The Mapping Class Group of the n-punctured Sphere

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Abstract

We give a brief foray into the theory of mapping class groups of surfaces, before discussing the case of n-punctured sphere in greater detail. Section 1 is an introduction to mapping class groups, section 2 deals with concrete examples of the mapping class group for low n, section 3 discusses a presentation of the mapping class group of the n-punctured sphere for $n \geq 2$ based on Joan Birman's *Braids, Links, and Mapping Class Groups* [1], and 4 considers a linear representation of the mapping class group of the n-punctured sphere for general n based on Stephen Bigelow and Ryan Budney's '*The mapping class group of a genus two surface is linear*' [2].

Keywords: Mapping Class Groups, N-punctured Sphere

1. An Introduction to Mapping Class Groups

For a given surface S with genus g, b boundary components and n punctures, we denote the *mapping class group* - the group of isotopy (or homotopy [3] - though it also evident that in this case we are always dealing with homeomorphisms) classes of orientation preserving diffeomorphisms (or homeomorphisms [4]) of S that restrict to the identity on ∂S (if ∂S is not empty) - by Mod(S). Thus,

$$\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^+(S, \partial S))$$

[1] *Birman, Joan.* Braids, Links, and Mapping Class Groups, Princeton Press, 1974.

[2] *Bigelow & Budney.* The mapping class group of a genus two surface is linear, Algebraic & Geometric Topology - Volume 1 (2001) 699708

[3] For orientation preserving homeomorphisms of compact surfaces, these are equivalent.

[4] For orientation preserving homeomorphisms of compact surfaces, we can switch freely - up to isotopy - between the two.

Equivalently,

$$\text{Mod}(S) = \text{Homeo}^+(S, \partial S)/\text{Homeo}_0(S, \partial S)$$

Remark 1. There is a 'disclaimer' for our definition: in the literature, one can also find Mod(S) defined for homeomorphisms that do not fix the boundary pointwise.

Remark 2. One is advised to think of punctures on a surface as marked points. Then each mapping class is an isotopy class of self-diffeomorphisms that preserves the set of marked points.

Now, let us compute some basic mapping class groups - that of the closed disk D^2, the punctured disk $D^2_{0,1}$ [5], the sphere $\mathbb{S}^2$, and the punctured sphere $\mathbb{S}^2_{0,1}$.

Lemma 1. *Alexander Lemma: Mod(D^2) is trivial*

Proof.
Identifying D^2 naturally with the closed unit disk in $\mathbb{R}^2$, we pick $\phi \in \text{Mod}(D^2)$. Defining $F : D^2 \times I \to D^2$ to be :
$F(x,t) = \begin{cases} (1-t)\phi(\frac{x}{1-t}) & 0 \leq |x| < 1-t \\ x & 1-t \leq |x| \leq 1 \end{cases}$
for $t \in [0,1)$ and $F(x,1)$ is the identity on D^2. Then F is an isotopy from ϕ to the identity on D^2 (see figure 1).

□

The mapping class group of the once punctured disk follows easily.

Corollary 1. *Mod($D^2_{0,1}$) is trvial*

Proof. One simply recreates the proof for Mod(D^2), except in our identification of D^2 with the closed unit disk in $\mathbb{R}^2$, we send the puncture to the origin. □

Proposition 1. *Mod($\mathbb{S}^2_{0,1}$) and Mod($\mathbb{S}^2$) are trivial*

[5] Here 0,1 denotes 0 boundary components and 1 punctures

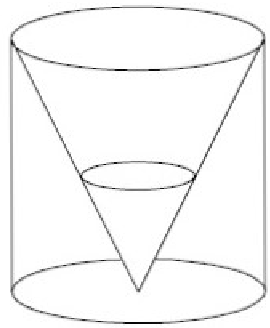

Figure 1: At time t, we apply ϕ on the disk of radius $1-t$ and the identity outside the inner disk (thus at time $t=0$ we have the image of ϕ and at $t=1$ we have the identity.)

Proof. For $\mathrm{Mod}(\mathbb{S}^2_{0,1})$, we know that $\mathbb{S}^2_{0,1}$is topologically equivalent to $\mathbb{R}^2$, and so, given an orientation preserving homeomorphism ϕ of $\mathbb{R}^2$, using the straight line homotopy $F(x,t) = (1-t)\phi(x) + tx$, we see that ϕ is homotopic to the identity. Thus, $\mathrm{Mod}(\mathbb{S}^2_{0,1})$ is trivial.
For $\mathrm{Mod}(\mathbb{S}^2)$, we use the fact that any orientation preserving homeomorphism ϕ of the sphere can be modified by isotopy to fix a point, and so, since $\mathrm{Mod}(\mathbb{S}^2_{0,1})$ is trivial, we see that $\mathrm{Mod}(\mathbb{S}^2)$ is trivial as well.

□

Let us now compute our first non-trivial mapping class group - that of the Annulus A.

Proposition 2. *Mod(A)* $\cong \mathbb{Z}$

Proof.

We begin by constructing a homomorphism $\rho : \mathrm{Mod(A)} \to \mathbb{Z}$. Pick $f \in \mathrm{Mod}(A)$, and let ϕ be a representative of f. Additionally, let δ be simple proper arc in A that connects the two boundary components. Since ϕ preserves the boundary, $\phi(\delta(0)) = \delta(0)$, and $\phi(\delta(1)) = \delta(1)$. Thus, $\phi(\delta) * \delta^{-1}$ is a loop in A, and $\rho(f) = [\phi(\delta) * \delta^{-1}] \in \pi_1(A, \delta(0)) \cong \mathbb{Z}$. This is a homomorphism since compositions of mapping classes are sent to compositions of homotopy classes of loops.

