MTL603: Partial Differential Equations Project

The Mean Curvature Flow in $\mathbb{R}^2$

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Chapter 1

Curves and Differential Geometry

To begin, we require some basic concepts from differential geometry. Here, we freely assume that all curves are smooth (i.e. adequately differentiable in some C^k or even C^∞).

- A closed curve is one that has the same endpoints.
- An embedded curve is a curve that does not self-intersect and is a bijection (here a diffeomorphism) of an interval onto its image.
- An immersed or regular curve is one with non-zero derivative everywhere.
- For a curve parametrised as $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, the tangent fevtor is given by
$$\gamma'(t) = (\frac{d\gamma_1}{dt}(t), \frac{d\gamma_2}{dt}(t)), \ldots, \frac{d\gamma_n}{dt}(t)$$
- The length of a curve is defined to be $L(\gamma) = \int_I \|\gamma'(u)\| \, du$. The length of a curve does not depend on how it is parametrized.

A unit-speed curve is a reparametrization of a curve such that $\forall t, |\gamma'(t)| = 1$. A curve is parametrized proportionally to arclength if its derivative has constant non-zero magnitude everywhere. Any regular curve admits an arclength reparametrization.

Definition 1. *For a unit-speed curve $\varphi : I \to \mathbb{R}^2$, the Frenet-Serret frame is comprised of the two orthogonal vector fields: T (the tangent vector) and N (the normal vector). The tangent vector is representative of the derivative of a given point on the curve, while the normal vector is defined as the unique vector orthogonal to the tangent vector at a given point.*

The signed curvature of a curve parametrised by arclength is the quantity κ defined implicitly by the equation

$$\frac{D}{ds}\gamma'(s) = \kappa(s)n(s).$$

where $\gamma'(s)$ denotes the tangent vector, $n(s)$ denotes the principal normal vector, and $\frac{D}{ds}$ is the covariant derivative from Riemannian geometry. κ essentially measures the "twisting" of a curve inwards or outwards at a given point.

Proposition 1. *The Frenet frame can be found for an arbitrary speed curve by normalizing each vector field as follows:*

$$T = \frac{\gamma'}{\|\gamma'\|}$$

$$\kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}$$

Remark 1. Note: The curvature of a curve depends on the manifold it sits on. For instance, consider the curve $\gamma(u) = (\cos(u), \sin(u), 0)$, and let $M_1 = \mathbb{R}^3$ with the Euclidean metric, and $M_2 = S^2$ embedded in $\mathbb{R}^3$ centered at the origin with the metric induced from the Euclidean metric. Then γ has constant curvature $\kappa = 1$ everywhere on M_1 but zero curvature everywhere on M_2.

Finally, we will require the twin concepts of geodesics and convecity.

Definition 2. *A curve γ is a geodesic if its curvature $\kappa = 0$ everywhere along the curve.*

Riemannian geometry and ODE theory tells us that for each point on a manifold, we can find a unique maximal geodesic that passes through that point in any given direction (i.e the geodesic vector has a tangent vector equal to a tangent vector in the tangent space at the point). Now, as one may have realised, in Euclidean space, geodesics are straight lines. Thus geodesics are 'generalisations' of straight lines, in the sense that they are *local length minimisers* i.e. they are local minimums of the length functional between two points. Note that this does not mean they are the shortest path between two points.

Consider a connected manifold M. Define a distance function on it by $d(p,q) = \inf \{L(\gamma) : \gamma \text{ is piecewise smooth}, \gamma : [a,b] \to M, \gamma(a) = p, \gamma(b) = q\}$. Then with this distance function can obtain the metric space (M,d) (in general, the distance function will depend on the Riemannian metric on the manifold, with the induced metric topology equal to that of the e "original" topology of M as a smooth manifold).

