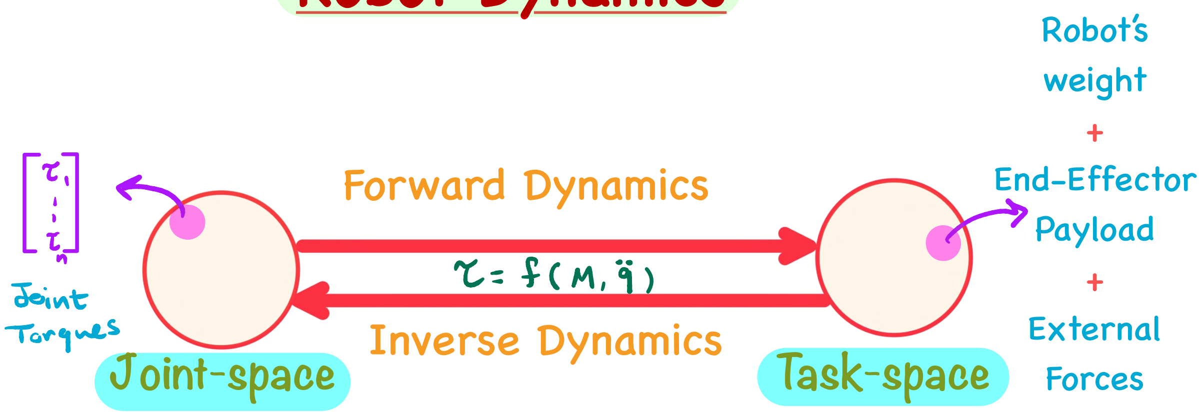
Intro To Robotics Part(2): Dynamics

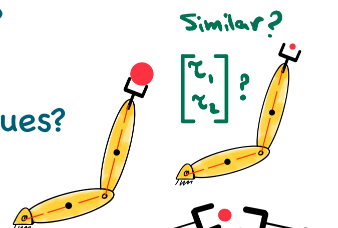
Robot Dynamics



Q: Why we study dynamics ? 🤔

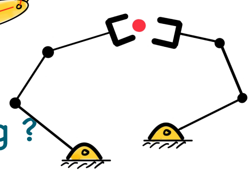
- kinematic model deals with how to move from one pose to another but it doesn't provide an we **answer to some questions :**

- What is the effect of robot material on the motion?
  Is the solution is the same for different materials?



- What is the effect of object weight on joint torques?
  how does this affect the final solution?

- What is the effect of external forces?
  Are these forces helping(robot collaboration) or resisting ?



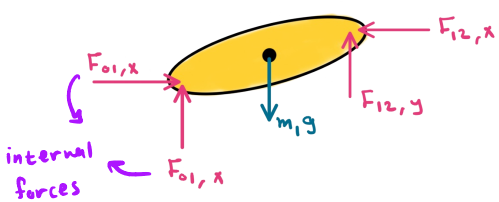
Dynamics helps us to answer all these questions! 😉

Q: But how? 🤔

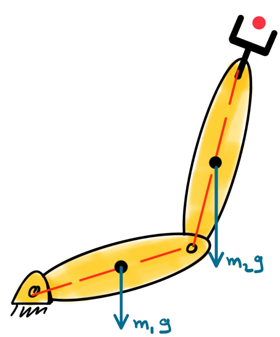
Newtonian Approach	Lagrangian Approach
- Uses Newton&Euler equations to derive body dynamics	- Uses Lagrange equations for energy analysis of the body to derive dynamics
- Depends on the analysis of each body's forces and moments	- Depends on calculating energies of the robot
- **Less common** in Robotics	- **More common** in Robotics

Q: But why we don't use Newtonian approach that much? 🤔

- Let's analyse **link(1)**



$$\boxed{\Sigma F = m a}$$



- For each link we have to compute:
  - **Internal Forces**
  - **Acceleration**

  } Complex!

Lagrangian Formulation

- Let's define the Lagrange(L) as the difference between:
  - Total Kinetic Energy (T)
  - Total Potential Energy (U)

$$\underbrace{L}_{\text{Lagrange}} = \underbrace{T}_{\text{Kinetic Energy}} - \underbrace{U}_{\text{Potential Energy}}$$

- Where,
  - T is the **Total** Kinetic **Energy** of **all links** of the robot

$$T = \sum_{i=1}^{N} \underbrace{T_i}_{\text{Kinetic energy for link(i)}} \quad , \quad i = 1, ---, \underbrace{N}_{\text{number of links}}$$

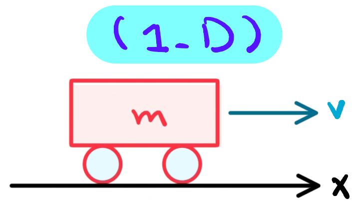
  - U is the **Total** Potential **Energy** of **all links** of the robot

$$U = \sum_{i=1}^{N} \underbrace{U_i}_{\text{Potential energy for link(i)}} \quad , \quad i = 1, ---, N$$

