Lecture: Symmetries and Group Theory

Duration: 1 hour

Overview

This lecture introduces the mathematical foundations of **symmetry** and its realization through **group theory**. We explore both discrete and continuous (Lie) groups, their representations, and their crucial role in understanding physical systems.

1 What is a Symmetry? (5 min)

A **symmetry** of a system is a transformation that leaves certain properties invariant. For example:

- Rotating a square by 90° leaves it unchanged.
- Translating a crystal lattice by one unit cell preserves its periodic structure.

Mathematically, a symmetry transformation T satisfies:

$$T(\mathcal{O}) = \mathcal{O}$$

for an observable $\mathcal{O}$. Mathematically, a **symmetry transformation** is an operation T that acts on a system's observables or configurations in such a way that certain essential properties remain unchanged. We say that T is a symmetry of an observable $\mathcal{O}$ if

$$T(\mathcal{O}) = \mathcal{O}.$$

This condition means that when the transformation T is applied to the observable, its measurable content or outcome does not change.

Interpretation

- $\mathcal{O}$ represents any measurable quantity of a physical system—such as energy, charge distribution, or probability density.
- T denotes a transformation acting on the space in which the system is defined—for instance, a rotation, reflection, translation, or time-reversal.

If after performing T, the observable remains the same, then the system possesses a **symmetry with respect to T**.

Example:

Consider a square lying in the plane. A rotation by 90° around its center, denoted R_{90}, leaves the square unchanged:

$$R_{90}(\text{square}) = \text{square}.$$

Hence, R_{90} is a symmetry of the square.

Why We Use This Formalism

Expressing symmetry as $T(\mathcal{O}) = \mathcal{O}$ allows us to:

1. **Abstract symmetry beyond geometry:** It generalizes geometric notions of symmetry (like reflection and rotation) to any kind of transformation in mathematics or physics, including those acting in internal spaces (such as spin, charge, or gauge spaces).

2. **Define invariant quantities:** By identifying transformations that leave observables unchanged, we can determine conserved quantities via **Noether's theorem**. For instance:

$$\text{Time invariance} \Rightarrow \text{Energy conservation}, \quad \text{Spatial translation invariance} \Rightarrow \text{Momentum conservatio}$$

3. **Classify physical systems:** Systems that share the same symmetry transformations belong to the same **symmetry class**. This provides a unifying language for studying particles, fields, crystals, and even solutions to differential equations.

4. **Reduce complexity:** Knowing that certain operations leave a system invariant reduces the number of independent variables or possible states we must consider. For example, in rotationally symmetric systems, functions depend only on the radius r, not on angular coordinates.

In Operator Language (Quantum Perspective)

In quantum mechanics, a symmetry transformation T acts on a state $|\psi\rangle$ and an observable operator $\hat{O}$ as

$$T\hat{O}T^{-1} = \hat{O}.$$

This expresses the same idea: the observable's action on the transformed state yields the same measurable outcome. If $\hat{O}$ commutes with T,

$$[T, \hat{O}] = 0,$$

then $\hat{O}$ is invariant under the transformation T, and the associated quantity is **conserved**.

Summary

The condition $T(\mathcal{O}) = \mathcal{O}$ is therefore not merely a definition—it encapsulates the profound idea that:

> *Symmetry is invariance under transformation.*

It allows us to identify conservation laws, classify systems, and build the mathematical structure of physics on a foundation of invariance principles.

2 Groups (10 min)

A **group** is a set G with an operation $\circ$ satisfying:

1. Closure: $a, b \in G \Rightarrow a \circ b \in G$
2. Identity: $\exists e \in G : e \circ a = a$
3. Inverse: $\forall a \in G, \exists a^{-1}$
4. Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$

2.1 Definition of a Group (Expanded Explanation)

A **group** is one of the most fundamental structures in mathematics. It provides a way to describe a set of elements together with an operation (often called "composition" or "multiplication") that combines any two elements of the set to produce another element of the same set.