- A subset $V \subset M$ is compact if it is compact in the metric space (M,d). Closed curves, being the image of a compact S^1 under a continuous map, are compact.
- A manifold M is complete if every geodesic can be extended indefinitely in both directions (that is, its domain I can be extended to $\mathbb{R}$).
- The Hopf-Rinow theorem from differential geometry tells us that any two points on a connected and complete manifold can be joined by a length-minimising geodesic.

Definition 3. *Let (M,g) be a connected and complete Riemannian manifold. A subset $K \subset M$ is convex if for any two pointsthe shortest geodesic between the points is entirely contained in K. he convex hull H of a subset $U \subset M$ is the intersection of all convex sets K containing U.*

Chapter 2

The Curve Shortening Flow

The curve shortening flow (CSF) equation is defined as

$$\frac{\partial\gamma(t)}{\partial t} = k(t)n(t),$$

Definition 4. *Consider the curve $\gamma_0(u) : I \to \mathbb{R}^2$, and let $T > 0$. We say that a family of curves $\gamma(u,t) = \gamma_t(u) : I \times [0,T) \to M$ evolves under curve shortening flow if it satisfies the initial value problem*

$$\frac{\partial\gamma}{\partial t} = \kappa n,$$

with

$$\gamma(u,0) = \gamma_0(u),$$

where $\kappa(u,t)$ is the curvature of γ, and n is the principal unit normal vector to γ.

We now consider a basic example. Let our initial $\gamma_0 = S61$, centred at the origin. Parametrising γ_0 by $\gamma_0(u) = (\cos(u), \sin(u)), u \in [0, 2\pi]$, we can compute the curvature and the normal vector :

$$\kappa(u) = 1; n(u) = -(\cos(u), \sin(u)) = -\gamma_0(u)$$

Since S^1 is radially symmetric, each γ_t retains its shape under the CSF, and so $\gamma(u,t) = r(t)(\cos(u), \sin(u))$. Then for any t we get

$$\kappa(u,t) = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} = \frac{(r(t))^2\|1\|}{(r(t))^3\|(-cos(u), sin(u))\|^3} = \frac{1}{r(t)}$$

and normal vector

$$n(u,t) = (-\cos(u), -\sin(u)) = -\gamma_0(u).$$

Then under the CSF, we get an ODE for the radius of the form $r'(t) = -\frac{1}{r(t)}$ with $r(0) = 1$. This can be solved to obtain $r(t) = \sqrt{1-2t}$. Thus under the CSF the unit circle evolves by $\gamma(u,t) = \sqrt{1-2t}(\cos(u), \sin(u))$, vanishing at $t = \frac{1}{2}$.

The theory of parabolic partial differential equations has a wide variety of applications in mathematics, physics and engineering. Indeed, the famous Ricci Flow, used to solved the Poincare Conjecture, is also a parabolic PDE. We only note here that since the curve shortening flow is a nonlinear parabolic PDE, some of the standard theory can be applied to it. The most prototypical parabolic PDE is the heat equation or diffusion equation, which is utilized to describe the conduction of heat in different situations and is is given by

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) u = 0. \tag{10}$$

The heat equation, and more generally parabolic equations, have properties that are useful in understanding the curve shortening flow.

1. If the domain of a solution to the Heat equation is given by $U_T := U \times (0,T]$, we define the closed parabolic cylinder $\bar{U}_T = U \times [0,T] \cup (\partial U \times [0,T])$ with boundary $\partial U_T := U \times \{t = 0\} \cup \partial U \times (0,T)$. Then any solution of the heat equation achievs its maximum on the boundary i.e. on ∂U_T. (Weak maximum Principle)

2. If U is connected and there exists a point $(x_0, t_0) \in U_T$ such that $u(x_0, t_0) = \max_{\bar{U}_T} u$, then u is constant within $\bar{U}_T$. (Strong maximum Principle)

3. The boundary value problem on U has a unique solution.

Now, the key here is to rewrite the the term $k(t)n(t)$ on the right-hand side of the CSF equation can be rewritten as $k(t)n(t) = \frac{\partial^2 \gamma}{\partial s^2}$ so that the CSF equation can be rewritten as $\frac{\partial}{\partial t}\gamma - \frac{\partial^2 \gamma}{\partial s^2} = 0$ which looks similar to the heat equation, though of course this is nonlinear.