Q: How to calculate the energies for each link?🤔

Step(1) : Kinetic Energy

I) Translational motion



- For 1-D motion

$$T = \frac{1}{2} m v^2$$

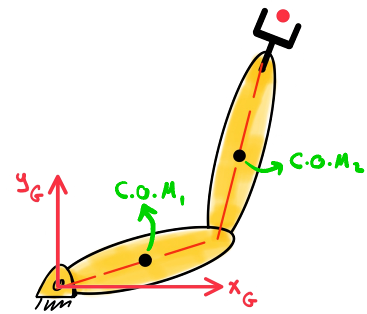
m : body mass
v : body velocity

- In robotics **velocity** is (3x1) vector

$$P_{P_1} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \Rightarrow \upsilon_{P_1} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Position of Point P_1

velocity of Point P_1



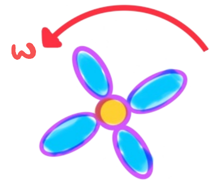
m : mass of the link which is concentrated at its center of mass(c.o.m)
v : 3-D velocity which can be written as follows:

$$v^2 = v^T v = \underset{(1\times3)}{\begin{bmatrix} v_x & v_y & v_z \end{bmatrix}} \underset{(3\times1)}{\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}} = v_x^2 + v_y^2 + v_z^2$$

$$T_{i,trans} = \frac{1}{2} m_i \, v_{cm_i}^T \, v_{cm_i}$$

Step(1) : Kinetic Energy

(1-D)



II) Rotational motion

- For 1-D motion

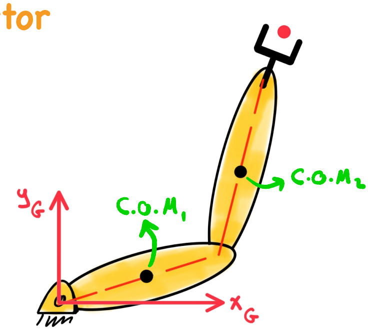
$$T = \frac{1}{2} I \omega^2$$

I : body moment of inertia about the axis of rotation
w : body angular velocity

- In robotics **angular velocity** is (3x1) vector

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \begin{matrix} \rightarrow \omega_x \\ \rightarrow \omega_y \\ \rightarrow \omega_z \end{matrix}$$

angular velocity of link (1) w.r.t frame {0}



- Moment of inertia differs based on which axis is the body rotating about. So, let's define Inertia Tensor

inertia matrix w.r.t global frame

$$I_i = \begin{bmatrix} I_{x_i} & I_{xy_i} & I_{xz_i} \\ I_{yx_i} & I_{y_i} & I_{yz_i} \\ I_{zx_i} & I_{zy_i} & I_{z_i} \end{bmatrix}$$

Depends on:
- shape of the link
- density of the link material

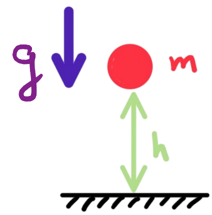
$${}^{0}I_i = R \, I \, R^T$$

inertia matrix w.r.t body frame

$$T_{i,rot} = \frac{1}{2} \underset{(1\times3)}{\omega_i^T} \; \underset{(3\times3)}{I_i} \; \underset{(3\times1)}{\omega_i}$$

Step(2) : Potential Energy

- Is the energy due to gravitational forces on masses when the are certain height from the reference frame.



- For 1-D

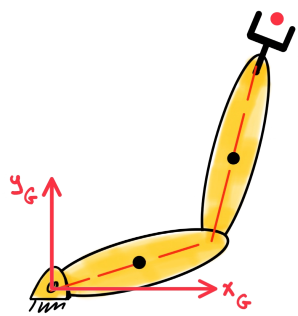
$$\boxed{U = m g h}$$

m :	mass of the body
g :	gravity
h :	height of the body w.r.t ref.frame

- In robotics we have:

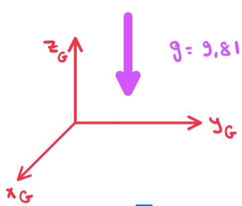
Potential energy of link (i)

$$\boxed{U_i = m_i \vec{g} P_{cm_i}}$$



Where,

- $\vec{g}$ is the gravitational acceleration vector w.r.t **global frame**



$$\vec{g} = \underset{(1\times3)}{\begin{bmatrix} 0 & 0 & -g \end{bmatrix}}$$

- P_{cm_i} is the position vector of c.o.m of link(i) w.r.t **global frame**

Putting All together

Step(1): Calculate Lagrange based on total K.E and total P.E

- For each link compute:

  - Kinetic Energy

$$T_{i,trans} = \frac{1}{2} m_i \underset{(1\times3)}{v_{cm_i}^T} \underset{(3\times1)}{v_{cm_i}}$$

$$T_{i,rot} = \frac{1}{2} \underset{(1\times3)}{w_i^T} \underset{(3\times3)}{I_i} \underset{(3\times1)}{w_i}$$

  - Potential Energy

$$U_i = m_i \vec{g} P_{cm_i}$$

- Compute total K.E (T)and P.E (U):

$$T = \sum_{i=1}^{N} T_i \quad , \quad U = \sum_{i=1}^{N} U_i$$

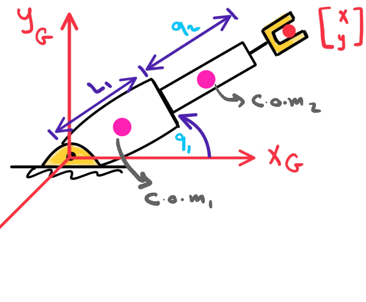
- Compute Lagrange (L):

$$L = T - U$$

Step(2): Calculate required torque for each joint

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$$

Example: RP-Robot using "Lagrangian Approach"