Formally, a group is a pair $(G, \circ)$, where G is a set and $\circ$ is a binary operation satisfying four key properties:

1. **Closure:** For any two elements $a, b \in G$, their combination $a \circ b$ is also in G.

$$a, b \in G \Rightarrow a \circ b \in G$$

This means you can “stay inside” the group when performing the operation—nothing escapes the set. *Example:* In the set of integers under addition, $2 + 3 = 5$ is still an integer, so it satisfies closure.

2. **Identity Element:** There exists a special element $e \in G$ (called the **identity**) such that combining e with any element leaves it unchanged:

$$e \circ a = a \circ e = a \quad \forall a \in G$$

Example: For integers under addition, the identity is 0, since $0 + a = a$.

3. **Inverse Element:** For every element $a \in G$, there exists another element $a^{-1} \in G$ (called the **inverse**) such that

$$a \circ a^{-1} = a^{-1} \circ a = e$$

The inverse “undoes” the effect of the original element. *Example:* For addition on integers, the inverse of a is $-a$, since $a + (-a) = 0$.

4. **Associativity:** The way in which elements are grouped during the operation does not matter:

$$(a \circ b) \circ c = a \circ (b \circ c)$$

This means operations can be performed in any order without ambiguity. *Example:* $(2 + 3) + 4 = 2 + (3 + 4) = 9$.

Why Groups Matter

Groups allow us to describe symmetry and transformations in a unified way. Whenever we have a collection of operations that can be composed, have an identity, and can be undone, we are dealing with a group.

Examples:

- The set of integers $\mathbb{Z}$ with addition $(+)$ is a group.
- The set of nonzero real numbers $\mathbb{R}^*$ with multiplication $(\times)$ is a group.
- The set of rotations around a fixed point in the plane forms a group under composition.

Groups therefore appear everywhere—from number systems and geometric symmetries to quantum mechanics and particle physics—because they encode the essential idea of combining transformations while preserving structure.

2.2 Conjugacy Classes

Two elements $a, b \in G$ are conjugate if $\exists g \in G$ such that $b = gag^{-1}$. They form **conjugacy classes**, fundamental for understanding representations.

2.3 Conjugacy Classes (Expanded Explanation)

In group theory, the idea of **conjugacy** captures the notion that certain elements of a group are "similar" in structure, even though they may look different.

Definition

Two elements $a, b \in G$ are said to be **conjugate** if there exists some $g \in G$ such that

$$b = gag^{-1}.$$

We then say that b is a **conjugate** of a by g.

Interpretation: The element g acts as a "change of viewpoint" inside the group. Applying g and its inverse g^{-1} around a transforms it into another element b that behaves in the same way within the structure of the group.

Conjugacy Class

The set of all elements that are conjugate to a forms the **conjugacy class** of a:

$$\text{Class}(a) = \{gag^{-1} \mid g \in G\}.$$

Thus, the group G is partitioned into distinct conjugacy classes—no two classes overlap, and together they cover the entire group.

Example 1: The Symmetric Group S_3

Consider the group S_3, the set of all permutations of three objects. Its elements are:

$$\{e, (12), (13), (23), (123), (132)\}.$$

The conjugacy classes are:

$$\begin{aligned} &\{e\}, \\ &\{(12), (13), (23)\}, \\ &\{(123), (132)\}. \end{aligned}$$

Here, all transpositions are conjugate to each other, and all 3-cycles are conjugate to each other.

Interpretation: Even though the individual permutations differ, they share the same "cycle structure"—the same type of action—so they belong to the same class.

Example 2: Rotations of a Square

For the cyclic group $C_4 = \{e, R_{90}, R_{180}, R_{270}\}$:

$$gag^{-1} = a \quad \forall g \in C_4,$$

so each element forms its own conjugacy class. In abelian groups (where $ab = ba$), every element commutes with every other, so:

$$gag^{-1} = a,$$

and each element stands alone in its class.