We require a general definition of parabolicity for a geometric flow:

$$\frac{\partial}{\partial t}\gamma = F(\phi, \zeta, k)n, \quad (p,t) \in I \times (0,T), \quad T > 0,$$

where $F = F(x, y, \zeta, q)$ is a function given in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$. We can say that F is parabolic in a set E if

$$\frac{\partial F}{\partial q}(x, y, \zeta, q) > 0 \quad \text{for all } (x, y, \zeta, q) \in E.$$

Furthermore, we may say that F is uniformly parabolic if there exist two positive numbers such that

$$\alpha \leq \frac{\partial F}{\partial q} \leq \beta.$$

Thus the CSF satisfies uniform parabolicity.

Now, it can be seen that the maximum principle straight away gives us something: let γ_1, γ_2 be embedded curves that do not intersect. By uniqueness we get two flows, $\gamma_{1,t}, \gamma_{2,t}$ on $[0,T) \times [0,2\pi)$ with $\gamma_{1,0} = \gamma_1 \quad \text{and} \gamma_{2,0} = \gamma_2$.. Then $\gamma_{1,t}$ does not intersect $\gamma_{2,t} \forall t$, since this would contradict the maximum principle.

A key result for the CSF is as the Gage-Hamilton-Grayson theorem, which concerns the existence and behaviour of curve shortening flow on closed embedded curves. Gage and Hamilton showed in 1983 that all smooth convex curves shrink to a point without any other singularities, and then in 1987 Grayson showed that the same holds for non-convex curves.

Theorem 2.0.1 (Gage-Hamilton-Grayson Theorem)**.** *Let the curve* $\gamma_0 : S^1 \to \mathbb{R}^2$ *be embedded. Then equation (2.1) has a solution up to some maximal finite time* T*. Moreover, the curve* γ_t *is smooth for all* $t \in [0,T)$*, and it converges to a round point as* $t \to T$*.*

Chapter 3

Properties of Curve Shortening Flow

First, we prove some critical lemmas for the CSF.

Lemma 1. *Let s parametrize γ by arclength. Then as operators we have*

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u},$$

where $v = \|\frac{\partial \gamma}{\partial u}\|$.

Proof. By definition of arclength, we have:

$$\frac{\partial s}{\partial u} = \frac{\partial}{\partial u}\int_0^u \|\frac{\partial \gamma}{\partial u'}\| du' = \|\frac{\partial \gamma}{\partial u}\| = v.$$

Thus,

$$\frac{\partial}{\partial s} = \frac{\partial u}{\partial s}\frac{\partial}{\partial u} = \left(\frac{\partial s}{\partial u}\right)^{-1}\frac{\partial}{\partial u} = \frac{1}{v}\frac{\partial}{\partial u}.$$

□

Lemma 2. *If $\gamma(u,t)$ evolves under curve shortening flow, then*

$$\frac{\partial v}{\partial t} = -\kappa^2 v.$$

Proof. Here we use the Serret-Frenet equations:

$$\frac{\partial T}{\partial u} = v\kappa n,$$
$$\frac{\partial n}{\partial u} = -v\kappa T,$$

where T is the unit tangent vector to γ. Here, we note that $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute. Thus, we can differentiate v^2:

$$\frac{\partial}{\partial t}\left(v^2\right) = 2\frac{\partial v}{\partial t}v.$$

and

$$\frac{\partial}{\partial t}\left(v^2\right) = \frac{\partial}{\partial t}\left(\left|\frac{\partial \gamma}{\partial u}\right|^2\right) = 2\frac{\partial}{\partial u}\left(\frac{\partial \gamma}{\partial t}\cdot\frac{\partial \gamma}{\partial u}\right)$$

$$= 2\frac{\partial}{\partial u}\left(\frac{\partial}{\partial t}\frac{\partial \gamma}{\partial u}\right) = 2\frac{\partial}{\partial u}\left(\frac{\partial^2 \gamma}{\partial u \partial t}\right) = 2vt\frac{\partial}{\partial u}\left(\kappa n\right) = 2vT\left(\frac{\partial \kappa}{\partial u}n - v\kappa^2 T\right)$$