Step(1): Calculate Lagrange

- For link(1):

    - Translational Kinetic Energy

$$P_{cm_1} = \begin{bmatrix} \frac{L_1}{2}\cos(q_1) \\ \frac{L_1}{2}\sin(q_1) \\ 0 \end{bmatrix} \Rightarrow v_{cm_1} = \begin{bmatrix} -\frac{L_1}{2}\sin(q_1)\,\dot{q}_1 \\ \frac{L_1}{2}\cos(q_1)\,\dot{q}_1 \\ 0 \end{bmatrix} \begin{array}{l} \rightarrow \dot{x} \\ \rightarrow \dot{y} \\ \rightarrow \dot{z} \end{array}$$

$$T_{1,trans} = \frac{1}{2} m_1 v_{cm_1}^T v_{cm_1}$$

$$= \frac{1}{2} m_1 \begin{bmatrix} -\frac{L_1}{2} s_1 \dot{q}_1 & \frac{L_1}{2} c_1 \dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{L_1}{2} s_1 \dot{q}_1 \\ \frac{L_1}{2} c_1 \dot{q}_1 \\ 0 \end{bmatrix}$$

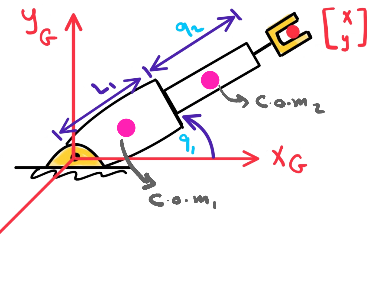
$$= \frac{1}{2} m_1 \left[\frac{L_1^2}{4} \sin(q_1)^2 \dot{q}_1 + \frac{L_1^2}{4} \cos(q_2)^2 \dot{q}_1 + 0 \right]$$

$$= \frac{1}{2} m_1 \frac{L_1^2}{4} \dot{q}_1 \left[\underbrace{\sin(q_1)^2 + \cos(q_1)^2}_{=1} \right]$$

$$\boxed{T_{1,trans} = \frac{m_1 L_1^2}{8} \dot{q}_1}$$

$$\boxed{\begin{array}{l} c_1 = \cos(q_1) \\ s_1 = \sin(q_1) \end{array}}$$

Step(1): Calculate Lagrange



- For link(1):

  - Rotational Kinetic Energy

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \begin{array}{l} \to \omega_x \\ \to \omega_y \\ \to \omega_z \end{array}, \quad I_1 = \begin{bmatrix} I_{x_1} & I_{xy_1} & I_{xz_1} \\ I_{yx_1} & I_{y_1} & I_{yz} \\ I_{zx_1} & I_{zy} & I_{z_1} \end{bmatrix}$$

$$T_{1,rot} = \frac{1}{2}\,\omega_1^T\, I_1\, \omega_1$$

$$= \underbrace{\begin{bmatrix} 0 & 0 & \dot{q}_1 \end{bmatrix}}_{\omega_1^T} \underbrace{\begin{bmatrix} I_{x_1} & I_{xy_1} & I_{xz_1} \\ I_{yx_1} & I_{y_1} & I_{yz_1} \\ I_{zx_1} & I_{zy_1} & I_{z_1} \end{bmatrix}}_{I_1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}}_{\omega_1}$$

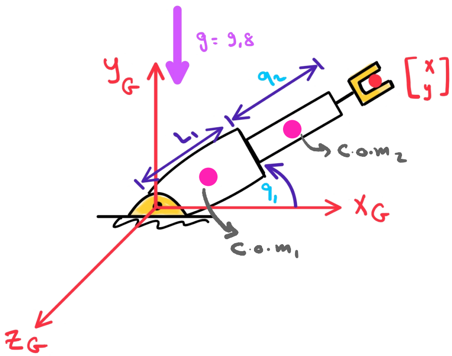
$$= \begin{bmatrix} 0 & 0 & \dot{q}_1 \end{bmatrix} \begin{bmatrix} I_{xy_1}\,\dot{q}_1 \\ I_{yz_1}\,\dot{q}_1 \\ I_{z_1}\,\dot{q}_1 \end{bmatrix}$$

$$\boxed{T_{1,rot} = \frac{1}{2} I_{z_1} \dot{q}_1^2}$$

$$T_1 = T_{r,trans} + T_{1,rot}$$

$$\boxed{T_1 = \left[\frac{mL^2}{8} + \frac{1}{2} I_{z_1}\right] \dot{q}_1}$$ → Total K.E of link(1)

Step(1): Calculate Lagrange



- For link(1):

  - Potential Energy

$$U_1 = m_1 \, \vec{g} \, P_{cm_1}$$

$$= m_1 \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \begin{bmatrix} \frac{L_1}{2} c_1 \\ \frac{L_1}{2} s_1 \\ 0 \end{bmatrix}$$

$$\boxed{U_1 = - m_1 \, g \, \frac{L_1}{2} \, \sin(q_1)} \rightarrow \text{P.E of link(1)}$$

