Model Theory Notes

Rohin Garg

April 10, 2018

These are my notes for Cornell's MATH 6830: Model Theory, based on David Marker's *Introduction to Model Theory*(Springer-Verlag, 2002).

Chapter 1

An Introduction

1.1 Basics

A language $\mathcal{L}$ is a set of function symbols and relation symbols, each of which has an 'arity' α in $\{0, 1, 2, ...\} = \omega$ (we will let constants be 0-ary functions).
A formula is a finite first order formula (more on this soon). $\varphi(v_1, ...v_n)$ is a n-ary formula φ where v_i is the ith free variable, and all free variables are among $\{v_1, ...v_n\}$, while $\varphi(a_1, ...a_n)$ is the result of substituting constants or terms $a_1, ...a_n$ into φ simultaneously here a_i replaces v_i. A sentence is just a formula with no free variables.
Remark: Sometimes 'the language $\mathcal{L}$' refers to a set of formulas in which case $\mathcal{L}$ is the signature.

An $\mathcal{L}$-structure will be denoted by a calligraphic letter ($\mathcal{M}, \mathcal{N}, \mathcal{A}, \mathcal{B}$), and will consist of an underlying set (denoted by the same letter in upper case roman - M, N, A, B) called the universe together with interpretations $f^{\mathcal{M}}$ and $R^{\mathcal{M}}$ for each function symbol $f \in \mathcal{L}$ and each relation symbol $R \in \mathcal{L}$.
So $f^{\mathcal{M}} : M^n \to M$ if f is n-ary, and $R^{\mathcal{M}} \subseteq M^n$ if R is n-ary.
Example: Consider the language of groups : $\mathcal{L}_g = \{\cdot, e\}$, where $\cdot$ is a binary function symbol, and e is a constant symbol. Consider the $\mathcal{L}_g$-structure $\mathcal{G} = \{G, \cdot^{\mathcal{G}}, e^{\mathcal{G}}\}$: it has a set G equipped with a binary relation $\cdot^{\mathcal{G}}$ and a distinguished constant $e^{\mathcal{G}}$.

It is only natural to now speak of maps between structures. An $\mathcal{L}$-embedding $\mu : \mathcal{M} \to \mathcal{N}$ is a map $\mu : M \to N$ that preserves the interpretation of symbols in $\mathcal{L}$:

- $\mu(f^{\mathcal{M}}(a_1, ...a_n)) = f^{\mathcal{N}}(\mu(a_1), ...\mu(a_n))$ for all $a_1, ...a_n$ and any function $f \in \mathcal{L}$
- $(a_1, ...a_n) \in R^{\mathcal{M}} \Leftrightarrow (\mu(a_1), ...\mu(a_n)) \in R^{\mathcal{N}}$ for all $a_1, ...a_n$ and any relation $R \in \mathcal{L}$

A bijective $\mathcal{L}$-embedding is an $\mathcal{L}$-isomorphism. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, then $\mathcal{M}$ is a substructure of $\mathcal{N}$.

1.2 Terms, Formulas and Theories

The set of $\mathcal{L}$-terms is the smallest $\mathcal{T}$ such that:

1. For all variables v_i with $i = 1, 2, ..., v_i \in \mathcal{T}$
2. If $t_1, ...t_n \in \mathcal{T}$ and f is a formula in $\mathcal{L}$, then $f(t_1, ...t_n) \in \mathcal{T}$.

Consider the language of rings $\mathcal{L}_r = \{+, -, 0, 1, \cdot\}$. Then $+(1, +(1, 1)$, and $\cdot(v_1, -(v_2, 1)$ are terms (we generally know them as 1+1+1 and $v_1(v_2 - 1)$).
Consider an $\mathcal{L}$-structure $\mathcal{M}$. Let t be a term built by using variables from $\overline{v} = (v_{i_1}, ..., v_{i_m})$; we will interpret

t as a function $t^{\mathcal{M}} : M^m \to M$. Let s be subterm of t and $\overline{a} = (a_{i_1}, ..., a_{i_m}) \in M$. We inductively define $s^{\mathcal{M}}(\overline{a})$ by:

1. If s is a constant symbol c, then $s^{\mathcal{M}}(\overline{a}) = c^{\mathcal{M}}$
2. If s is a variable v_{i_j}, then $s^{\mathcal{M}}(\overline{a}) = a_{i_j}$
3. If s is the term $f(t_1, ...t_n)$, then $s^{\mathcal{M}}(\overline{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\overline{a}), ...t_n^{\mathcal{M}}(\overline{a}))$

Thus, we define the function $t^{\mathcal{M}}$ by $\overline{(}a) \mapsto t^{\mathcal{M}}(\overline{a})$ For an example, consider the language $\mathcal{L} = \{f, g, c\}$ where f, g are functions (unary and binary respectively), and c is a constant. $t_1 = g(v_1, c), t_2 = f(g(c, f(v_1))$ are $\mathcal{L}$-terms. Let $\mathcal{M}$ be the $\mathcal{L}$-structure $\{\mathbb{R}, exp, +, 1\}$ i.e. $f^{\mathcal{M}} = exp, g^{\mathcal{M}} = +, c^{\mathcal{M}} = 1$. Then

$$t_1^{\mathcal{M}}(a_1) = a_1 + 1, \text{ and } t_2^{\mathcal{M}}(a_1) = e^{1+e_1^a}$$

Correspondingly, ϕ is an atomic $\mathcal{L}$-formula if ϕ is either:

- $t_1 = t_2$, where t_1, t_2 are terms

or

- $R(t_1, ...t_n)$, where $t_1, ..., t_n$ are terms and R is a relation in $\mathcal{L}$

Thus the set of $\mathcal{L}$-formulas is the smallest set $\mathcal{W}$ containing the atomic formulas such that:

1. $\phi \in \mathcal{W} \Rightarrow \neg\phi \in \mathcal{W}$
2. $\phi_1, \phi_2 \in \mathcal{W} \Rightarrow (\phi_1 \wedge \phi_2), (\phi_1 \vee \phi_2) \in \mathcal{W}$
3. $\phi \in \mathcal{W} \Rightarrow \exists v_i\phi, \forall v_i\phi \in \mathcal{W}$

We now define what it means for a formula to be true in a structure. For a formula ϕ, constants $\overline{a} \in M^m$, we we inductively define $\mathcal{M}$ satisfies $\phi(\overline{a})$, or $\mathcal{M} \models \phi(\overline{a})$ as:

1. If ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\overline{a})$ if $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$.
2. If ϕ is $R(t_1, ...t_n)$, then $\mathcal{M} \models \phi(\overline{a})$ if $(t_1^{\mathcal{M}}, ...t_n^{\mathcal{M}}) \in R^{\mathcal{M}}$.
3. If ϕ is $\neg\psi$, then $\mathcal{M} \models \phi(\overline{a})$ if $\mathcal{M} \not\models \psi(\overline{a})$.
4. If ϕ is $\psi \wedge \theta$, then $\mathcal{M} \models \phi(\overline{a})$ if $\mathcal{M} \models \psi(\overline{a})$ and $\mathcal{M} \models \theta(\overline{a})$.
5. If ϕ is $\psi \vee \theta$, then $\mathcal{M} \models \phi(\overline{a})$ if $\mathcal{M} \models \psi(\overline{a})$ or $\mathcal{M} \models \theta(\overline{a})$.
6. If ϕ is $\exists v_i\psi(\overline{v}, v_i)$, then $\mathcal{M} \models \phi(\overline{a})$ if there is a $b \in M$ such that$\mathcal{M} \models \psi(\overline{a}, b)$.
7. If ϕ is $\forall v_i\psi(\overline{v}, v_i)$, then $\mathcal{M} \models \phi(\overline{a})$ if for all $b \in M$, $\mathcal{M} \models \psi(\overline{a}, b)$.

