Chapter 6: Eigenfunction Expansions
Expanded Lecture Notes with Equation Examples

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1 6.1 Poisson's Equation and Series

Poisson's equation (and the related inhomogeneous Laplace equation) is the prototypical elliptic boundary-value problem:

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{1}$$

with appropriate boundary conditions on $\partial\Omega$. Eigenfunction expansions use the eigenpairs of $-\Delta$ (with chosen BCs) to represent u and f.

1.1 6.1.1 Inhomogeneous PDE

Consider $-\Delta u = f$ on a domain Ω with homogeneous boundary conditions (for definiteness, Dirichlet). Let $\{\varphi_n\}$ be the L^2-orthonormal eigenfunctions of $-\Delta$:

$$-\Delta\varphi_n = \lambda_n\varphi_n, \qquad \varphi_n|_{\partial\Omega} = 0,$$

with $0 < \lambda_1 \le \lambda_2 \le \cdots$. Expand

$$u = \sum_{n=1}^{\infty} u_n\varphi_n, \qquad f = \sum_{n=1}^{\infty} f_n\varphi_n, \qquad f_n = \langle f, \varphi_n\rangle.$$

Plugging into (1) yields

$$\lambda_n u_n = f_n \quad \Rightarrow \quad u_n = \frac{f_n}{\lambda_n},$$

so

$$u(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n}\,\varphi_n(\mathbf{x}). \tag{2}$$

Comment: convergence and regularity follow from elliptic regularity and decay of f_n/λ_n.

Example: 1D Poisson (revisited)

On $0 < x < L$, $u(0) = u(L) = 0$,

$$-u''(x) = f(x), \qquad \varphi_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Hence

$$u(x) = \sum_{n=1}^{\infty} \frac{f_n}{(n\pi/L)^2}\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} \frac{L^2 f_n}{(n\pi)^2}\sin\frac{n\pi x}{L},$$

with $f_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$.

1.2 6.1.2 Inhomogeneous boundary conditions

If boundary conditions are non-homogeneous (e.g. $u|_{\partial\Omega} = g$), one commonly *lifts* the boundary data by writing

$$u = w + v,$$

where w is any sufficiently smooth function with $w|_{\partial\Omega} = g$ and v satisfies homogeneous BCs:

$$-\Delta v = f + \Delta w, \qquad v|_{\partial\Omega} = 0.$$

Then solve for v by eigenfunction expansion as in 6.1.1. Choosing a simple polynomial or harmonic w that matches g reduces the problem to a homogeneous-BC problem.

Simple example

On $[0, L]$ with $u(0) = A$, $u(L) = B$, choose linear lift

$$w(x) = A + \frac{B - A}{L} x,$$

so $v = u - w$ satisfies homogeneous Dirichlet BCs and $-v'' = f + w'' = f$ (since $w'' = 0$). Solve for v with sine series.

1.3 6.1.3 General inhomogeneities: Green's functions and spectral Green's function

The eigenfunction expansion gives a spectral representation of the Green's function $G(\mathbf{x}, \mathbf{y})$ for $-\Delta$ with chosen BCs:

$$G(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} \frac{\varphi_n(\mathbf{x})\varphi_n(\mathbf{y})}{\lambda_n}.$$

Then the solution is

$$u(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}. \tag{3}$$

This formula is particularly useful for point sources and singular right-hand sides.

2 6.2 Stationary and Steady-State Solutions

A stationary (steady-state) solution of a time-dependent PDE is one for which time derivatives vanish. For the general parabolic equation

$$u_t = Lu + S(\mathbf{x}, t),$$

the steady-state solves $0 = Lu + S(\mathbf{x}, \infty)$ (if the source tends to a limit).

2.1 6.2.1 Removing inhomogeneities by decomposition

A standard approach: decompose u as the sum of a steady state u_s solving

$$Lu_s = -S_s(\mathbf{x})$$

and a transient part v solving the homogeneous-in-space equation:

$$v_t = Lv + \big(S(\mathbf{x}, t) - S_s(\mathbf{x})\big), \qquad v(\mathbf{x}, 0) = u(\mathbf{x}, 0) - u_s(\mathbf{x}).$$

If S is time-independent ($S = S_s$) then $v_t = Lv$ and $v \to 0$ as $t \to \infty$ under appropriate spectral assumptions (negative eigenvalues for L).

