1 Partial Differential Equations and Modelling

1.1 A Quick Note on Notation

When working with partial differential equations (PDEs), it is essential to establish clear notation. A PDE involves a function of multiple independent variables, and we often distinguish between:

- Partial derivatives, denoted ∂, used when a function depends on multiple variables.
- Total derivatives, denoted d, used when the function depends on a single variable.

For example, if u depends on space x and time t, we write:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},$$

where D is a diffusion coefficient. **Discussion Point:** Emphasize the difference between ∂ and d, since confusing them can lead to incorrect derivations.

1.2 Intensive and Extensive Properties

- **Intensive properties:** independent of the system size, e.g., temperature T, pressure p.
- **Extensive properties:** dependent on system size, e.g., mass m, total energy E.

Example: In the heat equation,

$$q = -k\nabla T,$$

T is intensive, but the total heat content $Q = \int_V \rho c T \, dV$ is extensive. **Lecture Tip:** Stress that PDEs often relate extensive quantities with intensive properties.

1.3 The Continuity Equation

The continuity equation represents a fundamental principle: conservation of mass. The continuity equation is a fundamental principle in fluid dynamics and other fields, stating that mass is conserved.

In simple terms, for a fluid flowing through a system (like a pipe or channel), it means that the rate at which mass enters a volume must equal the rate at which mass leaves, plus any accumulation of mass within that volume over time. For an incompressible fluid (where density ρ is constant) in steady flow. Where A is the cross-sectional area and v is the fluid velocity at points 1 and 2. This shows that the volume flow rate (Volume/Time) is constant. Consequently, where the area is smaller, the velocity must be faster (and vice versa) to keep the flow rate consistent.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- ρ is density, $\mathbf{v}$ is velocity.
- Physical interpretation: any change in mass within a volume is due to the net flux across its boundary.

Example: For incompressible fluids, $\rho =$ constant, so we have:

$$\nabla \cdot \mathbf{v} = 0$$

1.4 The Diffusion and Heat Equations

Both the **Diffusion Equation** and the **Heat Equation** are linear partial differential equations (PDEs) that are mathematically identical and are used to model the same physical process: the **spreading of a quantity** (like heat, particles, or chemical concentration) from regions of higher concentration/temperature to regions of lower concentration/temperature. This process is driven by random motion and is fundamentally about the **conservation of the transported quantity**.

1.4.1 The Equation

The most common form in one dimension is:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

where:

- $u(x, t)$ is the quantity being diffused (e.g., concentration or temperature) at position x and time t.
- $\frac{\partial u}{\partial t}$ is the **rate of change** of the quantity over time.
- $\frac{\partial^2 u}{\partial x^2}$ represents the **curvature** or **non-uniformity** of the quantity's profile (the Laplacian in higher dimensions).
- α is the **diffusivity** (or **thermal diffusivity**), a constant that measures how quickly the quantity spreads.

1.4.2 Meaning

The equation states that the rate at which the quantity changes at any point ($\frac{\partial u}{\partial t}$) is directly proportional to the second spatial derivative of that quantity ($\frac{\partial^2 u}{\partial x^2}$).

- **Diffusion Equation:** Models the transport of **mass** (e.g., perfume in a room, solute in a solvent). The mass flow is proportional to the concentration gradient (Fick's Law).
- **Heat Equation:** Models the transport of **thermal energy** (heat) in a material. The heat flow is proportional to the temperature gradient (Fourier's Law of Heat Conduction).

In essence, the equation means that **high curvature smooths out quickly**: if a point is much hotter/more concentrated than its neighbors, the quantity will rapidly flow away from that point until the quantity is evenly distributed.

1.5 The Wave Equation

The **Wave Equation** is a second-order, linear partial differential equation (PDE) that describes the behavior of a wide variety of waves, such as sound waves, light waves, and water waves, as well as vibrations of strings or membranes. It is fundamentally an expression of **Newton's Second Law** ($\mathbf{F} = m\mathbf{a}$) applied to a continuous medium, governing how a disturbance propagates in space and time.

1.5.1 The Equation

The standard form of the one-dimensional wave equation is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where:

- $u(x,t)$ is the displacement or amplitude of the wave at position x and time t.
- $\frac{\partial^2 u}{\partial t^2}$ is the **acceleration** (or rate of change of the rate of change) of the displacement over time. This represents the inertial term, analogous to $\mathbf{a}$ in $\mathbf{F} = m\mathbf{a}$.
- $\frac{\partial^2 u}{\partial x^2}$ represents the **spatial curvature** of the wave profile. This is related to the internal restoring forces (like tension in a string or pressure gradients in a gas), analogous to $\mathbf{F}$.
- c is the **constant speed** at which the wave propagates.