In model theory, when proving satisfaction results, we often use induction on the complexity of formulas. Here, we demonstrate a basic use of the same.

Proposition 1. *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$, $\overline{a} \in M$, and $\phi(\overline{v})$ is a quantifier free formula. Then*

$$\mathcal{M} \models \phi(\overline{a}) \textit{ if and only if } \mathcal{N} \models \phi(\overline{a})$$

Proof.
If $t(\overline{v})$ is a term and $\overline{b} \in M$, then $t^{\mathcal{M}}(\overline{b}) = t^{\mathcal{N}}(\overline{b})$, since:

- If t is the constant symbol c, then $c^{\mathcal{M}} = c^{\mathcal{N}}$, since $\mathcal{M} \subseteq \mathcal{N}$.
- If t is the variable v_i, then $t^{\mathcal{M}}(\overline{b}) = \overline{b} = t^{\mathcal{N}}(\overline{b})$, since $\mathcal{M} \subseteq \mathcal{N}$.

- If $t = f(t_1, ..., t_n)$, then $t^{\mathcal{M}}(\bar{b}) = t^{\mathcal{N}}(\bar{b})$. Firstly, since $\mathcal{M} \subseteq \mathcal{N}$, $f^{\mathcal{M}} = f^{\mathcal{N}}|_{M^n}$.
  Thus, $t^{\mathcal{M}}(\bar{b}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{b}), ..., t_n^{\mathcal{M}}(\bar{b})) = f^{\mathcal{N}}(t_1^{\mathcal{M}}(\bar{b}), ..., t_n^{\mathcal{M}}(\bar{b})) = f^{\mathcal{N}}(t_1^{\mathcal{N}}(\bar{b}), ..., t_n^{\mathcal{N}}(\bar{b})) = t^{\mathcal{N}}(\bar{b})$

Now, we prove the proposition by induction on formulas.
If ϕ is $t_1 = t_2$, then

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a}) \Leftrightarrow t_1^{\mathcal{N}}(\overline{a}) = t_2^{\mathcal{N}}(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(\overline{a})$$

If ϕ is $R(t_1, ...t_n)$, then

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow (t_1^{\mathcal{M}}, ...t_n^{\mathcal{M}}) \in R^{\mathcal{M}} \Leftrightarrow (t_1^{\mathcal{M}}, ...t_n^{\mathcal{M}}) \in R^{\mathcal{N}} \Leftrightarrow (t_1^{\mathcal{N}}, ...t_n^{\mathcal{N}}) \in R^{\mathcal{N}} \Leftrightarrow \mathcal{N} \models \phi(\overline{a})$$

Thus, the proposition is true for all atomic formulas.
If ϕ is $\neg\psi$, then

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{M} \not\models \psi(\overline{a}) \Leftrightarrow \mathcal{N} \not\models \psi(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(\overline{a})$$

If ϕ is $\psi \wedge \theta$, then

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{M} \models \psi(\overline{a}) \text{ and } \mathcal{M} \models \theta(\overline{a}) \Leftrightarrow \mathcal{N} \models \psi(\overline{a}) \text{ and } \mathcal{N} \models \theta(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(\overline{a})$$

If ϕ is $\psi \vee \theta$, then

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{M} \models \psi(\overline{a}) \text{ or } \mathcal{M} \models \theta(\overline{a}) \Leftrightarrow \mathcal{N} \models \psi(\overline{a}) \text{ or } \mathcal{N} \models \theta(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(\overline{a})$$

Thus, the proposition holds for all quantifier free formulas. □

Now for a couple of definitions.

Definition 2. *Two $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$ are elementarily equivalent (denoted $\mathcal{M} \equiv \mathcal{N}$) if*

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$$

for all $\mathcal{L}$-sentences ϕ.

Definition 3. *The full theory of $\mathcal{M}$ (denoted Th($\mathcal{M}$)), is the set of $\mathcal{L}$-sentences ϕ such that $\mathcal{M} \models \phi$*

Thus, trivially, $\mathcal{M} \equiv \mathcal{N} \Leftrightarrow \mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

Theorem 4. *If there exists an isomorphism $j : \mathcal{M} \to \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.*

Proof. This is proved first by showing terms behave well (i.e. by induction on terms) and then by induction on formulas. □

An $\mathcal{L}$-theory T is just a set of $\mathcal{L}$-sentences. $\mathcal{M}$ is a model of T (written $\mathcal{M} \models T$) if and only if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$. A theory is *satisfiable* if it has a model.
$T \vdash \varphi$ if and only if there is a logical proof of φ from T. We will now give an example of a theory (For more examples, the reader is referred to David Marker's *Introduction to Model Theory*, chapter 1).

Example 5. Infinite Sets

Let $\mathcal{L} = \emptyset$.
Consider the $\mathcal{L}$-theory T where we have, for each n, the sentence

$$\phi_n = \exists x_1 \exists x_2 ... \exists x_n \bigwedge_{i<j\leq n} x_i \neq x_j$$

ϕ_n states that there are at least n distinct elements. Thus, $\mathcal{M}$ is a model for T if and only if $\mathcal{M}$ is infinite.

Lastly, we have the definition of *logical consequence.*

Definition 6. *ϕ is a logical consequence of T (written $T \models \phi$) if $\mathcal{M} \models \phi$ whenever $\mathcal{M} \models T$.*

For a further introduction to some of the prerequisites, we refer the reader to Lou van den Dries' Notes on logic.

Chapter 2

Basic Techniques

2.1 Compactness and Henkin Constructions

Theorem 7. ***Gödel's Completeness Theorem:*** *Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence, then $T \models \phi$ if and only if $T \vdash \phi$.*

A theory T is consistent if $\nexists \varphi$ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Corollary 7.1. *A theory is consistent if and only if it is satisfiable*

Proof. Take a theory T.
If T is satisfiable, then it has a model. If it has a model, it cannot satisfy contradictory statements - the definition of consistency.
If T is consistent. Assume it isn't satisfiable. Then it has no models. Thus, every model of T is a model of $\phi \wedge \neg\phi$ for some ϕ. Thus, $T \models \phi \wedge \neg\wedge$, and by the Completeness theorem $T \vdash \phi \wedge \neg\wedge$. But T is consistent, so this is a contradiction. Thus, T is satisfiable. □

Definition 8. *A theory T is finitely satisfiable if every finite subset is satisfiable.*

Theorem 9. ***Compactness Theorem:*** *T is satisfiable if and only if T is finitely satisfiable.*

Proof.
If T is satisfiable, then, by definition, T is finitely satisfiable.
If T is finitely satisfiable, but not satisfiable, then T is inconsistent by the Compactness theorem. Let $\phi \wedge \neg\phi$ be the contradiction. Then, by the definition of a proof (proofs are finite), there exists some finite subset T_0 of T such that $T_0 \models \phi \wedge \neg\phi$. Thus, T_0 is not satisfiable. But we know that T is finitely satisfiable. This is a contradiction. Thus, T is satisfiable. □

We will now give an interesting application of the Compactness theorem.

