Lecture Plan: Function Spaces (1-hour session)

October 25, 2025

Overview

This lecture introduces the mathematical framework of **function spaces**, which is fundamental for understanding operators, eigenvalues, and the Sturm–Liouville theory. The session emphasizes intuition, formal structure, and examples linking abstract vector spaces to physical problems.

Lecture Outline (60 minutes total)

1. Abstract Vector Spaces (10 min)

Goals: Establish the connection between finite-dimensional vector spaces and spaces of functions.

- **Explain:** A set V (of functions) is a vector space over a field $\mathbb{F}$ (e.g., $\mathbb{R}$ or $\mathbb{C}$) if, for all $f, g \in V$ and $\alpha \in \mathbb{F}$:
  1. $f + g \in V$ (Closure under addition).
  2. $\alpha f \in V$ (Closure under scalar multiplication).
  3. Standard axioms hold (associativity, commutativity, existence of zero vector and inverse, distributivity, etc.).

  Emphasize that functions can form vector spaces too, e.g., the set of all continuous functions $C[a, b]$ on an interval $[a, b]$:

  $$f(x) + g(x), \quad \alpha f(x)$$

  are again continuous functions.

- **Define:** Inner product in function spaces, typically for the space of square-integrable functions $L^2(a, b)$:

  $$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

  and explain its analogy with the dot product in $\mathbb{R}^n$, $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^n u_i v_i$. The inner product allows us to define **length** (norm) and **angle** (orthogonality) in the function space: $\|f\| = \sqrt{\langle f, f \rangle}$.

- **Demonstrate:** **Completeness** (a space is complete if every Cauchy sequence in the space converges to a limit that is also in the space; a complete inner product space is called a **Hilbert space**), and **orthogonality** concepts. Two functions f and g are orthogonal if $\langle f, g \rangle = 0$.
- **Activity:** Ask students to verify orthogonality of $\sin(nx)$ and $\sin(mx)$ on $(0, \pi)$ for integers $n \neq m$.

$$\begin{aligned}\langle \sin(nx), \sin(mx) \rangle &= \int_0^\pi \sin(nx)\sin(mx)\,dx \\ &= \frac{1}{2}\int_0^\pi [\cos((n-m)x) - \cos((n+m)x)]\,dx \\ &= \frac{1}{2}\left[\frac{\sin((n-m)x)}{n-m} - \frac{\sin((n+m)x)}{n+m}\right]_0^\pi = 0\end{aligned}$$

Key message: Inner product spaces generalize Euclidean geometry to functions.
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2. Operators and Eigenvalues (10 min)

Goal: Bridge from vector algebra to functional operators.

- **Introduce:** A **Linear Operator** L is a mapping from a function space V to itself, $L : V \to V$, such that $L[\alpha f + \beta g] = \alpha L[f] + \beta L[g]$. Examples:
  - Differentiation: $L[f] = \frac{d}{dx} f(x) = f'(x)$.
  - Multiplication by x: $L[f] = x f(x)$.
  - Second derivative: $L[f] = f''(x)$.
- **Show:** Eigenvalue problems in finite vs infinite-dimensional settings. In matrix algebra ($\mathbb{R}^n$): $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$. In function spaces:

  $$Lf = \lambda f$$

  Here, λ is the **eigenvalue** (a scalar) and f is the non-trivial **eigenfunction** (the "eigenvector" of the function space).
- **Example:** The operator $L = -\frac{d^2}{dx^2}$ with $f(0) = f(\pi) = 0$. The eigenvalue problem $Lf = \lambda f$ becomes:

  $$-f''(x) = \lambda f(x) \quad \text{or} \quad f''(x) + \lambda f(x) = 0$$

  The solutions are $f_n(x) = \sin(nx)$ with eigenvalues $\lambda_n = n^2$ for $n = 1, 2, 3, \ldots$. Interpret λ as a characteristic value (e.g., related to energy levels in quantum mechanics or frequencies in vibration problems).
- **Concept:** A self-adjoint operator (also called Hermitian in quantum mechanics) is one for which $\langle f, Lg \rangle = \langle Lf, g \rangle$ holds for all suitable functions f, g. **Key Properties:** Self-adjoint operators have **real eigenvalues** and their **eigenfunctions corresponding to distinct eigenvalues are orthogonal**. This is crucial for creating an orthonormal basis.

