On the Complexity of Lens Spaces

Rohin Garg

IIT Delhi

May 2, 2025

Outline

Goal and Approach

Primary Goal

Provide complexity bounds on classes of lens spaces $L(p, q)$.

Our Approach

1. Consider the linking form on $L(p, q)$.
2. Extend it to the intersection form on a 4-manifold X bounded by $L(p, q)$ ($\partial X = L(p, q)$).
3. Triangulate the 4-manifold X with triangulation T.
4. Let T descend to a triangulation $\mathcal{T}'$ of $L(p, q)$.
5. Use previously studied tools (e.g., discrete Morse theory, simplification moves) to reduce $\mathcal{T}'$ towards a minimal triangulation.

Definition of Lens Spaces

Geometric Construction

For coprime integers $p > 1$ and $0 < q < p$, the **lens space** $L(p, q)$ is:

$$S^3/\mathbb{Z}_p \text{ where } \zeta \cdot (z_1, z_2) = (\zeta z_1, \zeta^q z_2)$$

with $\zeta = e^{2\pi i/p}$.

- Isometric: Preserves the standard round metric on S^3
- Free: Has no fixed points when p and q are coprime
- Orientable: Preserves orientation since q is coprime to p

Lens Spaces: Alternative Constructions I

Lens-Shaped Solid Model

Consider a solid 3-ball. First, mark p equally spaced points on the equator of the boundary sphere of the ball, and label them $a_0, a_1, \ldots, a_{p-1}$. On the boundary sphere, draw geodesic arcs connecting each point a_i to both the north and south poles of the sphere. Next, perform identifications on the boundary as follows:

1. Identify the north pole with the south pole.
2. For each i, identify the point a_i with $a_{i+q} \mod p$, and simultaneously identify a_{i+1} with $a_{i+q+1} \mod p$, where all indices are taken modulo p.

These identifications pair up spherical triangles on the boundary of the ball. The resulting quotient space is homeomorphic to the lens space $L(p; q)$.

Lens Spaces: Alternative Constructions II

Heegaard Splitting (Genus 1)

Lens spaces admit genus 1 Heegaard splittings $L(p, q) = V_0 \cup_\phi V_1$where V_0, V_1 are solid tori ($D^2 \times S^1$). The gluing map $\phi : \partial V_0 \to \partial V_1$ identifies meridians (μ) and longitudes (λ) as:

$$\phi_*(\mu_0) = q\mu_1 + p\lambda_1$$

The gluing is determined by the matrix $\begin{pmatrix} q & p \\ r & s \end{pmatrix} \in SL(2, \mathbb{Z})$ (where $qs - pr = 1$).

Examples and Properties

Key Examples

- $L(1,0) \cong S^3$
- $L(2,1) \cong \mathbb{RP}^3$
- $L(3,1)$ and $L(3,2)$ homotopy equivalent but not homeomorphic

Homology Groups

$$H_k(L(p,q)) \cong \begin{cases} \mathbb{Z} & k = 0, 3 \\ \mathbb{Z}/p\mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Reidemeister Torsion

For $L(p, q)$, the Reidemeister torsion is:

$$\tau(L(p,q)) = \prod_{k=1}^{p-1} (t^{qk} - 1)^{-1} \in \mathbb{Q}(\zeta_p)/\pm t^n$$

Example

$L(5, 1)$ and $L(5, 2)$ have different Reidemeister torsion:

$$\tau(L(5,1)) \neq \tau(L(5,2)) \text{ in } \mathbb{Q}(\zeta_5)/\pm t^n$$

Lens Spaces: Classification Theorems

Classification Criteria

Let $L(p,q)$ and $L(p,q')$ be two lens spaces.

1. **Homotopy equivalence:** $L(p,q) \simeq L(p,q')$ iff $q' \equiv \pm k^2 q \pmod{p}$ for some integer k.
2. **Homeomorphism:** $L(p,q) \cong L(p,q')$ iff $q' \equiv \pm q^{\pm 1} \pmod{p}$.
3. **Orientation-preserving homeomorphism:** $L(p,q) \cong L(p,q')$ iff $q' \equiv q^{\pm 1} \pmod{p}$.

- Example: $L(7,1) \simeq L(7,2)$ since $2 \equiv 3^2 \cdot 1 \pmod{7}$.
- Example: $L(7,1) \not\cong L(7,2)$ since $2 \not\equiv \pm 1^{\pm 1} \pmod{7}$.