Example: steady heat with constant source

For $ku_{xx} + Q_0 = 0$ on $0 < x < L$, $u(0) = u(L) = 0$:

$$u_s''(x) = -\frac{Q_0}{k}, \qquad u_s(x) = \frac{Q_0}{2k}\,(xL - x^2),$$

as previously. Then $u(x,t) = u_s(x) + v(x,t)$ with $v_t = kv_{xx}$.

3 6.3 Diffusion and Heat Equations

The diffusion (heat) equation on Ω:

$$u_t = k\Delta u + S(\mathbf{x}, t), \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}),$$

is solved by expanding in the eigenfunctions of $-\Delta$ (with chosen BCs). Let φ_n and λ_n be as before. Expand

$$u(\mathbf{x}, t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n(\mathbf{x}), \quad S(\mathbf{x}, t) = \sum_{n=1}^{\infty} S_n(t)\varphi_n(\mathbf{x}),$$

then modal ODEs:

$$\dot{u}_n(t) = -k\lambda_n u_n(t) + S_n(t), \qquad u_n(0) = u_n^0.$$

3.1 6.3.1 Initial conditions (modal decay)

For homogeneous source $S \equiv 0$, modes decay exponentially:

$$u_n(t) = u_n^0 e^{-k\lambda_n t},$$

so

$$u(\mathbf{x}, t) = \sum_{n=1}^{\infty} u_n^0 e^{-k\lambda_n t}\varphi_n(\mathbf{x}). \tag{4}$$

Higher modes (larger λ_n) decay faster — smoothing effect.

3.2 6.3.2 Constant source terms

If $S_n(t) = S_n^0$ (time-independent), modal ODE steady-state value is $u_n^\infty = S_n^0/(k\lambda_n)$ and solution:

$$u_n(t) = u_n^\infty + \big(u_n^0 - u_n^\infty\big)e^{-k\lambda_n t}.$$

Thus $u \to u_s = \sum_n \frac{S_n^0}{k\lambda_n}\varphi_n$ as $t \to \infty$.

3.3 6.3.3 Critical systems (marginal modes)

A *critical* or marginal mode occurs when $\lambda_n = 0$ (for Neumann BCs on connected domains $\lambda_1 = 0$ with constant eigenfunction). For Neumann Laplacian:

$$\lambda_1 = 0, \qquad \varphi_1 = \text{const.}$$

If S has a nonzero projection onto the constant mode, mass conservation matters: integrating the heat equation over Ω yields an ODE for the mean:

$$\frac{d}{dt}\overline{u}(t) = \overline{S}(t),$$

so the mean can drift (grow/decay) — no exponential decay for the zero mode. One must treat the zero mode separately in the spectral decomposition.

3.4 6.3.4 Time-dependent sources and Duhamel's principle

For general $S_n(t)$ the solution of

$$\dot{u}_n(t) = -k\lambda_n u_n(t) + S_n(t)$$

is given by variation of constants (Duhamel):

$$u_n(t) = u_n^0 e^{-k\lambda_n t} + \int_0^t e^{-k\lambda_n(t-s)} S_n(s)\, ds. \tag{5}$$

Hence

$$u(\mathbf{x}, t) = \sum_{n=1}^{\infty} \Big(u_n^0 e^{-k\lambda_n t} + \int_0^t e^{-k\lambda_n(t-s)} S_n(s)\, ds \Big) \varphi_n(\mathbf{x}).$$

This is the modal form of Duhamel's principle. In physical terms: each eigenmode is forced by the projection $S_n(t)$ with exponential memory kernel.

4 6.4 Wave Equation

The (forced) wave equation on $0 < x < L$ (Dirichlet BCs) is

$$u_{tt} = c^2 u_{xx} + F(x, t), \qquad u(0, t) = u(L, t) = 0,$$

with initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$. Expand in sine modes:

$$u(x, t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi x}{L}, \quad F(x, t) = \sum_{n=1}^{\infty} F_n(t) \sin \frac{n\pi x}{L}.$$

Then modal ODEs are forced harmonic oscillators:

$$\ddot{q}_n(t) + \omega_n^2 q_n(t) = F_n(t), \qquad \omega_n = \frac{n\pi c}{L}, \tag{6}$$

with initial conditions $q_n(0) = f_n$, $\dot{q}_n(0) = g_n$.