- For link(2):

  - Translational Kinetic Energy

$$P_{cm_2} = \begin{bmatrix} (L_1 + q_2/2)\, c_1 \\ (L_1 + q_2/2)\, s_1 \\ 0 \end{bmatrix} \Rightarrow v_{cm_2} = \begin{bmatrix} -(L_1 + q_2/2) s_1 \dot{q}_1 + \frac{1}{2} c_{q_1} \dot{q}_2 \\ (L_1 + q_2/2) c_1 \dot{q}_1 + \frac{1}{2} s_{q_1} \dot{q}_2 \\ 0 \end{bmatrix}$$

$$T_{2,trans} = \frac{1}{2} \, m_2 \, v_{cm_2}^T \, v_{cm_2}$$

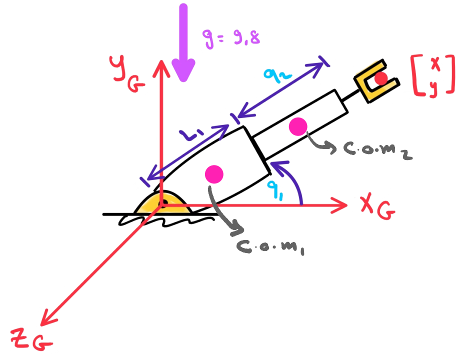
$$= \begin{bmatrix} -(L_1 + q_2/2) s_1 \dot{q}_1 + \frac{1}{2} c_{q_1} \dot{q}_2 & (L_1 + q_2/2) c_1 \dot{q}_1 + \frac{1}{2} s_{q_1} \dot{q}_2 & 0 \end{bmatrix} \begin{bmatrix} -(L_1 + q_2/2) s_1 \dot{q}_1 + \frac{1}{2} c_{q_1} \dot{q}_2 \\ (L_1 + q_2/2) c_1 \dot{q}_1 + \frac{1}{2} s_{q_1} \dot{q}_2 \\ 0 \end{bmatrix}$$

$$\boxed{T_{2,trans} = \frac{m_2}{2} \left[\left(L_1 + \frac{q_2}{2}\right)^2 \dot{q}_1^2 + \frac{1}{4} \, \dot{q}_2^2 \right]}$$

Step(1): Calculate Lagrange

- For link(2):

  - Rotational Kinetic Energy



$$I_2 = \begin{bmatrix} I_{x_2} & I_{xy_2} & I_{xz_2} \\ I_{yx_2} & I_{y_2} & I_{yz_2} \\ I_{zx_2} & I_{zy_2} & I_{z_2} \end{bmatrix}$$

Q: What about (w2) ?

- we represent **angular velocities** $^0w_2$ **w.r.t global frame**.So, we need to **consider all previous angular velocities** that might contribute to rotation.

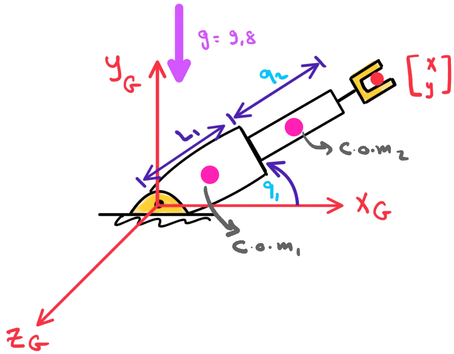
$$\omega_2 = {}^0\omega_2 = {}^0\omega_1 + {}^0R_1\,{}^1\omega_2$$

$^0w_1$: link(1) angular velocity
$^1R_2$: Rotation matrix of frame{1} w.r.t frame{0}
$^1w_2$: link(2) angular velocity w.r.t frame{1}

$$w_2 = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}}_{^0\omega_1} + \cancel{\underbrace{\begin{bmatrix} -S_1 & 0 & C_{q_1} \\ C_1 & 0 & S_{q_1} \\ 0 & 1 & 0 \end{bmatrix}}_{^0R_1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{^1\omega_2}} = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

"Prismatic Joint"

Step(1): Calculate Lagrange



- For link(2):

  - Rotational Kinetic Energy

$$T_{2,rot} = \frac{1}{2}\, \omega_2^T\, I_2\, \omega_2$$

$$= \underbrace{\begin{bmatrix} 0 & 0 & \dot{q}_1 \end{bmatrix}}_{\omega_2^T} \underbrace{\begin{bmatrix} I_{x_2} & I_{xy_2} & I_{xz_2} \\ I_{yx_2} & I_{y_2} & I_{yz} \\ I_{zx_2} & I_{zy} & I_{z_2} \end{bmatrix}}_{I_2} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}}_{\omega_2}$$

$$= \begin{bmatrix} 0 & 0 & \dot{q}_1 \end{bmatrix} \begin{bmatrix} I_{xy_2}\dot{q}_1 \\ I_{yz_2}\dot{q}_1 \\ I_{z_2}\dot{q}_1 \end{bmatrix}$$