Thus since the normal vector is orthogonal to the tangent vector,

$$2\frac{\partial v}{\partial t}v = -2v^2\kappa^2 \implies \frac{\partial v}{\partial t} = -\kappa^2 v$$

□

Lemma 3. *Under the CSF, as operators, we have that*

$$\frac{\partial^2}{\partial t \partial s} = \frac{\partial^2}{\partial s \partial t} + \kappa^2\frac{\partial}{\partial s}.$$

Proof. We straightaway compute:

$$\frac{\partial^2}{\partial t \partial s} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial t}\left(\frac{1}{v}\frac{\partial}{\partial u}\right) = \kappa^2\frac{1}{v}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial t}\frac{\partial}{\partial u} = \kappa^2\frac{1}{v}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial u}\frac{\partial}{\partial t}$$

$$= \kappa^2\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t} = \frac{\partial^2}{\partial s \partial t} + \kappa^2\frac{\partial}{\partial s}$$

□

Lemma 4. *If $\gamma(u,t)$ evolves under curve shortening flow, then*

$$\frac{\partial t}{\partial t} = \frac{\partial \kappa}{\partial s} n$$

and

$$\frac{\partial n}{\partial t} = -\frac{\partial \kappa}{\partial s} T$$

Proof. By the definition of the unit tangent T:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 \gamma}{\partial t \partial s} = \frac{\partial^2 \gamma}{\partial s \partial t} + \kappa^2 \frac{\partial \gamma}{\partial s} = \frac{\partial(\kappa n)}{\partial s} + \kappa^2 T = \frac{\partial \kappa}{\partial s} n + \frac{\partial n}{\partial s}\kappa + \kappa^2 T = \frac{\partial \kappa}{\partial s} n + (-\kappa T)\kappa + \kappa^2 T = \frac{\partial \kappa}{\partial s} n$$

For the second equation, we take the inner product of the tangent with the normal and differentiate with respect to t:

$$T \cdot n = 0 \implies 0 = \frac{\partial t}{\partial t} \cdot n + t \cdot \frac{\partial n}{\partial t} \implies \frac{\partial n}{\partial t} = -\frac{\partial \kappa}{\partial s} t$$

□

Lemma 5. *The curvature changes with time following the equation*

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 \kappa}{\partial s^2} + \kappa^3$$

Proof. Here, wet n be the unit normal vector given by an anticlockwise rotation of the unit tangent vector by an angle of $\frac{\pi}{2}$, and take κ to be the curvature induced by this choice of normal vector. Let $\theta(s,t)$ to be the angle between the unit tangent vector $\gamma'(s,t)$ and the x-axis. Then, in coordinates, we have

$$T = (\cos\theta, \sin\theta)$$

Differentiating,

$$\frac{\partial T}{\partial t} = \frac{\partial \theta}{\partial t}(-\sin\theta, \cos\theta) = \frac{\partial \theta}{\partial t} n$$

But as we have shown

$$\frac{\partial t}{\partial t} = \frac{\partial \kappa}{\partial s} n$$

Thus

$$\frac{\partial \theta}{\partial t} = \frac{\partial \kappa}{\partial s}$$

and similarly

$$\frac{\partial T}{\partial s} = \frac{\partial \theta}{\partial s} n$$

And we already know

$$\frac{\partial t}{\partial s} = \kappa n.$$

Thus,

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 \theta}{\partial t \partial s} = \frac{\partial^2 \theta}{\partial s \partial t} + \kappa^2 \frac{\partial \theta}{\partial s} = \frac{\partial}{\partial s}\frac{\partial \theta}{\partial t} + \kappa^2 \frac{\partial \theta}{\partial s} = \frac{\partial^2 \kappa}{\partial s^2} + \kappa^3$$

□

Finally, two more key lemmas.

