The Tetrad Formulation of General Relativity

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Motivation for Noncoordinate Bases

- Traditional approach uses coordinate bases $\hat{e}_{(\mu)} = \partial_\mu$
- Noncoordinate bases $\hat{e}_{(a)}$ are not derived from any coordinate system
- Advantages:
  - Reveals connection to gauge theories
  - Simplifies calculations in certain metrics
  - Enables description of spinor fields

Orthonormal Basis Definition

Basis Vectors

At each point, introduce basis vectors $\hat{e}_{(a)}$ (Latin indices) satisfying:

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab}$$

where η_{ab} is the canonical form of the metric.

Terminology

- Tetrad (4D), vielbein ("many legs"), dreibein (3D), zweibein (2D)
- Relation to coordinate basis:

$$\hat{e}_{(\mu)} = e^a_\mu \hat{e}_{(a)}$$

$$\hat{e}_{(a)} = e^\mu_a \hat{e}_{(\mu)}$$

Vielbein Relationships

Metric Relations

$$g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab}$$

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$$

Basis One-Forms

Orthonormal one-forms $\hat{\theta}^{(a)}$ compatible with basis vectors:

$$\hat{\theta}^{(a)}(\hat{e}_{(b)}) = \delta^a_b$$

Relations to coordinate basis:

$$\hat{\theta}^{(\mu)} = e^\mu_a \hat{\theta}^{(a)}$$

$$\hat{\theta}^{(a)} = e^a_\mu \hat{\theta}^{(\mu)}$$

Component Transformations

Vector Components

For vector $V = V^\mu \hat{e}_{(\mu)} = V^a \hat{e}_{(a)}$:

$$V^a = e^a_\mu V^\mu$$

$$V^\mu = e^\mu_a V^a$$

Tensor Components

Mixed components transform as:

$$V^a_b = e^a_\mu e^\nu_b V^\mu_\nu$$

Index Interpretation

- Greek indices: "curved" spacetime indices
- Latin indices: "flat" tangent space indices

Basic Concept

- Tetrad: Set of four orthonormal basis vectors $\{e^{\mu}_{(a)}\}$ at each spacetime point
- In Minkowski space: $\eta_{ab} = e^{\mu}_{(a)} e^{\nu}_{(b)} \eta_{\mu\nu}$
- Relates local frame to global coordinates

Inertial Observer Tetrad

For standard Minkowski coordinates (t, x, y, z):

$$e_{(0)} = (1, 0, 0, 0),\ e_{(1)} = (0, 1, 0, 0),\ e_{(2)} = (0, 0, 1, 0),\ e_{(3)} = (0, 0, 0, 1)$$

Satisfies $e_{(a)} \cdot e_{(b)} = \eta_{ab}$

Tetrad for Accelerated Observer

Rindler Coordinates

For observer with constant acceleration a in x-direction:

$$t = \frac{1}{a} e^{a\xi} \sinh(a\eta)$$
$$x = \frac{1}{a} e^{a\xi} \cosh(a\eta)$$

Rindler Tetrad

$$e_{(0)} = \frac{1}{a\sqrt{x^2 - t^2}}(x\partial_t + t\partial_x)$$
$$e_{(1)} = \frac{1}{a\sqrt{x^2 - t^2}}(t\partial_t + x\partial_x)$$
$$e_{(2)} = \partial_y, \quad e_{(3)} = \partial_z$$

Satisfies $e_{(a)} \cdot e_{(b)} = \eta_{ab}$ along hyperbolic trajectory

Spin Connection Definition

Covariant Derivative

For tensor with Latin indices:

$$\nabla_\mu X^a_b = \partial_\mu X^a_b + \omega_\mu{}^a{}_c X^c_b - \omega_\mu{}^c{}_b X^a_c$$

Tetrad Postulate

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega_\mu{}^a{}_b e^b_\nu = 0$$

Relation to Christoffel Symbols

$$\Gamma^\nu_{\mu\lambda} = e^\nu_a \partial_\mu e^a_\lambda + e^\nu_a e^b_\lambda \omega_\mu{}^a{}_b$$

$$\omega_\mu{}^a{}_b = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda$$

Transformation Properties

Under Local Lorentz Transformations

$$\omega_{\mu}^{a}{}_{b} \to \Lambda^{a}_{a'} \Lambda^{b'}_{b} \omega_{\mu}^{a'}{}_{b'} - \Lambda^{b'}_{c} \partial_{\mu} \Lambda^{a'}_{c}$$

where $\Lambda^{a}_{a'}$ satisfies:

$$\Lambda^{a}_{a'} \Lambda^{b}_{b'} \eta_{ab} = \eta_{a'b'}$$

Under General Coordinate Transformations

Transforms as a one-form in the Greek index:

$$\omega_{\mu}^{a}{}_{b} \to \frac{\partial x^{\nu}}{\partial x^{\mu'}} \omega_{\nu}^{a}{}_{b}$$

Cartan Structure Equations

Basis One-Forms and Connection

Define:

$$e^a = e^a_\mu dx^\mu$$

$$\omega^a{}_b = \omega^a_{\mu b} dx^\mu$$

Torsion and Curvature

$$T^a = de^a + \omega^a{}_b \wedge e^b$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

Bianchi Identities

$$dT^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b$$

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0$$

Cartan Structure Equations

Geometric Interpretation

- Torsion measures failure of infinitesimal parallelograms to close
- First structure equation defines torsion as "twisting" of frame fields
- Second structure equation relates curvature to connection rotation