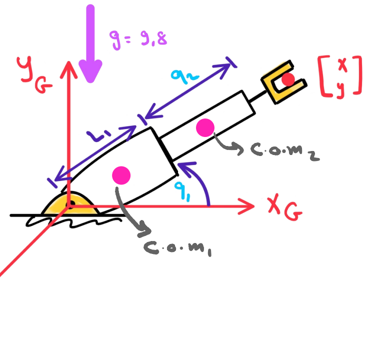
$$\boxed{T_{,rot} = \frac{1}{2} I_{z_2} \dot{q}_1^2}$$

  - Total Kinetic Energy

$$T_2 = T_{2,trans} + T_{2,rot}$$

$$\boxed{T_2 = \frac{1}{2} m_2 \left[\left(L_1 + \frac{q_2}{2}\right)^2 \dot{q}_1 + \frac{1}{4} \dot{q}_2^2 \right] + \frac{1}{2} I_{z_2} \dot{q}_1}$$

Step(1): Calculate Lagrange



- For link(2):

  - Potential Energy

$$U = m_2 \vec{g} P_{cm_2}$$

$$= m_2 \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \begin{bmatrix} (L_1 + q_2/2) c_1 \\ (L_1 + q_2/2) s_1 \\ 0 \end{bmatrix}$$

$$\boxed{C_1 = \cos(q_1) \quad S_1 = \sin(q_1)}$$

$$\boxed{U_2 = -m_2 g \left(L_1 + \frac{q_2}{2}\right) \sin(q_1)} \rightarrow \text{P.E of link(2)}$$

- Compute Lagrange (L):

$$T = T_1 + T_2 \quad , \quad U = U_1 + U_2$$

$$L = T - U$$

$$L = \underbrace{\left[\frac{m_1 L_1^2}{8} + \frac{1}{2} I_{z_1} \right] \dot{q}_1}_{T_1} + \underbrace{\frac{m_2}{2} \left[\left(L_1 + \frac{q_2}{2}\right)^2 \dot{q}_1 + \frac{1}{4} \dot{q}_2^2 \right] + \frac{1}{2} I_{z_2} \dot{q}_1^2}_{T_2}$$

$$+ \underbrace{\frac{m_1 L_1 g}{2} s_1}_{U_1} + \underbrace{m_2 g \left(L_1 + \frac{q_2}{2}\right) s_1}_{U_2}$$

Step(2): Calculate required torque for each joint

$$\boxed{\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}}$$

- For joint(1):

    - Lagrange equation

$$L = \left[\frac{m_1 L_1^2}{8} + \frac{I_{z_1}}{2}\right]\dot{q}_1^2 + \frac{m_2}{2}\left[\left(L_1 + \frac{q_2}{2}\right)\dot{q}_1^2 + \frac{1}{4}\dot{q}_2\right] + \frac{I_{z_2}}{2}\dot{q}_1^2 + \frac{m_1 L_1 g}{2} S_1 + m_2 g\left(L_1 + \frac{q_2}{2}\right) S_1$$

$$\frac{\partial L}{\partial \dot{q}_1} = 2\left[\frac{m_1 L_1^2}{8} + \frac{I_{z_1}}{2} + \frac{m_2}{2}\left(L_1 + \frac{q_2}{2}\right) + \frac{I_{z_2}}{2}\right]\dot{q}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = \frac{d}{dt}\left[\frac{m_1 L_1^2}{4} + I_{z_1} + m_2\left(L_1 + \frac{q_2}{2}\right) + I_{z_2}\right]\dot{q}_1$$

$$= \left[\frac{m_1 L_1^2}{4} + I_{z_1} + m_2\left(L_1 + \frac{q_2}{2}\right) + I_{z_2}\right]\ddot{q}_1 + 2 m_1 \left(L_1 + \frac{q_2}{2}\right)\dot{q}_1 \dot{q}_2$$

$$\frac{\partial L}{\partial q_1} = \frac{m_1 L_1 g}{2} + m_2 g\left(L_1 + \frac{q_2}{2}\right) c_1$$

$c_1 = \cos(q_1)$
$S_1 = \sin(q_1)$

Step(2): Calculate required torque for each joint

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$$

- For joint(2):

  - Lagrange equation

$$L = \left[\frac{m_1 L_1^2}{8} + \frac{I_{z_1}}{2}\right]\dot{q}_1^2 + \frac{m_2}{2}\left[\left(L_1 + \frac{q_2}{2}\right)\dot{q}_1^2 + \frac{1}{4}\dot{q}_2\right] + \frac{I_{z_2}}{2}\dot{q}_1^2 + \frac{m_1 L_1 g}{2} S_1 + m_2 g\left(L_1 + \frac{q_2}{2}\right) S_1$$

$$\frac{\partial L}{\partial \dot{q}_2} = 2\boxed{\frac{m_2}{8}}\dot{q}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = \frac{d}{dt}\left[\frac{m}{4}\right]\dot{q}_2 = \left[\frac{m_2}{4}\right]\ddot{q}_2$$

$C_1 = \mathrm{Cos}(q_1)$
$S_1 = \mathrm{Sin}(q_1)$

$$\frac{\partial L}{\partial q_2} = \frac{m_2 \dot{q}_1^2}{2} \cdot 2\left(L_1 + \frac{q_2}{2}\right)\left(\frac{1}{2}\right) + m_2 g\left(\frac{1}{2}\right) S q_1$$