Linking Forms on 3-Manifolds

The linking form captures how torsion 1-cycles link each other within a closed, oriented 3-manifold M. It measures the failure of these cycles to be boundaries in a subtle way. It is a bilinear pairing on the torsion subgroup of the first homology:

$$\lambda : \mathrm{Tors}(H_1(M;\mathbb{Z})) \times \mathrm{Tors}(H_1(M;\mathbb{Z})) \to \mathbb{Q}/\mathbb{Z}$$

Given torsion classes $[\alpha], [\beta]$:

1. Choose representatives α, β (1-cycles).
2. Since $[\alpha]$ is torsion, $n\alpha = \partial S$ for some $n \in \mathbb{Z} \setminus \{0\}$ and 2-chain S.
3. Define $\lambda([\alpha],[\beta]) = \frac{1}{n}(S \cdot \beta) \pmod 1$, where $S \cdot \beta$ is the algebraic intersection number.

This is well-defined, symmetric, and non-degenerate.

Linking Form for $L(p, q)$

The linking form of the lens space $L(p, q)$ is isometric to the form:

$$\lambda_{L(p,q)} : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$$

$$(a, b) \mapsto -\frac{q}{p} \cdot a \cdot b \pmod 1$$

Proof.

Using the Heegaard splitting $L(p, q) = X_g \cup_\phi Z_g$ (here $g = 1$), and the gluing matrix $\begin{pmatrix} A_\phi & B_\phi \\ C_\phi & D_\phi \end{pmatrix} = \begin{pmatrix} q & p \\ s & r \end{pmatrix}$ (with $qr - ps = -1$), a result for the linking form for a rational homology sphere $M(\phi)$ gives u $(v, w) \mapsto -v^T B_\phi^{-1} A_\phi w$. For $L(p, q)$, $g = 1$, $A_\phi = q$, $B_\phi = p$. $H_1(L(p, q)) \cong \mathbb{Z}^1 / B_\phi^T \mathbb{Z}^1 = \mathbb{Z}/p\mathbb{Z}$. The form becomes $(v, w) \mapsto -v(p^{-1})qw = -v\frac{q}{p}w \pmod 1$. ∎

Intersection Forms on 4-Manifolds

For a compact, oriented 4-manifold W, the intersection form measures how 2-dimensional homology classes (represented by surfaces) intersect each other within W. It is a symmetric bilinear form:

$$Q_W : H_2(W;\mathbb{Z})/\text{Torsion} \times H_2(W;\mathbb{Z})/\text{Torsion} \to \mathbb{Z}$$

Given classes $[A], [B] \in H_2(W;\mathbb{Z})$ represented by transversely intersecting surfaces A, B:

$$Q_W([A],[B]) = \#(A \cap B)$$

counting intersection points with signs based on orientation.

Properties

- **Symmetric:** $Q_W(x, y) = Q_W(y, x)$.
- **Unimodular:** If W is closed, $\det(Q_W) = \pm 1$.
- **Rank:** $b_2(W) = \text{rank}(H_2(W;\mathbb{Z}))$.
- **Signature:** $\text{sign}(W) = b_2^+(W) - b_2^-(W)$.
- **Type:** Even if $Q_W(x, x)$ is always even, otherwise Odd.

Intersection Forms: Examples

- S^4: $H_2(S^4) = 0$. Q_{S^4} is the zero form on the zero group.
- $\mathbb{CP}^2$: $H_2(\mathbb{CP}^2) \cong \mathbb{Z}[\mathbb{CP}^1]$. $Q_{\mathbb{CP}^2} = \langle +1 \rangle$. Rank=1, Sign=1, Type=Odd.
- $\overline{\mathbb{CP}^2}$: $H_2(\overline{\mathbb{CP}^2}) \cong \mathbb{Z}$. $Q_{\overline{\mathbb{CP}^2}} = \langle -1 \rangle$. Rank=1, Sign=-1, Type=Odd.
- $S^2 \times S^2$: $H_2(S^2 \times S^2) \cong \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$. $Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = H$ (Hyperbolic form). Rank=2, Sign=0, Type=Even.
- $K3$ Surface: Simply connected, $b_2 = 22$. $Q_{K3} \cong H \oplus H \oplus H \oplus (-E_8) \oplus (-E_8)$. Rank=22, Sign=-16, Type=Even.

Significance

- Freedman (1982): Q_W (+ Kirby-Siebenmann invariant) classifies simply connected closed topological 4-manifolds.
- Donaldson (1983): Smooth 4-manifold classification is much more restrictive. E.g., definite forms must be diagonalizable over $\mathbb{Z}$.