For surjectivity, consider the matrix $M = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. Now, the universal cover

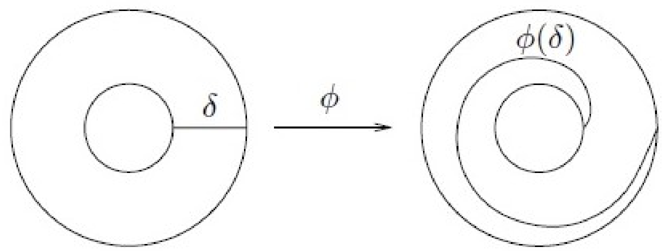

Figure 2: For $n = -1$; this is also known as a Dehn twist.

of A is the strip $\widetilde{A} \approx \mathbb{R} \times I$; clearly, $\widetilde{A}$ is preserved by M. Additionally, since M is a linear transformation, M is equivariant with respect to the group of deck transformations of $\widetilde{A}$ (isomorphic to $\pi_1(A) \cong \mathbb{Z}$). Thus, the restriction of M to $\widetilde{A}$ descends to a homeomorphism ϕ of A. An example is shown for $n = -1$. Thus, $\rho(\phi) = n$.

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "M"] \arrow[d, "i"'] & \mathbb{R}^2 \arrow[d, "i"] \\
\widetilde{A} \arrow[r, "\widetilde{\phi}", "\tau"'] \arrow[d, "p"'] & \widetilde{A} \arrow[d, "p"] \\
A \arrow[r, "\phi"] & A
\end{tikzcd}
$$

It remains to be shown that ρ is injective. We take an $f \in \mathrm{Mod}(A)$ such that $f \in ker(\rho)$, and let ϕ be a representative of f, with $\widetilde{\phi}$ being the preferred lift of ϕ to $\widetilde{A}$. Since $\phi(f) = 0$, $\phi(\delta) \sim \delta$ for all simple proper arcs in A, and so $\widetilde{\phi}$ acts as the identity on $\partial\widetilde{A}$.

Let $\widetilde{H}$ denote the straight line homotopy from $\widetilde{\phi}$ to the identity; we claim that $\widetilde{H}$ is equivariant with respect to $\mathbb{Z}$ (the group of deck transformations of $\widetilde{A}$). Here, it is enough to show that for a deck transformation τ

$$\widetilde{\phi}(\tau \cdot x) = \tau \cdot \phi(x)$$

Now, we know from covering space theory that

$$\widetilde{\phi}(\tau \cdot x) = \phi_*(\tau) \cdot \phi(x)$$

where ϕ_* is the induced map $\pi_1(\widetilde{A}, \widetilde{x_0}) \to \pi_1(A, x_0)$. Therefore $\phi_* \in Aut(\mathbb{Z})$, and since ϕ fixes ∂A pointwise, it follows that $\phi_*(\tau) = \tau$, and thus

$$\widetilde{\phi}(\tau \cdot x) = \tau \cdot \phi(x)$$

i.e. $\widetilde{H}$ is equivariant.

As $\widetilde{H}$ is equivariant with respect to the deck transformations of $\widetilde{A}$, and it fixes the boundary of $\widetilde{A}$, it descends to a homotopy on A between ϕ and the identity on A that fixes the boundary of A pointwise, which implies that f is trivial, and thus we have injectivity for ρ. $\square$

2. Mod(S) for $\mathbb{S}^2_{0,2}, \mathbb{S}^2_{0,3}, \mathbb{S}^2_{0,4}$

Some definitions are in order.

Definition 1. *A* ***proper*** *arc is an arc α such that $\alpha^{-1}(M \cup \partial S)$, where M is the set of marked points. A* ***simple*** *arc is an arc that is an embedding on its interior i.e. it is injective. An isotopy (or homotopy) of an arc is* ***relative to the boundary*** *if its endpoints stay fixed through the isotopy. An arc is* ***essential*** *if it is neither nullhomotopic nor homotopic to a boundary component or a marked point.* [6]

The methods for computing Mod(S) for $\mathbb{S}^2_{0,2}$ and $\mathbb{S}^2_{0,3}$ are similar, so we will just compute Mod($\mathbb{S}^2_{0,3}$). For this, we will need the following proposition:

Proposition 3. *Any two essential simple proper arcs in $\mathbb{S}^2_{0,3}$ with the same endpoints are isotopic. Any two essential arcs that both start and end at the same marked point of $\mathbb{S}^2_{0,3}$ are isotopic.*

Proof.
Let α, β be two simple proper arcs in $\mathbb{S}^2_{0,3}$ connecting the same two marked points. Then we can modify α by isotopy such that α and β have disjoint interiors. Now, if we cut $\mathbb{S}^2_{0,3}$ along $\alpha \cup \beta$, we the disjoint union of a disk with two marked points on the boundary and a once punctured disk with two marked points on the boundary, and so, since α and β bound an embedded disk in $\mathbb{S}^2_{0,3}$, they are isotopic.
The proof for two essential simple proper arcs follows similarly. □

We can now compute Mod($\mathbb{S}^2_{0,3}$).

Theorem 2.1. *Mod($\mathbb{S}^2_{0,3}$) $\cong S_3$*

Proof.
Let η : Mod($S^2_{0,3}$) $\to$ §$_3$ be given the effect of the mapping class on the marked points of $S^2_{0,3}$. Since an orientation preserving homeomorphism must take marked points to marked points, we know that η is surjective. And it is a homomorphism since compositions of mapping classes are mapped to

[6] *Farb, Benson & Margalit, Dan.* A Primer on Mapping Class Groups, Princeton University Press, 2011

compositions of permutations.