Example 10. *$\mathcal{L} = \{+, \cdot, 0, 1, \leq\}$, $T = Th(\mathbb{N}, +, \cdot, 0, 1, \leq) = \{\varphi | \varphi$ is an $\mathcal{L}$-sentence and $\mathbb{N} \models \varphi\}$ i.e. T is the full theory of the arithmetic on the natural numbers. Then there is an $\mathcal{N} \models T$, and and a $c \in M$ such that c is larger than all the natural numbers.*

Let c be a constant symbol not in $\mathcal{L}$, Let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$.
Let $T^* = T \cup \{c \neq \underbrace{1 + 1.... + 1}_{n} | n \in \mathbb{N}\}$.

Note: T^* is finitely satisfiable, since if $T_0 \subseteq T^*$ is finite, then there is an $\omega \in \mathbb{N}$ such that $c \neq \underbrace{1 + 1... + 1}_{\omega}$ is not in T_0.
Define $\mathcal{N}$ to be the $\mathcal{L}^*$ structure whose reduct(projection) to $\mathcal{L}$ is $\mathbb{N}$ and $c^{\mathcal{N}} = \omega$. Clearly $\mathcal{N} \models T_0$. By the compactness theorem, T^* is satisfiable. Let $\mathcal{N} \models T^*$.
Now, $\mathcal{N}$ (or rather its reduct to $\mathcal{L}$) is a model of true arithmetic. So $\mathcal{N}$ satisfies all the axioms in the

induction scheme for instance.
Thus, $\mathcal{N} \models T$, but $\mathcal{N}$ contains ω, which is bigger than all the natural numbers.

Note: $\{\omega, \omega-1, \omega-2..\}$ is an infinite descending sequence i.e. there is no formula that defines this sequence.
We'll write $\varphi(\mathcal{M})$ to denote $\{(a_1, a_2, ..., a_n) = \overline{a} \in M^n | \mathcal{M} \models \varphi(\overline{a})\}$. The sets $\varphi(\mathcal{M})$ are said to be definable.
Eg. $\{\omega, \omega-1, \omega-2..\} \subset \mathcal{N}$ is not definable.

Henkin's Construction
Given a language $\mathcal{L}$ and a theory T, we want to construct a model $\mathcal{M}$ of T. Henkin's construction gives us a good way of doing this.

Definition 11. *T is maximal for all $\mathcal{L}$ sentences φ if either $\varphi \in T$ or $\neg\varphi \in T$.*

Fact: A maximal consistent theory is complete.
Reason: If T is a maximally consistent theory, then T is closed under logical implication. Eg. If $\varphi \notin T$, then $T \cup \{\varphi\}$ is inconsistent. It follows that $T \cup \{\neg\varphi\}$ is consistent. So $\neg\varphi \in T$.

Lemma 12. *Any consistent $\mathcal{L}$ theory is contained in a complete, consistent $\mathcal{L}$ theory.*

Proof. By Zorn's Lemma, there is a maximal superset $\mathcal{L}$ theory of $T \subseteq T^*$. Assume T^* is not consistent. Then $T^* \vdash \phi \wedge \neg\phi$. Since $T^* \vdash \neg\phi$, $\phi \in T^*$, and since $T^* \vdash \phi$, $T^* \cup \{\phi\}$ is consistent. But then $T^* \cup \{\phi\}$ is a consistent superset of T^*, contradicting the maximality of T^*. Thus, T^* is maximally consistent. By the previous fact T^* is complete. □

Note: Given a structure $\mathcal{M}$, $TH(\mathcal{M})$ is complete. So enlarging T to a complete theory first is natural.

Definition 13. *A theory T in a language $\mathcal{L}$ has the witness property if whenever $T \vdash \exists x\varphi(x)$ for some $\mathcal{L}$ sentence $\exists x\varphi(x)$, there is a constant c in the language such that $T \vdash \varphi(c)$*

Lemma 14. *If $\mathcal{L}$ is a language and T is a consistent $\mathcal{L}$ theory, then there is a language $\mathcal{L} \subseteq \mathcal{L}'$ and a consistent $\mathcal{L}'$ theory $T \subseteq T'$ such that:*

1. *T' has the witness property.*
2. *$|\mathcal{L}'| \leq |\mathcal{L}| + \aleph_0$*

Reason: For each $\exists x\varphi(x)$ such that $T \vdash \exists x\varphi(x)$, introduce a new constant c_φ and let
$\mathcal{L}_{n+1} = \mathcal{L} \cup \{c_\varphi | \exists x\varphi(x)$ is an $\mathcal{L}_n$sentence such that $T_n \vdash \exists x\varphi(x)\}$ and $T_{n+1} = T_n \cup \{\varphi(c_\varphi) | \exists x\varphi(x)$ is an $\mathcal{L}_n$sentence such that $\exists x\varphi(x)\}$, where $\mathcal{L}_0 = \mathcal{L}$ and $T_0 = T$. If T_n is consistent, so is T_{n+1}. In general $|\mathcal{L}_{n+1}| \leq |\mathcal{L}_n| + \aleph_0$.
Set $\mathcal{L}' = \cup_{n=0}^{\infty} \mathcal{L}_n, T' = \cup_{n=0}^{\infty} T_n$. Then T' has the witness property and is consistent.

Proposition 15. *If T is a consistent $\mathcal{L}$ theory, then there is a language $\mathcal{L} \subseteq \mathcal{L}'$ and a consistent $\mathcal{L}'$ theory $T \subseteq T'$ such that:*

1. *T' is consistent and complete, and has the witness property.*
2. *$|\mathcal{L}'| \leq |\mathcal{L}| + \aleph_0$*

Reason:
For (1) Construct T_n and $\mathcal{L}_n$ such that:

- $T_n \subseteq T_{n+1}$, $\mathcal{L}_n \subseteq \mathcal{L}_{n+1}$, $|\mathcal{L}_{n+1}| \leq |\mathcal{L}_n| + \aleph_0$
- T^{2n+1} is complete and consistent, $\mathcal{L}_{2n+1} = \mathcal{L}_{2n}$
- T_{2n+2} has the witness property
- $T_0 = T$, $\mathcal{L}_0 = \mathcal{L}$

Set $\mathcal{L}' = \cup_{n=0}^{\infty}\mathcal{L}_n, T' = \cup_{n=0}^{\infty}T_n$.
Then T' is consistent and complete, and has the witness property.
(2) is trivial.

Suppose now that T is an $\mathcal{L}$ theory which is complete, consistent and has the witness property, define $\sim$ on the set of $\mathcal{L}$ constants by $a \sim b$ if $'a = b' \in T$. Define $\mathcal{M}_T$ to be the set of all equivalence classes. We now define for each n-ary function $f \in \mathcal{L}$, with $f^{\mathcal{M}} : M_T^n \to M_T$ defined by

$$f^{\mathcal{M}_T}([a_1], [a_2]...., [a_n]) = [c] \text{ if } 'f(a_1, a_2...., a_n) = c' \in T$$

.