Bianchi Identities from Cartan Formalism

$$DT^a = R^a{}_b \wedge e^b \quad \text{(First identity)}$$
$$DR^a{}_b = 0 \quad \text{(Second identity)}$$

where D is exterior covariant derivative:

$$DT^a = dT^a + \omega^a{}_b \wedge T^b$$

Tetrad Formalism in GR

General Definition

At each point p in spacetime:

$$g_{\mu\nu} e^{\mu}_{(a)} e^{\nu}_{(b)} = \eta_{ab}$$

Inverse relation:

$$g_{\mu\nu} = \eta_{ab} e^{(a)}_{\mu} e^{(b)}_{\nu}$$

Physical Interpretation

- $e^{\mu}_{(a)}$ gives local inertial frame at each point
- $e^{(a)}_{\mu}$ converts between coordinate and local frames
- Spin connection ω^{ab}_{μ} describes how frames rotate during parallel transport

Example: FLRW Metric in Vielbein Form

Metric and Vielbein Choice

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

Choose orthonormal basis:

$$e^0 = dt, \quad e^i = a(t)dx^i$$

Solving for Spin Connection

From torsion-free condition $\omega^a{}_b \wedge e^b = -de^a$:

$$de^0 = 0$$
$$de^i = \dot{a}dt \wedge dx^i$$

With antisymmetry: $\omega^0{}_0 = 0$, $\omega^0{}_j = \omega^j{}_0$, $\omega^i{}_j = -\omega^j{}_i$

FLRW Spin Connection Solution

Solution

$$\omega^0{}_j = \dot{a} dx^j, \quad \omega^i{}_0 = \dot{a} dx^i, \quad \omega^i{}_j = 0$$

Verification:

$$\omega^0{}_j \wedge e^j = \dot{a} dx^j \wedge a dx^j = 0$$

$$\omega^i{}_0 \wedge e^0 + \omega^i{}_j \wedge e^j = \dot{a} dx^i \wedge dt + 0 = -\dot{a} dt \wedge dx^i$$

Curvature Two-Form (Key Components)

$$R^0{}_j = \ddot{a} dt \wedge dx^j$$

$$R^i{}_0 = \ddot{a} dt \wedge dx^i$$

$$R^i{}_j = \dot{a}^2 dx^i \wedge dx^j$$

Converting FLRW Curvature to Coordinate Basis

Vielbein Components

$$e^a_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad e^\nu_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & \frac{1}{a} \end{pmatrix}$$

Riemann Tensor Components

Using $R^\rho{}_{\sigma\mu\nu} = e^\rho_a e^b_\sigma R^a{}_{b\mu\nu}$:

$$R^0{}_{j0i} = a\ddot{a}\delta_{ji}$$

$$R^i{}_{0k0} = -\frac{\ddot{a}}{a}\delta^i_k$$

$$R^i{}_{jkl} = \dot{a}^2(\delta^i_k\delta_{jl} - \delta^i_l\delta_{jk})$$

Canonical Tetrad

For Schwarzschild metric:

$$ds^2 = -(1 - 2M/r)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2$$

Choose:

$$\begin{aligned} e_{(0)} &= \sqrt{1 - 2M/r}\,\partial_t \\ e_{(1)} &= \frac{1}{\sqrt{1 - 2M/r}}\,\partial_r \\ e_{(2)} &= \frac{1}{r}\,\partial_\theta \\ e_{(3)} &= \frac{1}{r\sin\theta}\,\partial_\phi \end{aligned}$$

Schwarzschild Solution: Spin Connection

Non-zero Spin Connection Components

$$\omega^0{}_1 = \frac{M}{r^2} dt$$

$$\omega^1{}_2 = -\sqrt{1 - 2M/r}\, d\theta$$

$$\omega^1{}_3 = -\sqrt{1 - 2M/r} \sin\theta \, d\phi$$

$$\omega^2{}_3 = -\cos\theta \, d\phi$$

Advantages of Tetrad Approach

- Separates gravitational and coordinate effects
- Reveals physical structure of spacetime
- Simplifies calculation of curvature invariants

- Extension of GR allowing for torsion $T^a \neq 0$
- Spin connection ω_μ^{ab} becomes independent dynamical field
- Field equations:

$$G^{ab} = 8\pi T^{ab}$$
$$T^a{}_{bc} = 8\pi \tau^a{}_{bc}$$

where $\tau^a{}_{bc}$ is spin density tensor

Tetrad Action

Action in terms of tetrad and spin connection:

$$S = \frac{1}{16\pi} \int e^a \wedge e^b \wedge R^{cd} \epsilon_{abcd} + S_{\text{matter}}$$

where $e = \det\left(e_\mu^a\right)$ and ϵ_{abcd} is Levi-Civita tensor

Field Equations Derivation

Varying with respect to tetrad:

$$\epsilon_{abcd} e^b \wedge R^{cd} = 16\pi\tau^{(a)}$$

where $\tau^{(a)}$ is energy-momentum 3-form

Varying with respect to spin connection:

$$\epsilon_{abcd} T^c \wedge e^d = 16\pi\sigma_{ab}$$

where σ_{ab} is spin current 3-form

- Torsion propagates spin density through spacetime
- Avoids singularities in some cases
- Reduces to GR in vacuum (no spin sources)

References

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