Now, we pick $f \in ker(\eta)$, and let ϕ be a representative of f. Then ϕ fixes the three marked points. Let α be an arc with two of the marked points as endpoints. Since ϕ fixes the marked points, $\phi(\alpha)$ has the same endpoints as α. By proposition 5, we know that $\alpha \sim \phi(\alpha)$. From differential topology, we then know that ϕ is isotopic to a map that fixes α pointwise. Next, we cut $\mathbb{S}^2_{0,3}$ along α, obtaining a once punctured disk (with boundary given by α). Since ϕ is orientation preserving, ϕ induces an orientation preserving homeomorphism $\widetilde{\phi}$ of the punctured disk that is the identity on the boundary i.e. $\widetilde{\phi} \in \mathrm{Mod}(D^2_{0,1})$. However, we already know that the mapping class group of the once punctured disk is trivial. Thus $\widetilde{\phi}$ is isotopic to the identity, which implies that ϕ is isotopic to the identity (on $\mathbb{S}^2_{0,3}$).

$\square$

One should pause here and consider a more 'concrete' perspective. One can easily see that $\mathbb{S}^2_{0,3}$ is topologically equivalent to $\mathbb{R}^2$ with two punctures (say, p_1, p_2). Now, if we identify $\mathbb{R}^2$ in the natural way with $\mathbb{C}$, one can then think of the mapping classes as Möbius transformations of the extended complex plane that permute the points $\{p_1, p_2, \infty\}$ (here we assumed that p_3 was the puncture 'pulled out' to infinity).
Thus, we may write $\mathrm{Mod}(\mathbb{S}^2_{0,3})$ here as $\mathrm{Mod}(\mathbb{S}^2_{0,3}) = \{z, 1-z, \frac{z}{z-1}, \frac{1}{z}, \frac{-1}{z-1}, \frac{z-1}{z}\}$, where the elements are functions of z. Additionally, if we denote p_1 by '1', p_2 by '2' and ∞ by '3', then the above set corresponds to the elements $\{(), (12), (23), (13), (123), (132)\}$ in S_3.

Proposition 4. *Mod(*$\mathbb{S}^2_{0,2}$) $\cong S_2$

The proof for $\mathrm{Mod}(\mathbb{S}^2_{0,2})$ is analogous to the proof of proposition 6, except we use the fact the $\mathrm{Mod}(D^2)$ is trivial.

Now, before we compute $\mathrm{Mod}(\mathbb{S}^2_{0,4})$, we will compute $\mathrm{Mod}(\mathbb{T}^2)$, the mapping class group of the torus.

Theorem 2.2. *Mod*$(\mathbb{T}^2) \cong$ *SL(2,*$\mathbb{Z}$*)*

Proof.

Let $f \in \mathrm{Mod}(T^2)$ be represented by ϕ. ϕ induces a map $\phi_* : H_1(T^2, \mathbb{Z}) \to H_1(T^2, \mathbb{Z})$ on homology. Since ϕ is a homeomorphism (and so has an inverse), ϕ_* is an automorphism of $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2$. We define a map $\sigma : \mathrm{Mod}(T^2) \to Aut(\mathbb{Z}^2) \cong GL(2, \mathbb{Z})$ by $\sigma(\phi) = \phi_*$.

$$\begin{tikzcd}
T^2 \cong \mathbb{R}^2/\mathbb{Z}^2 \arrow[rr, "\phi"] & \arrow[d, "\sigma"] & T^2 \cong \mathbb{R}^2/\mathbb{Z}^2 \\
H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2 \arrow[rr, "\phi_*"'] & & H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2
\end{tikzcd}$$

So why is $Im(\sigma) \subseteq SL(2, \mathbb{Z})$?

Well, we know that $GL(2, \mathbb{Z}) = SL(2, \mathbb{Z}) \cup \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} SL(2, \mathbb{Z})$. Since $\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ = -1, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} SL(2, \mathbb{Z})$ is orientation reversing, while $SL(2, \mathbb{Z})$ is orientation preserving by definition. Thus, $\sigma(f)$ is an element of $SL(2, \mathbb{Z})$.

Next, we prove surjectivity. For any $A \in SL(2, \mathbb{Z})$, A is an orientation preserving homeomorphism of $\mathbb{R}^2$. From basic linear algebra, we know that A is equivariant with respect to the group of deck transformations $\cong \pi_1(T^2) \cong \mathbb{Z}^2$ (in fact A preserves $\mathbb{Z}^2$), and so it descends to a homeomorphism ϕ_A of the torus (which we identify with $\mathbb{R}^2/\mathbb{Z}^2$). Using the fact that the primitive elements of $\mathbb{Z}^2$ are in bijective correspondence with homotopy classes of essential, oriented, simple, closed curves [7], we find that $\sigma(\phi_A) = A$.