Relating Linking and Intersection Forms

The linking form λ_M on the boundary $M = \partial X$ of a compact, oriented 4-manifold X is related to the intersection form Q_X on X. Assume $H_1(X;\mathbb{Z}) = 0$. The long exact sequence for (X, M) gives:

$$H_2(X) \xrightarrow{j_*} H_2(X, M) \xrightarrow{\partial_*} H_1(M) \to 0$$

Poincaré-Lefschetz duality: $H_2(X, M) \cong H^2(X)$. Universal Coefficient Theorem $+$ $H_1(X) = 0 \implies H_2(X)$ is free. Duality identifies j_* with the map $x \mapsto Q_X(x, \cdot) : H_2(X) \to H^2(X) \cong \mathrm{Hom}(H_2(X), \mathbb{Z})$. Therefore, $H_1(M) \cong \mathrm{coker}(j_*) \cong \mathrm{coker}(Q_X)$. If $H_2(X) \cong \mathbb{Z}^k$ and Q_X is represented by matrix Q, then $H_1(M) \cong \mathbb{Z}^k/\mathrm{im}(Q)$.

Relating Linking and Intersection Forms: The Formula

Theorem

Let X be a compact, oriented 4-manifold with boundary $M = \partial X$. Assume $H_1(X;\mathbb{Z}) = 0$ and $H_2(X;\mathbb{Z}) \cong \mathbb{Z}^k$ is free. Let Q_X be the intersection form represented by the $k \times k$ integer matrix Q. Let Q^{-1} be the inverse matrix (with rational entries). The linking form $\lambda_M : Tors(H_1(M)) \times Tors(H_1(M)) \to \mathbb{Q}/\mathbb{Z}$ is given by: Identify $H_1(M) \cong coker(Q : \mathbb{Z}^k \to \mathbb{Z}^k)$. For $\alpha, \beta \in Tors(H_1(M))$ represented by vectors $v_\alpha, v_\beta \in \mathbb{Z}^k$ (viewed in the cokernel),

$$\lambda_M(\alpha, \beta) \equiv -v_\alpha^T Q^{-1} v_\beta \pmod 1$$

Sketch of Proof Idea

Uses duality, the boundary map $\partial_* : H_2(X, M) \to H_1(M)$, and the definition of the linking form involving intersections with surfaces whose boundaries represent multiples of the torsion cycles. The formula arises naturally from these relationships.

4-Manifolds Bounded by $L(p, q)$

What kind of 4-manifolds X have $\partial X = L(p, q)$? This is crucial for using the $Q_X \leftrightarrow \lambda_{\partial X}$ relationship.

Existence Results

- **Sarkar & Su:** Every lens space bounds a simply connected 4-manifold.
- **Lisca:** $L(p, q)$ bounds a rational homology ball ($\mathbb{Q}$-ball) iff p/q has a continued fraction expansion of a specific form. Not all topological $\mathbb{Q}$-ball fillings admit smooth structures (obstructions from gauge theory / Heegaard Floer homology).
- **Greene:** $L(p, q)$ bounds a smooth, positive definite 4-manifold iff q is a quadratic residue modulo p.
- **Jo, Park, & Park:** $L(p, q)$ bounds a smooth 4-manifold with $b_2 = 1$ iff $q \equiv \pm k^2 \pmod{p}$ for some k.

Constructing X via Continued Fractions

We want to onstruct a 4-manifold X with $\partial X = L(p,q)$ such that its intersection form Q_X recovers the linking form $\lambda_{L(p,q)}(1,1) = -q/p \pmod 1$. (Using the text's formula). We need $(Q_X^{-1})_{11} \equiv q/p \pmod 1$.
First, we compute the **negative** continued fraction expansion:

$$-\frac{p}{q} = [a_1, a_2, \ldots, a_n] = a_1 - \frac{1}{a_2 - \frac{1}{\cdots - \frac{1}{a_n}}}$$

where $a_i \leq -2$. Next, we define the intersection form Q_X for a candidate X (with $H_2(X) \cong \mathbb{Z}^n$) as the matrix:

$$Q_X = \begin{pmatrix} a_1 & 1 & 0 & \ldots & 0 \\ 1 & a_2 & 1 & \ldots & 0 \\ 0 & 1 & a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 1 & a_n \end{pmatrix}$$

