Chapter 5
Transformations

For the purpose of controlling a robot, it is necessary to know the relationships between the joints motion (input) and the end-effector motions (output), because the joint motions control the end-effector movements. Thus, the study of kinematics is important, where transformations between the coordinate frames attached to different robot links of the robot need to be performed

5.1 Robot Architecture

A robot is made up of several links connected serially by joints. The robot's degree of freedom (DOF) depends on the number of links and joints, their types, and the kinematic chain of the robot.

5.1.1   Links and joints

The individual bodies that make up a robot are called 'links.' Here, unless otherwise stated, all links are assumed to be rigid, i.e., the distance between any two points within the body does not change while it is moving. A rigid body in the three dimensional Cartesian space has six DOF. This implies that the position of the body can be described by three translational, and the orientation by three rotational coordinates. For convenience, certain non-rigid bodies, such as chains, cables, or belts, which when serve the same function as the rigid bodies, may be considered as links. From the kinamatic point of view, two or more members when connected together without any relative motion between them is considered as a single link. For example, an assembly of two gears connected by a common shaft is treated as one link.

Links of a robot are coupled by kinematic pairs or joints. A 'joint' couples two links and provides physical constraints on the relative motion between the links. It is not a physical entity but just a concept that allows one to specify how one link moves with respect to another one. For example, a hinge joint of a door allows it to move relative to the fixed wall about an axis. No other motion is possible. Type of relative motion permitted by a joint is governed by the form of the contact surface between the members, which can be a surface, a line, or a point. Accordingly, they are termed as either 'lower' or 'higher' pair joint. If two mating links are in surface contact, the joint is referred to as a lower pair joint. On the contrary, if the links are in line or point contact, the joint is called higher pair joint. As per the definition, the hinge joint of the door is a lower pair joint, whereas a ball rolling on a plane makes a higher pair joint. Frequently used lower and higher pairs are listed in Table 5.1.

(i) *Revolute Joint, R*

A revolute joint, also known as a 'turning pair' or a 'hinge' or a 'pin joint,' permits two paired links to rotate with respect to each other about the axis of the joint, say, *Z*, as shown in Fig. 5.1. Hence, a revolute joint imposes five constraints, i.e., it prohibits one of

the links to translate with respect to the other one along the three perpendicular axes, *X*, *Y*, *Z*, along with the rotation about two axes, *X* and *Y*. This joint has one-degree- of-freedom (DOF).
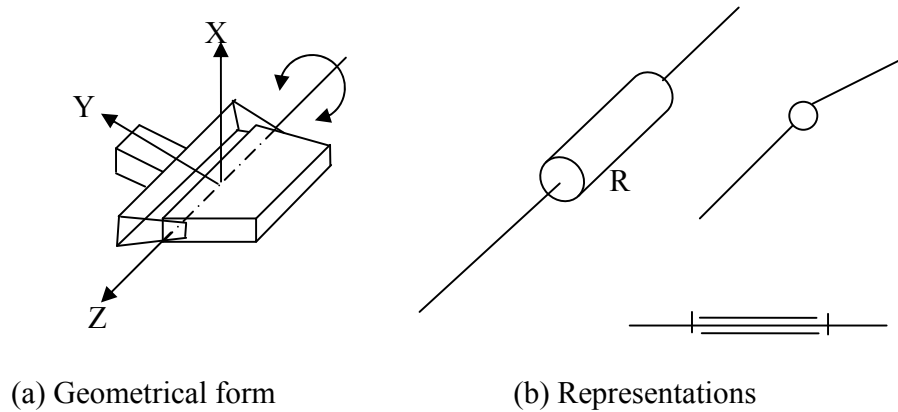

(a) Geometrical form   (b) Representations

Figure 5.1 A revolute joint

(ii) *Prismatic Joint, P*

A 'prismatic joint' or a 'sliding pair' allows two paired links to slide with respect to each other along its axis, as shown in Fig. 5.2. It also imposes five constraints and, hence, has one-DOF.

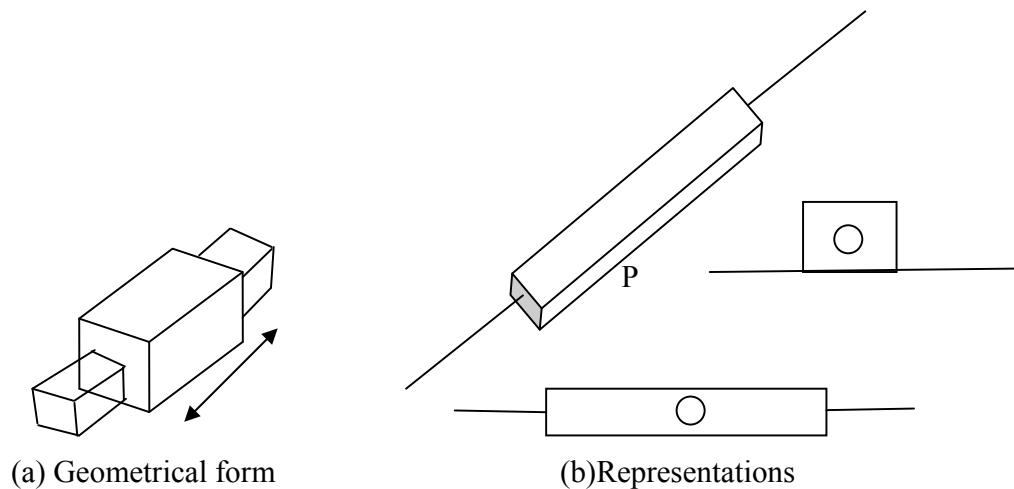(a) Geometrical form   (b)Representations

Figure 5.2 A prismatic joint

(iii) *Helical Joint, H*

As shown in Fig. 5.3, a helical joint allows two paired links to rotate about and translate at the same time along the axis of the joint. The translation is, however, not independent. It is related to the rotation by the pitch of the screw. Thus, the helical joint also has five constraints, and accordingly one-DOF.

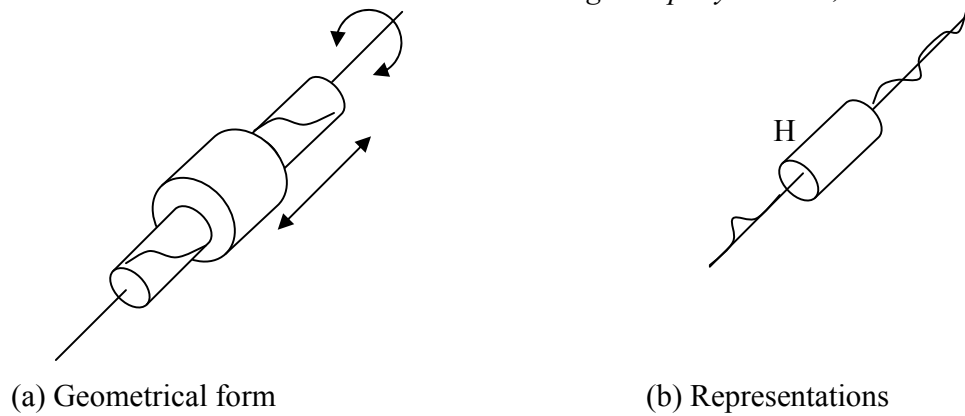| (a) Geometrical form	(b) Representations

Figure 5.3 A helical joint

(iv) *Cylindrical Joint, C*

It permits rotation about, and independent translation along, the axis of the joint, as shown in Fig. 5.4. Hence, a cylindrical joint imposes four constraints on the paired links, and has two-DOF.

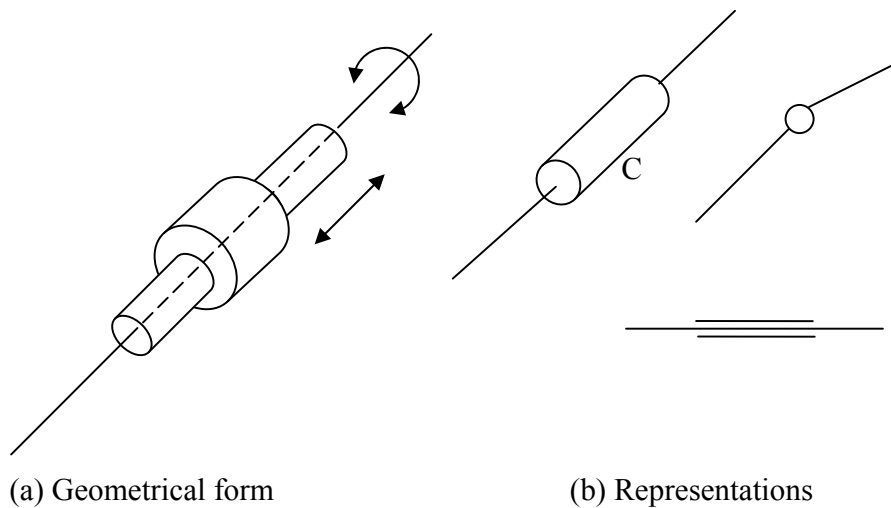| (a) Geometrical form	(b) Representations

Figure 5.4 A cylindrical joint

 (v) *Spherical Joint, S*

It allows one of the coupled links to rotate freely in all possible orientation with respect to the other one about the center of a sphere. No relative translation is permitted. Hence, it imposes three constraints and has three-DOF.
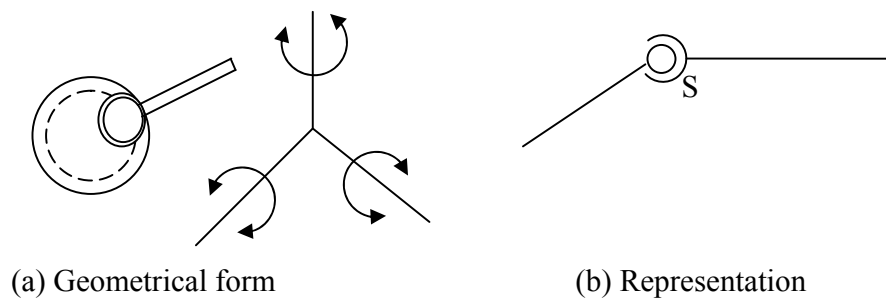

(a) Geometrical form	(b) Representation

Figure 5.5 A spherical joint

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(vi) *Planar Joint, L*

This three-DOF joint allows two translations along the two independent axes of the plane of contact and one rotation about the axis normal to the plane, Fig. 5.6.
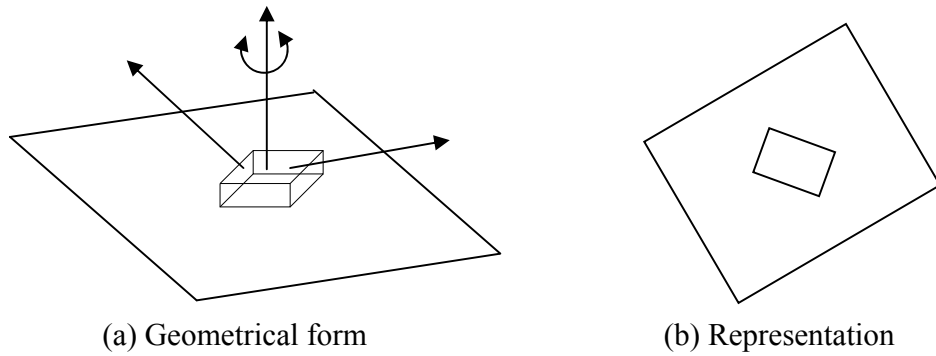

(a) Geometrical form   (b) Representation

Figure 5.6 A planar joint

Table 5.1 Lower pair joints

Name	Symbol	Geometric form and Representations	DOF	Common Surface
Revolute	*R*	Fig. 5.1	1	Cylinder
Prismatic	*P*	Fig. 5.2	1	Prism
Helical	*H*	Fig. 5.3	1	Screw
Cylindrical	*C*	Fig. 5.4	2	Cylinder
Spherical	*S*	Fig. 5.5	3	Sphere
Planar	*L*	Fig. 5.6	3	Plane

Table 5.1 summerizes the basic lower pair joints, where all the joints have surface contact between the interconnecting links. Another commonly used lower pair joint in robotics is the two-DOF 'universal joint,' as shown in Fig. 5.7. This is the combination of two intersecting revolute joints. Examples of higher pair joints in robots are 'gears' and 'cams with roller followers,' where they make line contacts.
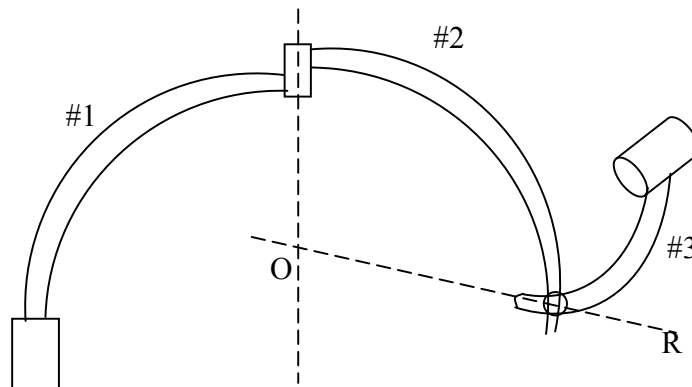
Figure 5.7 A universal joint

5.1.2 Kinematic chain

A 'kinematic chain' is a series of links connected by joints. When each and every link in a kinematic chain is coupled to at most two other links the chain is referred to as 'simple kinematic chain.' A simple kinematic chain can be either 'closed' or 'open.' It is closed if each and every link is coupled to two other links as shown in Fig. 5.8. A kinematic chain is open if it contains exactly two links, namely, the end ones, that are coupled to only one link. A robotic manipulator shown in Fig. 5.9 falls in this category.