Lastly, we prove injectivity. Suppose $\sigma(f) = I$. Let f be represented by a homeomorphism ϕ. Using our correspeondence, let α and β be $(1, 0)$ and $(0, 1)$ in $\pi_1(T^2)$ respectively. Since ϕ acts via I, $\phi(\alpha) \sim \alpha$ and $\phi(\beta) \sim \beta$. Again, from differential topology, we get that ϕ is isotopic to a map that

[7] *Farb & Margalit.* A Primer on Mapping Class Groups, section 1.2.2

fixes α pointwise.
Next, we cut T^2 along α to obtain an annulus. Since ϕ is an orientation preserving homeomorphism that preserves α, it induces a homeomorphism $\widetilde{\phi}$ on the annulus, with $[\widetilde{\phi}] \in \mathrm{Mod}(A)$. Since $\phi(\beta) \sim \beta$, using proposition 3, we see that $\rho([\widetilde{\phi}]) = 0$. Since ρ is injective, $\widetilde{\phi}$ is isotopic (via an isotopy that fixes the boundary) to the identity on the annulus, which implies that ϕ is isotopic to the identity i.e. $f = I$. □

From this, we also get the following result:

Theorem 2.3. *Mod*$(T^2_{0,1}) \cong$ *SL(2,$\mathbb{Z}$)*

The proof is this is similar to the proof for the torus. We use the fact that $H_1(T^2_{0,1}, \mathbb{Z}) \cong H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2$. For injectivity, instead of just cutting along α, we cut along $\alpha \cup \beta$ (we can do this because, again $\phi(\alpha) \sim \alpha$ and $\phi(\beta) \sim \beta$) to get the once punctured disk, whose mapping class group is trivial.

In our quest for $Mod(\mathbb{S}^2_{0,4})$, we will also make use of the following fact from hyperbolic geometry: for a compact hyperbolic surface S, there is a bijective correspondence between the conjugacy classes in $\pi_1(S)$ and oriented geodesic curves in S.

So, for the torus and closed, simple, oriented curves on it, we get a bijective correspondence between homotopy classes of oriented, essential, simple, closed curves on T^2 and the primitive elements of $\mathbb{Z}^2 \cong \pi_1(T^2)$.
A geometric description of the same arises in the following way: the primitive elements of $\mathbb{Z}^2$ are of the form $(\pm 1, 0), (0, \pm 1)$ or (p, q) where $gcd(p, q) = 1$. Now, given a primitive element of $\mathbb{Z}^2$, we obtain a (p, q)-curve by projecting a line of slope $\frac{q}{p}$ to $\mathbb{T}^2$.

Another way of constructing (p, q) curves is the following: identify $(1, 0), (0, 1) \in \mathbb{Z}^2$ with α, β on T^2 respectively. Then a general (p, q) curve is made by taking p parallel copies of α, then giving a twist this collection of curves a $2\pi/q$ twist around β.

But how does all this help us with the four punctured sphere? Well, firstly, the torus forms a cover of $S^2_{0,4}$(see figure 4). [8]

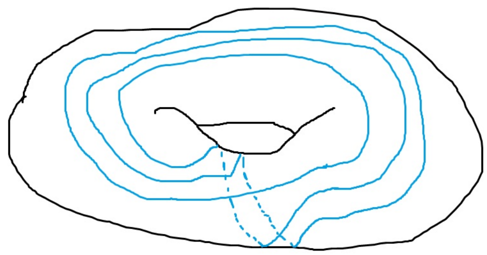

Figure 3: A (3,2) curve on the torus

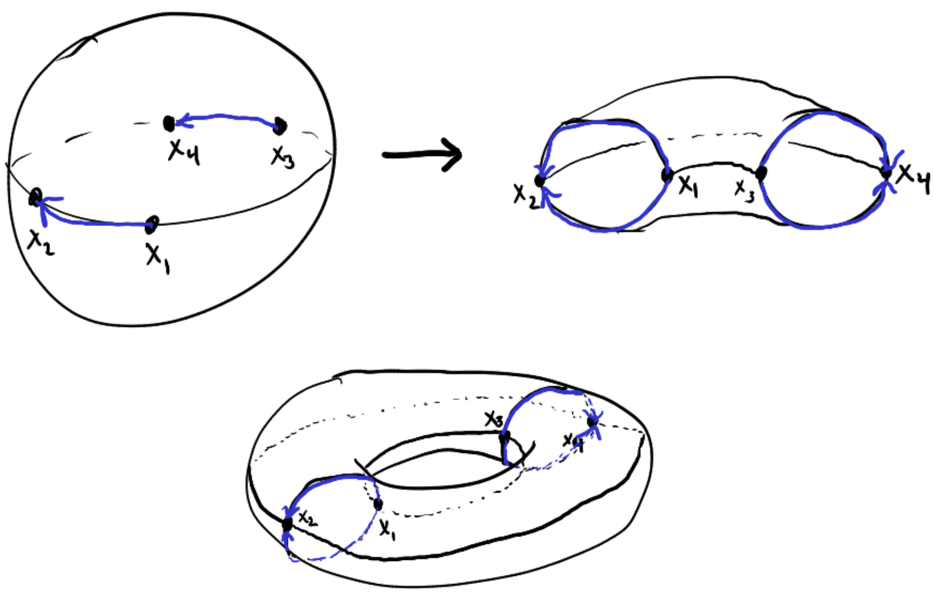

Figure 4: The torus as a 2-cover of the 4-punctured sphere

A second (and more revealing) reason is the following proposition:

Proposition 5. *The hyperelliptic involution induces a bijection between the*

[8]https://math.stackexchange.com/questions/2833072/isotopy-classes-of-essential-simple-closed-curves-in-a-4-punctured-sphere

set of homotopy classes of essential simple closed curves in T^2 and the set of homotopy classes of essential simple closed curves in $\mathbb{S}^2_{0,4}$. [9]

Proof.
Let i denote the hyperelliptic involution. Using i, we project the curves α and β to simple closed curves $\tilde{\alpha}$ an $\tilde{\beta}$ respectively (shown in figure 5). To construct a (p, q) curve, we simply do the analogue of what we did for the torus : take p parallel copies of $\tilde{\alpha}$, then giving a twist this collection of curves a $2\pi/q$ twist around $\tilde{\beta}$.