Continued Fraction Construction

For the matrix Q_X constructed from $[a_1, \ldots, a_n] = -p/q$, we have: $\det(Q_X) = (-1)^n p \quad$ (up to sign depending on conventions) and $(Q_X^{-1})_{11} = \frac{\det(Q_{n-1})}{\det(Q_X)}$ where Q_{n-1} is the bottom-right $(n-1) \times (n-1)$ minor (or top-left, depending on indexing). The continued fraction convergents relate p/q to these determinants. Specifically, for the negative expansion:

$$\frac{\det(Q_X)}{\det(Q_{n-1})} = -\frac{p}{q} \quad \text{(where } Q_{n-1} \text{ is } (n-1) \times (n-1) \text{ top-left)}$$

Thus, $(Q_X^{-1})_{11} = \frac{\det(Q_{n-1})}{\det(Q_X)} = -\frac{q}{p}$. Using the generator [1] for $H_1(L(p,q)) \cong \mathbb{Z}_p$, represented by $v = (1, 0, \ldots, 0)^T$ in the cokernel basis corresponding to the continued fraction:

$$\lambda_{L(p,q)}([1],[1]) = -v^T Q_X^{-1} v = -(Q_X^{-1})_{11} = -(-q/p) = q/p \pmod 1$$

This matches the required linking form (up to the sign convention difference noted earlier).

Realizing X via Plumbing

The 4-manifold $X_{p,q}$ with the intersection form Q_X derived from $[a_1, \ldots, a_n] = -p/q$ can be constructed via plumbing.

1. Start with a weighted graph Γ (here, a linear graph A_n). Vertices v_i have weights $e_i = a_i$.
2. Associate to each vertex v_i a D^2-bundle over S^2 with Euler number e_i.
3. For each edge (v_i, v_j) in Γ, "plumb" the bundles together by identifying a disk neighborhood $D^2 \times D^2$ in the base of one bundle with a similar neighborhood in the other, swapping base and fiber coordinates.

The resulting smooth 4-manifold $P(\Gamma)$ has $\partial P(\Gamma) = L(p, q)$ for the linear graph with weights a_i. The intersection form $Q_{X_{p,q}}$ in the basis given by the zero-sections is exactly the matrix Q_X derived from the continued fraction.

Example: Plumbing for $L(5,2)$

1. **Target Linking Form:** $\lambda(1,1) \equiv 2/5 \pmod 1$.
2. **Continued Fraction:** Compute $-p/q = -5/2$.
$$-\frac{5}{2} = -3 + \frac{1}{2} = -3 - \frac{1}{-2} = [-3, -2]$$
So, $a_1 = -3$, $a_2 = -2$.
3. **Intersection Form:**
$$Q_X = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}$$
4. **Plumbing Graph:** A linear graph with two vertices (-3) and (-2)
5. **Verification:** $\det(Q_X) = (-3)(-2) - (1)(1) = 6 - 1 = 5 = p$. Now, $Q_X^{-1} = \frac{1}{5}\begin{pmatrix} -2 & -1 \\ -1 & -3 \end{pmatrix}$. Let the generator [1] correspond to the basis vector $v = (1,0)^T$. Then
$$\lambda(g,g) = -\begin{pmatrix} 1 & 0 \end{pmatrix} Q^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2}{5} \mod 1$$

Example: Triangulation for $L(4,1)$ - Part 1

We need a plumbing manifold X with boundary $\partial X = L(4,1)$, defined by one D^2-bundle over S^2 with Euler number -4. Its intersection form is $Q_X = \langle -4 \rangle$. We triangulate X using two 4-simplices:
$T_1 = (A_1, B_1, C_1, D_1, E_1)$, $T_2 = (A_2, B_2, C_2, D_2, E_2)$.
Gluings:

1. Primary 3-face: $(A_1, B_1, C_1, D_1) \sim (A_2, B_2, C_2, D_2)$.
2. Twisted (framing):

$$(A_1, B_1, C_1, E_1) \sim (B_2, C_2, A_2, E_2)$$
$$(A_1, B_1, D_1, E_1) \sim (C_2, A_2, D_2, E_2)$$
$$(A_1, C_1, D_1, E_1) \sim (B_2, A_2, D_2, E_2)$$

Example: Triangulation for $L(4,1)$ - Part 2

Boundary tetrahedra: $\tau_1 = (A_1, B_1, C_1, E_1)$, $\tau_2 = (A_2, B_2, C_2, E_2)$. The 4D gluings induce identifications on ∂T:

- **Vertices:** $A_1 \sim B_2,\ B_1 \sim C_2,\ C_1 \sim A_2,\ E_1 \sim E_2$.
- **Edges:** E_1E_2 forms core. A_1B_1, B_1C_1, C_1A_1 map to B_2C_2, C_2A_2, A_2B_2 with twist.
- **Faces:** $(A_1, B_1, C_1) \sim (B_2, C_2, A_2); (A_1, B_1, E_1) \sim (B_2, C_2, E_2); (B_1, C_1, E_1) \sim (C_2, A_2, E_2); (C_1, A_1, E_1) \sim (A_2, B_2, E_2)$.