> *Mechanism vs. Manipulator*
>
> Whereas a mechanism can be open- and closed-chain mechanical system, a manipulator is treated here as an open-chain system.
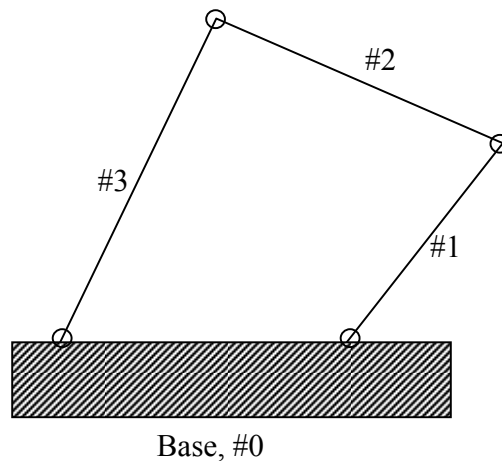

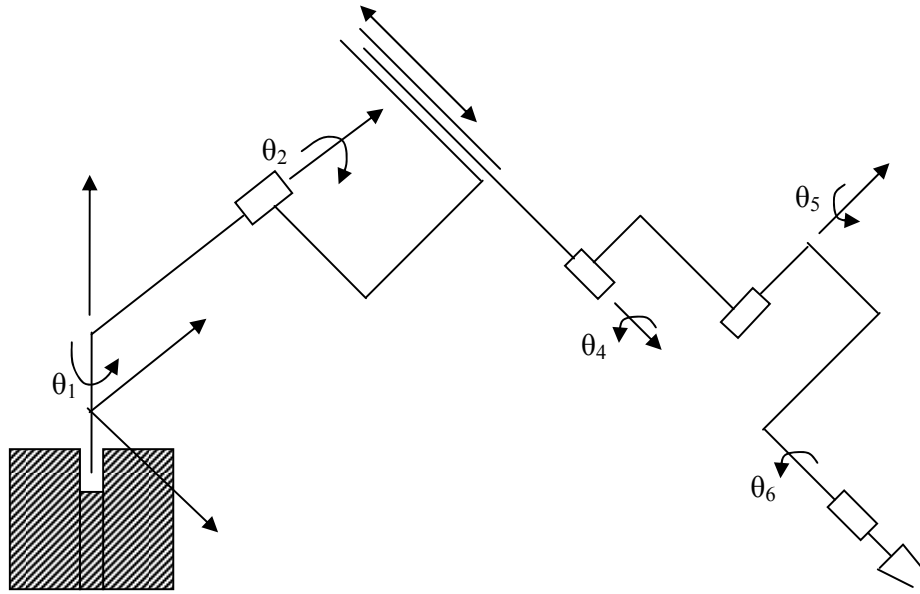Base, #0

Figure 5.8 A four-bar mechanism

Figure 5.9 A robot manipulator

5.1.3 Degree of freedom

Formally, the degree of freedom (DOF) of a mechanical system is defined as the number of independent coordinates or minimum coordinates required to fully describe its pose or configuration. Thus, a rigid body moving in the three dimensional Cartesian space has six-DOF, three each for position and orientation. Several methodologies exist to determine the DOF. One such method is given by Grubler in 1917 for planar mechanisms, which is later generalized by Kutzbach in 1929 for spatial mechanisms. Combinely, they are known as Grubler-Kutzbach citerion, which is derived below:

Assume,

s : dimension of working space (For planar, $s = 3$; spatial, $s = 6$);

r : number of rigid bodies or links in the system;

p : number of kinematic pairs or joints in the system;

c_i : number of constraints imposed by each joint;

c : total number of constraints imposed by p joints;

n_i : relative degree of freedom of each joint;

n : degree of freedom of the whole system.

The DOF of the whole system, n, is then determined from the total DOF associated with all the moving links minus the total number of constraints imposed by all the joints, i.e.,

$$n = s\,(r - 1) - c, \qquad \text{where } c \equiv \sum_{i=1}^{p} c_i \qquad \qquad \dots (5.1)$$

Note that -1 in eq. (5.1) corresponds to the fixed link of the system which has zero-DOF. Moreover, for the coupled links, the sum of the number of constraints imposed by each joint, c_i, and the relative degree of freedom permitted by that joint, n_i, is equal to the

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dimension of the working space, s, i.e., $s = c_i + n_i$. Hence, the total number of constraints imposed by all the joints can be re-written as

$$c = \sum_{i=1}^{p} c_i = \sum_{i=1}^{p} (s - n_i) = s\,p - \sum_{i=1}^{p} n_i \qquad \ldots (5.2)$$

Upon substitution of eq. (5.2) into eq. (5.1), one obtains the DOF, n, as

$$n = s(r - p - 1) + \sum_{i}^{p} n_i \qquad \ldots (5.3)$$

which is the well-known Grubler-Kutzbach criterion for the determination of a system's degree of freedom. Note that the link dimensions are not entered in the formula, which sometimes affect the DOF of the mechanism due to its geometrical arrangement of the links.

Example 5.1 Four-bar Mechanism

Figure 5.8 shows a planar four bar mechanism whose links, including the ground, are connected in a loop by four revolute joints. Here, $s = 3$, $r = 4$, and $p = 4$. Equation (5.3) gives

$$n = 3\,(4 - 4 - 1) + (1 + 1 + 1 + 1) = 1 \qquad \ldots (5.4)$$

Hence, the planar four-bar linkage has one DOF.

Example 5.2 A Robot Manipulator

Figure 5.9 shows the schematic diagram of a Stanford robot (Angeles, 2003). Its DOF can be determined using eq. (5.3) as $s = 6$, $r = 7$ (including the fixed base), $p = 6$. Then, from Eq. (5.3), the DOF of the robot is

$$n = 6\,(7 - 6 - 1) + 6 \times 1 = 6 \qquad \ldots (5.5)$$

Example 5.3 Five-bar Mechanism

Figure 5.10 shows a planar five-bar mechanism. There are five links connected in a loop by five revolute joints. Hence, $s = 3$, $r = 5$ (including the base), and $p = 5$. Equation. (5.3) yields

$$n = 3\,(5 - 5 - 1) + 5 \times 1 = 2 \qquad \ldots (5.6)$$

The planar five-bar mechanism has two DOF, which is often used as a robot manipulator also.
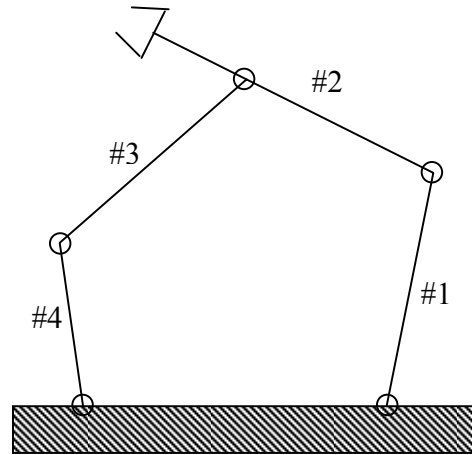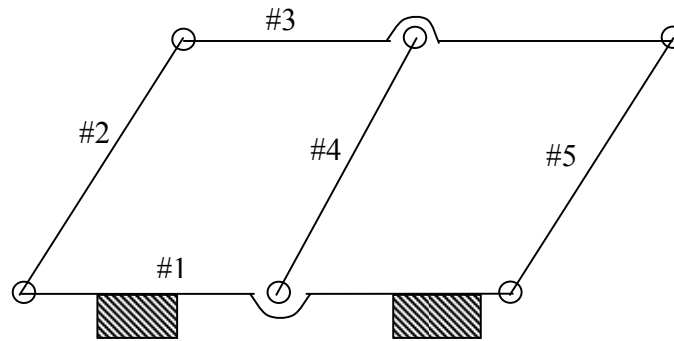
Figure 5.10 A five-bar mechanism

Figure 5.11 A double parallelogram mechanism

Example 5.4 A Double Parallelogram Mechanism

Figure 5.11 shows a double parallelogram mechanism, where $s = 3$, $r = 5$ (including the fixed base), $p = 6$. Then, eq. (5.3), gives

$$n = 3 (5 - 6 - 1) + 6 \times 1 = 0 \qquad \qquad \dots (5.7)$$

which interprets the system as a structure with zero DOF. This is not true, as the mechanism is known to have one-DOF. Actually, the link 5 has length equal to that of link 2. Hence, link 5 is kinematically redundant and can be removed without destroying the basic kinematic chain and subsequently the input-output behaviour of the mechanism.

In general, if the Grubler-Kutzbach criterion, eq. (5.3), yields $n > 0$, the mechanism has n-DOF. If the criterion yields $n = 0$, it is a structure with zero DOF, and if $n < 0$, the system is a statically indeterminate structure with redundant constraints. Exceptions, however, occur as the one shown in Fig. 5.11 does not obey the Grubler-Kutzback criterion. For such systems, different mobility criterion, e.g., Loop mobility criterion, as described by Tsai (1999), etc. can be employed to determine their DOF.

5.2 Pose of a Rigid Body

Rigid body motion in the three-dimensional Cartesian space comprises of translation and rotation. Whereas translation is defined using three Cartesian coordinates, the rotation needs three angular coordinates. Hence, the rigid body motion can be defined completely using six coordinates. In the study of the kinematics of robot manipulators, one constantly deals with the position and orientation of several bodies in space. The bodies of interest include the links of the manipulator, tools, and work piece. To identify the position and orientation of a body, i.e., its 'pose' or 'configuration,' a fixed reference coordinate system is established, which is called the 'fixed frame.' Next, a Cartesian coordinate system attached to the moving body is employed to describe its pose.

The pose or the position and orientation of a rigid body with respect to the reference coordinate system is known from the six independent parameters. As shown in Fig. 5.12, let *X-Y-Z-*coordinate system be the 'fixed

Difference between Pose and Configuration
They mean the same in this book, i.e., the position of a point on a rigid body and the orientation of the rigid body.

reference frame'. The *U-V-W*-coordinate system is attached to the moving body and referred as the 'moving frame.' Clearly, the pose or configuration of the rigid body is known if the pose of the moving frame with respect to the fixed frame is known. This pose is determined from the 'position' of any point on it, say, the origin, *O*, or point *P*, and the 'orientation' of the moving frame with respect to the fixed frame.

5.2.1 Position description

The 'position' of any point, *P*, on a rigid body in motion with respect to the fixed reference frame can be described by the 3-dimentional Cartesian vector, **p**, as indicated in Fig. 5.12. If the coordinates of point *P* or the components of vector **p** are, p_x, p_y, p_z, in the fixed frame *F*, it is denoted as

$$[\mathbf{p}]_F \equiv \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad \qquad \ldots (5.8)$$

where subscript *F* stands for the reference frame where the vector, **p**, is represented.

The subscripts, *x, y* and *z,* represent the projections of the position vector, **p**, onto the coordinate axes of the fixed reference frame, namely, along *X, Y* and *Z,* respectively. Vector **p** can alternatively be expressed as

$$\mathbf{p} = p_x \mathbf{x} + p_y \mathbf{y} + p_z \mathbf{z} \qquad \qquad \ldots (5.9)$$

where **x**, **y** and **z** denote the unit vectors along the axes, *X, Y* and *Z* of frame *F*, respectively, as indicated in Fig. 5.12. Their representations in frame *F*, namely $[\mathbf{x}]_F$, $[\mathbf{y}]_F$ and $[\mathbf{z}]_F$, are as follows :

$$[\mathbf{x}]_F \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{y}]_F \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad [\mathbf{z}]_F \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \ldots (5.10)$$
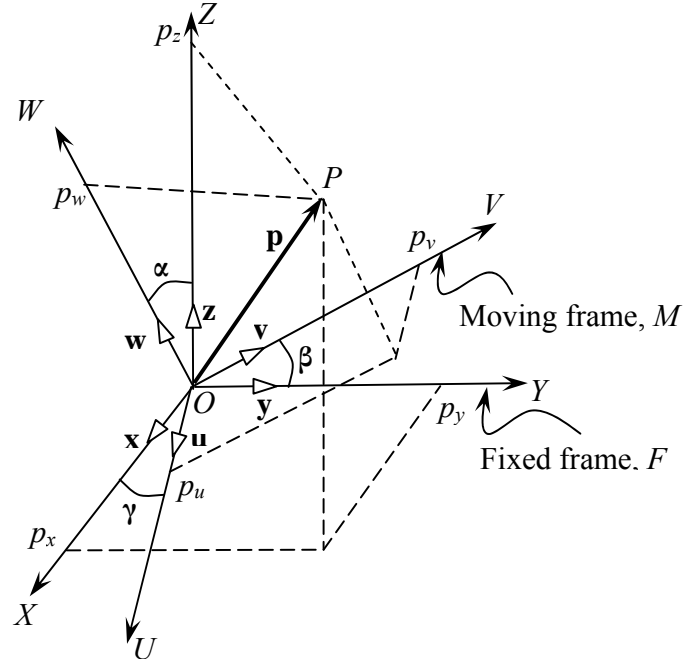
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Figure 5.12 Spatial description

Substituting eq. (5.10) into eq. (5.9), it can be shown that the expression of vector **p** in frame F, i.e., $[\mathbf{p}]_F$, is same as that of given in eq. (5.8). Note that if vector **p** is represented in another fixed frame different from frame F then the vector, **p,** will have different components along the new coordinate axes even though the actual position of point P has not changed. Thus, a vector will normally be written without mentioning any frame, e.g., as done in eq. (5.9) which, in contrast to eqs. (5.8) and (5.10), is termed as frame invariant representation, i.e., independent of the choice of any particular reference frame. It is, however, important to note that when a numerical problem is solved one must choose a suitable coordinate frame.