Step(2): Calculate required torque for each joint

- Torque of joint(1):

$$\tau_1 = \left[\frac{m_1 L_1^2}{4} + I_{z_1} + m_2\left(L_1 + \frac{q_2}{2}\right)^2 + I_{z_2}\right]\ddot{q}_1 + 2 m_2 \left(L_1 + \frac{q_2}{2}\right)\dot{q}_1 \dot{q}_2 - \left[\frac{m_1 L_1 g}{2} + m_2 g \left(L_1 + \frac{q_2}{2}\right)\right] c q_1$$

- Torque of joint(2):

$$\tau_2 = \left[\frac{m_2}{4}\right]\ddot{q}_2 - \frac{m_2 \dot{q}_1^2}{2}\left(L_1 + \frac{q_2}{2}\right) - \frac{m_2 g}{2} s q_1$$

- Rewrite equations in **Matrix Form**

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \frac{m_1 L_1^2}{4} + I_{z_1} + m_2\left(L_1 + \frac{q_2}{2}\right)^2 & 0 \\ 0 & \frac{m_2}{4} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ -\frac{m_2}{2}\left(L_1 + \frac{q_2}{2}\right) & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 2 m_2 \left(L_1 + \frac{q_2}{2}\right) \\ 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \dot{q}_2 \end{bmatrix}$$

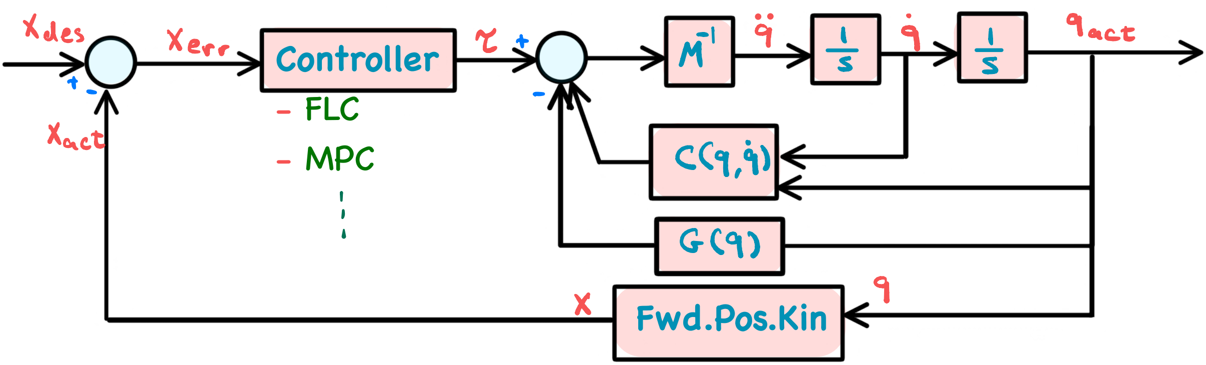
$$- \begin{bmatrix} \left(\frac{m_1 L_1 g}{2} + m_2 g \left(L_1 + \frac{q_2}{2}\right)\right) c_1 \\ \frac{m_2 g}{2} s_1 \end{bmatrix}$$

- General form

$$\tau = \underbrace{D(q)}_{\text{Inertia/Mass Matrix}}\ddot{q} + \underbrace{c(q,\dot{q})\dot{q}}_{\text{Non-Linear Forces (Centrifugal + Coriolis)}} + \underbrace{G(q)}_{\text{Gravitational Forces}}$$

- Similar to: $F = m\ddot{x} + b\dot{x} + kx \Rightarrow \Sigma F = ma$ (1-D)

Dynamics Simulation



- This wasn't easy 🙁, but the good news that we don't need to go through all these calculations again. 😁

Q: But how? 🤔

- We can **construct** the matrices of general form D,C,G using velocity kinematics.

- To do this we have to move from :

Task-Space velocity(v,w) -> Joint-Space velocity ($\dot{q}$)

Q: But how to compute matrix D ? 🤔

(Optional proof)

- Kinetic Energy:

  - For single Link

$$T = \underbrace{\frac{1}{2} m v^T v}_{\text{Translational K.E}} + \underbrace{\frac{1}{2} w^T I w}_{\text{Rotational K.E}}$$

  - For n-Link

$$T = \underbrace{\frac{1}{2} \sum_{i=1}^{n} m_i v_i^T v_i}_{\text{Total Translational K.E}} + \underbrace{\frac{1}{2} \sum_{i=1}^{n} w_i^T I_i w_i}_{\text{Total Rotational K.E}}$$

- Using Velocity Kinematics

$$\begin{array}{c} v \\ \\ \omega \end{array}\left\{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \hline \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_{6\times1}\right. = \begin{bmatrix} \underset{(3\times1)}{J_{v_1}} & \cdots & \underset{(3\times1)}{J_{v_n}} \\ \hline \underset{(3\times1)}{J_{\omega_1}} & \cdots & \underset{(3\times1)}{J_{\omega_n}} \end{bmatrix}_{6\times n} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times1}$$