Why Conjugacy Classes Matter

1. **Structural Similarity:** Conjugate elements represent the same "type" of transformation, viewed from different reference frames within the group.

2. **Representation Theory:** In any finite group, the **number of irreducible representations** equals the **number of conjugacy classes**. This connection makes conjugacy classes central in constructing character tables.

3. **Physical Insight:** In physics, conjugate elements often correspond to operations that have the same physical effect but are oriented differently in space (for example, rotations about different axes of the same magnitude).

Summary

Conjugacy classes divide a group into families of elements that share identical internal structure. Studying these classes provides deep insight into how the group behaves and how it can be represented through matrices or transformations.

2.4 Subgroups

A subset $H \subseteq G$ that is itself a group under the same operation.

2.5 Subgroups (Expanded Explanation)

A **subgroup** is a smaller group that lives inside a larger one, following the same operation and satisfying all the group properties on its own.

Definition

If G is a group with operation $\circ$, then a subset $H \subseteq G$ is called a **subgroup** of G if:

$$a, b \in H \Rightarrow a \circ b^{-1} \in H,$$

and H is itself a group under the same operation $\circ$.

In other words, H must:

1. Contain the identity element of G,
2. Be closed under the group operation,
3. Contain inverses of all its elements.

If these three properties hold, then H automatically satisfies associativity (inherited from G), and therefore forms a group on its own.

Intuitive Meaning

A subgroup represents a **smaller symmetry** or **restricted set of operations** that still behaves consistently. If G is the "full set of symmetries" of an object or system, then H might represent a subset of those symmetries that leave some additional property unchanged.

Example 1: Rotations of a Square

Let $G = D_4$, the dihedral group of symmetries of a square, containing rotations and reflections:

$$G = \{e, R_{90}, R_{180}, R_{270}, M_x, M_y, M_{d_1}, M_{d_2}\}.$$

The set of only rotations forms a subgroup:

$$H = \{e, R_{90}, R_{180}, R_{270}\}.$$

This subgroup H is closed under composition (rotating twice gives another rotation), contains the identity, and each rotation has an inverse (e.g., $R_{90}^{-1} = R_{270}$).

Example 2: Integers and Even Integers

Consider $G = (\mathbb{Z}, +)$, the group of all integers under addition. The set of even integers $H = \{2k \mid k \in \mathbb{Z}\}$ forms a subgroup:

$$\begin{aligned}
&\text{Closure: } 2m + 2n = 2(m+n) \in H,\\
&\text{Identity: } 0 \in H,\\
&\text{Inverse: } -(2k) = 2(-k) \in H.
\end{aligned}$$

Hence, H is a subgroup of G.

Proper and Improper Subgroups

- If $H = G$, it is called the **improper subgroup**.
- If H is strictly smaller than G, i.e. $H \subsetneq G$, it is called a **proper subgroup**.

Physical Interpretation

In physics, subgroups often represent **reduced symmetry conditions**. For instance:

- A perfect sphere has symmetry group $SO(3)$.
- If the sphere is slightly deformed into an ellipsoid, the remaining symmetry might only be rotations around one axis, forming a subgroup $SO(2) \subset SO(3)$.

Thus, subgroups describe situations where some, but not all, symmetries of a system remain.

Summary

- Every subgroup is a group within a group, following the same rules.
- Subgroups help us analyze complex symmetries by breaking them into simpler, manageable parts.
- They are the foundation for more advanced structures such as **cosets**, **normal subgroups**, and **quotient groups**.

2.6 Homomorphisms

A function $\phi : G \to H$ satisfying $\phi(ab) = \phi(a)\phi(b)$. These preserve structure between groups.