4.1 6.4.1 Inhomogeneous sources and initial conditions

The solution of (6):

$$q_n(t) = f_n \cos(\omega_n t) + \frac{g_n}{\omega_n} \sin(\omega_n t) + \int_0^t \frac{\sin\big(\omega_n(t-s)\big)}{\omega_n} F_n(s)\, ds, \tag{7}$$

by variation of parameters (Green's function for oscillator). Reassembling:

$$u(x, t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi x}{L}.$$

This displays how initial data and forcing excite modes.

4.2 6.4.2 Damped systems

Add damping (viscous) to the string:

$$u_{tt} + 2\gamma u_t = c^2 u_{xx} + F(x, t),$$

with modal equation

$$\ddot{q}_n + 2\gamma \dot{q}_n + \omega_n^2 q_n = F_n(t).$$

Characteristic roots $\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_n^2}$ determine under-, critical-, or over-damping. Homogeneous solution decays with envelope $e^{-\gamma t}$ (for underdamped case oscillatory with decaying amplitude).

4.3 6.4.3 Driven systems and resonance

For periodic forcing $F_n(t) = A_n \cos(\Omega t)$, steady-state modal response (after transients die out) is

$$q_n^{(\mathrm{ss})}(t) = \frac{A_n}{\sqrt{(\omega_n^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \cos(\Omega t - \phi_n),$$

where ϕ_n is a phase shift. Resonance occurs when $\Omega \approx \omega_n$; damping $\gamma > 0$ controls amplitude growth.

5 6.5 Terminating the Series

If the right-hand side (forcing or initial data) lies in the span of finitely many eigenfunctions, the series reduces to a finite sum (terminates). This often happens with polynomial or trigonometric forcing specially tailored to the eigenbasis.

Example: single-mode source

If $f(x) = \sin(\pi x/L)$ and homogeneous BCs, then only the $n = 1$ coefficient is nonzero, so the Poisson solution (or modal evolution) involves only the first eigenfunction:

$$u(x) = \frac{L^2}{\pi^2} \sin \frac{\pi x}{L} \quad \text{(Poisson example)}.$$

Finite-dimensional reductions are useful for modal truncation and model-order reduction.

6 6.6 Infinite Domains

On $\mathbb{R}^d$ the Laplacian has continuous spectrum. Eigenfunction expansions are replaced by Fourier transforms. For the heat equation $u_t = ku_{xx}$ on $\mathbb{R}$ with initial data u_0,

$$\widehat{u}(k, t) = \widehat{u_0}(k) e^{-k^2 kt} \quad \text{(Fourier variable } k\text{)},$$

leading to the convolution representation with the heat kernel:

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} u_0(y)\, dy. \tag{8}$$

Spectral integral representation

More generally, if $-\Delta$ has continuous spectral measure $\rho(\lambda)$, the expansion becomes an integral of the form

$$u(x) = \int_0^{\infty} \frac{\varphi_\lambda(x)}{\lambda} \langle f, \varphi_\lambda \rangle\, d\rho(\lambda),$$

where φ_λ are generalized eigenfunctions (plane waves in $\mathbb{R}^d$).

Summary: practical recipe for modal methods

1. Choose eigenbasis $\{\varphi_n\}$ for spatial operator consistent with BCs.
2. Expand all spatial functions (initial data, sources) in that basis.
3. Reduce PDE to ODEs for modal amplitudes $u_n(t)$.
4. Solve modal ODEs (closed form when possible: exponentials, convolution integrals).

5. Reconstruct $u(\mathbf{x}, t)$ by superposition; treat zero (critical) modes separately.

References and further reading: standard PDE texts treat eigenfunction expansions, Duhamel's principle, and Green's functions in detail (e.g. Evans, Strauss, Boyce & DiPrima).