Example 5.5 Position of a point in the fixed frame

If coordinates of a point P in the fixed coordinate frame are, $p_x = 3$, $p_y = 4$, and $p_z = 5$, then the position vector, **p**, from eq. (5.8) can be given by

$$[\mathbf{p}]_F \equiv \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

5.2.2   Orientation description

The 'orientation' of a rigid body with respect to the fixed frame can be described in different ways. For example,

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(i)   Direction Cosine Representation;

(ii)   Euler Angle Representation; and others.

Each one has its own limitations. If necessary, one may switch from one to other representation during a robot's motion control to avoid former's limitations. Here, the above two will be presented which are sufficient to understand the underlying concepts and their limitations. Anybody interested in other representations may refer to the books by Tsai (1999) or Angeles (2003).

(i) *Direction Cosine Representation*

To describe the orientation or rotation of a rigid body, consider the motion of a moving frame M with respect to a fixed frame F, with one point fixed, say, the origin of the fixed frame, O, as shown in Fig. 5.12. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ denote the three unit vectors pointing along the coordinates axes, U, V and W of the moving frame M, respectively, similar to the unit vectors, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, along X, Y and Z of the fixed frame F, respectively. Since each of the unit vectors, $\mathbf{u}$, $\mathbf{v}$ or $\mathbf{w}$, denotes the position of a point at a unit distance from the origin on the axes of frame M, they are expressed using their projections on the X, Y, Z axes of the frame F as

$$\mathbf{u} = u_x\,\mathbf{x} + u_y\,\mathbf{y} + u_z\,\mathbf{z} \qquad\qquad \ldots (5.11\text{a})$$

$$\mathbf{v} = v_x\,\mathbf{x} + v_y\,\mathbf{y} + v_z\,\mathbf{z} \qquad\qquad \ldots (5.11\text{b})$$

$$\mathbf{w} = w_x\,\mathbf{x} + w_y\,\mathbf{y} + w_z\,\mathbf{z} \qquad\qquad \ldots (5.11\text{c})$$

where u_x, u_y and u_z are the components of the unit vector, $\mathbf{u}$, along X, Y and Z axes, respectively. Similarly v_x, v_y, v_z, and w_x, w_y, w_z are defined for the unit vectors, $\mathbf{v}$ and $\mathbf{w}$, respectively. Now, point P of the rigid body, as shown in Fig. 5.12 and given by eq. (5.8), is expressed in the moving frame, M, as

$$\mathbf{p} = p_u\mathbf{u} + p_v\mathbf{v} + p_w\mathbf{w} \qquad\qquad \ldots (5.12)$$

where p_u, p_v, and p_w are the components of the vector, $\mathbf{p}$, along U, V, W axes of the moving frame, M. Upon substitution of eqs. (5.11a-c) into eq. (5.12) yields

$$\mathbf{p} = (p_uu_x + p_vv_x + p_ww_x)\mathbf{x} + (p_uu_y + p_vv_y + p_ww_y)\mathbf{y} + (p_uu_z + p_vv_z + p_ww_z)\mathbf{z} \quad \ldots (5.13)$$

Comparing the right hand sides of eqs. (5.8) and (5.13), the following identities are obtained:

$$p_x = u_xp_u + v_xp_v + w_xp_w \qquad\qquad \ldots (5.14\text{a})$$

$$p_y = u_yp_u + v_yp_v + w_yp_w \qquad\qquad \ldots (5.14\text{b})$$

$$p_z = u_zp_u + v_zp_v + w_zp_w \qquad\qquad \ldots (5.14\text{c})$$

Equations (5.14a-c) are written in a matrix form as

$$[\mathbf{p}]_F = \mathbf{Q}\,[\mathbf{p}]_M \qquad\qquad \ldots (5.15)$$

where $[\mathbf{p}]_F$ and $[\mathbf{p}]_M$ are the representations of the 3-dimensional vector, $\mathbf{p}$, in frames F and M, respectively, and $\mathbf{Q}$ is the 3×3 rotation or orientation matrix transforming the representation of vector $\mathbf{p}$ from frame M to F. They are given as follows:

$$[\mathbf{p}]_F \equiv \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \ [\mathbf{p}]_M \equiv \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}, \ \text{and} \ \mathbf{Q} \equiv \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} = \begin{bmatrix} \mathbf{u^T x} & \mathbf{v^T x} & \mathbf{w^T x} \\ \mathbf{u^T y} & \mathbf{v^T y} & \mathbf{w^T y} \\ \mathbf{u^T z} & \mathbf{v^T z} & \mathbf{w^T z} \end{bmatrix} \quad \dots (5.16)$$

Note the columns of matrix $\mathbf{Q}$. They are nothing but the components of the orthogonal (meaning 90° to each other) unit vectors, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in the fixed frame F, which must satisfy the following six orthogonal conditions:

$$\mathbf{u^T u} = \mathbf{v^T v} = \mathbf{w^T w} = 1, \text{ and } \mathbf{u^T v}(\equiv \mathbf{v^T u}) = \mathbf{u^T w}(\equiv \mathbf{w^T u}) = \mathbf{v^T w}(\equiv \mathbf{w^T v}) = 0 \quad \dots (5.17)$$

Moreover, for the three orthogonal vectors, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, the following hold true:

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}, \quad \mathbf{v} \times \mathbf{w} = \mathbf{u}, \quad \text{and} \quad \mathbf{w} \times \mathbf{u} = \mathbf{v} \quad \dots (5.18)$$

Hence, the 3×3 'rotation matrix,' $\mathbf{Q}$, denoting the orientation of the moving frame, M, with respect to the fixed frame, F, is called 'orthogonal.' It satisfies the following properties due to eq. (5.17):

$$\mathbf{Q^T Q} = \mathbf{QQ^T} = \mathbf{1}; \ \text{where} \ \det(\mathbf{Q}) = 1, \quad \text{and} \quad \mathbf{Q}^{-1} = \mathbf{Q}^T \quad \dots (5.19)$$

$\mathbf{1}$ being the 3×3 identity matrix. Moreover, if one is interested to find the rotation description of the frame, F, with respect to frame M denoted by $\mathbf{Q'}$, it can be derived similarly. It can be shown that $\mathbf{Q'} = \mathbf{Q}^T$.

Also note from eq. (5.16), that the (1,1) element of $\mathbf{Q}$ is the cosine of the angle between the vectors $\mathbf{u}$ and $\mathbf{x}$, i.e., $\mathbf{u^T x}$. The same holds true with other elements of $\mathbf{Q}$. Hence, this rotation matrix is known as the 'Direction Cosine Representation' of the rotation matrix. Such representation requires nine parameters, namely, the elements of the 3×3 matrix, $\mathbf{Q}$. However, not all the nine parameters are independent as they must satisfy the six conditions of eq. (5.17). Thus, only three parameters are independent and should be sufficient to define the three-DOF rotational motion. It is, however, difficult to choose the set of three independent parameters. This is the drawback of the Direction Cosine Representation.

Example 5.6 Elementary Rotations

Suppose that a reference frame M coincides with the fixed frame F. Now, frame M is rotated by an angle α about the axis Z, as shown in Fig. 5.13(a). The unit vectors of the new frame M can be described in terms of their components in the reference frame, F, as
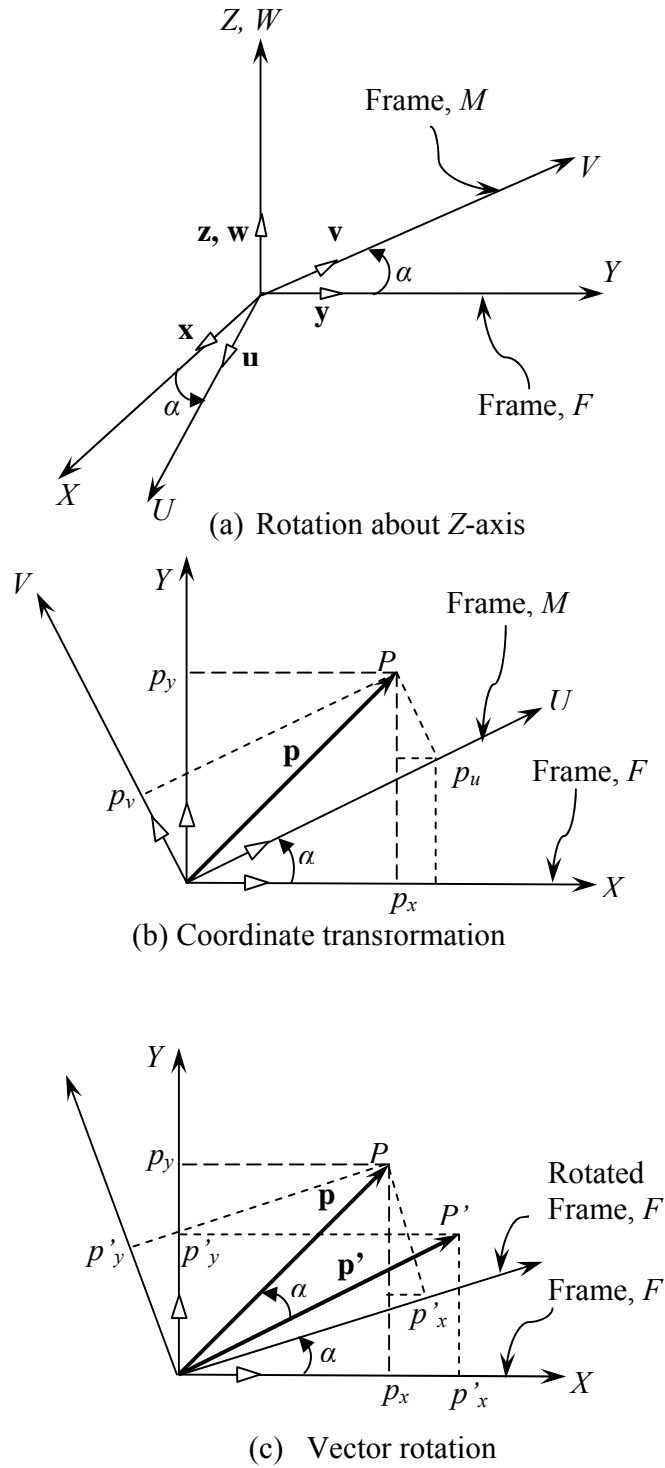
(a) Rotation about *Z*-axis

(b) Coordinate transformation

(c) Vector rotation

Figure 5.13 Alternative interpretations of rotation matrix.

$$[\mathbf{u}]_F \equiv \begin{bmatrix} C\alpha \\ S\alpha \\ 0 \end{bmatrix}, \quad [\mathbf{v}]_F \equiv \begin{bmatrix} -S\alpha \\ C\alpha \\ 0 \end{bmatrix}, \quad \text{and} \quad [\mathbf{w}]_F \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \ldots (5.20)$$

where, $S \equiv \sin$; $C \equiv \cos$. Hence, the rotation matrix denoted by $\mathbf{Q}_Z$ is given by

$$\mathbf{Q}_Z \equiv \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.21)$$

In a similar manner, it can be shown that the rotations of angle β about axis *Y,* and by an angle γ about axis *X* are, respectively, given by

$$\mathbf{Q}_Y \equiv \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix}, \quad \text{and} \quad \mathbf{Q}_X \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix} \qquad \ldots (5.22)$$

The above matrices in eqs. (5.21) and (5.22) are called the 'elementary rotations' and are useful to describe any arbitrary rotation while the angles with respect to the coordinate axes are known.

Example 5.7 Properties of Elementary Rotation Matrices

From eqs. (5.21) and (5.22), matrix multiplication of, say, $\mathbf{Q}_z^T$ and $\mathbf{Q}_z$ results the following:

$$\mathbf{Q}_z^T \mathbf{Q}_z = \begin{bmatrix} C\alpha & S\alpha & 0 \\ -S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the definition of determinant given in Appendix A, it can be easily calculated that $\det(\mathbf{Q}_z)=1$. Hence, $\mathbf{Q}_z$ satisfies both the properties of a rotation matrix. Similarly, the other two matrices can be shown to have the properties of a rotation matrix as well.

Example 5.8 Coordinate Transformation

Consider two coordinate frames with a common origin. One of them is rotated by an angle α about the axis *Z*. Let $[\mathbf{p}]_F$ and $[\mathbf{p}]_M$ be the vector representations of point *P* in frames *F* and *M*, respectively, as shown in Fig. 5.13(b). On the basis of simple geometry, the relationships between the coordinates of point *P* in the two coordinate frames are given by

$$p_x = p_u\, C\alpha - p_v\, S\alpha \qquad \ldots (5.23)$$

$$p_y = p_u\, S\alpha + p_v\, C\alpha \qquad \ldots (5.24)$$

$$p_z = p_w \qquad \ldots (5.25)$$

where p_x, p_y, p_z, and p_u, p_v, p_w are the three coordinates of point *P* along the axes of frames *F* and *M*, respectively. It is easy to recognize from eqs. (5.23) to (5.25) that the vector $[\mathbf{p}]_F \equiv [p_x, p_y, p_z]^T$ is nothing but

$$[\mathbf{p}]_F = \mathbf{Q}_Z\, [\mathbf{p}]_M \qquad \ldots (5.26)$$

where $[\mathbf{p}]_M \equiv [p_u, p_v, p_w]^T$ and $\mathbf{Q}_Z$ is given by eq. (5.21). Matrix $\mathbf{Q}_Z$ not only represents the orientation of the frame, *M,* with respect to the fixed frame, *F,* as in Fig. 5.13(a), but

also transforms the representation of a vector from frame *M*, say, $[\mathbf{p}]_M$, to another frame *F* with the same origin, i.e., $[\mathbf{p}]_F$.