Lemma 6. *Let γ be a closed embedded curve which evolves by curve shortening flow. Let $L(t)$ be the length of γ_t and $A(t)$ the area it encloses. Then:*

- *(i)* $\frac{dL}{dt} = -\int_{\gamma_t} \kappa^2 \, ds$,
- *(ii)* $\frac{dA}{dt} = -2\pi$.

Proof. For (i), we simply use the definition of length:

$$\frac{dL}{dt} = \frac{d}{dt}\int_0^{2\pi} v \, du = \int_0^{2\pi} \frac{\partial v}{\partial t} \, du$$

Since $\frac{\partial v}{\partial t} = -\kappa^2 v$, we get:

$$\frac{dL}{dt} = -\int_0^{2\pi} \kappa^2 v \, du = -\int_{\gamma_t} \kappa^2 \, ds$$

To prove (ii), let $\gamma(u,t) = (\gamma_1(u,t), \gamma_2(u,t))$ and choose n to be the normal vector pointing inwards. We then use Green's theorem to get an expression for $A(t)$:

$$A(t) = \iint dA = \frac{1}{2}\int_0^{2\pi} \left(\gamma_1 \frac{\partial \gamma_2}{\partial u} - \gamma_2 \frac{\partial \gamma_1}{\partial u}\right) du = -\frac{1}{2}\int_0^{2\pi} v\gamma \cdot n \, du$$

We now differentiate both sides with respect to t:

$$\frac{dA}{dt} = -\frac{1}{2}\int_0^{2\pi} \left(\frac{\partial \gamma}{\partial t} vn + \gamma \frac{\partial v}{\partial t} n + \gamma v \frac{\partial n}{\partial t}\right) du = -\frac{1}{2}\int_0^{2\pi} \left(\kappa v - \gamma \kappa^2 vn - \gamma \frac{\partial \kappa}{\partial u} t\right) du$$

$$= -\frac{1}{2}\int_0^{2\pi}\left(\kappa v - \varphi\kappa^2 vn + \kappa v + \varphi v\kappa^2 n\right) du = -\int_0^{2\pi} \kappa v \, du = -\int_{\gamma_t} \kappa ds$$

Since each γ_t is a closed curve, we get:

$$\frac{dA}{dt} = -2\pi.$$

$\square$

These last two lemmas are of special importance. They directly show that under the CSF, both length and enclosed area are decreasing functions with respect to time. Furthermore, the area enclosed by a curve is necessarily non-negative, the flow cannot exist beyond $T = \frac{A_0}{2\pi}$. But, we also know the flow exists until the curve shrinks to a point, at which point it must have zero area. Thus, the flow cannot stop before $T = \frac{A_0}{2\pi}$, and so the maximal T is $T = \frac{A_0}{2\pi}$.

We end this section by stating a concrete form of a result we had shown earleir, namely, that the CSF keeps embedded curves embedded.

Theorem 3.0.1. *Let $\gamma : S^1 \times [0, T_0) \to \mathbb{R}^2$ be a family of closed curves satisfying equation (2.1). If the initial curve γ_0 is embedded and if there exists $c \in \mathbb{R}$ such that $\kappa(u,t) \leq c$ for all $(u,t) \in S^1 \times [0, T_0)$, then γ_t is an embedded curve for each $t \in [0, T_0)$.*

Chapter 4

Applications of Curve Shortening Flow

The CSF can be a wonderful tool for proving statements about curves and geodesics on Riemannian Manifold. Indeed, it has been used to prove two major theorems:

- Every smooth Riemannian metric g on the 2-sphere S^2 admits at least 3 closed embedded geodesics.
- Any smooth embedded curve on the round sphere S^2 which divides the sphere into two parts of equal area must have at least four inflection points.