Note: This is well defined since T has the witness property, $T \vdash \exists x(f(\overline{a}) = x)$. So there is a c such that $T \vdash f(\overline{a}) = c$.

Given an n-ary relation $R \in \mathcal{L}$, we define $([a_1]_{\mathcal{N}}, ..., [a_n]_{\mathcal{N}}) \in R^{\mathcal{M}_T}$ if and only if $T \vdash Ra_1a_2...a_n$.

Then one verifies by induction on the complexity of a sentence φ that $T \vdash \varphi$ if and only if $\mathcal{M}_T \models \varphi$, with $|\mathcal{M}_T| \leq |\mathcal{L}| + \aleph_0$.

Note: We've seen that if T is consistent, then there is a $T', \mathcal{L}', T \subseteq T'$ consistent with a model $\mathcal{M} \models T'$.
T is a consistent $\mathcal{L}$ theory; why is T contained in a maximally consistent theory? By Zorn's Lemma, where the partial ordering used is $\subseteq$.

Theorem 16. *If T is a finitely satisfiable $\mathcal{L}$-theory, and κ is an infinite cardinal with $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality at most κ.*

Proof.
Enlarge $\mathcal{L}$ to a $\mathcal{L}'$ by adding a constant c_η for each $\eta < \kappa$.
Define $T' = T \cup \{''c_{\eta_1} \neq c''_{\eta_2} | \eta_1 \neq \eta_2 \text{ in } \kappa\}$. Is T' consistent? If T' is inconsistent, then there is a finite $F \subseteq \kappa$ such that $T \cup \{''c_{\eta_1} \neq c''_{\eta_2} | \eta_1 \neq \eta_2 \text{ in } F\}$ is inconsistent (by compactness).
Note, however, that if $\mathcal{M} \models T$ and $|\mathcal{M}| \geq |F|$, then we can interpret $\{c_\eta | \eta \in F\}$ in $\mathcal{M}$ distinctly. In particular, if T has an infinite model, T' is consistent.
By the Henkin construction, there is a model $\mathcal{M}$ of T' of cardinality at most $|\mathcal{L}| + \aleph_0 = \kappa$. Of course, any model of T' has cardinality at least κ. ☐

2.2 Complete Theories

Chapter 3

Algebraic Examples

3.1 Quantifier Elimination

Definition 17. *An $\mathcal{L}$-theory T has quantifier elimination if for each formula $\phi(\overline{v})$, there is a quantifier free formula $\psi(\overline{v})$ such that $T \vdash \phi(\overline{v}) \Leftrightarrow \psi(\overline{v})$.*

A structure $\mathcal{M}$ has quantifier elimination if $\mathrm{Th}(\mathcal{M})$ has quantifier elimination.

Example 18. $(\mathbb{C}, +, \cdot, 0, 1, -1, (\cdot)^{-1})$ *has QE;* $(\mathbb{R}, +, \cdot, 0, 1)$ *does not*

$\{x \in \mathbb{R} | \exists y(x = y^2)\}$ is $[0, \infty)$.
Note: If $A \subseteq \mathbb{R}$ is definable via an atomic formula, then A is the zero set of some polynomials, and thus A is either finite orall of $\mathbb{R}$. If A is definable via a QF formula, then A is either finite or cofinite. Since $[0, \infty)$ is infinite and con-infinite, it is not definable via a QF formula. Hence $(\mathbb{R}, +, \cdot, 0, 1)$ does not have QE.

A *quasi order* is a reflexive relation $\preceq$. A quasi order is linear if for all x, y, $x \preceq y$ or $y \preceq y$. If both, we write $x \equiv y$.
Fix a countable linear order L and $\varphi(\overline{v})$, $\overline{v} = (v_1, ..., v_n)$. If $\leq$ is a quasi order on $\{v_i | i \leq n\}$, then define $\chi_{\underline{\alpha}}(\overline{v})$ to be $\bigwedge_{v_i \leq v_j} (v_i \leq v_j) \wedge \bigwedge_{v_i \not\leq v_j} \neg(v_i \leq v_j)$.

Lemma 19. *Suppose that L_0, L_1 are two countable models of DLO (the theory of dense linear orders), and f is a finite order preserving map from L_0 to L_1. Then f extends to an isomorphism.*

Proof. TO BE FILLED $\square$

In order to prove that DLO has QE, it is enough to produce a QF $\psi(\overline{v})$ such that $L \models \forall \overline{v}(\psi(\overline{v}) \Leftrightarrow \varphi(\overline{v}))$. Why is this enough? If $L \models \forall \overline{v}(\psi(\overline{v}) \Leftrightarrow \varphi(\overline{v}))$, then DLO proves this (since it is complete).

Theorem 20. *DLO has quantifier elimination*

Proof.
$\Lambda = \{\leq \{v_1, .., v_n\} | \leq$ is a linear order and $L \models \exists \overline{v}(\varphi(\overline{v}) \wedge \chi_{\leq} \overline{v})\}$
Claim: $\psi(\overline{v}) = \vee\{\chi_{\leq}(\overline{v}) | \preceq \in \Lambda\}$
Clearly, $L \models \forall \overline{v}(\psi(\overline{v}) \Rightarrow \varphi(\overline{v}))$. To prove that $L \models \forall \overline{v}(\psi(\overline{v}) \Leftarrow \varphi(\overline{v}))$, suppose that $\overline{a} \in L$ is arbitrary such that $L \models \psi(\overline{a})$. Then $L \models \chi_{\leq}(\overline{a})$. It follows that $v_i \preceq v_j$ if and only if $a_i \preceq a_j$. Since $\alpha \in \Lambda$, there is a $\overline{b}$ such that $L \models \chi_{\leq}(\overline{b}) \wedge \varphi(\overline{b})$.
Observe that $a_i \mapsto b_i$ defines a partial order preserving map L to L. By lemma, this extends to an automorphism of L which maps $\overline{a}$ to $\overline{b}$. Automorphisms preserve truth, so $L \models \varphi(\overline{a})$ if and only if $L \models \varphi(\overline{b})$, which is true. ASK PROF ABOUT THIS $\square$

A Criterion for Quantifier Elimination

Theorem 21. *Suppose $\mathcal{L}$ is a language and T is an $\mathcal{L}$-theory. T has quantifier elimination provided: for every quantifier free formula $\varphi(\overline{v}, w)$, every $\mathcal{L}$-structure $\mathcal{A}$ and all models $\mathcal{M}, \mathcal{N} \models T$, both of which have $\mathcal{A}$ as a substructure and all $\overline{a} \in A$,*

$$\mathcal{M} \models \exists w \varphi(\overline{a}, w) \text{ if and only if } \mathcal{N} \models \exists w \varphi(\overline{a}, w)$$

Lemma 22. *Suppose $\mathcal{L}$ is a language and T is an $\mathcal{L}$-theory. The following are equivalent for an $\mathcal{L}$-formula $\varphi(\overline{v})$:*