This initially gives a 2-tetrahedron triangulation $\mathcal{T}'$ of $L(4,1)$, which collapses to a minimal 1-tetrahedron triangulation $\tau = (A, B, C, D)$ with face pairings: $(A, B, C) \sim (A, B, D); (A, B, D) \sim (A, C, D); (A, C, D) \sim (B, C, D); (B, C, D) \sim (A, B, C)$ (Matches known minimal triangulation).

Example: Triangulation for $L(5,2)$ - Part 1

We need a plumbing manifold X with boundary $\partial X = L(5,2)$, defined by 2 D^2-bundle over S^2 with Euler number -3 and -2. Its intersection form is $Q_X = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}$. To triangulate X, we use four 4-simplices: T_1, T_2, T_3, T_4 with the following gluings

1. Core identifications:
   $(A_1, B_1, C_1, D_1) \sim (A_2, B_2, C_2, D_2); (A_2, B_2, C_2, E_2) \sim (A_3, B_3, C_3, D_3); (A_3, B_3, C_3, E_3) \sim (A_4, B_4, C_4, D_4)$
2. Framing adjustments:
   $(A_1, B_1, C_1, E_1) \sim (B_2, C_2, A_2, E_2); (A_1, B_1, D_1, E_1) \sim (C_2, A_2, D_2, E_2); (A_2, B_2, D_2, E_2) \sim (B_3, C_3, A_3, E_3); (A_3, B_3, D_3, E_3) \sim (B_4, C_4, A_4, E_4); (A_2, B_2, C_2, D_2) \sim (C_3, A_3, D_3, E_3)$

Example: Triangulation for $L(5, 2)$ - Part 2

The boundary ∂T inherits a complex triangulation $\mathcal{T}'$ from the 4D gluings. This $\mathcal{T}'$ collapses to the minimal 1-tetrahedron triangulation $\tau = (A, B, C, D)$ with face pairings: $(A, B, C) \sim (B, A, D); (A, B, D) \sim (C, B, A); (A, C, D) \sim (A, C, B); (B, C, D) \sim (A, D, C)$ This minimal triangulation correctly produces $L(5, 2)$ with:

- $H_1(L(5, 2)) = \mathbb{Z}/5\mathbb{Z}$
- $\pi_1(L(5, 2)) = \langle x \mid x^5 = 1 \rangle$
- Linking form $\lambda(1, 1) = 2/5 \pmod 1$ (matches calculation).

Conclusion and Future Work

We explored finding the complexity (minimal triangulation size) of lens spaces $L(p,q)$ by:

- Developed method to triangulate $L(p,q)$ via bounded 4-manifolds
- Related linking forms to intersection forms of bounding manifolds
- Explicit triangulations for $L(4,1)$ and $L(5,2)$

For the future:

- Find a general method to simplify the induced triangulation $\mathcal{T}'$ for arbitrary $L(p,q)$.
- Conjecture: For $L(p,q)$ with a suitably 'small' continued fraction expansion for $-p/q$, a minimal triangulation might be readily obtainable from the induced boundary triangulation of the plumbed manifold.
- Develop explicit triangulation schemes for general plumbed manifolds $X_{p,q}$.

References

- Lisca, P. *Lens Spaces, Rational Balls and the Ribbon Conjecture*. Annals of Mathematics, 2008.
- Greene, J.E. *The Lens Space Realization Problem*. Annals of Mathematics, 2013.
- Sarkar, S., & Suh, D. Y. *A new construction of lens spaces*. Topology and its Applications, 2018.
- Jo, W., Park, J., & Park, K. *On Lens Spaces Bounding Smooth 4-Manifolds*. arXiv, 2024.
- Gompf, R., & Stipsicz, A. *4-Manifolds and Kirby Calculus*. AMS, 1999.
- Milnor, J. *Topology from the Differentiable Viewpoint*. Princeton, 1965.