000 hits.
  - “Lagrangian” returns 5820000 hits.
- “Lagrangian” still wins!

Obtaining and Using Relaxations

- Properties of relaxations
  - If a relaxation of IP is infeasible, then so is IP.
  - If the optimal solution to the relaxation is feasible for IP, then it is optimal for IP.
- The easiest way to obtain relaxations of IP is to drop some of the constraints defining the feasible set S.
- We have two choices
  - LP relaxation: Drop the integrality constraints to obtain an LP.
  - Combinatorial relaxation: Drop a set of inequality constraints that make the resulting IP "easy."
- It is "obvious" how to obtain an LP relaxation, but combinatorial relaxations are not as obvious.

Example: Traveling Salesman Problem

The TSP is a combinatorial problem $(E, \mathcal{F})$ whose ground set is the edge set of a graph $G = (V, E)$.

- V is the set of customers.
- E is the set of travel links between the customers.

A feasible solution is a subset of E consisting of edges of the form $\{i, \sigma(i)\}$ for $i \in V$, where σ is a simple permutation V specifying the order in which the customers are visited.

IP Formulation:

$$\begin{array}{rcll} \sum_{j=1}^{n} x_{ij} & = & 2 & \forall i \in N^{-} \\ \sum_{\substack{i \in S \\ j \notin S}} x_{ij} & \geq & 2 & \forall S \subset V,\ |S| > 1. \end{array}$$

where x_{ij} is a binary variable indicating whether $\sigma(i) = j$.

Combinatorial Relaxations of the TSP

- The Traveling Salesman Problem has several well-known combinatorial relaxations.

- Assignment Problem
  - The problem of assigning n people to n different tasks.
  - Can be solved in polynomial time.
  - Obtained by dropping the subtour elimination constraints.

- Minimum 1-tree Problem
  - A *1-tree* in a graph is a spanning tree of nodes $\{2, \ldots n\}$ plus exactly two edges incident to node one.
  - A minimum 1-tree can be found in polynomial time.
  - This relaxation is obtained by dropping all subtour elimination constraints involving node 1 and also all degree constraints not involving node 1.

Lagrangian Relaxation

- The idea is again based on relaxing a set of constraints from the original formulation.
- We try to push the solution towards feasibility by penalizing violation of the dropped constraints.
- Suppose our IP is defined by

$$\begin{aligned} \max\ & cx \\ \text{s.t. } A^1x &\leq b^1 \\ A^2x &\leq b^2 \\ x &\in \mathbb{Z}^n_+ \end{aligned}$$

where optimizing over $Q = \{x \in \mathbb{Z}^n_+ \mid A^2x \leq b^2\}$ is "easy."

- Lagrangian Relaxation:

$$LR(\lambda) : z_{LR}(\lambda) = \max_{x \in Q}\{(c - \lambda A^1)x + \lambda b^1\}.$$

Properties of the Lagrangian Relaxation

- For any $\lambda \geq 0$, $LR(\lambda)$ is a relaxation of IP (why?).
- Solving $LR(\lambda)$ yields an upper bound on the value of the optimal solution.
- Because of our assumptions, $LR(\lambda)$ can be solved easily.
- Recalling the development of LP duality in 406, one can think of λ as a vector of "dual variables."
- What is the obvious next step?