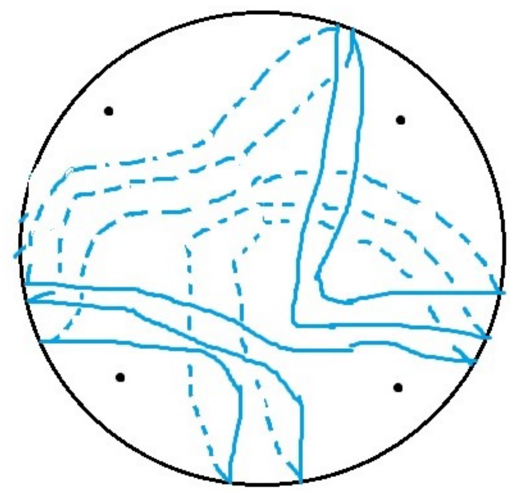

Figure 5: A (3,2) curve on the torus

Now, if γ is an arbitrary simple, essential, closed curve on $\mathbb{S}^2_{0,4}$, then we can (assume that we can) homotope γ to be in minimal position with respect to α. Cutting along β, we obtain 2 twice punctured disks, and α an γ give a collection of disjoint arcs on these disks. By our earlier assumption of minimality, these arcs are essential. Using an analogue of Proposition 5 for $\mathbb{S}^2_{0,2}$, we find that γ can be freely homotoped (as can α), and so the homotopy

[9] *Farb & Margalit.* A Primer on Mapping Class Groups

class of γ corresponds to a primitive element of $\mathbb{Z}^2$.
Additionally, the lift of a (p,q) curve in $\mathbb{S}^2_{0,4}$ is a $(2p,2q)$ curve on T, where the identification is induced by i.

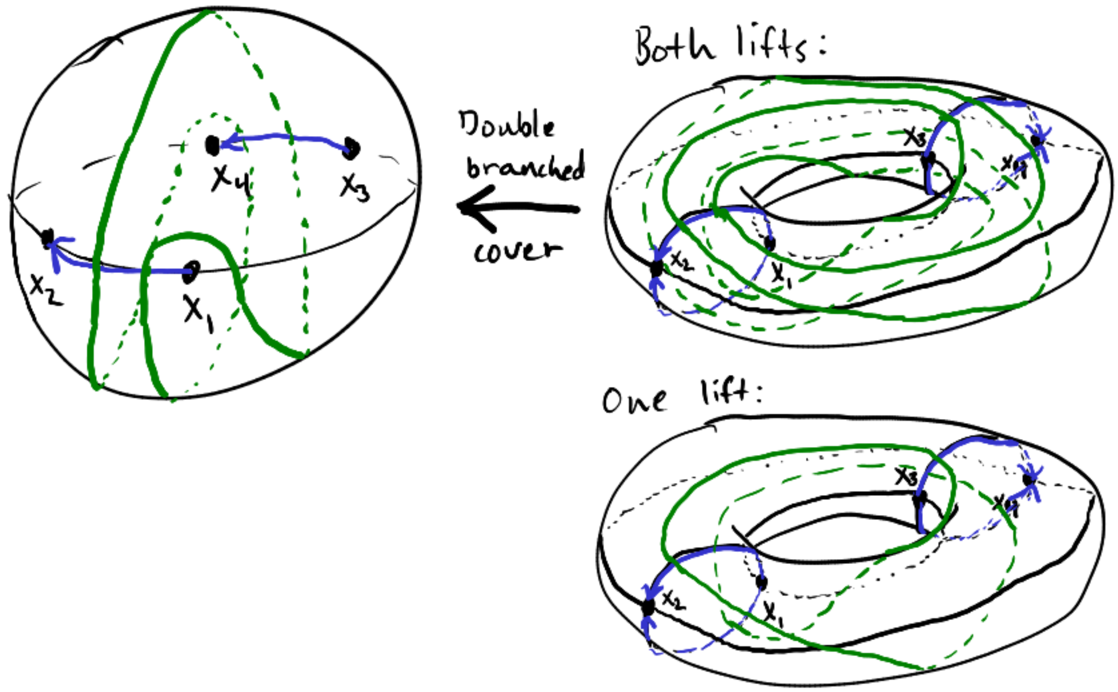

Figure 6: A (3,2) curve on the torus

□

We now have the tools necessary to find $\mathrm{Mod}(\mathbb{S}^2_{0,4})$.

Theorem 2.4. *Mod*$(\mathbb{S}^2_{0,4}) \cong$ *PSL(2,*$\mathbb{Z}$*)* $\ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$

Proof.
Let $f \in \mathrm{Mod}(S^2_{0,4})$ be represented by ϕ. For the lift of ϕ to $\mathrm{Homeo}^+(T^2)$, we have two choices: $\widetilde{\phi}$ and $i\widetilde{\phi}$. So, using the map from theorem 8, we define $\overline{\sigma}$: $\mathrm{Mod}(\mathbb{S}^2_{0,4}) \to \mathrm{PSL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ by $\overline{\sigma}(f) = [\sigma([\widetilde{\phi}])]$ (since $\sigma(i) = -I$, the map is well defined).

$$\begin{tikzcd}
T^2 \arrow[r, "\widetilde{\phi}"] \arrow[d, "p"'] & T^2 \arrow[d, "p"] \\
\mathbb{S}^2_{0,4} \arrow[r, "\phi"] & \mathbb{S}^2_{0,4}
\end{tikzcd}$$

Now, for $A \in \mathrm{PSL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, there is an induced orientation preserving homeomorphism $\widetilde{\phi_A}$ of T^2 which commutes with i, which further induces an orientation preserving homeomorphism ϕ_A of $\mathbb{S}^2_{0,4}$. Then the map $A \mapsto \phi_A$ is a right inverse of $\overline{\sigma}$.

