Lecture: Chapter 7 – Green's Functions

1 Introduction

Green's functions are a powerful mathematical tool used to solve inhomogeneous differential equations. They provide a systematic way to construct solutions for linear differential operators subject to specific boundary or initial conditions. Applications include electrostatics, quantum mechanics, heat conduction, and wave propagation.

2 What are Green's Functions?

2.1 Definition

A Green's function, $G(x, x')$, for a linear operator $\mathcal{L}$ satisfies

$$\mathcal{L}G(x, x') = \delta(x - x'),$$

where $\delta(x - x')$ is the Dirac delta function. The solution to an inhomogeneous equation

$$\mathcal{L}u(x) = f(x)$$

can then be written as

$$u(x) = \int G(x, x')f(x')\,dx'.$$

3 Green's Functions in One Dimension

3.1 Inhomogeneous Initial Conditions

For a one-dimensional operator, we can construct solutions with inhomogeneous initial conditions by integrating the Green's function over the source term.

3.2 Sturm-Liouville Operators and Inhomogeneities in the Boundary

For Sturm-Liouville problems of the form

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = f(x),$$

Green's functions can be constructed to satisfy specific boundary conditions, allowing the systematic solution of the differential equation.

3.3 General Structure of Green's Function Solutions

The general solution using Green's functions can be expressed as

$$u(x) = \sum_i u_i^{\text{hom}}(x) + \int G(x, x') f(x') dx',$$

where u_i^{hom} are solutions to the homogeneous equation.

4 Poisson's Equation

Poisson's equation,

$$\nabla^2 \phi(\mathbf{r}) = -\rho(\mathbf{r}),$$

arises in electrostatics and potential theory.

4.1 Hadamard's Method of Descent

Hadamard's method of descent is a technique used to construct Green's functions in lower-dimensional spaces from known Green's functions in higher-dimensional spaces. This method is particularly useful for solving partial differential equations like Laplace's or the wave equation.

Idea: If the Green's function $G_{n+1}(\mathbf{r}, \mathbf{r}')$ is known in $(n+1)$-dimensional space, the Green's function in n-dimensions can be obtained by integrating over the extra coordinate:

$$G_n(\mathbf{r}_n, \mathbf{r}'_n) = \int_{-\infty}^{\infty} G_{n+1}((\mathbf{r}_n, x_{n+1}), (\mathbf{r}'_n, 0))\, dx_{n+1}.$$

Example: To derive the 2D Green's function for Laplace's equation from the 3D Green's function:

$$\nabla_{3D}^2 G_3(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \quad \Rightarrow \quad G_3(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}.$$

By integrating over the third coordinate z:

$$G_2(x, y; x', y') = \int_{-\infty}^{\infty} G_3((x, y, z), (x', y', 0))\, dz = \int_{-\infty}^{\infty} \frac{dz}{4\pi\sqrt{(x - x')^2 + (y - y')^2 + z^2}}.$$

Evaluating the integral gives the 2D Green's function:

$$G_2(x, y; x', y') = -\frac{1}{2\pi} \ln \sqrt{(x - x')^2 + (y - y')^2}.$$

Remarks:

- The method works because the higher-dimensional Green's function already satisfies the PDE, and integration over one coordinate reduces the dimensionality.
- It provides a systematic way to derive Green's functions in 1D or 2D from 3D results, which are often simpler or already known.

5 Heat and Diffusion

Green's functions solve the heat equation

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = f(\mathbf{r}, t)$$

by representing the evolution of temperature (or concentration) from a point source.

6 Wave Propagation

6.1 One-dimensional Wave Propagation

For the 1D wave equation,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x, t),$$

the Green's function describes the response to a point impulse in space and time. Simplified Analysis: One-dimensional models are mathematically simpler, allowing us to derive exact solutions and understand fundamental wave behavior.

Understanding Fundamental Concepts: Concepts like reflection, transmission, standing waves, interference, and resonance can be studied clearly in 1D before moving to more complex 2D or 3D systems.

Green's Function Foundation: The 1D wave equation is often the first step in constructing Green's functions for wave problems. This helps in understanding how a point source disturbance propagates over time.

Practical Applications: Many physical systems, such as strings in musical instruments, sound in narrow tubes, or electrical signals in transmission lines, can be effectively modeled as 1D.

Pedagogical Value: Learning 1D wave propagation builds intuition that can later be generalized to higher dimensions, making it easier to tackle complex wave phenomena.

6.2 Three-dimensional Wave Propagation

The 3D wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = f(\mathbf{r}, t)$$

can be solved with the retarded Green's function:

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta(t - t' - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi |\mathbf{r} - \mathbf{r}'|}.$$

6.3 Two-dimensional Wave Propagation

The 2D case requires a logarithmic Green's function due to the geometry of space:

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{H(t - t' - |\mathbf{r} - \mathbf{r}'|/c)}{2\pi \sqrt{(t - t')^2 - |\mathbf{r} - \mathbf{r}'|^2/c^2}}.$$

6.4 Physics Discussion

Green's functions describe how disturbances propagate and spread in different geometries and dimensions.