1.5.2 Meaning

The equation states that the local acceleration of the medium ($\frac{\partial^2 u}{\partial t^2}$) is proportional to the local curvature of the wave ($\frac{\partial^2 u}{\partial x^2}$).

- **High Curvature $\rightarrow$ High Acceleration:** If a section of the medium (like a point on a string) is sharply curved, it experiences a strong net restoring force, causing it to accelerate rapidly back toward its equilibrium position.
- **Propagation:** The key feature is that any initial disturbance ($u(x,0)$) and initial velocity ($\frac{\partial u}{\partial t}(x,0)$) will split into two parts that travel indefinitely in opposite directions at the fixed speed c.

In essence, the wave equation is the mathematical foundation for phenomena where a disturbance travels without being diffused (like heat) or dissipated (in the ideal case).

1.6 Boundary and Initial Conditions

To obtain a unique solution for a Partial Differential Equation (PDE) that models a physical system, we need to specify both **Initial Conditions (ICs)** and **Boundary Conditions (BCs)**. Without these constraints, the solution would represent a family of functions, not the single physical reality.

1.6.1 Initial Conditions (ICs)

Initial Conditions specify the state of the system at the starting time, $t = 0$. They are required for problems that evolve over time (time-dependent PDEs, like the Diffusion and Wave equations).

- **Purpose:** To fix the unknown integration constants that arise from the time-derivative component of the PDE.
- **Example (Heat/Diffusion Equation):** You must specify the entire temperature or concentration distribution across the domain at $t = 0$.

- **Example (Wave Equation):** Since the wave equation has a second-order time derivative ($\frac{\partial^2 u}{\partial t^2}$), you must specify **two** ICs: the initial displacement $u(x,0)$ and the initial velocity $\frac{\partial u}{\partial t}(x,0)$.

1.6.2 Boundary Conditions (BCs)

Boundary Conditions specify the state or behavior of the solution at the spatial edges (boundaries) of the domain for all time $t \geq 0$. They fix the integration constants arising from the spatial derivative component of the PDE.

There are three main types of BCs:

1. **Dirichlet (First Type):** Specifies the **value** of the function u directly at the boundary.
   - *Physical Meaning:* Fixed temperature (e.g., a wall is kept at 0°C), fixed concentration, or fixed displacement (e.g., a vibrating string is tied down to $u = 0$).
2. **Neumann (Second Type):** Specifies the **value of the derivative** (the slope or flux) of u at the boundary.
   - *Physical Meaning:* Fixed heat flux or mass flux across the boundary. A common example is an **insulated** (no heat flow) or **sealed** (no mass flow) boundary, where the derivative is zero: $\frac{\partial u}{\partial x} = 0$.
3. **Robin (Third Type):** Specifies a **linear combination** of the value and the derivative at the boundary.
   - *Physical Meaning:* Convective heat transfer (e.g., heat loss to an ambient fluid, where the heat flux is proportional to the temperature difference at the surface).

1.7 PDEs in Space Only (Steady-State)

A Partial Differential Equation (PDE) that depends only on spatial variables (like x, y, z) is a special case derived from a time-dependent PDE by setting the **time derivative to zero**. This simplification is known as the **steady-state** or **equilibrium** assumption.

1.7.1 Concept and Transformation

For a time-dependent PDE:

$$\frac{\partial u}{\partial t} = \text{Spatial Derivative Term}$$

The steady-state solution $u_s(x, y, z)$ is found by assuming the system has reached a stable condition where the quantity u is no longer changing with time, meaning:

$$\frac{\partial u}{\partial t} = 0$$

This transforms the time-dependent PDE into a simpler, purely spatial PDE:

$$0 = \text{Spatial Derivative Term} \quad \Longrightarrow \quad \text{Spatial Derivative Term} = 0$$

1.7.2 Key Examples

- **From the Heat/Diffusion Equation to Laplace's Equation:** The Heat Equation ($\frac{\partial u}{\partial t} = \alpha \nabla^2 u$) leads to **Laplace's Equation:** $\nabla^2 u = 0$.
- **From the Heat/Diffusion Equation with Source to Poisson's Equation:** The steady-state equation is $\nabla^2 u = -\frac{f}{\alpha}$.