1. *There exists a quantifier free formula $\psi(\overline{v})$ such that $T \models \forall \overline{v}(\varphi(\overline{v}) \Leftrightarrow \psi(\overline{v}))$*
2. *For all $\mathcal{L}$-structures $\mathcal{A}$, and all models $\mathcal{M}, \mathcal{N} \models T$, and all $\overline{a} \in A$, if $\mathcal{A}$ is a substructure of $\mathcal{M} \cap \mathcal{N}$, then $\mathcal{M} \models \varphi(\overline{a})$ if and only if $\mathcal{N} \models \varphi(\overline{a})$*

Proof.
$1 \Rightarrow 2$ is the easy direction. Suppose 1. Let $\mathcal{A}, \mathcal{M}, \mathcal{N}$, $\overline{a} \in \mathcal{A}$ as in 2. If $\mathcal{M} \models \varphi(\overline{a})$, then $\mathcal{N} \models \psi(\overline{a})$, since $\mathcal{M} \models T$. Then $\mathcal{A} \models \psi(\overline{a})$ (since ψ is quantifier free). So $\mathcal{N} \models \psi(\overline{a})$ since ψ is quantifier free.
So $\mathcal{N} \models \varphi(\overline{a})$, since $\mathcal{N} \models T$.

For $2 \Rightarrow 1$, let φ be given and let $d_1, ...d_n$ be constants not in $\mathcal{L}$. Define $\Gamma = \{\psi(\overline{d}) | \psi(\overline{d}) \text{ is quantifier free and } T \vdash \forall \overline{v}(\varphi(\overline{v}) \Rightarrow \psi(\overline{v}))\}$ Suppose $T \cup \Gamma \models \varphi(\overline{d})$. Then by compactness, there are $\psi_1(\overline{v}), ..., \psi_m(\overline{v})$ such that $\psi_i(\overline{v}) \in \Gamma$ for all $i \leq m$ and $T \cup \{\psi_i(\overline{v}) | i \leq m\} \vdash \varphi(\overline{d})$. But then $T \models \bigwedge_{i \leq m}(\psi_i(\overline{d}) \Leftrightarrow \varphi(\overline{d}))$, and so $T \models \forall \overline{v} \bigwedge_{i \leq m}(\psi_i(\overline{v}) \Leftrightarrow \varphi(\overline{v}))$.
Suppose for contradiction that $T \cup \Gamma \cup \neg\varphi(\overline{d})$ is consistent. Let $\mathcal{M}$ model $T \cup \Gamma \cup \neg\varphi(\overline{d})$, and let $\mathcal{A}$ be substructure of $\mathcal{M}$ generated by $\overline{d}$: $\mathcal{A} = \{t^{\mathcal{M}}(\overline{d}) | t(\overline{v}) \text{ is a an } \mathcal{L} - \text{terms}\}$.
Claim: $T \cup \Gamma \cup diag(\mathcal{A})$ (ASK ABOUT THIS)
Observe: $diag(\mathcal{A}) \subseteq \Gamma$ i.e. adding it is redundant.
Suppose not. If the claim is false, then there exist finitely many $\psi_1(\overline{v}), ..., \psi_m(\overline{v}) \in \Gamma$ such that $T \cup \{\psi_i(\overline{v}) | i \leq m\} \vdash \neg\varphi(\overline{d})$. So then $T \models \bigwedge_{i \leq m}(\psi_i(\overline{d}) \Leftrightarrow \neg\varphi(\overline{d}))$, so $T \vdash \varphi(\overline{d}) \Rightarrow \bigwedge_{i \leq m}(\psi_i(\overline{d}))$, and thus $T \vdash \forall \overline{v}(\varphi(\overline{v}) \Rightarrow \bigwedge_{i \leq m}(\psi_i(\overline{v})))$.
But $\mathcal{M} \models T$, and $\varphi(\overline{a}), \bigwedge_{i \leq m}(\psi_i(\overline{a}))$, which is a contradiction.
Thus, the claim is true. (ASK PROF) □

Lemma 23. *T is an $\mathcal{L}$-theory, and for each $\varphi(\overline{v})$ which is quantifier free, there is a quantifier free $\psi(\overline{v})$ such that*

$$T \vdash \forall \overline{v}(\exists \varphi(\overline{v}) \Leftrightarrow \psi(\overline{v}))$$

Then T has quantifier elimination.

We prove this by induction on quantifier depth of $\theta(\overline{v})$ such that $T \vdash \forall \overline{v}(\theta(\overline{v}) \Leftrightarrow \psi(\overline{v}))$.

Proof.

If $\theta(\overline{v})$ has quantifier depth 0, then it is quantifier free and we can take $\psi(\overline{v}) = \theta(\overline{v})$.
If $\theta(\overline{v}) = \exists w \theta_0(\overline{v}, w)$, then by induction, there is quantifier free $\varphi_0(\overline{v}, w)$ such that $T \vdash \forall v \forall w(\theta_0(v, w) \Leftrightarrow \varphi_0(\overline{v}, w))$.
By hypothesis, there is a $\psi(\overline{v})$ such that $T \vdash \forall \overline{v}(\psi(\overline{v}) \Leftrightarrow (\exists w \varphi_0(\overline{v}, w)))$ i.e. $T \vdash \forall \overline{v}(\psi(\overline{v}) \Leftrightarrow (\exists w \theta_0(\overline{v}, w)))$

We are now finished: observe that the set of $\theta(\overline{v})$ which are equivalent to quantifier free formulas is closed under negation, conjunction and disjunction. □

Divisible Abelian Groups have quantifier elimination
Recall, DAG is the theory of torsion free divisible Abelian groups. If $G \vdash DAG$ and $g \in G$ and $m \in \mathbb{Z} \backslash \{0\}$,

there exists a unique $h \in G$ such that $m \cdot h = g$. We write h as g/m ($m \cdot h = \underbrace{h + h + ... + h}_{m}$).

Any torsion free Abelian group can be enlarged to a divisible Abelian Group. This is called the divisible hull of G. That is, if G is torsion free Abelian, there is a $H \models DAG$ and an embedding $\eta : G \to H$ such that if H' is any other DAG, with $\eta' : G \to H'$, then there is a $\kappa : H \to H'$ such that $\eta' = \kappa \circ \eta$.

The construction looks like this:
Start with $G \times \mathbb{Z}\backslash\{0\}$ (intending that (g, m) corresponds to g/m) and take the equivalence relation $(g, p) \sim (h, q)$if and only if $g \cdot q - p \cdot h = 0$. One has to check that it all works out (it does) - PROVE THIS

Define $i : G \to G \times (\mathbb{Z}\backslash\{0\})/\sim$ by $g \mapsto [(g, 1)]_\sim$.
Proposition: The universal theory of DAG is TAG (torsion free Abelian groups).
$T_\forall$ (universal theory) is all sentences $\forall \overline{v} q(\overline{v})$, where $q(\overline{v})$ is quantifier free, so TAG $\subseteq$ DAG$_\forall$.
One check that the axioms of torsion free Abelian groups are universal.
Proof: On the other hand suppose that φ is a universal sentence which is not in TAG. Then $TAG \cup \{\neg\varphi\}$ has a model G which embeds into H which satisfies DAG. But now $H \vdash \neg\varphi$, which is a contradiction - DAG $\vdash \varphi$.

