Q1: Given $f(z) = \ln(1+z)$, where $z = X^T x$, $x \in R^d$

Sol^n:

if $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$, then $x^T = [x_1\ x_2 \cdots x_d]$

$\therefore X^T X = [x_1^2 + x_2^2 + x_3^2 \cdots + x_d^2]$

applying chain rule,

$$\frac{\partial f}{\partial x} = \frac{\delta f}{\delta z} \cdot \frac{\delta z}{\delta x}$$

$$= \frac{\delta}{\delta z}(\ln[1+z]) \cdot \frac{\partial}{\partial x}(x^T \cdot x)$$

$$= \frac{1}{1+z} \cdot \frac{\delta}{\delta z}(z) \cdot \frac{\delta}{\delta x}(x_1^2 + x_2^2 \cdots x_d^2)$$

$$= \frac{1}{1+z}(2x_1 + 2x_2 \cdots 2x_d)$$

$$= \frac{2}{1+z}(x_1 + x_2 + \cdots x_d)$$

$$\therefore \boxed{\frac{\delta}{\delta x} \ln(1+z) = \frac{2}{1+z} \cdot \sum_{i=1}^{d} x_i}$$

$\underline{Q2}$: $f(z) = e^{-z/2}$, where $z = g(z)$, $g(y) = y^T S^{-1} y$,

$y = h(x)$, $h(x) = x - \mu$

$\underline{Sol^n}$:

$$\frac{\delta f}{\delta x} = \frac{\delta f}{\delta z} \cdot \frac{\delta z}{\delta y} \cdot \frac{\delta y}{\delta x}$$

now,

(1) $$\frac{\delta f}{\delta z} = \frac{\delta}{\delta z}\left(e^{-z/2}\right) = -\frac{e^{-z/2}}{2}$$

(2) $$\frac{\delta z}{\delta y} = \frac{\delta}{\delta y}\left(y^T S^{-1} y\right) = \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{h(y^T S^{-1} + S^{-1} y + h S^{-1})}{h}$$

$$\cancel{= y^T} = y^T S^{-1} + S^{-1} y$$

(3) $$\frac{\delta y}{\delta x} = \frac{\delta}{\delta x}(x - \mu) = 1$$

$$\therefore \boxed{\frac{\delta f}{\delta x} = -\frac{e^{-z/2}}{2} \times \left(y^T S^{-1} + S^{-1} y\right) \times 1}$$