Example 5.9 Vector Rotation

Consider the vector, **p**, which is obtained by rotating a vector **p′** in the *X-Y*-plane by an angle α about the axis, *Z,* of the reference frame, F of Fig. 5.13(c). Let p'_x, p'_y, p'_z be the coordinates of vector **p′** in frame *F*, i.e., $[\mathbf{p}']_F \equiv [p'_x, p'_y, p'_z]^T$. Then, vector **p** in frame *F*, $[\mathbf{p}]_F$, has the following components.

$$p_x = p'_x\, C\alpha - p'_y\, S\alpha \qquad\qquad \dots (5.27)$$

$$p_y = p'_x\, S\alpha + p'_y\, C\alpha \qquad\qquad \dots (5.28)$$

$$p_z = p'_z \qquad\qquad \dots (5.29)$$

which can be obtained by rotating the fixed frame, *F*, with which vector **p'** is attached by an angle α counter clockwise about the *Z*-axis so that vector **p'** reaches p, as indicated in Fig. 5.13(c). Equations (5.27)-(5.29) can be written in compact form as

$$[\mathbf{p}]_F = \mathbf{Q}_Z\, [\mathbf{p}']_F \qquad\qquad \dots (5.30)$$

where $\mathbf{Q}_Z$ is given by eq. (5.21).

(ii) *Euler Angles Representation*

The Euler angles constitute a 'minimal' representation of orientation obtained by composing three elementary rotations with respect to the axes of current frames. There is a possibility of twelve distinct sets of Euler angles with regard to the sequence of possible elementary rotations, namely, *XYZ, XZY, XZX, XYX, YXZ, YZX, YXY, YZY, ZXY, ZYX, ZXZ,* and *ZYZ*. Amongst all, the *ZYZ* set is commonly used in the Euler angles representation. This implies that the fixed frame, *F*, first rotates about its *Z*-axis to reach an intermediate frame A, then about *Y*-axis of the rotated frame, i.e., *Y'* of frame *A*, to reach another intermediate frame *B*, and, finally, about *Z*-axis of the twice rotated frame, i.e., *Z''* of frame *B*, to reach the desired frame *M*, as shown in Fig. 5.14. Let ϕ, θ, and φ be the angles about *Z*, *Y'*, and *Z''*, respectively. The overall rotation described by these angles is then obtained as the composition of elementary rotations, as explained below: Referring to Fig. 5.14,
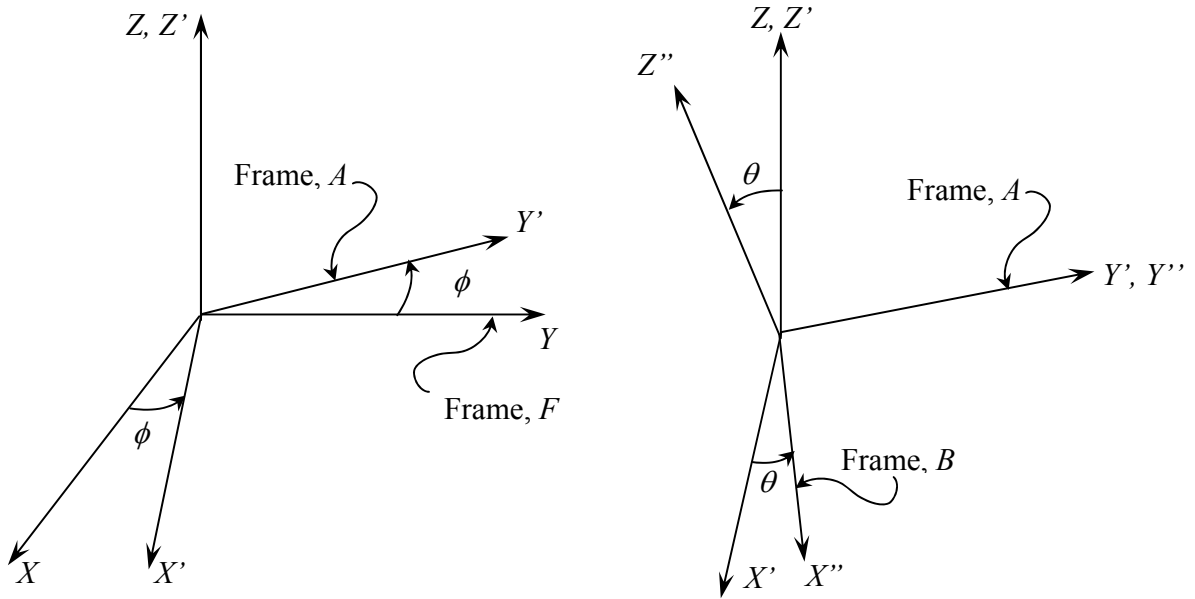
(a) Rotation about *Z*-axis   (b) Rotation about new *Y*-axis, i.e., *Y'*-axis
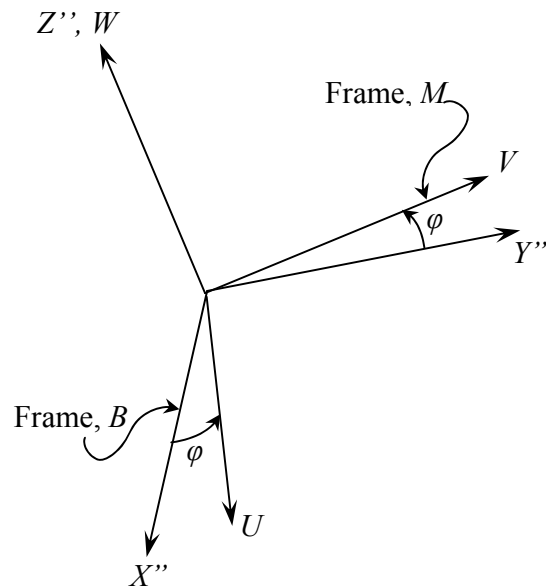
(c) Rotation about new *Z*-axis, i.e., *Z''*-axis

Figure 5.14 The ZYZ Euler angles

- Rotate the fixed frame, *F*, by the angle, ϕ, about axis Z, as indicated in Fig. 5.14(a). This rotation is described by the rotation matrix $\mathbf{Q}_Z$, as derived in eq. (5.21), i.e.,

$$\mathbf{Q}_Z \equiv \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

...(5.31a)

- Rotate the current frame *A* by an angle θ about its *Y'* axis, Fig. 5.14(b). This rotation is denoted by $\mathbf{Q}_{Y'}$ and described by the rotation matrix, $\mathbf{Q}_Y$ of eq. (5.22), i.e.,

$$\mathbf{Q}_{Y'} \equiv \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix}$$

...(5.31b)

- Rotate the current frame *B* by angle φ about its *Z''* axis, Fig. 5.14(c). This rotation is denoted by $\mathbf{Q}_{Z''}$ and described by the rotation matrix, $\mathbf{Q}_Z$ of eq. (5.21), i.e.,

$$\mathbf{Q}_{Z''} \equiv \begin{bmatrix} C\varphi & -S\varphi & 0 \\ S\varphi & C\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

...(5.31c)

The resulting orientation of frame *M*, denoted by $\mathbf{Q}$, is obtained from the composition of the three elementary rotations, $\mathbf{Q}_Z$, $\mathbf{Q}_{Y'}$ and $\mathbf{Q}_{Z''}$, with respect to their current frames. It is obtained via postmultiplications of the successive rotation matrices, i.e.,

$$\mathbf{Q} = \mathbf{Q}_Z\mathbf{Q}_{Y'}\mathbf{Q}_{Z''}$$

. . . (5.31d)

whose elements are computed below:

$$\mathbf{Q} \equiv \begin{bmatrix} C\phi C\theta C\varphi - S\phi S\varphi & -C\phi C\theta S\varphi - S\phi C\varphi & C\phi S\theta \\ S\phi C\theta C\varphi + C\phi S\varphi & -S\phi C\theta S\varphi + C\phi C\varphi & S\phi S\theta \\ -S\theta C\varphi & S\theta S\varphi & C\theta \end{bmatrix}$$

. . . (5.31e)

The drawback of the minimal Euler angles representation, eq. (5.31e), is that it sometimes fails in finding the solution of an inverse problem, i.e., for a given rotation matrix find the equivalent Euler rotation angles. For example, if $\mathbf{Q}$ is given as follows:

$$\mathbf{Q} \equiv \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

...(5.32a)

then the angle, ϕ, can be obtained from the comparison of (1,3) and (2,3) elements of $\mathbf{Q}$ given by in eqs. (5.31e) and (5.32b), i.e.,

$$\varphi = atan2(\frac{q_{23}}{S\theta}, \frac{q_{13}}{S\theta})$$

. . $X^{//}$

where *"atan2(y,x)"* is the two-argument function that yields one unique solution for the angle. The solution of eq. (5.32b) exists provided $S\theta \neq 0$, i.e., when $\theta \neq 0$ or $n\pi$, for $n = 1, 2, \ldots$ An important aspect of the rotation representation, either direction cosine or Euler angle or any other, is that unlike vectors it is non-commutative, i.e., the order of rotations are important in deriving the correct rotation matrix. As illustrated in Fig. 5.15(a-c), the rotation of a box about *Z* and current *Y* axes is different from the rotation about *Y* and current *Z* axes, as shown in Fig. 5.16(a-c). In order to show it mathematically, the resultant matrix for the rotations about *Z* and *Y*-axes, denoted by $\mathbf{Q}_{ZY}$, can be given by

$$\mathbf{Q}_{ZY} = \mathbf{Q}_Y\mathbf{Q}_Z = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \dots (5.33a)$$

where $\mathbf{Q}_Z$, and $\mathbf{Q}_Y$ are as follows:

$$\mathbf{Q}_Y \equiv \begin{bmatrix} C90^o & 0 & S90^o \\ 0 & 1 & 0 \\ -S90^o & 0 & C90^o \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}; \mathbf{Q}_Z \equiv \begin{bmatrix} C90^o & -S90^o & 0 \\ S90^o & C90^o & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \dots(5.33b)$$

Note the order of multiplication in eq. (5.33a) is reverse to that of eq. (5.31d). This is due to the fact that the rotations in eq. (5.31d) are with respect to the current frames, whereas in eq. (5.33a) they are with respect to the fixed frame. The result of eq. (5.33a) can be verified from Fig. 5.15(c) using the concept of direction cosine representation of a rotation matrix explained earlier. Similarly, the resultant rotation matrix for the rortations about *Y* and *Z*-axes, denoted by $\mathbf{Q}_{YZ}$, can be obtained as

$$\mathbf{Q}_{YZ} = \mathbf{Q}_Z\mathbf{Q}_Y = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \qquad \dots (5.33c)$$
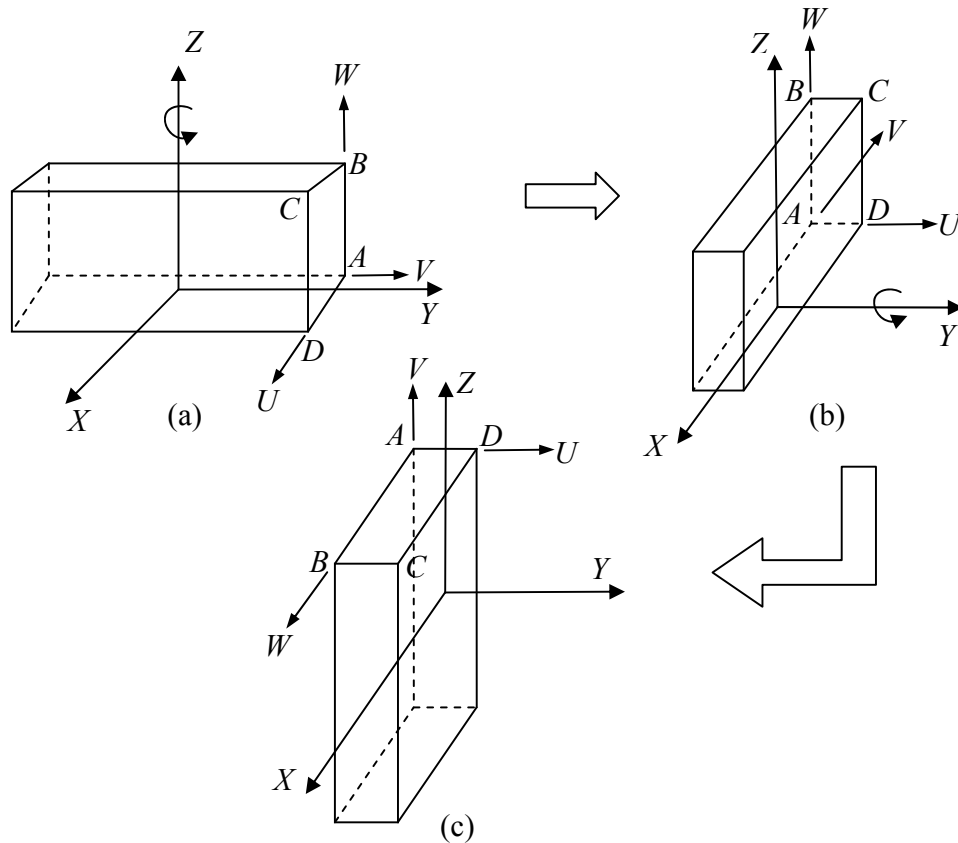