Theorem 24. *DAG has algebraically prime models.*

A theory T has algebraically prime models if and only if whenever $\mathcal{A} \models T_\forall$, there exists a $\mathcal{B} \models T$ and an embedding $\epsilon : \mathcal{A} \to \mathcal{B}$, such that if $\mathcal{C} \models T$ and $f : \mathcal{A} \to \mathcal{C}$ is an embedding, $f = g \circ \epsilon$ for some g.

Lemma 25. *Suppose $G \leq H$ are models of DAG. Suppose that $\theta(\overline{v}, w)$ is quantifier free. If $\overline{a} \in G^{<\omega}$ and $H \models \exists w \theta(\overline{a}, w)$, then $G \models \exists w \theta(\overline{a}, w)$.*

Proof. Since we may assume that θ is in disjunctive normal form, and since H satisfies one of the conjuncts of the disjunct, we may assume that $\theta(\overline{v}, w)$ is a large conjunction:
$\bigwedge_{i=1}^{p}(k_{ij} v_j + l_i w = 0)^{\epsilon_i}$, where ϵ_i is ± 1, and the meaning of the sentence is negated if $\epsilon_i = -1$.
If any $\epsilon_j \neq -1$, then $\sum_{j=1}^{n}(k_{ij} a_j + l_i w = 0)$ has a unique solution in G.
If all ϵ_j are -1, then the conjunction just asserts that w is not among finitely many elements $g_1, ..., g_p$ of G since any model of DAG is infinite. $\square$

Theorem 26. *DAG as quantifier elimination*

Eg. of Algebraically prime models: DAG$_\forall$ is the theory of TAG.. That DAG has algebraically prime models is witnessed by the divisible hull construction.

Definition 27. *If $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M} \subseteq \mathcal{N}$, then we say that $\mathcal{M}$ is simply closed in $\mathcal{N}$, written $\mathcal{M} \prec_S \mathcal{N}$, if whenever $\varphi(\overline{v}, w)$ is quantifier free and $\overline{a} \in \mathcal{M}$, $\mathcal{N} \models \exists w \varphi(\overline{a}, w) \Rightarrow \mathcal{M} \models \exists w \varphi(\overline{a}, w)$*

We showed that if $G \leq H$ are models of DAG, then G is simply closed in H.

Theorem 28. *If T satisfies:*

1. *whenever $\mathcal{M} \subseteq \mathcal{N}$, both modelling T, $\mathcal{M} \prec_S \mathcal{N}$*
2. *T has algebraically prime models.*

then, T has quantifier elimination.

Corollary 28.1. *DAG has quantifier elimination.*

Proof. Theorem 27
Suppose T is given as in the hypothesis. Let $\mathcal{A}$ be an $\mathcal{L}$ structure be contained in $\mathcal{M}$ and $\mathcal{N}$, both of which model T, Suppose $\varphi(\overline{v}, w)$ is quantifier free and $\overline{a} \in A$.

Let $\mathcal{B}$ be a minimal expansion of $\mathcal{A}$ such that $\mathcal{B} \models T$.
$\mathcal{B}$ embeds into both $\mathcal{M}$ and $\mathcal{N}$, and so by relabelling, we may assume $\mathcal{B} \subseteq \mathcal{M} \cap \mathcal{N}$. Now, $\mathcal{B} \prec_S \mathcal{M}$, and $\mathcal{B} \prec_S \mathcal{N}$. So $\mathcal{M} \models \exists w(\varphi(\overline{a}, w))$ if and only if $\mathcal{B} \models \exists w(\varphi(\overline{a}, w))$ if and only if $\mathcal{N} \models \exists w(\varphi(\overline{a}, w))$.
Thus, $\mathcal{M} \models \exists w(\varphi(\overline{a}, w))$ if and only if $\mathcal{N} \models \exists w(\varphi(\overline{a}, w))$.
We are now done by our quantifier elimination test(**Theorem 20**). □

Definition 29. *T is strongly minimal if T has only infinite models and whenever $\mathcal{M} \models T$ and $\mathcal{D} \subseteq \mathcal{M}$ is definable, $\mathcal{D}$ is finite or cofinite. $\mathcal{D}$ is definable if there is $\varphi(\overline{v}) \in \mathcal{L}_{\mathcal{M}}$ such that $\mathcal{D} = \{d \in \mathcal{M} | \mathcal{M} \models \varphi(d)\}$.*

Sometimes we write $\varphi(\mathcal{M})$ for $\mathcal{D} = \{d \in \mathcal{M} | \mathcal{M} \models \varphi(d)\}$.

Observe: If $\mathcal{L}$ has no relation, then any $\mathcal{L}$ theory with quantifier elimination and only infinite models is strongly minimal. Eg. DAG is strongly minimal.
Note: If a theory has quantifier elimination and there is a model $\mathcal{A} \vdash T$ which embeds into all other models of T, then T is complete. If $\mathcal{M} \models T$, $\mathcal{A} \rightarrow_{\text{embeds}} \mathcal{M}$, so $\mathcal{A} \prec \mathcal{M}$, and so $Th(\mathcal{A}) = Th(\mathcal{M})$. So any model of DAG contains $(\mathbb{Q}, +, 0, 1)$.

Ordered Divisible Abelian Groups (ODAG) and Ordered Abelian Groups (OAG)
The language of OAG is $\mathcal{L} = \{+, 0, -, \leq\}$. An ordered Abelian group $(G, +, -, 0, \leq)$ is an $\mathcal{L}$ structure such that $(G, +, -, 0) \models AG$, $(G, \leq) \models LO$ and why $\forall x \forall y \forall z((x \leq y) \Rightarrow (x + z \leq y + z))$ - translation preserves the order.
ODAG adds that G is a divisible group (Ordered groups are always torsion free).

FACT:ODAG$_\forall$ is OAG and ODAG has algebraically models (attempt to extend the field of fractions works).
Proof: The algebraically prime models are the divisible hulls given the obvious orderings.

Lemma 30. *If $G \leq H$ are models of ODAG, then $G \prec_S H$*

Note: When checking that $G \prec_S H$, it suffices to consider formulas which are conjuncts of atomic or negation atomic formulas - If $\varphi(\overline{v}, w)$ is quantifier free, assume that $\overline{a} \in G, H \models \exists w \varphi(\overline{a}, w)$.
The above formulas have the form $g + mw \leq h + nw$, or $g + mw = h + nw$, where g is a sum/difference of the a_is. Fix $\overline{a} \in G$.
These are all equivalent to one of the form $w \leq h, g \leq w$ or $g = w$.

Negation atomic formulas are equivalent to $w < h, g < w$ or $g \neq w$. Since $g \neq w$ is $(g < w) \cup (g > w)$, we may assume all conjuncts have the form $(g < w)$ or $(w < h)$.
So $\varphi(\overline{a}, w)$ is asserting that w is strictly between g and h for some $g < h$ in G.

FACT: If $G \models ODAG$ and $g < h$ are in G, then there is a $x \in G$ such that $g < x < h$. Check that $x = \frac{g+h}{2}$ works.
Thus **Theorem**: ODAG is quantifier free.