- Linear velocity of Joint(i)

$$\boxed{v_i = J_{v_i} \cdot \dot{q}}$$

- Angular velocity of Joint(i)

$$\boxed{\omega_i = J_{\omega_i} \cdot \dot{q}}$$

- Rewrite Kinetic Energy Formula

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^T v_i + \frac{1}{2}\sum_{i=1}^{n} \omega_i^T I_i \omega_i$$

$$= \frac{1}{2}\sum_{i=1}^{n} m_i \dot{q}^T J_{v_i}^T J_{v_i} \dot{q} + \frac{1}{2}\sum_{i=1}^{n} \dot{q}^T J_{\omega_i}^T R_i I R_i^T J_{\omega_i} \dot{q}$$

$$= \frac{1}{2}\dot{q}^T \underbrace{\left[\sum_{i=1}^{n} m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T R_i I R_i^T J_{\omega_i}\right]}_{D(q)\text{: Inertia Matrix}} \dot{q}$$

$$= \frac{1}{2}\sum_{i,j} d_{ij}(q)\, \dot{q}_i \dot{q}_j = \frac{1}{2}\dot{q}^T D(q)\, \dot{q}$$

$$v_i = J_{v_i} \cdot \dot{q}$$

$$\omega_i = J_{\omega_i} \dot{q}$$

$$I_i = R_i I_i R_i^T$$

$$D(q) \rightarrow \text{Symmetric},\ \text{Positive Definite}$$

Notes:

- If we **plug in** the new representation of K.E in Lagrange equation we will find out that we can **determine the matrix** C **using inertia matrix** D

- We already have $\dot{q}$ since there is an **encoder** for each joint.

Equation of Motion

$$\sum_j d_{kj}(q)\ddot{q}_j + \sum_{i,j} c_{ijk}(q)\,\dot{q}_i\,\dot{q}_j + \phi_k(q) = \tau_i$$

$$c_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_i} - \frac{\partial d_{ij}}{\partial q_k}\right.$$

$$\phi_k = \frac{\partial P}{\partial q_k}$$

Putting All together

Step(1): Calculate **Velocity Kinematics**

- For each **center of mass of each link** compute:

$$Jw_{c_i} = ? \ , \ Jv_{c_i} = ?$$

Step(2): Calculate **Inertia Matrix (D)**

$$D = \sum_{i=1}^{n} m_i Jv_i^T Jv_i + Jw_i^T R_i I_i R_i^T Jw_i$$

Step(3): Calculate **Centrifugal** + **Coriolis** Matrix (C)

$i = j$ $\qquad$ $i \neq j$

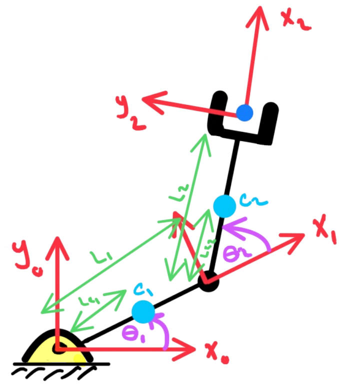
$$C_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} + \frac{\partial d_{ij}}{\partial q_k}\right\}$$

Step(4): Calculate **Gravitational Forces** Matrix (G)

- Compute **Total Potential Energy (P)**

$$\phi_k = \left\{\frac{\partial P}{q_k}\right\}$$

Example(2): RR-Robot



Step(1): Calculate Velocity Kinematics

I) For center of mass (1)

$$J_{w_1} = {}^0Z_{i-1} = {}^0Z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{v_1} = {}^0Z_{i-1} \times ({}^0O_N - {}^0O_{i-1})$$

$$= {}^0Z_0 \times ({}^0O_{c_1} - {}^0O_0)$$

$$= \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 0 & 0 & 1 \\ L_{c_1} C\theta_1 & L_{c_1} S\theta_1 & 0 \end{vmatrix} = \begin{bmatrix} -L_{c_1} S\theta_1 \\ L_{c_1} C\theta_1 \\ 0 \end{bmatrix}$$

$S\theta = Sin(\theta_1)$

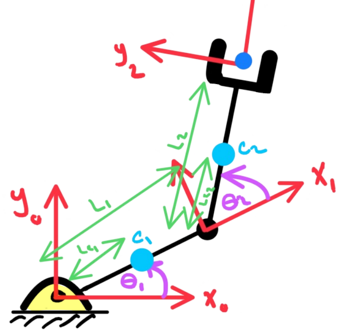
$C\theta = Cos(\theta)$

$$J_{w_{c_1}}^T J_{w_{c_1}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$J_{v_{c_1}}^T J_{v_{c_1}} = \begin{bmatrix} L_{c_1} S\theta_1 & 0 \\ L_{c_1} C\theta_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L_{c_1} S\theta_1 & L_{c_1} C\theta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} L_{c_1}^2 & 0 \\ 0 & 0 \end{bmatrix}$$

