Lecture: Symmetries and Group Theory

Duration: 1 hour

Overview

This lecture introduces the mathematical foundations of **symmetry** and its realization through **group theory**. We explore both discrete and continuous (Lie) groups, their representations, and their crucial role in understanding physical systems.

1 What is a Symmetry? (5 min)

A **symmetry** of a system is a transformation that leaves certain properties invariant. For example:

- Rotating a square by 90° leaves it unchanged.
- Translating a crystal lattice by one unit cell preserves its periodic structure.

Mathematically, a symmetry transformation T satisfies:

$$T(\mathcal{O}) = \mathcal{O}$$

for an observable $\mathcal{O}$.

2 Groups (10 min)

A **group** is a set G with an operation $\circ$ satisfying:

1. Closure: $a, b \in G \Rightarrow a \circ b \in G$
2. Identity: $\exists e \in G : e \circ a = a$
3. Inverse: $\forall a \in G, \exists a^{-1}$
4. Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$

2.1 Conjugacy Classes

Two elements $a, b \in G$ are conjugate if $\exists g \in G$ such that $b = gag^{-1}$. They form **conjugacy classes**, fundamental for understanding representations.

2.2 Subgroups

A subset $H \subseteq G$ that is itself a group under the same operation.

2.3 Homomorphisms

A function $\phi : G \to H$ satisfying $\phi(ab) = \phi(a)\phi(b)$. These preserve structure between groups.

3 Discrete Groups (10 min)

3.1 The Cyclic Group

Generated by a single element r:

$$C_n = \{e, r, r^2, \ldots, r^{n-1}\}, \quad r^n = e$$

Common in rotational symmetries.

3.2 The Dihedral Group

Symmetry group of an n-gon:

$$D_n = \{r^k, sr^k \mid k = 0, \ldots, n-1\}, \quad s^2 = e,\, srs = r^{-1}$$

Includes rotations and reflections.

3.2.1 Dihedral Groups in Three Dimensions

Extension to 3D solids (tetrahedral, octahedral, icosahedral) leads to complex point group structures.

3.3 The Symmetric Group

The group of all permutations of n objects, denoted S_n. Foundation for abstract algebra and particle exchange symmetry.

4 Lie Groups (10 min)

Continuous groups characterized by differentiable parameters.

4.1 Rotations

The rotation group in 3D, $SO(3)$, consists of all 3×3 orthogonal matrices with $\det R = 1$.

4.2 Translations

Represented by additive continuous parameters:

$$T(a) : x \to x + a$$

Essential in defining momentum conservation.

4.3 Matrix Groups

Groups represented by invertible matrices such as $GL(n, \mathbb{R})$, $SU(2)$, $SO(3)$, etc. Their Lie algebras capture infinitesimal transformations.

5 Representation Theory (10 min)

A **representation** of a group G is a map $D : G \to GL(V)$ associating each group element with a matrix acting on a vector space V.

5.1 Tensor Products and Direct Sums

Representations can combine:

$$D_1 \otimes D_2, \quad D_1 \oplus D_2$$

Used to build higher-dimensional states (e.g., spin coupling).

5.2 Reducible Representations

A representation is **reducible** if it can be written as a direct sum of smaller representations.

6 Physical Implications and Examples (7 min)

6.1 Reduction of Solution Forms

Symmetries reduce the number of independent variables or solutions of equations — for instance, rotational symmetry simplifies the Schrödinger equation.

6.2 Important Transformations in Physics

Common symmetry operations:

- Parity (P)
- Time reversal (T)
- Gauge transformations

7 Irreducible Representations and Characters (7 min)

7.1 Irreducible Representations (Irreps)

Representations that cannot be decomposed further. They describe fundamental "building blocks" of states.

7.2 Schur's Lemmas and Orthogonality

Key results:

$$\text{If } D_1, D_2 \text{ are irreps and } AD_1(g) = D_2(g)A, \\ \text{then } A = 0 \text{ or invertible.}$$

Orthogonality relations of characters enable classification of irreps.

7.3 Characters

Trace of a representation:

$$\chi(g) = \mathrm{Tr}[D(g)]$$

The character table encapsulates all representation information for finite groups.

7.4 Physical Insights

Irreps correspond to conserved quantities:

- $SO(3)$ irreps $\Rightarrow$ angular momentum states.
- $SU(3)$ irreps $\Rightarrow$ quark symmetries in particle physics.

8 Outlook (1 min)

Group theory unifies diverse physical phenomena through symmetry. From crystal lattices to quantum field theory, understanding symmetry means understanding nature's constraints and possibilities.