$$\begin{tikzcd}
Mod(T^2) \arrow[r, "\sigma"] \arrow[d, "\overline{p}"'] & SL(2,\mathbb{Z}) \arrow[d, "/(I \sim -I)"] \\
Mod(\mathbb{S}^2_{0,4}) \arrow[r, "\overline{\sigma}"] & PSL(2,\mathbb{Z})
\end{tikzcd}$$

The two hyperelliptic involutions (pictured below as i_1, i_2) of $\mathbb{S}^2_{0,4}$ are order 2 homeomorphisms of $\mathbb{S}^2_{0,4}$. The subgroup $< i_1, i_2 >$ generated by the two hyperelliptic involutions is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Each involution lifts to a homeomorphism of T^2 that rotates the torus by π, and so $\sigma(\widetilde{i_i}) = -I$, which means that $< i_1, i_2 >$ is in the kernel of $\overline{\sigma}$.

Now, let $f \in ker(\overline{\sigma})$ be represented by ϕ. ϕ lifts to a homeomorphism $\overline{\phi}$ of T^2, which acts by $\pm I$ on $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2$, and, using our earlier correspondence, $\overline{\phi}$ acts trivially on the set of homotopy classes of essential, simple, closed

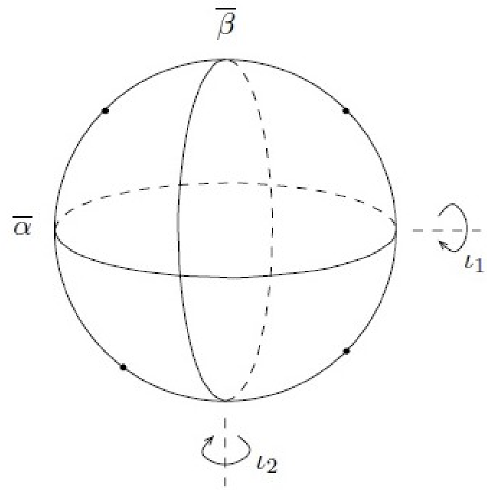

Figure 7: The hyperelliptic involutions of the four punctured sphere

curves in $S^2_{0,4}$. This implies that f fixes $[\overline{\alpha}]$ and $[\overline{\beta}]$, from which it follows that we get a $k \in < i_1, i_2 >$ such that fk fixes the four marked points of $\mathbb{S}^2_{0,4}$. Let φ be a representation of fk. Since φ also fixes $\overline{\alpha}$ and $\overline{\beta}$, we cut along $\overline{\alpha} \cup \overline{\beta}$ to obtain 4 once punctured disks, whose mapping class group we know to be trivial, which implies that fk is trivial, and so $ker(\overline{\sigma}) \cong < i_1, i_2 >$. $\square$

3. A Presentation for Mod($S^2_{0,n}$)

Joan Birman's *Braids, Links, and Mapping Class Groups* contains many important results, but our focus will be on theorem 4.5 [10]:

Theorem 3.1. *For $n \geq 2$, Mod($\mathbb{S}^2_{0,n}$) admits a presentation with generators $x_1, , , x_{n-1}$ and relations:*

1. $x_i x_j = x_j x_i$ *for* $|i - j| \geq 2$
2. $x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$
3. $x_1 \cdots x_{n-2} x_{n-1}^2 x_{n-2} \cdots x_1 = 1$
4. $(x_1 \cdots x_{n-1})^n = 1$

Remark 3. In 1938, Max Dehn proved the finite generation of Mod(S) for arbitrary S (using Dehn twists), and his original proof required $2g(g-1)$ generators, where g is the genus of the surface. Lickorish gave another proof in 1964 that required only $3g - 1$ twists, and finally in 1979 Stephen Humphries showed that those $3g - 1$ could be reduced to $2g + 1$ twists (in fact he showed that any generating set must have cardinality at least $2g+1$).

To begin, we need a presentation of the n-braid group on $\mathbb{S}^2$. Luckily, this is just the standard presentation of the braid group along with one extra relation:

Fact 3.1.1. The n-braid group on $\mathbb{S}^2$, denoted $B_n(\mathbb{S}^2)$ admits a presentation with $\sigma_1, , \sigma_{n-1}$

1. $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$
2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
3. $\sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1 = 1$

As is evident, this bears a striking resemblance to the presentation for Mod($\mathbb{S}_{0,n}$), and we will soon provide a reason why this is the case.

Our first lemma [11] is the following:

[10] Birman's notation is slightly different from that which prevails in contemporary literature, and so we hope this will serve as a partial 'cleaning up'

[11] Theorem 4.3 in Birman's book

Lemma 2. *Let $j_* = j_{gh*}$ $Mod(S^g_n) :\to Mod(S^g)$ be the homomorphism induced by the inclusion j :$PHomeo^+_n(S^g) \subseteq Homeo^+(S^g)$. Then $ker(j_*) \cong B_n(S^g)$ for $g \geq 2$.*
If $g = 1, n \geq 2$ or $g = 0, n \geq 3$, then $ker(j_*) \cong B_n(S^g)/Z(B_n(S^g))$.

Here, $\text{PHomeo}^+_n(S^g)$ is the set of all orientation preserving self-homeomorphisms of a surface S^g with genus g that preserve a set of n distinguished [12] points.

While this lemma will not be proved here, it certainly deserves a few remarks.