Activity: Write an example operator $L[y] = \frac{d^2 y}{dx^2}$ and ask what properties (boundary conditions, interval) make it self-adjoint (e.g., on $[0, 1]$ with $y(0) = 0, y(1) = 0$).
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3. Sturm–Liouville Theory (10 min)

Goal: Show how Sturm–Liouville systems generalize eigenvalue problems.

- **Explain:** The Sturm–Liouville (SL) operator $L[y]$ is defined by the equation:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + [q(x)]y = -\lambda w(x)y$$

The full Sturm-Liouville form of the eigenvalue problem is:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + [\lambda w(x) - q(x)]y = 0$$

Here, $p(x)$, $q(x)$, and $w(x)$ are continuous functions, and $w(x) > 0$ is the **weight function**. The SL operator $L[y] = \frac{d}{dx}(p(x)y') - q(x)y$ is always self-adjoint with respect to the weighted inner product $\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,dx$, provided the boundary conditions are suitable.

- Discuss **regular vs singular problems** and **boundary conditions (BCs)**. A problem is **regular** on $[a, b]$ if $p(x) > 0$ on $[a, b]$ and BCs are of the form $\alpha_1 y(a) + \alpha_2 y'(a) = 0$ and $\beta_1 y(b) + \beta_2 y'(b) = 0$. A problem is **singular** if $p(x)$ vanishes at an endpoint, or the interval is infinite, requiring $y(x)$ to be bounded or finite at those points (e.g., Legendre's equation on $[-1, 1]$).

- **Insight:** The eigenfunctions $\{\phi_n(x)\}$ generated by a regular SL system are **orthogonal** with respect to the weight function $w(x)$ and form a **complete basis**. Any sufficiently smooth function $f(x)$ can be expanded in a generalized Fourier series:

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \quad c_n = \frac{\langle f, \phi_n\rangle_w}{\langle \phi_n, \phi_n\rangle_w}$$

- **Physical example:** The heat conduction equation $\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}$ in a non-uniform rod leads to an SL system if κ is a function of x. A vibrating string $\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$ with variable density $\rho(x)$ and tension $T(x)$ is also an SL problem.

Tip: Draw analogy with Fourier series—each eigenfunction behaves like a "basis vector" that captures a fundamental mode (like a frequency) of the physical system.

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4. Separation of Variables (10 min)

Goal: Show how PDEs reduce to Sturm–Liouville form.

- Outline method: For a Partial Differential Equation (PDE), assume the solution is a product of functions, each depending on a single variable: $u(x, t) = X(x)T(t)$. Substituting this into the PDE, one separates the variables to obtain independent Ordinary Differential Equations (ODEs) for $X(x)$ and $T(t)$, both equal to a separation constant, $-\lambda$.

- Derive two ODEs — one temporal, one spatial. For the Heat Equation $u_t = u_{xx}$:

$$X(x)T'(t) = X''(x)T(t) \quad \Rightarrow \quad \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

  This yields:

$$\begin{aligned} T'(t) + \lambda T(t) = 0 \quad &\text{(Temporal)} \\ X''(x) + \lambda X(x) = 0 \quad &\text{(Spatial)} \end{aligned}$$

- Show spatial part is a Sturm–Liouville problem: The spatial equation $X''(x) + \lambda X(x) = 0$ is a special case of the SL equation with $p(x) = 1$, $q(x) = 0$, and $w(x) = 1$. The resulting solutions $X_n(x)$ are the eigenfunctions.

- Emphasize: Applying separation of variables to PDEs in different coordinate systems (Cartesian, cylindrical, spherical) gives rise to different special functions as solutions to the spatial SL problem.