These are major theorems and so we will not prove them here. Instead, we will focus on an another much celebrated results: the isoperimetric inequality.

Theorem 4.0.1 (Planar Isoperimetric Inequality)**.** *Let γ_0 be a closed embedded curve in $\mathbb{R}^2$. Let A denote the area it encloses and L denote its length. Then the following inequality holds:*

$$4\pi A \leq L^2$$

Proof. Consider the evolution of γ_0 under CSF and let γ_t be the curves obtained. We have already shown the time evolution of the are and the length under the CSF:

$$\frac{dA}{dt} = -2\pi; \frac{dL}{dt} = -\int_{\gamma_t} k^2 \, ds.$$

Thus,

$$-4\pi\frac{dA}{dt} = 2(2\pi)^2 = 2\left(\int_{\gamma_t} k\,ds\right)^2$$

where $\int_{\gamma_t} k\,ds = 2\pi$ since each γ_t is a closed curve. Then by the Cauchy-Schwarz Inequality

$$-4\pi\frac{dA}{dt} \leq 2\int_{\gamma_t} 1\,ds \int_{\gamma_t} k^2\,ds = -2L(t)\frac{L(t)}{dt} = -\frac{d}{dt}(L(t))^2$$

By the Gage-Hamilton-Grayson theorem, γ_t shrinks to a point after time T. Then by integrating both sides between $t = 0$ and $t = T$, we get

$$4\pi A \leq L^2$$

□

Remark 2. The planar Isoperimetric inequality can be extended to arbitrary curves on a manifold that enclose a surface homeomorphic to the closed 2-disc.

Finally, the mean curvature flow also has several practical applications across many fields:

1. A 1956 paper showed the application of the CSF to materials science, where Mullins used it to show how grooves develop on the surfaces of hot polycrystals by obtaining a version of the CSF which described changes due to evaporation and condensation.[1]

2. The CSF has been shown to be remarkably useful in image processing: it can be used to smooth images, by removing unnecessary noise without compromising the amount of information conveyed, and to enhance images, by emphasising particular parts of them. A paper from 1995 used a particular version of the CSF to restore corrupted images. Here, a variant of the CSF is applied to curves that are the level sets of the intensity function of black and white images, with the κ term in of the CSF either being retained as the curvature or being replaced by $\max(\kappa, 0)$ or $\min(\kappa, 0)$ as necessary. The properties of the CSF make

[1]Mullins, W. W. (1957). Theory of Thermal Grooving. Journal of Applied Physics, 28(3), 333–339. doi:10.1063/1.1722742

this process computationally efficient, since, firstly, it is based on only one parameter (the curvature κ), and secondly, the flow terminates after a finite amount of time.

3. By the late 80s algorithms that could numerically compute the propagation of surfaces moving with a speed dependent on the curvature. These algorithms can help to quickly solve various surface motion problems and thus can be used in modelling such as flame stretching, vortex sheet rollup, Hele-Shaw cells, and crystal growth.[2]

4. Yet another application of CSF is to chemical reactions modelled by a specific form of the reaction-diffusion equation. Here the CSF equation arises for systems with a very high reaction rate and a very slow diffusion rate which naturally lead to the formation of fronts—effectively curves in the two-dimensional case. These fronts i.e. curves change as the reaction takes place, and can be modelled accurately by the CSF, using which it can be shown that the fronts can shrink to a point in finite time.[3]

[2]Osher, S., & Sethian, J. A. (1988). Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations. Journal of Computational Physics, 79(1), 12–49. doi:10.1016/0021-9991(88)90002-2

[3]Rubinstein, J., Sternberg, P., & Keller, J. B. (1989). Fast Reaction, Slow Diffusion, and Curve Shortening. SIAM Journal on Applied Mathematics, 49(1), 116–133. doi:10.1137/0149007