Remark 4. We define the space $b_{0,n}(S^g)$ to be $b_{0,n}(S^g) = \{(a_1, , , a_n) \in \prod_{i=1}^n S^g/(a_i \neq a_j \text{ if } i \neq j)\}/\sim$, where $(a_1, , a_n) \sim (a'_1, , a'_n)$ if there is a permutation taking one to other. Then $\pi_1(b_{0,n}(S^g)) = B_n(S^g)$. Using an earlier result of Birman, we then get that the evaluation map ϵ :$\text{PHomeo}^+_0(S^g) \to b_{0,n}(S^g)$ is a locally trivial fibering with fiber $\text{PHomeo}^+_n(S^g)$ (the evaluation maps evaluates each $f \in \text{PHomeo}^+_0(S^g)$ on the n distinguished points).

Remark 5. Using this, we obtain the following exact sequence:

$$\to \pi_1(PHomeo^+_0(S^g)) \xrightarrow{\epsilon_*} \pi_1(b_{0,n}(S^g)) \xrightarrow{d_*} Mod(S^g_n) \to j_*Mod(S^g) \to \pi_0(b_{0,n}(S^g)) \cong 1$$

where ϵ_* is the map induced by the evaluation map, and d_* is a map we will now define.
Pick a loop $\gamma = (\gamma_1, , \gamma_n) \in \pi_1(b_{0,n}(S^g))$. Then there exists an isotopy $F : S^g \times [0,1] \to S^g$ such that $F_0 = \text{id}$, $F_t(x_i) = \gamma_i(t)$, and so $F_1 \in \text{PHomeo}^+_n(S^g)$. Then $d_*(\gamma) = [F_1]$, the isotopy class of F_1.

The next lemma is quite important, as it is used in the proof of lemma 2 as well as for our main theorem.

Lemma 3. *Let $n \geq 3$ & $\sigma_1, , \sigma_{n-1}$ are the aforementioned generators for $B_n(\mathbb{S}^2)$. Then then centre $Z(B_n(\mathbb{S}^2))$ is a subgroup of order 2 generated by $(\sigma_1 \cdots \sigma_{n-1})^n$.*

Proof.
The map $\rho : B_n(\mathbb{S}^2) \to S_n$ defined by $\rho(\sigma_i) = (i, i+1)$ is a homomorphism (compositions of braids correspond to compositions of permutations). Thus, since S_n is centreless for $n \geq 3$, it must be the case that elements in $Z(B_n(\mathbb{S}^2))$

[12] Not marked !

are mapped to the identity permutation (this is a well known property of homomorphisms), and so it is enough to find the centre of the $X_n(B_n(\mathbb{S}^2))$, which is the subgroup of $B_n(\mathbb{S}^2)$ consisting of those elements for whom the corresponding permutation (under ρ) in S_n leaves the letter n invariant.

We can write down a presentation of $X_n(B_n(\mathbb{S}^2))$: it has the generators $\sigma_1, , \sigma_{n-2}$, and the relations

1. $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geq 2$
2. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$
3. $(\sigma_1 \cdots \sigma_{n-2})^{2(n-1)} = 1$

By comparing this with results about the euclidean braid group, one recognises $X_n(B_n(\mathbb{S}^2))$ as a quotient of $B_{n-1}(\mathbb{E}^2)$. Another result about $B_{n-1}(\mathbb{E}^2)$ states that its (infinite cyclic)centre is generated by $(\sigma_1 \cdots \sigma_{n-2})^{n-1}$. Since there are no new commutation relations, it stands to reason that $Z(X_n(B_n(\mathbb{S}^2)))$ is also generated by $(\sigma_1 \cdots \sigma_{n-2})^{n-1}$, and has order either 1 or 2. However, using the relations for the braid group, one obtains that $(\sigma_1 \cdots \sigma_{n-2})^{n-1} = (\sigma_1 \cdots \sigma_{n-1})^n$.

Finally, we must show that $Z(X_n(B_n(\mathbb{S}^2)))$ (and so $Z(B_n(\mathbb{S}^2))$ itself) has order 2. TBD □

Theorem 3.1 now follows almost immediately.

Proof.
Using proposition 1 ($\mathrm{Mod}(\mathbb{S}^2)$ is trivial), and lemma 2, it follows that ker $(j_*) = \mathrm{Mod}(\mathbb{S}^2_{0,n}) \cong B_n(\mathbb{S}^2)/Z(B_n(\mathbb{S}^2))$. We already have a presentation for the braid group on the sphere, and we know how to write its centre in terms of those generators. Now, let ϕ be the isomorphism $B_n(\mathbb{S}^2)/Z(B_n(\mathbb{S}^2))$ to $\mathrm{Mod}(\mathbb{S}^2_{0,n})$. Let $x_i = \phi(\sigma_i)$; these are the generators of the mapping class group. The first three relations are just the relations for the braid group, while the fourth relation follows from quotienting by the centre, which we know to be generated by $(\sigma_1 \cdots \sigma_{n-1})^n$. □

4. A linear representation of $\mathbf{Mod}(S^2_{0,n})$

In this section, we focus on Stephen Bigelow and Ryan Budney's paper '*The mapping class group of a genus two surface is linear*' [13]. Our goal is to prove theorem 2.1 in the paper, which states that:

Theorem 4.1. *There exists a faithful representation of the mapping class group of the n-punctured sphere.*

Some notation and facts: We know that $\mathrm{Mod}(D^2_{0,n}) \cong B_n$. Additionally, for $n \geq 3$, $Z(B_n) \cong \mathbb{Z}$, and is generated by the full twist braid Δ^2. In $\mathrm{Mod}(D^2_{0,n})$, this can be thought of as a Dehn twist around a curve paralell to $\partial D^2_{0,n}$.

For our proof of the theorem, we will need an extra proposition. Let $p_1, , p_n$ be n distinct points in S^2.