Demonstrate: Example from the Heat Equation or the Time-Independent Schrödinger Equation ($-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$). Applying separation of variables in a central potential $V(r)$ in spherical coordinates reduces the angular dependence to Legendre/Spherical Harmonics and the radial part to a radial SL-type equation.

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5. Special Functions (10 min)

Goal: Introduce the link between coordinate systems and special functions.

- **Explain:** The SL equations that arise from the spatial separation of variables in different coordinate systems yield classical **Special Functions**:
  - **Cylindrical/Polar** coordinates → **Bessel functions** $J_\nu(x)$ (solutions to Bessel's equation, which arises from $p(x) = x, q(x) = \nu^2/x, w(x) = x$).
  - **Spherical** coordinates → **Legendre polynomials** $P_l(x)$ (from the polar angle θ) and **Spherical Harmonics** $Y_l^m(\theta, \phi)$ (from the full angular part).
  - Cartesian coordinates with an infinite domain → **Hermite functions** (solutions to Hermite's equation, crucial for the quantum harmonic oscillator).

- Mention Hermite functions (quantum harmonic oscillator), whose eigenfunctions are $\psi_n(x) = H_n(x)e^{-x^2/2}$ where $H_n(x)$ are Hermite polynomials.

- Emphasize **orthogonality** and **completeness** in their respective domains: e.g., $\int_0^1 xJ_\nu(\lambda_n x)J_\nu(\lambda_m x)\,dx = 0$ for $n \neq m$, using the weight function $w(x) = x$.

Visual aid: Show plots of Bessel functions (decaying oscillations) and Hermite functions (Gaussian-weighted polynomials) to convey their characteristic oscillatory and decaying behavior.

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6. Function Spaces as Representations (5 min)

Goal: Touch on symmetry and reducibility.

- Explain: Function spaces can serve as **representations of group symmetries**. A group G acting on the physical system (e.g., rotations) induces a linear transformation (an operator) on the function space.
- Define **reducibility**: A representation (function space) is **reducible** if it can be decomposed into a direct sum of smaller, non-trivial, invariant subspaces under the group actions. It is **irreducible** if it cannot be decomposed further.

Example: Rotational symmetry in spherical harmonics $\to$ the function space of square-integrable functions on the sphere can be decomposed into subspaces V_l spanned by the spherical harmonics Y_l^m for fixed l and $m = -l, \ldots, l$. These subspaces V_l are the **irreducible representations of SO(3)** (the group of rotations). The $2l+1$ functions Y_l^m for a fixed l transform among themselves under rotation, but do not mix with functions from a different l.

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7. Distribution Theory (5 min)

Goal: Briefly introduce generalized functions and derivatives.

Deeper explanation of distributions, the Dirac delta, and weak derivatives

Test function spaces. Two common spaces of test functions are

$\mathcal{D}(\mathbb{R}) = \{\phi \in C^\infty(\mathbb{R}) \ : \ \phi \text{ has compact support}\}, \qquad \mathcal{S}(\mathbb{R}) = \{\phi \in C^\infty(\mathbb{R}) \ : \ \phi$

derivatives decay faster than any polynomial$\}$,

endowed with natural locally convex topologies (describing how sequences of test functions converge).

Definition of a distribution. A *distribution* T is a continuous linear functional on a test-space (typically $\mathcal{D}$ or $\mathcal{S}$):

$$T : \mathcal{D}(\mathbb{R}) \to \mathbb{R} \quad (\text{or } \mathbb{C}), \quad \text{linear, and continuous in the test-space topology.}$$

We write the action of T on ϕ as $\langle T, \phi \rangle$. Continuity means: if $\phi_n \to 0$ in the test-space sense then $\langle T, \phi_n \rangle \to 0$.

Regular distributions. Every locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R})$ defines a distribution T_f by

$$\langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\, \phi(x)\, dx.$$

Such distributions are called *regular*. Distributions that are not of this form (e.g. δ) are *singular*.