3.2 Algebraically Closed Fields

ACF has quantifier elimination
We need to show that ACF (1) has algebraically prime models and (2) for all $\mathcal{M} \subseteq \mathcal{N}, \mathcal{M} \prec_S \mathcal{N}$.

- (1) is "just" the assertion that the algebraic closure of a field exists. Actually, ACF$_\forall$ is just the theory of integral domains. Moreover, if D is an integral domain, once can form a field of fractions and take the algebraic closure. (TRY TO PROVE THIS)

- For(2), suppose $\theta(\overline{v}, w)$ is given. Suppose $\overline{a} \in M^N$ is given. If any of the equalities used to form the conjunction are not negated and are not trivial, then any solution to the equality is, by definition, algebraic over $\mathcal{M}$.

Note: However, $\mathcal{M}$ is algebraically closed, so any solution to the equality is in $\mathcal{N}$. Here, we are using the following fact:
if $\mathcal{M}$ is an algebraically closed field, and b is algebraic over $\mathcal{M}$, then $b \in \mathcal{M}$. So algebraic closure gives not only that all nontrivial polynomial equations have solutions but that they have **all** their solutions. Reason: Factorisation into linear factors. ALternatively, we could observed that $p(\overline{a}, w) = 0$ is equivalent to $\sum_{i=0}^{l}(c_i w + d_i) = 0$, which is equivalent to $\bigvee_{i=0}^{l}(c_i w + d_i)$. So WLOG, θ has only linear equalities/inequalities.

If θ has equalities, we are done. If θ has only inequalities, $(c_i w + d_i) \neq 0$, then $\theta(\overline{a}, w$ asserts that $w \in \{\frac{-d_i}{c_i} | i = 0, 1, 2, .., l\}$. Since any model of ACF is infinite, $\mathcal{M}$ has an element $b \in \{\frac{-d_i}{c_i} | i = 0, 1, 2, .., l\}$.

Zariski Closed Sets
Fix an n and a field K. $K[U_1, ..., U_n]$ is the ring of polynomials with variables $v_1, v_2...., v_k$ with coefficients in K.
$I \subseteq K[\overline{U}]$ is an ideal if $I = \phi$, I is closed under addition and if if $f \in I$ and $g \in K[\overline{U}]$, then $fg \in I$. I is a radical ideal if whenever $f^m \in I, f \in I$.

If$X \subseteq K^n$, then define $I(X) = \{f(\overline{u}) \in K[\overline{U}] | \ \forall \overline{a} \in X f(\overline{a}) = 0\}$. Thus if $X \subseteq Y$, then $I(Y) \subseteq I(X)$.
If $S \subseteq K[\overline{U}]$, we define $V(S) = \{\overline{a} \in K^n | \forall f \in S f(\overline{a}) = 0\}$. Thus, if $S \subseteq T$, then $V(T) \subseteq V(S)$.
We say that $X \subseteq K[\overline{U}]$ is Zariski closed if $X = V(S)$ for some S. We will see that these form closed sets in some topology in K^n. This topology is compact, in fact hereditarily (so non trivial). Points are closed, but typically, points can be separated by open sets. (T_1, but not T_2)

FACTS:

1. For $X \in K^n$, $I(X)$ is a radical ideal.
2. If X is Zariski closed, $V(I(X)) = X$.
3. If X is properly contained in Y, then $I(Y)$ is properly contained in $I(X)$
4. If $X, Y \in K^n$ are Zariski closed, then $X \cup Y = V(I(X) \cap I(Y))$, $X \cap Y = V(I(X) + I(Y))$. In particular, Zariski closed sets are closed under taking finite unions and intersection. (in fact we shall soon see that arbitrary intersections of Zariski closed sets are Zariski closed as well)

The first three are easily proven, and so if any clarification is required, the reader is referred to section 3.2 in Marker. The 4th is not that non-trivial either, however, we feel it's proof deserves a bit more attention that those of the others.

Proof.
4) To see that $X \cup Y \subseteq V(I(X) \cap I(Y))$, suppose that $\overline{x} \in X$ WLOG. Then for all $p(\overline{v}) \in I(X) \cap I(Y) \subseteq I(X)$, $p(\overline{x}) = 0$. So $\overline{x} \in V(I(X) \cap I(Y))$.
For the other way around, suppose $\overline{z}$ is such that $p(\overline{z}) = 0$ for all $p(\overline{v}) \in I(X) \cap I(Y)$. If $\overline{z} \notin X$, pick $p(\overline{v}) \in I(X)$ such that $p(\overline{z}) \neq 0$. If $\overline{z} \notin Y$, pick $q(\overline{v}) \in I(Y)$ such that $q(\overline{z}) \neq 0$.
Now pick $p(\overline{v})q(\overline{v}) \in I(X) \cap I(Y)$ and $p(\overline{v})q(\overline{v}) \neq 0$

Similarly for $X \cap Y = V(I(X) + I(Y))$. □

We will make use of the following theorem.

Theorem 31. ***Hilbert Basis Theorem***
For a field K, if I is an ideal in $K[\overline{V}]$, then I is finitely generated. Equivalently, there are no infinite ascending sequences of ideals i.e. $K[\overline{V}]$ is a Noetherian ring.

Corollary 31.1. *If $\mathcal{F}$ is a family of Zariski closed sets, then there is a finite $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\cap\mathcal{F}_0 = \cap\mathcal{F}$.*

Proof. Attempt to construct an infinite ascending sequence of ideals as follows: If $\nexists$ a finite $\mathcal{F}_0$, then there is an infinite sequence $X_0 \supseteq X_1 \supseteq ....$ of Zariski closed sets. But then $I(X_0) \subseteq I(X_1) \subseteq ....$ is an infinite ascending chain of ideals - this is a contradiction. □

Definition 32. *A subset E if K^n is constructible if E is a finite boolean combination of Zariski closed sets.*

Corollary 32.1. *In an algebraically closed field K, constructible sets are exactly the definable sets.*

If E is Zariski closed, $E = V(I(E))$. Since $I(E)$ is finitely generated, $E = \bigvee_{p(\overline{v}) \in S}(p(\overline{v}))$, where S is finite. So every Zariski closed set is definable via a conjunction of atomic formulas. In particular, every constructible set is definable.
On the other hand, if E is definable, E is quantifier free definable, and hence is constructible.

Theorem 33. ***Hilbert's Nullstellensatz***
If $I \subsetneq J$ are radical ideals in $K[\overline{V}]$, where K is an algebraically closed field, then $V(J) \subsetneq V(I)$, so $X \mapsto I(X)$ is a bijective correspondence between Zariski closed sets and radical ideals.

Proof.
Fix $I \subsetneq J$, and let $p(\overline{v}) \in J \backslash I$. let Q be a prime ideal containing I such that $p(\overline{v}) \notin Q$. Let L be the algebraic closure of $K[\overline{V}]/p(\overline{v})$. Let $a_i \in L$ be V_i/Q.
Observe that $K \hookrightarrow L$. Let $q_1, , ...q_n$ form a finite basis for I. Now, $L \models (p(\overline{a}) \neq 0) \bigwedge \wedge_{i=1}^{n}(q_i(\overline{a}) = 0)$ and thus, $L \models (\exists w(p(\overline{w}) \neq 0)) \bigwedge \wedge_{i=1}^{n}(q_i(w) = 0)$.