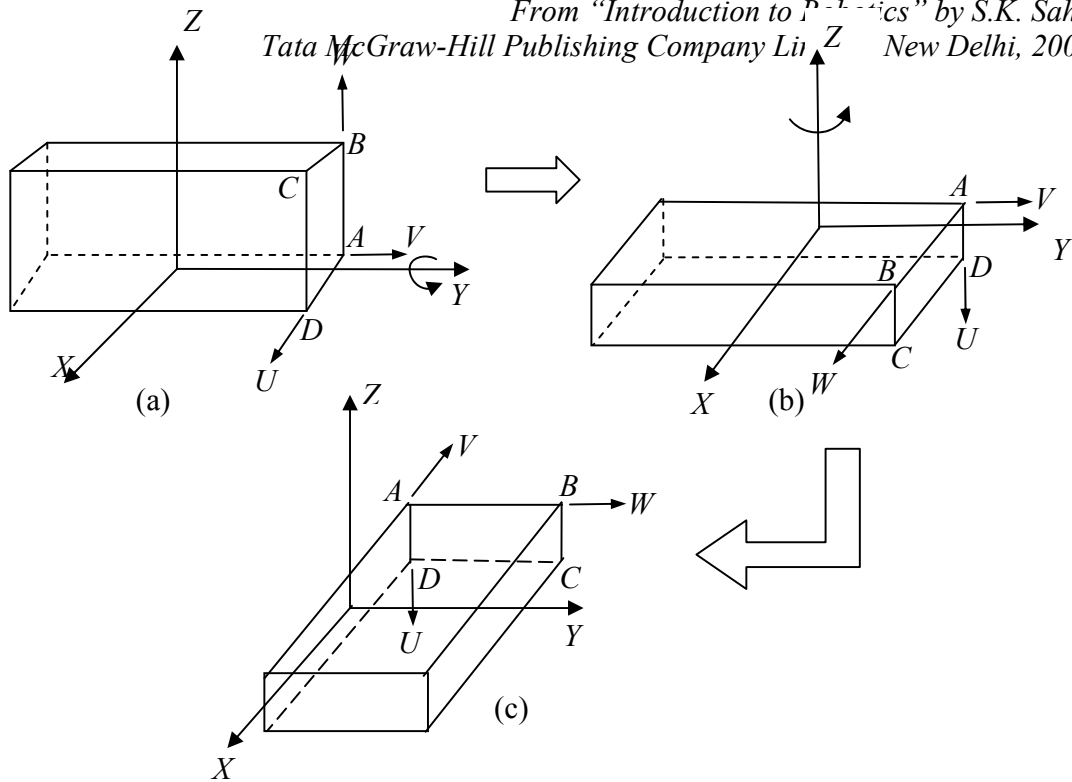Figure 5.15 Successive rotation of a box about *Z* and *Y*-axes

Figure 5.16 Successive rotation of a box about *Y* and *Z*-axes

Comparing eqs. (5.33a) and (5.33c), it is obvious that the two sequences of rotations are different. However, the same is not true for the vector representation of a point *P* with respect to *O*, one can either add vector **b** to **a** or **a** to **b**, as in Fig. 5.17. Both additions will result the same vector **p**.
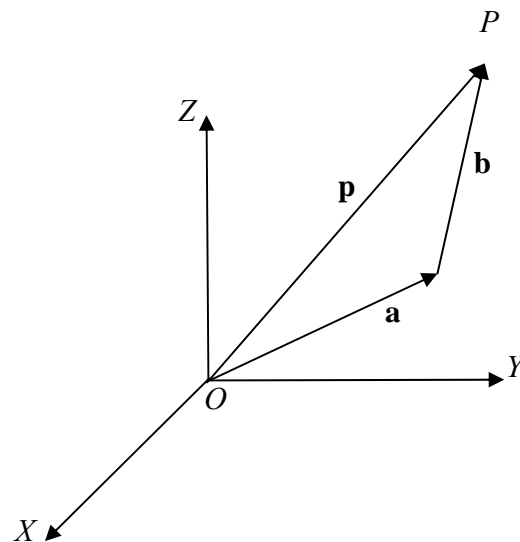
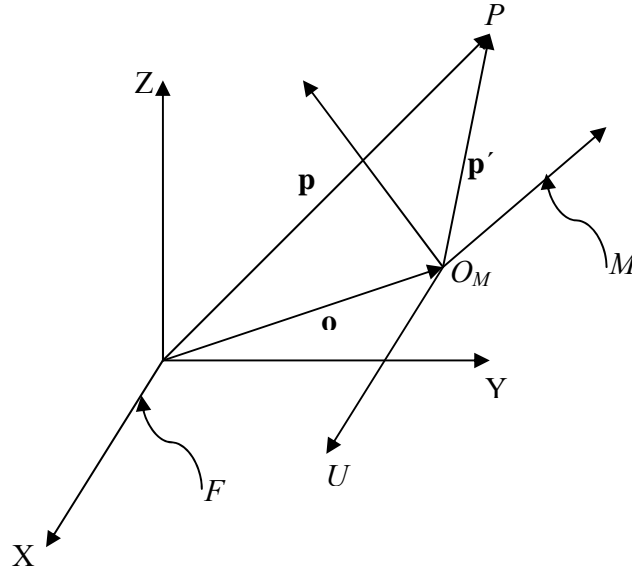Figure 5.17 Addition of two vectors

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Figure 5.18 Two coordinate frames

5.3 Coordinate Transformation

As illustrated in Sub-section 5.2.1, the position of a rigid body in space is expressed in terms of the position of a suitable point on the body, say, O_M in Fig. 5.18, with respect to the origin O, of the fixed frame, F, while its orientation or rotation is expressed in terms of the unit vector components of the moving frame, M, attached to the body, as explained in Sub-section 5.2.2. Referring to Fig. 5.18, the relation between the two coordinate frames, namely, F and M, are derived as follows: Consider an arbitrary point P on the rigid body with which the frame M is attached at O_M. Let **p** and **p′** be the vectors denoting the point, P, from the origin of the frames, F and M, respectively. In addition, let **o** be the position vector denoting the translation of the origin of frame M, O_M, from that of frame F, i.e, O. Thus,

$$\mathbf{p} = \mathbf{o} + \mathbf{p}' \qquad \qquad \dots (5.34)$$

Note that if **p′** is known in the moving frame M, then it is nothing but the coordinates of point P in frame M, i.e., $[\mathbf{p}']_M$. Moreover, if **Q** is the orientation of frame M with respect to frame F then vector **p'** in frame F, i.e., $[\mathbf{p}']_F = \mathbf{Q}[\mathbf{p}']_M$. Hence, the vector, **p,** in the fixed frame F, i.e., $[\mathbf{p}]_F$, can be obtained as

$$[\mathbf{p}]_F = [\mathbf{o}]_F + \mathbf{Q}[\mathbf{p}']_M \qquad \qquad \dots (5.35)$$

Equation (5.35) represents the 'coordinate transformation,' of point P of the rigid body from moving frame M to the fixed frame, F, while both translation and rotation are involved.

5.3.1   Homogeneous transformation

The coordinate transformation given by eq. (5.35), can be rearranged as

$$\begin{bmatrix} [\mathbf{p}]_F \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & [\mathbf{o}]_F \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} [\mathbf{p}']_M \\ 1 \end{bmatrix} \qquad \dots (5.36)$$

where $\mathbf{0} \equiv [0, 0, 0]^T$ is the three-dimensional vector of zeros. Equation (5.36) is written in a compact form as

$$[\overline{\mathbf{p}}]_F = \mathbf{T}[\overline{\mathbf{p}'}]_M \qquad \dots (5.37)$$

where $[\overline{\mathbf{p}}]_F$ and $[\overline{\mathbf{p}'}]_M$ are the four-dimensional vectors obtained by putting one at the bottom of the original three-dimensional vectors, $[\mathbf{p}]_F$ and $[\mathbf{p}']_M$, respectively, as the fourth element, whereas the 4×4 matrix, $\mathbf{T}$, is called 'homogeneous transformation matrix.' Equation (5.37) is simple in a sense that the transformation of a vector, which includes both translation and rotation from frame M to F, is done by just multiplying one 4×4 matrix, instead of a matrix multiplication and vector addition, as in eq. (5.35). However, from the view of computational complexity, i.e., the numbers of multiplications/divisions and additions/subtractions required in a computer program, eq. (5.35) is economical compared to eq. (5.36) or (5.37), as some unnecessary multiplications and additions with 1s and 0s will

> ### *Why Homogenous?*
>
> Matrix $\mathbf{T}$ of eq. (5.36) takes care of both the translation and rotation of the frame attached to the body with respect to the fixed frame.

be performed. It is pointed out here that, for the homogeneous transformation matrix, $\mathbf{T}$, the orthogonality property does not hold good, i.e.,
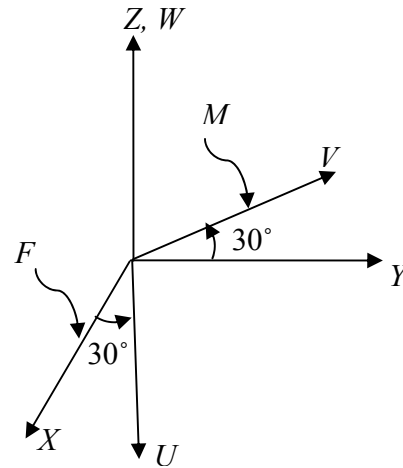
$$\mathbf{T}^T\mathbf{T} \neq \mathbf{1} \qquad \text{or} \qquad \mathbf{T}^{-1} \neq \mathbf{T}^T \qquad \dots (5.38)$$

However, the inverse of the homogeneous transformations matrix, $\mathbf{T}$, can be obtained easily from eq. (5.36) as

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{Q}^T & -\mathbf{Q}^T[\mathbf{o}]_F \\ \mathbf{0}^T & 1 \end{bmatrix} \qquad \dots (5.39)$$
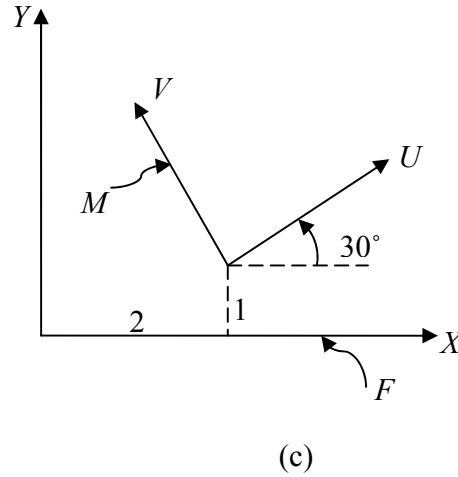

(a)　　　　　　　　　　　　　　　　(b)

(c)
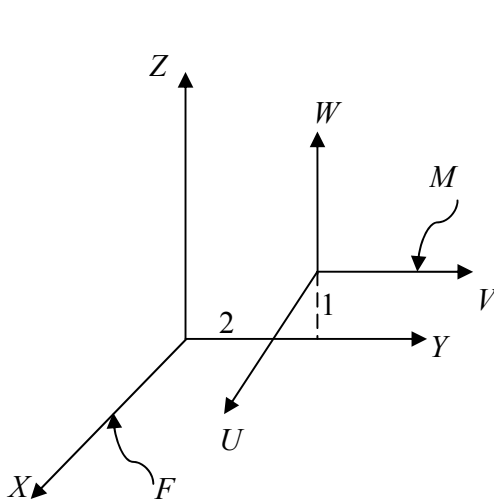
Figure 5.19 Different motions

Example 5.10 Pure Translation

Referring to Fig. 5.19(a), consider the frame, *M,* which is obtained from frame *F* by the translation it by two units along *Y,* and one unit along *Z.* Their relation is represented by a homogeneous transformation matrix, namely,

$$\mathbf{T} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \dots (5.40)$$

Example 5.11 Pure Rotation

Consider Fig. 5.19(b), where frame *M* is obtained from frame *F* by rotating it about its *Z* axis by an angle of 30°. The homogeneous transformation matrix for the pure rotation is then given by

$$\mathbf{T} \equiv \begin{bmatrix} C30° & -S30° & 0 & 0 \\ S30° & C30° & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{\dfrac{3}{2}} & -\dfrac{1}{2} & 0 & 0 \\ \dfrac{1}{2} & \sqrt{\dfrac{3}{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \dots (5.41)$$

Example 5.12 General Motion

As shown in Fig. 5.19(c), frame *M* is obtained from frame *F* by translating it along *X* and *Y* axes by 2 and 1 units, respectively, followed by a rotation of 30^0 about *W.* The corresponding homogeneous transformation matrix is given from the multiplication of the homogeneous transformation matrices representing the pure translation and rotation, respectively, with respect to the current frames, i.e.,

$$\mathbf{T} \equiv \mathbf{T}_t \mathbf{T}_r \qquad \dots (5.42)$$

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where subscripts '*t*' and '*r*' stand for 'translation' and 'rotation', respectively, whereas $\mathbf{T}_t$ and $\mathbf{T}_t$ are given by eqs. (5.40) and (5.41), respectively. The resultant matrix, $\mathbf{T}$, is given by

$$\mathbf{T} \equiv \begin{bmatrix} C30^\circ & -S30^\circ & 0 & 2 \\ S30^\circ & C30^\circ & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} & 0 & 2 \\ \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.43)$$