Proposition 6. *For $n \geq 4$, there is a short exact sequence*

$$0 \to \mathbb{Z} \to B_{n-1} \to Stab(p_n) \to 0$$

Here, the image of $\mathbb{Z}$ in B_{n-1} is $Z(B_{n-1})$, and $Stab(p_n)$ is the subgroup of $Mod(S^2_{0,n})$ consisting of the mapping classes that fix p_n.

Proof.
We begin by separating the sphere into two hemispheres D_1, D_2 such that $D_1 \cap D_2 = \partial D_1 = \partial D_2$. Without a loss of generality, we assume that $p_1, , p_{n-1} \in D_1$, and $p_n \in D_2$. Then $B_{n-1} \cong \mathrm{Mod}(D_1, n-1)$. Pick $f \in \mathrm{Mod}(D_1, n-1)$, an let ϕ be a representative for f. We can extend ϕ to a diffeomorphism ϕ' of S^2 be letting it be the identity on D_2 (since ϕ is the identity on ∂D_1). Let $\varphi : B_{n-1} \to \mathrm{Mod}(S^2_{0,n})$ be the homomorphism $\phi \mapsto \phi'$.

Firstly, we must show that $Im(\varphi) = Stab(p_n)$. Let g be a representative of the mapping class in $\mathrm{Mod}(S^2_{0,n})$ that fixes p_n. Now, consider $g_{|(D_2)}$ - the image of $g_{|(D_2)}$ is a closed tubular neighbourhood of p_n in $S^2_{0,n-1}$, which exists by the tubular neighbourhood theorem, and by uniqueness, we get that $g_{|(D_2)}$ - applied to D_2 - is (up to isotopy) the identity (relative to p_n), and we can extend this to an isotopy of $S^2_{0,n}$. Since $g_{|(D_2)}$ is the identity (up to isotopy)

[13] *Bigelow & Budney.* The mapping class group of a genus two surface is linear, Algebraic & Geometric Topology - Volume 1 (2001) 699708

on D^2, $g = \varphi(g_{|(D_2)})$.
Thus, $Im(\varphi) = Stab(p_n)$.

We begin by showing that $ker(\varphi) =< \Delta^2 >$. Let k be a representative of an element in $ker(\varphi)$, and let $l = \varphi(k)$ (which is the identity on D_2). Then there is an isotopy between l and the identity. Now, restricting this isotopy to D_2, we get an element in π_1 of the space NT of tubular neighbourhoods of p_n. From the proof of uniqueness for the tubular neighbourhoods theorem, we know that NT is homotopy equivalent to $GL(T_{p_n}) = GL^+(n, \mathbb{R})$, and so $\pi_1(GL^+(n, \mathbb{R})) = \pi_1(NT) = \mathbb{Z}$. We then isotope the diffeomorphisms in our isotopy to a map whose restriction to D_2 is some number of Dehn twists around the sphere i.e. rotations by 2π, and so we see that f is some power of Δ^2, the full twist braid.

$\square$

Let $\mathcal{L}_n : \mathcal{B}_n \to \mathrm{GL}\big(\binom{n}{2}, \mathbb{Z}[q^{\pm}, t^{\pm}]\big)$, the Lawrence-Krammer representation; we assign algebraically independent complex values to q and t (and so the image of $\mathcal{L}$ is in $\mathrm{GL}\big(\binom{n}{2}, \mathbb{C}\big)$). [14]

Let $\mathcal{L}(\Delta^2) = \lambda I$. Let $ab : B_n \to Z$ be the abelianisation map given by $ab(\sigma_i) = 1, ab(\sigma_i^{-1}) = -1$. Now, since $\Delta^2 = (\sigma_1\sigma_2 ... \sigma_{n-1})^2$, $ab(\Delta^2) = n(n-1) \neq 0$. Let $exp : \mathbb{Z} \to \mathbb{C}^*$ be the homomorphism such that $exp(\Delta^2) = \lambda^{-1}$. We now define a new representation $\mathcal{L}'$ by $\mathcal{L} : (b) = (exp \circ ab(b))\mathcal{L}(b)$. The claim is that $ker(\mathcal{L}') \cong \mathbb{Z}$ for $n \geq 3$. By our rescaling, $\mathcal{L}'(\Delta^2) = I$ (we already know that Δ^2 generates $\mathbb{Z}$). Now assume $b \in ker(\mathcal{L}')$. Then $\mathcal{L}(b)$ is of the form kI, and so is in the centre of our matrix group. Since $\mathcal{L}$ is faithful, b is in the centre of the braid group i.e. in $\mathbb{Z}$.

This completes all the heavy lifting - we can now prove the main theorem.

Proof.
We already know that $\mathrm{Mod}(\mathbb{S}^2_{0,1}) \cong 1$, $\mathrm{Mod}(\mathbb{S}^2_{0,2}) \cong S_2$, and $\mathrm{Mod}(\mathbb{S})^2_{0,3} \cong S_3$, so the result follows for $n \leq 3$.
For $n \geq 4$, from the sequence in proposition 13, we get that $\mathcal{L}'$ induces a

[14] Bigelow and Krammer have shown this to be faithful in *Braid Groups are Linear.* J. Amer. Math. Soc. 14 (2001),no.2, 471486

faithful representation of $Stab(p_n)$. Since the index of $Stab(p_n)$ in $\mathrm{Mod}(\mathbb{S}^2_{0,n})$ is finite, we can extend the induced representation to a representation $\mathcal{K}$ of $\mathrm{Mod}(\mathbb{S}^2_{0,n})$. The theorem follows by noting that extensions of faithful representations are faithful. $\square$

Acknowledgements

I would like to thank Professor Tulsi Srinivasan for her patient guidance and wonderful explanations, and also for introducing me to a such an intriguing branch of mathematics.