Since $K \prec L$, $K \models (\exists w(p(\overline{w}) \neq 0)) \bigwedge \wedge_{i=1}^{n}(q_i(w) = 0)$. Let $\overline{b}$ be the witness. Then $b_i \in V(I) \backslash V(J)$. □

Lemma 34. *If I is a radical ideal in $K[V]$, then I is an intersection of prime ideals.*

3.3 Real Closed Fields

Chapter 4

Realising and Omitting Types

4.1 Types

Definition 35. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and $A \subseteq M$. If $n \geq 1$, an* ***n-type*** *(over A in $\mathcal{M}$) is a collection P of $\mathcal{L}_A$ formulas in the free variables $v_1, .., v_n$ such that $P \cup Th(\mathcal{M})$ is satisfiable i.e. there ia an $\mathcal{N}$ and $c_1, ...c_n \in N$ such that $\mathcal{N} \models Th(\mathcal{M})$ and $\mathcal{N} \models \varphi(\overline{c})$ for all $\varphi(\overline{v}) \in P$.*

An n-type is said to be **realised** in $\mathcal{M}$ if $\exists a_a, ...a_n \in M$ such that $\mathcal{M} \models \varphi(\overline{a})$ for all $\varphi(\overline{v}) \in P$. It is useful to think of types as virtual tuples in a structure.

Example 36. *Let $\mathcal{M}$ be $(\mathbb{N}, \leq)$. The type of ∞ is the 1-type $\{"n \leq v" | n \in \mathbb{N}|\}$.*

The type of ∞ is not realised in this structure.

A type P is said to be **complete** if for all $\varphi(\overline{v})$ either $\varphi(\overline{v}) \in P$ or $\neg\varphi(\overline{v}) \in P$. $S_n^{\mathcal{M}}(A)$ is the set of all n-types over A in $\mathcal{M}$.
Note: A set of formulas in the free variables $v_1, , v_n$ is an n-type iff for all finite subsets $P_0 \subseteq P, \exists$ a sequence $a_1, , , a_n$ which realises P_0.

Example 37. *If F is an algebraically closed field, then there is a unique n-type which is not realised. This is the type of the transcendental over F. Note that by QE, any $\varphi(\overline{v})$ in P is equivalent to a formula of the form $F(v) = 0$ or $F(v) \neq 0$, where $f \in F[\overline{v}]$*

Notation: $tp^{\mathcal{M}}(\overline{a}/A) = \{\varphi(\overline{v}) | \mathcal{M} \models \varphi(\overline{a})\}$ is the type of $\overline{a}$ in A over $\mathcal{M}$.
Another quick example: let $\mathcal{M} = (\mathbb{Q}, \leq)$, $A = \mathbb{Z}$. Then $tp^{\mathcal{M}}(\frac{1}{2}/A) = tp^{\mathcal{M}}(\frac{1}{3}/A)$.

Proposition 38. *If P is an n-type over A in $\mathcal{M}$, then there is an elementary extension $\mathcal{M}$ of $\mathcal{M}$ in which P is realised.*

Automorphisms and Types:
If $\mathcal{M}$ is an $\mathcal{L}$-structure, $A \subseteq M$ and σ is an automorphism of $\mathcal{M}$ fixing A pointwise, then σ preserves types in the following way: $tp^{\mathcal{M}}(\overline{a}/A) = tp^{\mathcal{M}}(\sigma(\overline{a})/A)$, where $\sigma(\overline{a}) = (\sigma(a_1), , , , , \sigma(a_n))$.
food for thought: the converse is false - think of a counterexample! - however, we have the following result:

Theorem 39. *If $\mathcal{M}$ is a $\mathcal{M}$-structure, $A \subseteq M$ and $\overline{a}, \overline{b}$ are two n-types with the same type over A, then there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and an automorphism σ of $\mathcal{N}$ such that $\sigma(\overline{a}) = \overline{b}$, and $\sigma|_A = id_A$.*

Definition 40. *f is a partial elementary embedding from $\mathcal{M}$ to $\mathcal{N}$ if $dom(f) \subseteq M$, $range(f) \subseteq N$, and whenever $\varphi(\overline{v})$ is an $\mathcal{L}$-formula and $\overline{a} \in dom(f)$, $\mathcal{M} \models \varphi(\overline{a})$ iff $\mathcal{N} \models \varphi(\overline{a})$.*

Observe that $tp^{\mathcal{M}}(\overline{a}/A) = tp^{\mathcal{M}}(\overline{b}/A)$ iff $\{a_i \mapsto b_i\} \cup id_A$ (considered as a function) is a partial elementary embedding.

Lemma 41. *Suppose f is a partial elementary embedding from $\mathcal{M}$ to $\mathcal{N}$, and $a \in M - dom(f)$, then there is an elementary extension $\mathcal{N}'$ of $\mathcal{N}$ and an extension f' of f with f' a partial elementary embedding of $\mathcal{M}$ into $\mathcal{N}$ such that $a \in dom(f')$.*

Proof.
Let $\mathcal{N} \prec \mathcal{N}'$ realise the type $\{\varphi(v, f(c_1),,,, f(c_n))|c_1,,, c_n \in dom(f) \& \mathcal{M} \models \varphi(a, c_1,,, c_n)\}$.
Since the type of a over $dom(f)$ is satisfiable (namely, by a), P is satisfiable, and therefore the elementary extension exists.
Define $f' = f \cup \{a \mapsto b\}$, where b realises P in $\mathcal{N}'$. □

Lemma 42. *If f is a partial elementary embedding from $\mathcal{M}$ to $\mathcal{N}$, then there is an elementary extension $\mathcal{N}'$ of $\mathcal{N}$ and an elementary embedding $f' : \mathcal{M} \to \mathcal{N}'$ such that $f'|_{dom(f)} = f$.*

Proof.
Proof by transfinite induction.
Start by enumerating $M - dom(f)$ as $\{a_\zeta|\zeta < \kappa\}$ for some κ.
Construct $\{f_\zeta|\zeta < \kappa\}$ and & $\{\mathcal{N}_\zeta|\zeta < \kappa\}$ so that $f_0 = f$, $dom(f_\zeta) = dom(f) \cup \{a_\eta|\eta < \zeta\}$, and $\mathcal{N} = \mathcal{N}_0 \prec \mathcal{N}_\eta \prec \mathcal{N}_\zeta$ whenever $0 \leq \eta \leq \zeta \leq \kappa$; f_ζ extends f_η if $\zeta > \eta$ and $f_\zeta : \mathcal{M} \to \mathcal{N}_\zeta$ is a partial elementary embedding.
Suppose this construction has been done for all $\zeta < \alpha$. If α is a limit, set $f_\alpha = \cup_{\zeta<\alpha} f_\zeta$ and $\mathcal{N}_\alpha = \cup_{\zeta<\alpha}\mathcal{N}_\zeta$.
If $\alpha = \beta + 1$, then use the previous lemma with $a = b$ and $f = f_\beta$, with $f_\beta : \mathcal{M} \to \mathcal{N}_\beta$. Then f' is the desired $f_{\beta+1} = f_\alpha$. □