Note: Since time is eliminated, **Initial Conditions are no longer needed**. The solution is determined entirely by the **Boundary Conditions (BCs)**.

1.8 Linearisation

Nonlinear PDEs are often approximated by linearisation near equilibrium u_0:

$$f(u) \approx f(u_0) + f'(u_0)(u - u_0)$$

Example: Small perturbations in fluid flow, acoustics, or chemical reactions.

1.9 The Cauchy Momentum Equations

1.9.1 Inviscid Fluids (Euler Equations)

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \mathbf{f}$$

1.9.2 Navier-Stokes Equations

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f}$$

- Includes viscosity $\mu \nabla^2 \mathbf{v}$.
- Example: Flow in pipes, boundary layers.

1.9.3 Incompressible Flow

$$\nabla \cdot \mathbf{v} = 0$$

Simplifies Navier-Stokes and Euler equations for incompressible fluids.

1.10 Superposition and Inhomogeneities

1.10.1 The Principle of Superposition

The principle of superposition is a core concept in linear algebra and applies directly to **linear, homogeneous** Partial Differential Equations (PDEs).

- **Definition:** If u_1 and u_2 are both solutions to a given linear, homogeneous PDE, then any linear combination of these solutions, $u = c_1 u_1 + c_2 u_2$ (where c_1 and c_2 are constants), is also a solution.
- **Utility:** This principle is crucial for solving PDEs using techniques like **separation of variables** and **Fourier series**. It allows us to build the general solution by summing an infinite series of simpler, fundamental solutions (eigenfunctions) that satisfy the homogeneous equation and the homogeneous boundary conditions.

1.10.2 Inhomogeneities and the Breakdown of Superposition

An **inhomogeneity** is any term in a PDE or its boundary conditions that does not contain the dependent variable u or its derivatives. These terms are often called **source terms** or **non-homogeneous terms**.

- **Inhomogeneous PDE (Example):** $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \mathbf{f}(\mathbf{x}, \mathbf{t})$, where $f(x,t)$ is a non-zero source term (e.g., internal heat generation).
- **Inhomogeneous BCs (Example):** $u(0,t) = \mathbf{T_0}$, where T_0 is a non-zero fixed boundary temperature.
- **Impact:** The principle of superposition **does not apply** directly to inhomogeneous problems. If u_1 and u_2 satisfy the inhomogeneous equation, their sum $u_1 + u_2$ would satisfy the equation with **twice** the inhomogeneous term, not the original equation.

1.10.3 The Solution Strategy

To solve an inhomogeneous linear problem, we decompose the full solution u into two parts:

$$u = u_p + u_h$$

- **Particular Solution (u_p):** A solution that satisfies the **inhomogeneous equation** and/or **inhomogeneous boundary conditions**. This represents the fixed, non-transient part of the solution (often the steady-state solution).
- **Homogeneous Solution (u_h):** The remaining, transient solution that satisfies the **homogeneous equation** and the **homogeneous boundary conditions**. This is the part solved by traditional methods like superposition.

The full solution is then constructed by summing u_p and u_h.

1.11 Modelling Thin Volumes (Dimensional Reduction)

Modelling a **thin volume** involves simplifying a 3D physical problem into a lower-dimensional PDE (usually 1D or 2D) by exploiting the fact that the domain's size in one or two dimensions is negligible compared to the others. This process is essential for making complex problems analytically solvable or computationally efficient.

1.11.1 The Simplification Principle

The key assumption in dimensional reduction is that the variation of the dependent variable (e.g., temperature u) along the thin dimension is either negligible or can be approximated by a simple algebraic function.

- **Thin Rod/Wire (3D → 1D):** If a wire is very long compared to its diameter, heat primarily flows along the length (x-direction). The temperature variations across the tiny cross-section (y, z directions) are assumed to be uniform or negligible. The 3D Heat Equation simplifies to the 1D Heat Equation.
- **Thin Plate/Membrane (3D → 2D):** If a plate is very wide and long compared to its thickness (z-direction), the behavior is primarily described by the x and y coordinates. The variation through the thickness is often ignored or modelled as a simple boundary loss term.

1.11.2 Example: Heat Loss in a Thin Fin/Rod

Consider a long, thin rod (a cooling fin) where heat flows along its length (x) but is also lost from the surface to the surroundings (convection).