7 Problems with a Boundary

7.1 Green's Functions with Inhomogeneous Boundary Conditions

In many problems, the boundary conditions are not zero (inhomogeneous), for example:

$$u(a) = \alpha, \quad u(b) = \beta.$$

A standard Green's function $G(x, x')$ is constructed for homogeneous boundary conditions, so it does not automatically satisfy these inhomogeneous boundaries.

Modified Approach: Split the solution into two parts:

$$u(x) = v(x) + w(x),$$

where:

- $v(x)$ satisfies the homogeneous differential equation but fulfills the inhomogeneous boundary conditions:
$$\mathcal{L}v(x) = 0, \quad v(a) = \alpha,\, v(b) = \beta.$$
- $w(x)$ satisfies the inhomogeneous differential equation with homogeneous boundary conditions:
$$\mathcal{L}w(x) = f(x), \quad w(a) = w(b) = 0.$$

The function $w(x)$ can then be expressed using the standard Green's function:

$$w(x) = \int_a^b G(x, x') f(x')\, dx'.$$

Finally, the total solution is:

$$\boxed{u(x) = v(x) + \int_a^b G(x, x') f(x')\, dx'.}$$

Remarks:

- This decomposition ensures that both the differential equation and the inhomogeneous boundary conditions are satisfied.
- In some cases, the Green's function itself can be modified to satisfy the boundary conditions directly, often by adding an auxiliary function that only affects the boundaries.
- Applications include electrostatics, heat conduction, and wave propagation problems with fixed or prescribed boundary values.

7.2 Method of Images

The method of images constructs solutions that satisfy boundary conditions by introducing mirror sources.

7.3 Spherical Boundaries and Poisson's Equation

Poisson's equation in three dimensions is given by:

$$\nabla^2 \phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0},$$

where $\phi(\mathbf{r})$ is the potential, $\rho(\mathbf{r})$ is the charge density, and ϵ_0 is the permittivity of free space.

Spherical Symmetry: When the problem has spherical symmetry, it is convenient to use spherical coordinates (r, θ, ϕ). In this case, the Laplacian becomes:

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \phi^2}.$$

For a radially symmetric charge distribution $\rho = \rho(r)$, the potential ϕ depends only on r, and Poisson's equation reduces to:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = -\frac{\rho(r)}{\epsilon_0}.$$

Green's Function in Spherical Coordinates: The Green's function $G(\mathbf{r}, \mathbf{r}')$ satisfies

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'),$$

with appropriate boundary conditions on the spherical surface, e.g., $\phi(R) = 0$ for a grounded spherical shell of radius R.

For a point source inside a sphere, the solution can be written using the method of images or expansion in spherical harmonics:

$$\phi(\mathbf{r}) = \int_V G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\, d^3 r'.$$

Method of Images for a Spherical Boundary: To satisfy $\phi(R) = 0$, an image charge q' is placed outside or inside the sphere at a location determined by inversion with respect to the sphere:

$$r' = \frac{R^2}{r_0}, \quad q' = -q\frac{R}{r_0},$$

where r_0 is the distance of the real charge from the center. The total potential is the sum of contributions from the real and image charges.

Applications:

- Electrostatics: potential inside or outside spherical conductors.
- Gravitational potential problems with spherical mass distributions.
- Modeling spherical capacitors or spherical shells.

8 Perturbation Theory

Perturbation theory is used to find approximate solutions to problems that cannot be solved exactly, by treating a small part of the system as a *perturbation* to a solvable problem. Green's functions play a central role in this approach.

8.1 Basic Idea

Suppose we have a differential equation:

$$\mathcal{L}u(x) + \epsilon\mathcal{L}_1 u(x) = f(x),$$

where:

- $\mathcal{L}$ is a linear operator with a known Green's function $G_0(x, x')$,
- $\epsilon\mathcal{L}_1$ is a small perturbation ($\epsilon \ll 1$),
- $f(x)$ is the source term.

We can write the solution as a series expansion:

$$u(x) = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \dots,$$

where $u_0(x)$ solves the unperturbed equation:

$$\mathcal{L}u_0(x) = f(x), \quad u_0(x) = \int G_0(x, x') f(x')\, dx'.$$

8.2 Iterative Corrections

The first-order correction satisfies:

$$\mathcal{L}u_1(x) = -\mathcal{L}_1 u_0(x),$$

so

$$u_1(x) = -\int G_0(x, x')\mathcal{L}_1 u_0(x')\, dx'.$$

Higher-order corrections can be computed iteratively:

$$u_2(x) = -\int G_0(x, x')\mathcal{L}_1 u_1(x')\, dx', \quad \text{etc.}$$

8.3 Applications

- Quantum mechanics: perturbation of Hamiltonians.
- Electrodynamics: small changes in boundary conditions or dielectric properties.
- Mechanical systems: small forces or stiffness variations.

Key Point: Using Green's functions in perturbation theory allows systematic, iterative calculation of corrections, making complex problems tractable when exact solutions are unavailable.

9 Problems

Include exercises to calculate Green's functions for:

- 1D Poisson's equation with Dirichlet boundaries
- Heat equation in 1D and 2D
- Wave equation in 1D and 3D
- Method of images examples