Recall: $\boxed{Sin(\theta_1)^2 + Cos(\theta_1)^2 = 1}$

Step(1): Calculate Velocity Kinematics



II) For center of mass (2)

$$J_{\omega_2} = {}^0z_{i-1} = {}^0z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{v_2} = {}^0z_{i-1} \times ({}^0o_N - {}^0o_{i-1})$$

$$= {}^0z_1 \times ({}^0o_{c_2} - {}^0o_1)$$

$$\begin{array}{ccc} + & - & + \\ \end{array}$$

$$\begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ L_{c_2} C(\theta_1+\theta_2) & L_{c_2} S(\theta_1+\theta_2) & 0 \end{bmatrix} = \begin{bmatrix} -L_{c_2} \sin(\theta_1+\theta_2) \\ L_{c_2} C(\theta_1+\theta_2) \\ 0 \end{bmatrix}$$

$$J_{\omega_{c_2}}^T J_{\omega_{c_2}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$J_{v_{c_2}}^T J_{v_{c_2}} =$$

$$\begin{bmatrix} -L_1 S\theta_1 - L_{c_2} S(\theta_1+\theta_2) & L_1 C\theta_1 + L_{c_2} C(\theta_1+\theta_2) & 0 \\ -L_{c_2} S(\theta_1+\theta_2) & L_{c_2} (\theta_1+\theta_2) & 0 \end{bmatrix} \begin{bmatrix} -L_1 S\theta_1 - L_{c_2} \sin(\theta_1+\theta_2) & -L_{c_2} \sin(\theta_1+\theta_2) \\ L_1 C\theta_1 + L_{c_2} \cos(\theta_1+\theta_2) & L_{c_2} \cos(\theta_1+\theta_2) \\ 0 & 0 \end{bmatrix}$$

$$J_{v_{c_2}}^T J_{v_{c_2}} = \begin{bmatrix} L_1^2 + L_{c_2}^2 + 2 L_1 L_{c_2}^2 + 2 L_1 L_{c_2} c\theta_2 & L_{c_2}^2 + L_1 L_{c_2} \cos(\theta_2) \\ L_{c_2}^2 + L_1 L_{c_2} \cos(\theta_2) & m_2 L_{c_2}^2 \end{bmatrix}$$

Step(2): Calculate Inertia Matrix (D)

$$D = \sum_{i=1}^{n} m_i J_{v_i}^T J_{v_i} + J_{w_i}^T R_i I_i R_i^T J_{w_i}$$

Since I_i is **scalar** then $R_i R_i^T$ is **Identity**

$$\underset{(2\times 2)}{D} = m_1 J_{v_{c_1}}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}^T J_{v_{c_2}} + I_1 J_{w_{c_1}}^T J_{w_{c_1}} + I_2 J_{w_{c_2}}^T J_{w_{c_2}}$$

$$m_1 \begin{bmatrix} L_{c_1}^2 & 0 \\ 0 & 0 \end{bmatrix} + m_2 \begin{bmatrix} L_1^2 + L_{c_2}^2 + 2 L_1 L_{c_2}^2 + 2 L_1 L_{c_2} c\theta_2 & L_{c_2}^2 + L_1 L_{c_2} \cos(\theta_2) \\ L_{c_2}^2 + L_1 L_{c_2} \cos(\theta_2) & m_2 L_{c_2}^2 \end{bmatrix}$$

$$+ I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$d_{11} = m_1 L_{c_1}^2 + m_2 (L_1^2 + L_{c_2}^2 + 2 L_1 L_{c_2}^2 + 2 L_1 L_{c_2} c\theta_2) + I_1 + I_2$$

$$d_{12} = d_{21} = m_2 (L_{c_2}^2 + L_1 L_{c_2} c(\theta_2)) + I_2$$

$$d_{22} = m_2 L_{c_2}^2 + I_2$$

Step(3): Calculate Centrifugal + Coriolis Matrix (C)

$$C_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} + \frac{\partial d_{ij}}{\partial q_k}\right\}$$

Note:

- For fixed k , we have $C_{ijk} = C_{jik}$

$$C_{111} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0$$

$$C_{121} = C_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = \underbrace{-m_2 L_1 L_{c_2} \sin(\theta_2)}_{h}$$

$$C_{221} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h$$

$$C_{112} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h$$

$$C_{122} = C_{211} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0$$

$$C_{222} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0$$

Step(4): Calculate Gravitational Forces Matrix (G)

I) Potential Energy for link(1)

$$P_1 = m_1 \, g \, L_{c_1} \, S(\theta_1)$$

II) Potential Energy for link(2)

$$P_2 = m_2 \, g \, (L_1 S(\theta_1) + L_{c_2} S(\theta_1 + \theta_2))$$

=> Total Potential Energy

$$P = P_1 + P_2 = (m_1 L_{c_1} + m_2 L_1) g \, S(\theta_1) + m_2 L_{c_2} g \, S(\theta_1 + \theta_2)$$

- $\phi_1 = \dfrac{\partial P}{\partial q_1} = (m_1 L_{c_1} + m_2 L_1) g \, C(\theta_1) + m_2 L_{c_2} g \, C(\theta_1 + \theta_2)$
- $\phi_2 = \dfrac{\partial P}{\partial q_1} = m_2 L_{c_2} C(\theta_1 + \theta_2)$

Equation of motion

$$\boxed{\begin{aligned} \tau_1 &= d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + \phi_1 \\ \tau_2 &= d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + \qquad + c_{112}\dot{q}_1^2 + \phi_2 \end{aligned}}$$