- **Standard Heat Equation (Ideal):** $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$
- **Modified 1D Model for Thin Volume:**

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} - \beta(u - u_a)$$

  - The term $\beta(u - u_a)$ is added to the 1D equation.
  - u is the rod temperature, and u_a is the ambient temperature.
  - This term **lumps** the complex 2D/3D surface heat loss mechanism into a simple linear decay term, making the entire problem manageable in 1D. β is a constant related to the material properties and surface area.

In summary, modelling thin volumes is an effective engineering approximation where the complexity of the full spatial derivatives in the thin directions is replaced by simpler, effective source or sink terms in the remaining dimensions.

1.12 Dimensional Analysis, Scaling, and Advanced Modeling

1.12.1 Units

Ensure equations are dimensionally consistent.

1.12.2 Buckingham π Theorem

The **Buckingham π Theorem** is the central theorem in dimensional analysis. It is used to systematically reduce the number of variables required to describe a physical phenomenon. If a physical process involves n dimensional variables and these variables can be expressed in terms of m fundamental dimensions (e.g., mass M, length L, time T), the theorem states that the process can be described completely by a relationship among exactly **$k = n - m$ independent dimensionless groups (π groups)**. These groups, like the **Reynolds number** (Re), are constant regardless of the system of units used, allowing results from small models to be scaled to large prototypes.

1.12.3 Dimensional Analysis and Modelling: Scaling Laws and Non-dimensionalisation

- **Non-dimensionalisation to Reduce Parameters:** This is the process of normalizing all variables in a problem by characteristic scales (e.g., a characteristic length L, velocity V, or time T). This transforms a dimensional PDE into a **non-dimensional PDE** where physical parameters are combined into a few **dimensionless groups**. The key benefit is that a single non-dimensional solution applies to all geometrically similar systems.

- **Scaling Laws:** These are relationships derived from the dimensionless groups that allow experimental results from a model (e.g., in a wind tunnel) to be translated to a full-scale prototype. For instance, to ensure dynamic similarity between a model and prototype, all relevant dimensionless groups must be held equal.

1.12.4 Parameters as Units

Using **derived units** or characteristic scales from the problem to non-dimensionalize the variables effectively means treating these parameters as base units for the problem. This technique, often called **normalization**, simplifies PDEs by reducing the number of explicit constants to one (or zero) by selecting characteristic scales such that the resulting non-dimensional coefficients (the π groups) are unity or an important constant. This simplification reduces computational complexity and highlights the essential physics of the problem by revealing the governing dimensionless ratios.

1.13 Modelling with Delta Functions and Coordinate Transformations

1.13.1 Modelling with Delta Functions

The **Dirac Delta function** ($\delta(x)$) is a generalized function (or distribution) used to model physical quantities that are zero everywhere except at a single point, but whose integral over all space is unity.

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

- **Use in PDEs:** The δ-function models highly localized sources or initial conditions, such as a **point heat source**, an **instantaneous point impulse** (for the wave equation), or a **point charge** (for Poisson's equation). For example, a concentrated heat source at x_0 is modeled by adding a term like $Q\delta(x - x_0)$ to the heat equation.

1.13.2 Coordinate Transformations

A **Coordinate Transformation** involves rewriting a PDE from one coordinate system (e.g., Cartesian x, y, z) into another (e.g., Cylindrical r, θ, z or Spherical r, θ, ϕ).

- **Purpose:** The primary reason is to exploit the **geometry and symmetry** of the domain or the boundary conditions.
- **Utility:** If a problem's geometry is cylindrical or spherical, the PDE, when written in the corresponding coordinate system, simplifies significantly because derivatives with respect to coordinates where there is no variation (due to symmetry) become zero. This simplification is often essential for solving the PDE analytically.

Use δ-functions to represent point sources or concentrated forces.

1.14 Summary of Basic PDE Uses

- **Continuity Equation:** Used to enforce the **Conservation of Mass** in fluid dynamics; what flows in must flow out or accumulate.
- **Diffusion/Heat Equation:** Used to model the **Spreading** or equalization of quantities (like heat or concentration) from high to low regions.
- **Wave Equation:** Used to model the **Propagation** or transmission of disturbances (waves and vibrations) through a medium at a fixed speed.

- **Laplace's/Poisson's Equation (Steady-State):** Used to find the final, **Equilibrium** state of a system (like temperature or electric potential) governed entirely by its boundaries and fixed sources.