Example 5.13 Transfer of a Point

Referring to Fig. 5.20, consider a point P, which is expressed in frame F and denoted by vector $\mathbf{p}_1$. Let $\mathbf{p}_2$ denotes the new location of this point after rotating $\mathbf{p}_1$ by 30° about Z, and then translating 2 units along Y and -1 unit along Z axes. The relation between $\mathbf{p}_1$ and $\mathbf{p}_2$ can be described as

$$[\mathbf{p}_2]_F = \mathbf{T}[\mathbf{p}_1]_F, \text{ where } \mathbf{T} \equiv \mathbf{T}_t\mathbf{T}_r = \begin{bmatrix} C30^\circ & -S30^\circ & 0 & 0 \\ S30^\circ & C30^\circ & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.44a)$$

where $\mathbf{T}_t$ and $\mathbf{T}_r$ are corresponding to the pure translation and pure rotation, respectively, i.e.,

$$\mathbf{T}_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{T}_r = \begin{bmatrix} C30^\circ & -S30^\circ & 0 & 0 \\ S30^\circ & C30^\circ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.44b)$$

Note that the order of multiplications in eq. (5.44a), which is similar to eq. (5.33a) or (5.33c), as the motions are specified with respect to the fixed frame, F, instead of the current frames.
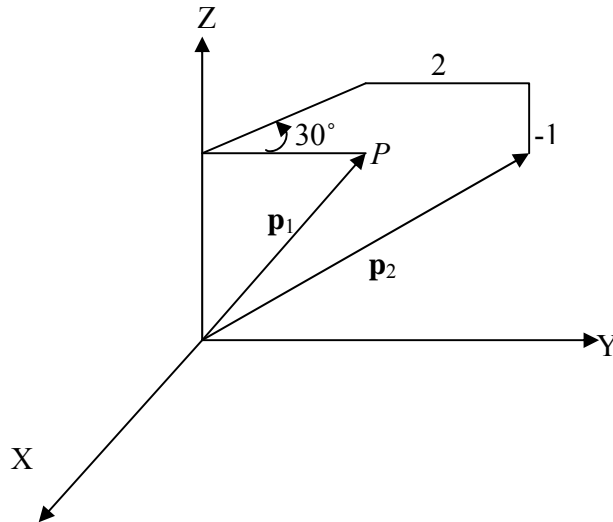

Figure 5.20 Transfer of a point

Example 5.14 Product of two Homogeneous Transformations

Assume that $\mathbf{T}_A$ and $\mathbf{T}_B$ are given by

$$\mathbf{T}_A = \begin{bmatrix} C\,30^o & -S\,30^o & 0 & 2 \\ S\,30^o & C\,30^o & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{T}_B = \begin{bmatrix} C\,45^o & S\,45^o & 0 & 1 \\ -S\,45^o & C\,45^o & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.45)$$

which are graphically represented in Fig. 5.21. Note that each of the transformations, $\mathbf{T}_A$ or $\mathbf{T}_B$, is a product of two elementary transformations representing translation and rotation with respect to the current frames. For example, matrix $\mathbf{T}_A$ can be written as,

$$\mathbf{T}_A = \mathbf{T}_{At}\mathbf{T}_{Ar} \qquad \ldots (5.46a)$$

wehre the 4×4 homogeneous transformation matrices, $\mathbf{T}_{At}$ and $\mathbf{T}_{Ar}$, are associated with the translation and rotation from the fixed frame, F, to frame A, respectively, i.e.,
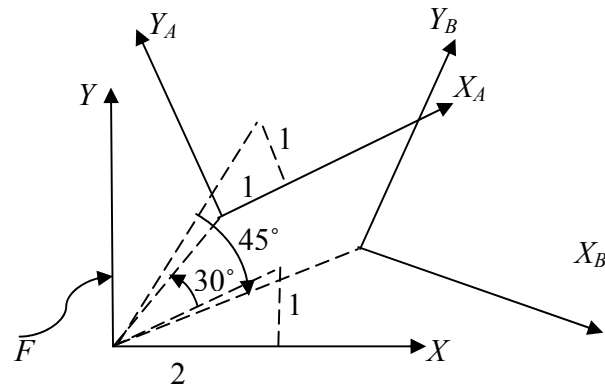
$$\mathbf{T}_{At} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{T}_{Ar} = \begin{bmatrix} C\,30^o & -S\,30^o & 0 & 0 \\ S\,30^o & C\,30^o & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.46b)$$
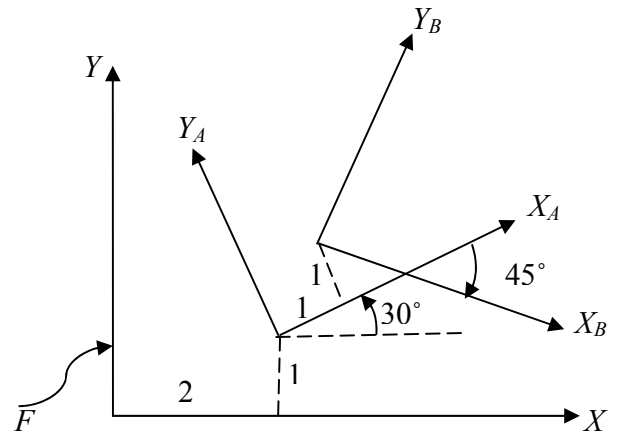
(a) Transformations with respect to current frames

(b) Transformations with respect to fixed frame

Figure 5.21 Graphical interpretations of the products of two homogeneous transformation matrices

The total transformation to reach frame *B* from frame *F* is now obtained as

$$\mathbf{T} = \mathbf{T}_A \mathbf{T}_B \qquad \qquad \text{... (5.47a)}$$

The 4 × 4 matrix, **T**, is expressed as

$$\mathbf{T} = \begin{bmatrix} \dfrac{\sqrt{3}+1}{2\sqrt{2}} & \dfrac{\sqrt{3}-1}{2\sqrt{2}} & 0 & \dfrac{3+\sqrt{3}}{2} \\ -\dfrac{\sqrt{3}-1}{2\sqrt{2}} & \dfrac{\sqrt{3}+1}{2\sqrt{2}} & 0 & \dfrac{3+\sqrt{3}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{... (5.47b)}$$

Note that if the same transformations are performed with respect to the fixed frame, *F*, as shown in Fig. 5.21(b), the order of multiplications has to be reversed, i.e.,

$$\mathbf{T} = [\mathbf{T}_B]_F [\mathbf{T}_A]_F \qquad \qquad \text{... (5.48a)}$$

where $[\mathbf{T}_A]_F$ and $[\mathbf{T}_B]_F$ are also obtained from the reverse order multiplications of the translations and rotations, e.g., $[\mathbf{T}_A]_F = [\mathbf{T}_{Ar}]_F [\mathbf{T}_{At}]_F$ --- $\mathbf{T}_{Ar}$ and $\mathbf{T}_{Ar}$ being the 4 × 4 matrices given in eq. (5.46b). Matrices $[\mathbf{T}_A]_F$, $[\mathbf{T}_B]_F$ and **T** are given as follows:

$$[\mathbf{T}_A]_F = \begin{bmatrix} C\,30^o & -S\,30^o & 0 & \sqrt{3}-\dfrac{1}{2} \\ S\,30^o & C\,30^o & 0 & 1+\dfrac{\sqrt{3}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; [\mathbf{T}_B]_F = \begin{bmatrix} C\,45^o & S\,45^o & 0 & \sqrt{2} \\ -S\,45^o & C\,45^o & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{...(5.48b)}$$

$$\mathbf{T} = \begin{bmatrix} \dfrac{\sqrt{3}+1}{2\sqrt{2}} & \dfrac{\sqrt{3}-1}{2\sqrt{2}} & 0 & \dfrac{\sqrt{6}+2\sqrt{3}-1}{2} \\ -\dfrac{\sqrt{3}-1}{2\sqrt{2}} & \dfrac{\sqrt{3}+1}{2\sqrt{2}} & 0 & \dfrac{\sqrt{2}+2+\sqrt{3}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{...(5.48c)}$$

Comparing eqs. (5.47b) and (5.47c) only the coordinates of the origin of frame B are different, as clear from Figs. 5.21(a-b).

Example 5.15 Verification of eq. (5.48c) using MATLAB

In order to verify the equation, the following commands are used:

```
> tam=[cos(pi/6),-sin(pi/6),0,sqrt(3)-1/2; sin(pi/6),cos(pi/6),0,1+sqrt(3)/2;0,0,1,0;0,0,0,1];
> tbm=[cos(pi/4),sin(pi/4),0,sqrt(2);-sin(pi/4),cos(pi/4),0,0;0,0,1,0;0,0,0,1];
> tm=tam*tbm

tm =

   0.9659   0.2588        0   2.4568
  -0.2588   0.9659        0   2.5731
        0        0   1.0000        0
        0        0        0   1.0000
```

where 'tam' and 'tbm' represent the matrices, $[\mathbf{T}_A]_F$ and $[\mathbf{T}_B]_F$, respectively, whereas the resultant matrix, $[\mathbf{T}_A]_F$, is denoted by 'tm.' Equation (5.48c) can be easily verified to be the expression given by 'tm.'

5.4 Denvit and Hartenberg (DH) Parameters

A robot manipulator consists of several links connected by, usually, single degree of freedom joints, say, a revolute or a prismatic joint. In order to control the end-effector with respect to the base, it is necessary to find the relation between the coordinate frames attached to the end-effector and the base. This can be obtained from the description of the coordinate transformations between the coordinate frames attached to all the links and forming the overall description in a recursive manner. For this purpose, the material presented in the previous section for describing the position and orientation of the rigid body is useful for obtaining composition of coordinate transformations

First appearance of DH parameters
The DH parameters were first appeared in 1955 (Denavit and Hartenberg, 1955) to represent a directed line which is nothing but the axis of a lower pair joint.

between the consecutive frames. As a first step, a systematic general method is to be derived to define the relative position and orientation of two consecutive links. The problem is to define two frames attached to two successive links and compute the coordinate transformation between them. In general, the frames are arbitrarily chosen as long as they are attached to the link they are referred to. Nevertheless, it is convenient to set some rules for the definition of the link frames. The convention adopted here for a serial chain robot shown in Fig. 5.22 is that it has $n + 1$ links, namely, link #0, . . . #n, coupled by n joints, i.e., joint 1, . . . n.

Figure 5.22 Serial manipulator

Now, referring to Fig. 5.23,

(a) Let axis i denotes the axis of the joint connecting link $i-1$ to link i.

(b) A coordinate system X_i, Y_i, Z_i is attached to the end of the link $i-1$ — not to the link i! — for $i = 1, \ldots n+1$.

(c) Choose axis Z_i along the axis of joint i, whose positive direction can be taken towards either direction of the axis.

(d) Locate the origin, O_i, at the intersection of axis Z_i with the common normal to Z_{i-1} and Z_i. Also, locate O'_i on Z_i at the intersection of the common normal to Z_i and Z_{i+1}.

(e) Choose axis X_i along the common normal to axes Z_{i-1} and Z_i with the direction from former to the later.

(f) Choose axis Y_i so as to complete a right handed frame.

Note that the above conventions do not give a unique definition of the link frames in the following cases:

- For frame 1 that is attached to the fixed base, i.e., link 0, only the direction of axes Z_1 is specified. Then O_1 and X_1 can be chosen arbitarily.
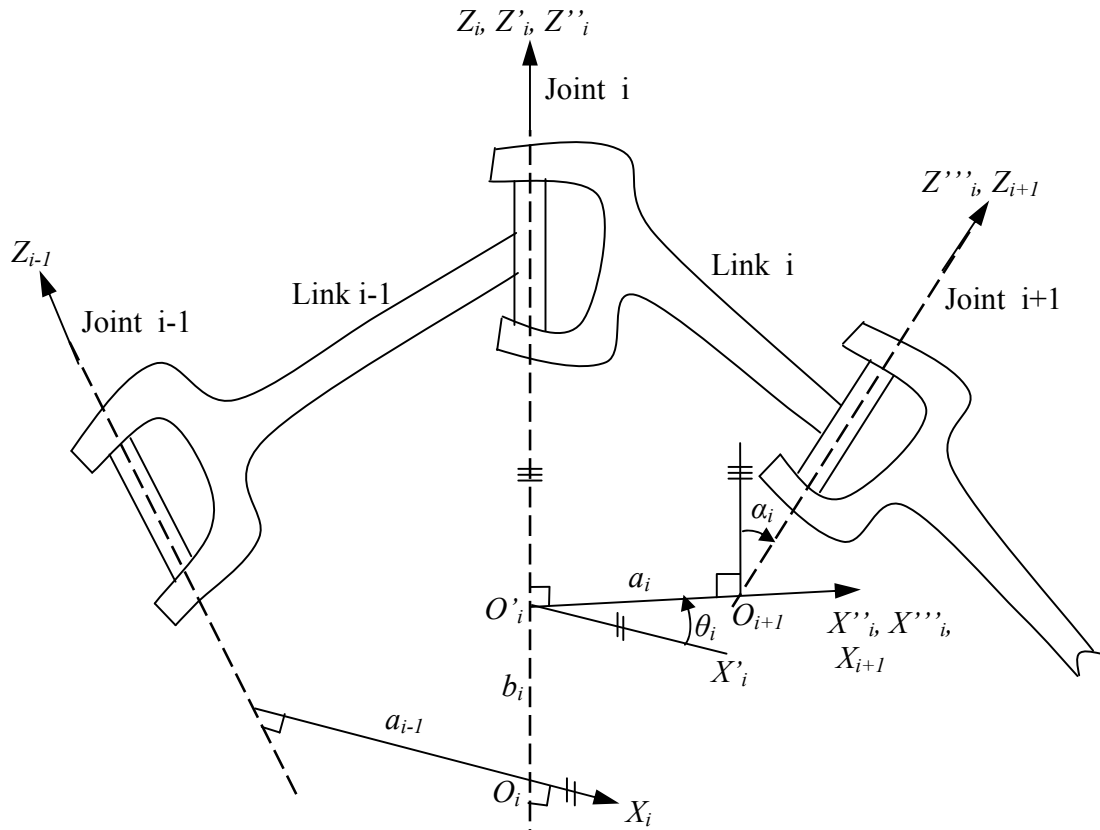
(a) The ith joint is revolute
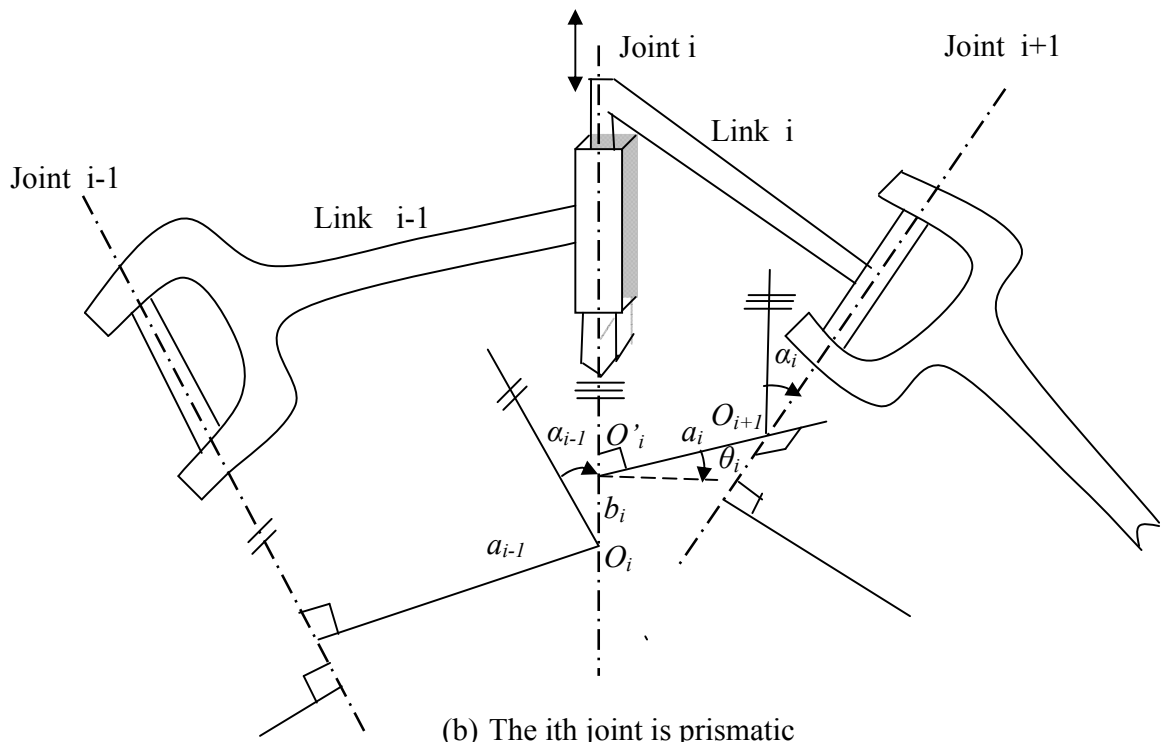
(b) The ith joint is prismatic

Figure 5.23 Frame convention and Denavit and Hartenberg (DH) parameters

- For the last frame $n + 1$ the foregoing convention do not apply since there is no link $n + 1$. Thus, frame $n + 1$ can be arbitrarily chosen.

- When two consecutive axes are parallel, the common normal between them is not uniquely defined.

- When two consecutive axes intersect, the direction of X_i is arbitrary.

When joint i is prismatic, only the direction of axis Z_i is determined, whereas the location of O_i is arbitrary.

In all such cases, the indeterminacy can be exploited to simplify the procedure. For instance, the axes of frame $n + 1$ can be made parallel to those of frame n. Once the link frames have been established, the position and orientation of frame i with respect to frame $i - 1$ are completely specified by four parameters known as the Denavit and Hartenberg (DH) parameters. Hence, these frames are also referred as DH frames. The four DH parameters are defined as follows:

(a) b_i (Joint offset)

Length of the intersections of the common normals on the joint axis Z_i, i.e., O_i and O'_i. It is the relative position of links $i - 1$ and i. This is measured as the distance between X_i and X_{i+1} along Z_i.

(b) θ_i (Joint angle)

Angle between the orthogonal projections of the common normals, X_i and X_{i+1}, to a plane normal to the joint axes Z_i. Rotation is positive when it is made counter clockwise. It is the relative angle between links $i - 1$ and i. This is measured as the angle between X_i and X_{i+1} about Z_i.

(c) a_i (Link length)

Length between the O'_i and O_{i+1}. This is measured as the distance between the common normals to axes Z_i and Z_{i+1} along X_{i+1}.

(d) α_i (Twist angle)

Angle between the orthogonal projections of joint axes, Z_i and Z_{i+1} onto a plane normal to the common normal. This is measured as the angle between the axes, Z_i and Z_{i+1}, about axis X_{i+1} to be taken positive when rotation is made counter clockwise.

Note that the above four parameters are defined sequentially as one moves from link $i - 1$ to link $i + 1$ though link i. Moreover, the first two parameters, namely, b_i and θ_i, define the relative position of links $i - 1$ and i, whereas the last two, a_i and α_i, describe the size and shape of link i that are always constant. Parameters, b_i and θ_i, are, however, variable depending on the type of joints in use. In particular,

- θ_i is variable if joint i is revolute; and

- b_i is variable if joint i is prismatic.

So, for a given type of joint, i.e., revolute or prismatic, one of the DH parameters is variable, which is called 'joint variable,' whereas the other three remaining parameters are constant that are called 'link parameters.'

Example 5.16   DH Parameters of a Three-link Planar Arm

Figure 5.24 shows a three link planar arm. The coordinate frames to define the DH parameters are shown in the figure. The DH parameters are tabulated in Table 5.2, where a_i and θ_i, for $i = 1, 2, 3$, are the link lengths and the joints angles, respectively. Axis Z_i is perpendicular to the plane of page and X_1 is chosen arbitrarily. Note that Frame 1, i.e., X_1, Y_1, and Z_1, is fixed to link denoted as #0. Since there is no link 4, frame 4 can be arbitrarily assigned so that its X-axis is placed along the link, as done for frames 2 and 3.
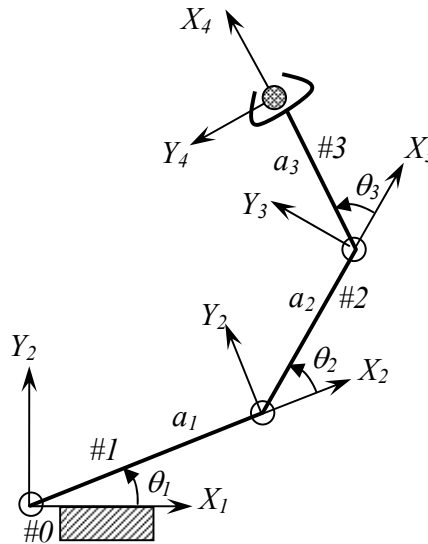

Figure 5.24 A three-link planar arm

Table 5.2 DH parameters of the three-link arm

Link	b_i	θ_i	a_i	α_i
1	0	θ_1 (JV)	a_1	0
2	0	θ_2 (JV)	a_2	0
3	0	θ_3 (JV)	a_3	0

JV: Joint Variable

Note that for a 2-link planar with both revolute joints, i.e., #3 is removed from Fig. 5.24, the DH parameters of Table 5.2 without the third row hold good.

Example 5.17   DH Parameters of a Revolute-Prismatic Planar Arm

Referring to a Revolute-Prismatic (RP) planar arm, Fig. 5.25, where the revolute and prismatic joints are indicated as R and P, respectively, the DH parameters are listed in Table 5.3.
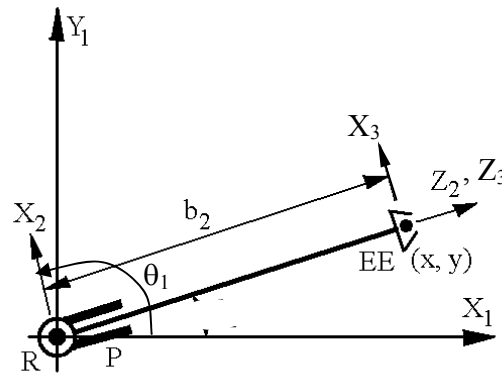
Figure 5.25 Revolute-Prismatic planar arm

Table 5.3 DH parameters of the RP arm

Link	b_i	θ_l	a_i	α_i
1	0	θ_l (JV)	0	$\pi/2$
2	b_2 (JV)	0	0	0

Example 5.18   DH Parameters of a Prismatic-Revolute Planar Arm

If the revolute and prismatic joints are interchange, the result is a Prismatic-Revolute (PR) arm, as shown in Fig. 5.26. Its DH parameters are shown in Table 5.4.

Figure 5.26 Prismatic-Revolute planar arm

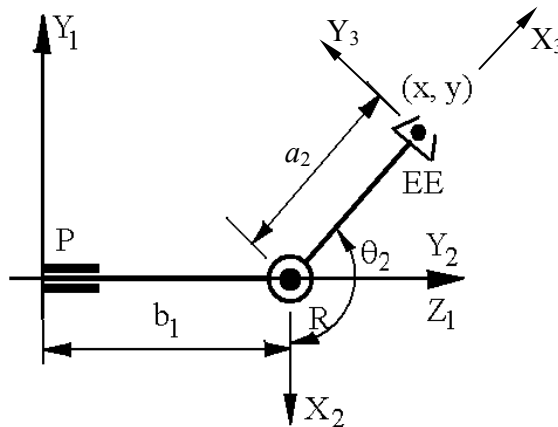Table 5.4 DH parameters of the PR arm

Link	b_i	θ_I	a_i	α_i
1	b_2 (JV)	$-\pi/2$	0	$\pi/2$
2	0	θ_2 (JV)	a_2	0

Example 5.19 DH Parameters of a Spherical Arm

Referring to the spherical type robot arm shown in Fig. 5.27, note that the first and second links, namely, #1 and #2, intersect and the first link length does not affect the end-effector motion due to the rotation of the first joint. So it is beneficial to put both the first and second frames at the intersection of the first two revolute axes, namely, at O_1 or O_2. The DH parameters are tabulated in Table 5.5, where b_2, b_3, and θ_i, for $i = 1, 2, 3$, are indicated in Fig. 5.27.
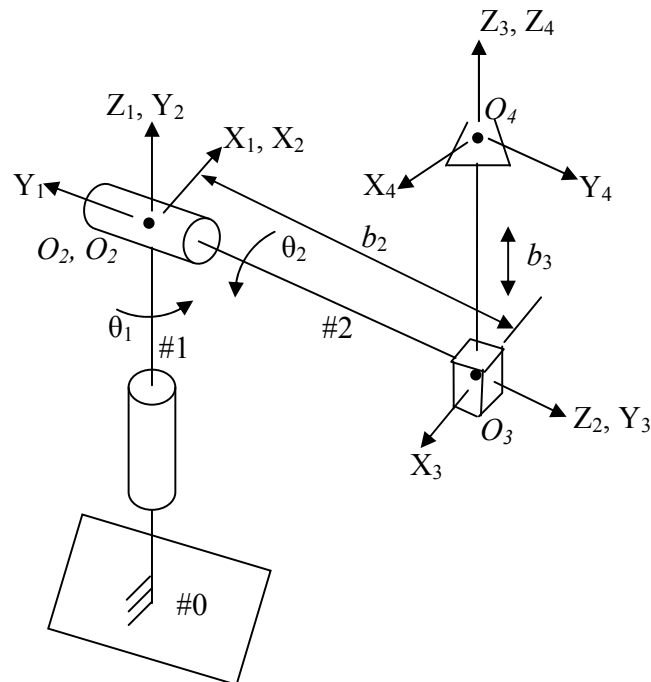

Figure 5.27 A spherical-type arm

Table 5.5 DH parameters of the spherical arm

Link	b_i	θ_i	a_i	α_i
1	0	θ_1 (JV)	0	$\pi/2$
2	b_2	θ_2 (JV)	0	$\pi/2$
3	b_3 (JV)	0	0	0

5.4.1 Transformation between DH frames

At this point, it is possible to express the coordinate transformation between the DH frames i and $i + 1$, which are attached to links $i - 1$ and i, respectively. Referring to Fig. 5.23,

(a) Translate frame i by b_i along axes Z_i. This brings the origin O_i of the frame, i, into coincidence with O_i'. Let O_i' be the origin of the displaced frame i'. The corresponding transformation matrix is

$$\mathbf{T}_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.49a)$$

where subscript b indicates the translation by amount b_i along the Z_i-axis.

(b) The displaced frame, i.e., frame i', is rotated by an angle θ_i about the axis Z_i, which brings axis X_i' (parallel to X_i) to an alignment with the axis, X_{i+1}. The new frame is denoted as frame i''. The corresponding transformation matrix is

$$\mathbf{T}_\theta = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.49b)$$

(c) Translate the frame, i'', by a_i along axis $X_{i''}$ ($\equiv X_{i+1}$). This brings the origin of frame i'', $O_{i''}$ ($\equiv O'_i$), into coincidence with O_{i+1}. The corresponding transformation matrix is

$$\mathbf{T}_a = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.49c)$$

The new frame is denoted with frame i'''.