The Dirac delta as a distribution. The Dirac delta δ is the distribution defined by

$$\langle \delta, \phi \rangle = \phi(0) \quad \text{for all } \phi \in \mathcal{D}.$$

Important properties and ways to think about δ:

- **Reproducing property:** $\int \delta(x)\,\phi(x)\,dx = \phi(0)$ (notation: the integral is interpreted distributionally).
- **Approximations:** δ is the limit of "approximate identities" δ_n in the distribution sense, e.g.

  $$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}, \qquad \langle \delta_n, \phi \rangle = \int \delta_n(x)\phi(x)\,dx \to \phi(0).$$

  Convergence here means $\langle \delta_n, \phi \rangle \to \langle \delta, \phi \rangle$ for every test ϕ.
- **Support:** $\operatorname{supp} \delta = \{0\}$ (a distribution can have very small support).

Differentiation of distributions (weak derivative). Differentiation is defined by duality: for a distribution T, its derivative T' is the distribution satisfying

$$\langle T', \phi \rangle := -\langle T, \phi' \rangle \qquad \text{for all } \phi \in \mathcal{D}.$$

This definition generalizes classical differentiation and is consistent with integration by parts (no boundary terms because test functions have compact support).

Examples:

1. If $T = T_f$ with $f \in C^1$, then $T' = T_{f'}$ because $\langle T_{f'}, \phi \rangle = \int f'(x)\phi(x)\,dx = -\int f(x)\phi'(x)\,dx = -\langle T_f, \phi' \rangle$.
2. **Heaviside step function $H(x)$.** Define $H(x) = 0$ for $x < 0$, $H(x) = 1$ for $x > 0$. As a distribution T_H (regular, from an L^1_{loc} function),

   $$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_0^\infty \phi'(x)\,dx = \phi(0).$$

   Thus $H' = \delta$ in the sense of distributions.
3. **Sign function:** $\operatorname{sgn}(x) = H(x) - H(-x)$. Its derivative is 2δ at the origin: $(\operatorname{sgn})' = 2\delta$.
4. **Principal value:** The distribution p.v. $\frac{1}{x}$ is defined by

   $$\left\langle \text{p.v.}\, \frac{1}{x}, \phi \right\rangle = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\phi(x)}{x}\,dx,$$

   and is useful when classical integrals diverge symmetrically.

Operations and extensions.

- **Linearity:** distributions form a vector space; differentiation is linear.
- **Convolution:** A distribution can be convolved with a compactly supported distribution or with a suitable test function; approximations $\delta_n * f \to f$ recover smoothing.
- **Fourier transform:** On the space of tempered distributions (continuous functionals on $\mathcal{S}$), the Fourier transform is well defined and exchanges differentiation with multiplication by polynomials, making distribution theory powerful for PDEs.

Why this is useful in practice.

- Model idealized sources (point charges, point masses) and initial conditions in physics.
- Define derivatives of discontinuous or non-differentiable functions; handle PDEs with singular data.
- Provide a rigorous setting for manipulations often done formally by physicists (e.g., Green's functions).

Very concise summary (one paragraph)

A *distribution* is a continuous linear functional on a space of smooth test functions ($\mathcal{D}$ or $\mathcal{S}$). Regular functions $f \in L^1_{\text{loc}}$ define distributions by $\langle T_f, \phi \rangle = \int f\phi$. The Dirac delta δ is the singular distribution $\langle \delta, \phi \rangle = \phi(0)$ and can be obtained as the limit of sharply peaked smooth functions. Differentiation is defined by duality: $\langle T', \phi \rangle = -\langle T, \phi' \rangle$. This *weak derivative* extends classical derivatives (e.g. the weak derivative of the Heaviside step is δ), and distribution theory lets us treat PDEs and singular sources rigorously.

(If you want, I can add three short worked examples you can present on the board: (1) limit of Gaussians to δ; (2) compute H' to show $H' = \delta$; (3) Fourier transform of δ and of a derivative.)

End with: How distribution theory extends the function space framework, allowing solutions to PDEs that are not classically smooth, thus unifying many problems in physics and engineering.