2.7 Homomorphisms (Expanded Explanation)

A **homomorphism** is a function that connects two groups while preserving their structure. Formally, a map

$$\phi : G \to H$$

between groups G and H is a **group homomorphism** if, for all $a, b \in G$,

$$\phi(ab) = \phi(a)\phi(b).$$

Meaning

This condition means that performing the operation in G and then mapping gives the same result as mapping first and performing the operation in H. In short, a homomorphism respects how elements combine.

Example

Let $G = (\mathbb{R}, +)$ and $H = (\mathbb{R}^+, \times)$. The function

$$\phi(x) = e^x$$

is a homomorphism because

$$\phi(x + y) = e^{x+y} = e^x e^y = \phi(x)\phi(y).$$

Why It Matters

Homomorphisms allow us to compare groups and transfer information between them. They reveal when two groups share the same underlying structure, leading to the ideas of **isomorphisms** (structure-preserving equivalences) and **kernels** (elements that map to the identity in H).

Summary

A homomorphism is a structure-preserving bridge between groups, ensuring that the "rules of combination" are maintained across different mathematical or physical contexts.

3 Discrete Groups (10 min)

3.1 The Cyclic Group

Generated by a single element r:

$$C_n = \{e, r, r^2, \ldots, r^{n-1}\}, \quad r^n = e$$

Common in rotational symmetries.

3.2 The Dihedral Group

Symmetry group of an n-gon:

$$D_n = \{r^k, sr^k \mid k = 0, \ldots, n-1\}, \quad s^2 = e,\ srs = r^{-1}$$

Includes rotations and reflections.

3.2.1 Dihedral Groups in Three Dimensions

Extension to 3D solids (tetrahedral, octahedral, icosahedral) leads to complex point group structures.

3.3 The Symmetric Group

The group of all permutations of n objects, denoted S_n. Foundation for abstract algebra and particle exchange symmetry.

4 Lie Groups (10 min)

Continuous groups characterized by differentiable parameters.

4.1 Rotations

The rotation group in 3D, $SO(3)$, consists of all 3×3 orthogonal matrices with $\det R = 1$.

4.2 Translations

Represented by additive continuous parameters:

$$T(a) : x \to x + a$$

Essential in defining momentum conservation.

4.3 Matrix Groups

Groups represented by invertible matrices such as $GL(n, \mathbb{R})$, $SU(2)$, $SO(3)$, etc. Their Lie algebras capture infinitesimal transformations.

5 Representation Theory (10 min)

A **representation** of a group G is a map $D : G \to GL(V)$ associating each group element with a matrix acting on a vector space V.

5.1 Tensor Products and Direct Sums

Representations can combine:

$$D_1 \otimes D_2, \quad D_1 \oplus D_2$$

Used to build higher-dimensional states (e.g., spin coupling).

5.2 Reducible Representations

A representation is **reducible** if it can be written as a direct sum of smaller representations.

6 Physical Implications and Examples (7 min)

6.1 Reduction of Solution Forms

Symmetries reduce the number of independent variables or solutions of equations — for instance, rotational symmetry simplifies the Schrödinger equation.

6.2 Important Transformations in Physics

Common symmetry operations:

- Parity (P)
- Time reversal (T)
- Gauge transformations

7 Irreducible Representations and Characters (7 min)

7.1 Irreducible Representations (Irreps)

Representations that cannot be decomposed further. They describe fundamental "building blocks" of states.

7.2 Schur's Lemmas and Orthogonality

Key results:

$$\begin{aligned}&\text{If } D_1, D_2 \text{ are irreps and } AD_1(g) = D_2(g)A,\\&\text{then } A = 0 \text{ or invertible.}\end{aligned}$$

Orthogonality relations of characters enable classification of irreps.

7.3 Characters

Trace of a representation:

$$\chi(g) = \mathrm{Tr}[D(g)]$$

The character table encapsulates all representation information for finite groups.

7.4 Physical Insights

Irreps correspond to conserved quantities:

- $SO(3)$ irreps $\Rightarrow$ angular momentum states.
- $SU(3)$ irreps $\Rightarrow$ quark symmetries in particle physics.