(d) Finally, rotate the frame i''' by an angle α_i about the axis Xi''' ($\equiv X_{i+1}$), which will now coincide with frame $i + 1$. The corresponding transformation matrix is

$$\mathbf{T}_{\alpha'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \dots (5.49d)$$

The resulting coordinate transformation between the frames connected with bodies i and $i - 1$, namely, $\mathbf{T}_i$, is then obtained by post-multiplying the above four elementary transformations, as done in eq. (5.31d), i.e.,

$$\mathbf{T}_i = \mathbf{T}_b \mathbf{T}_\theta \mathbf{T}_a \mathbf{T}_\alpha \qquad \dots (5.50a)$$

The expression, $\mathbf{T}_i$, can also be read as the transformation matrix of the frame attached to body i, i.e., frame $i + 1$, represented in the frame attached to body $i - 1$, i.e., frame i. Substituting the matrix expressions from eqs. (5.49a-d) to eq. (5.50a), the following expression is obtained:

$$\mathbf{T}_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & b_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \dots (5.50b)$$

Notice that the transformation matrix from frame i to frame $i + 1$ is a function of only the ith joint variable, i.e., θ_i for a revolute joint, and b_i for a prismatic joint, because other three DH parameters are constant.

Example 5.20 Homogeneous Transformation of the Three-link Planar Arm

Referring to Fig. 5.24 and the DH parameters given in Table 5.2, the homogenous transformation matrices, $\mathbf{T}_i$, can be derived from eq. (5.50b) as

$$\mathbf{T}_i = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_i C\theta_i \\ S\theta_i & C\theta_i & 0 & a_i S\theta_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{, for } i=1,2,3 \qquad \dots (5.51)$$

Example 5.21 Homogeneous Transformation of the RP Planar Arm

Referring to Fig. 5.25 and the DH parameters given in Table 5.3 the homogenous transformation matrix, $\mathbf{T}_i$, for $i=1,2$, can be derived from eq. (5.50b) as

$$\mathbf{T}_1 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{, and } \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \dots (5.52)$$

Example 5.22 Homogeneous Transformation of the PR Planar Arm

Referring to Fig. 5.26 and the DH parameters given in Table 5.4 the homogenous transformation matrix, $\mathbf{T}_i$, for $i=1,2$, can be derived from eq. (5.50b) as

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{T}_2 = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2 C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2 S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.53)$$

Example 5.23   Homogeneous Transformation of the Spherical Arm

Referring to Fig. 5.27 and the DH parameters given in Table 5.5 the homogenous transformation matrices, $\mathbf{T}_i$, for i=1,2,3, can be derived from eq. (5.50b) as

$$\mathbf{T}_1 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{T}_2 = \begin{bmatrix} C\theta_2 & 0 & -S\theta_2 & 0 \\ S\theta_2 & 0 & C\theta_2 & 0 \\ 0 & -1 & 0 & b_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{T}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (5.54)$$

Example 5.24 Obtain eq. (5.50b) using MATLAB

In order to obtain eq. (5.50b), the symbolic operations have to be performed in MATLAB. For that the following commands are used:

```
> syms bi thi ai ali;
> tbm=[1,0,0,0;0,1,0,0;0,0,1,bi;0,0,0,1];
> tthm=[cos(thi),-sin(thi),0,0;sin(thi),cos(thi),0,0;0,0,1,0;0,0,0,1];
> tam=[1,0,0,ai;0,1,0,0;0,0,1,0;0,0,0,1];
> talm=[1,0,0,0;0,cos(ali),-sin(ali),0; 0,sin(ali),cos(ali),0; 0,0,0,1];
> tim=tbm*tthm*tam*talm

tim =

[       cos(thi), -sin(thi)*cos(ali),  sin(thi)*sin(ali),      cos(thi)*ai]
[       sin(thi),  cos(thi)*cos(ali), -cos(thi)*sin(ali),      sin(thi)*ai]
[              0,           sin(ali),           cos(ali),               bi]
[              0,                  0,                  0,                1]
```

where command 'syms' are used to define the symbolic variables, 'bi,' 'thi,' 'ai,' and 'ali,' for the DH parameters, b_i, θ_i, a_i, and α_i, respectively. Moreover, the notations, 'tbm,' 'tthm,' 'tam,' and 'talm' are used to represent the matrices, $\mathbf{T}_b$, $\mathbf{T}_\theta$, $\mathbf{T}_a$, and $\mathbf{T}_\alpha$, given by eqs. (5.49a-d), respectively. Finally, the resultant matrix, $\mathbf{T}_i$, denoted as 'tim' in MATLAB environment, is evaluated above, which matches with eq. (5.50b).

5.5 Summary

In this chapter, first the rigid body pose or configuration is defined. Their representations are presented. Homogeneous transformation matrix is also defined which takes care of both translation and rotation of a coordinate frame with respect to another one. In the

later part, Denavit and Hartenberg (DH) parameters are introduced and corresponding homogeneous transformation matrix is derived.

Exercises

5.1 Prove the following :

    a. det $(\mathbf{Q}) = 1$ … (5.55a)

    b. $\mathbf{Q}^{-1} = \mathbf{Q}^{T}$ … (5.55b)

    c. $\mathbf{Q}' = \mathbf{Q}^{T}$ … (5.55c)

    d. For elementary rotation by α about any axis, *X, Y,* or *Z,* $\mathbf{Q}_k = \mathbf{Q}_k^{T}$, where $k = X,$ *Y,* or *Z.*

5.2 What are the equivalent meanings of a rotation matrix?

5.3 Let $\mathbf{Q}_A$ be the rotation of the fixed frame to a new frame *A* with respect to frame *F,* and $\mathbf{Q}_B$ be the rotation of frame *A* to another new frame *B* with respect to frame *A.* What is the resultant matrix representation of the frame *B* with respect to their original reference frame *F,* i.e., $\mathbf{Q}$?

5.4 Let $[\mathbf{Q}_A]_F$ be the rotation of the fixed frame to a new frame *A* with respect to frame *F,* and $[\mathbf{Q}_B]_F$ be the rotation of frame *A* to another new frame *B,* also with respect to frame *F.* Find the resultant matrix representation of frame *B* with respect to the original reference frame *F,* i.e., $\mathbf{Q}$? Should the result be same as obtained in question 3 above?

5.5 Assume that $\mathbf{Q}_A$ and $\mathbf{Q}_B$, as defined in question 3 above, are given by

$$\mathbf{Q}_A = \begin{bmatrix} C\,30^O & -S\,30^O & 0 \\ S\,30^O & C\,30^O & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{Q}_B = \begin{bmatrix} C\,45^O & -S\,45^O & 0 \\ S\,45^O & C\,45^O & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \ldots (5.56)$$

    Illustrate the above two transformations graphically, and find $\mathbf{Q} = \mathbf{Q}_A\,\mathbf{Q}_B$.

5.6 If $\mathbf{T}_A$ be the homogeneous matrix representing a transformation of the fixed frame to a new frame *A* with respect to fixed *F* and $\mathbf{T}_B$ be another transformation to a frame *B* with respect to frame *A,* find the resultant transformation matrix, $\mathbf{T}$.

5.7 If $[\mathbf{T}_A]_F$ be the homogeneous matrix representing a transformation of the fixed frame to a new frame *A* with respect to frame *F,* and $[\mathbf{T}_B]_F$ be another transformation to a frame *B,* also with respect to frame *F,* find the resultant transformation matrix. Comment on the results obtained in this question and in the previous question.

5.8 Find out the DH parameters of the SCARA robot shown in Fig. 5.28.

5.9 Figure 5.29 shows an anthropomorphic articulated arm. Find their DH parameters.

5.10 Find the homogeneous transformation matrices for the robot architecture shown in Fig. 5.29.

5.11 Find the DH parameters of a spherical arm shown in Fig. 5.30.

5.12 What are the homogeneous transformation matrices for the robot architecture shown in Fig. 5.30.

MATLAB Programming Based Exercises

5.13 Evaluate the numerical values of $\mathbf{Q}_A$ and $\mathbf{Q}_B$.

5.14 Prove eqs. (5.55a-c).

5.15 Using the numerical values in Exercise 5.10 show the non-commutative property of the rotations, i.e., $\mathbf{Q}_A\mathbf{Q}_B \neq \mathbf{Q}_B\mathbf{Q}_A$.

5.16 Using the functions, 'atan' and 'atan2,' find the angle formed by the line joining the origin (0,0) and a point (2,1).

5.17 Repeat Exercise 5.16 for the points (0,0) and (-2,-1). Comment on the results obtained using 'atan' and 'atan2.' Which one is the correct one?
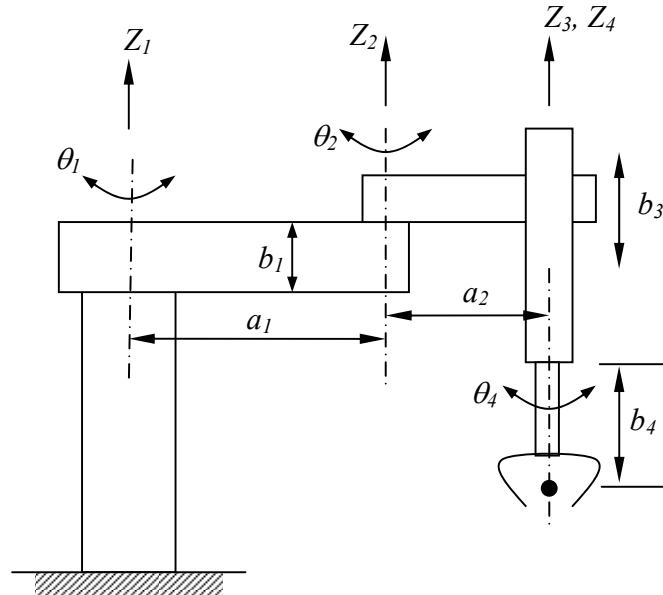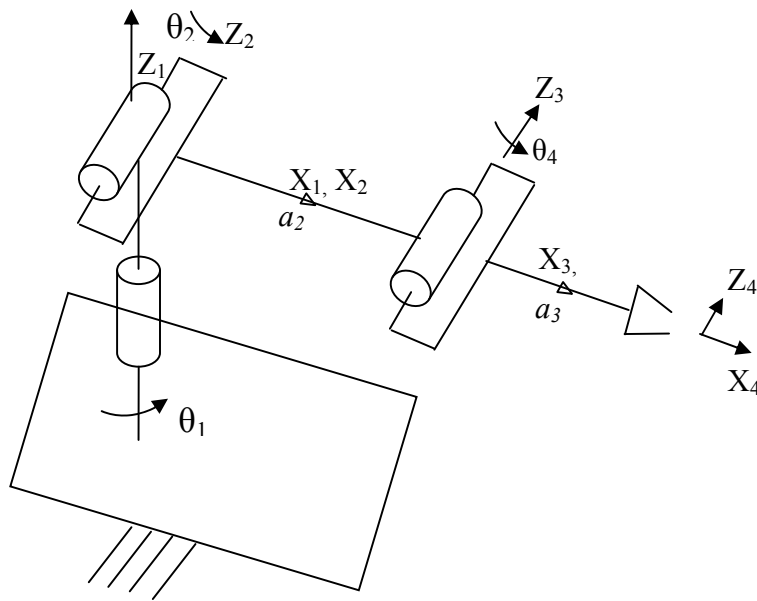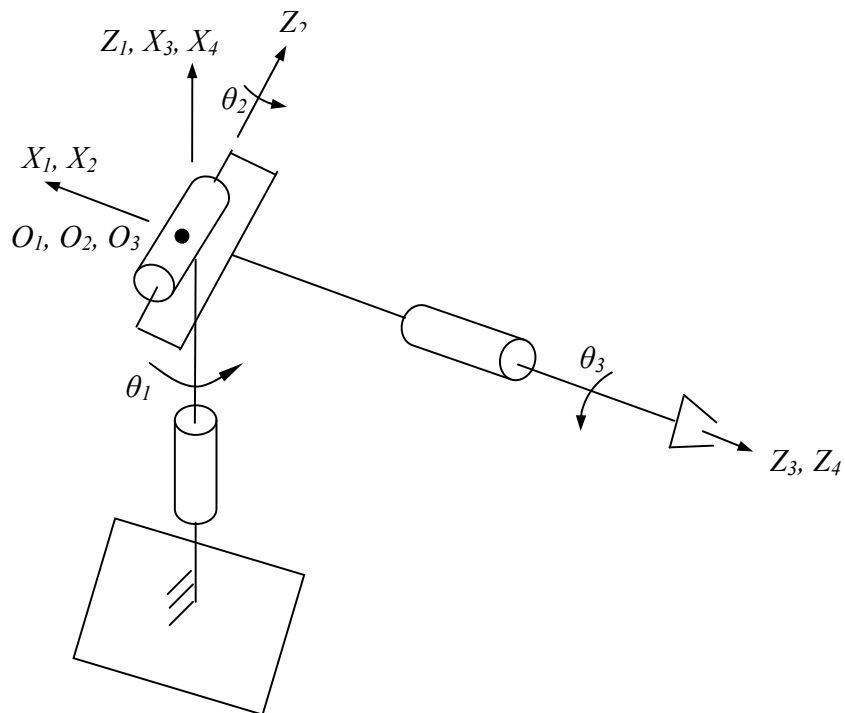
Figure 5.28 A SCARA robot

Figure 5.29 An articulated arm

Figure 5.30 A spherical wrist