Simulation of a Multistage Rocket

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1 Introduction

A multistage rocket is a rocket that is designed with different stages of thrust, where each stage has its own engines and propellant. After a stage of the rocket is completed, the section of the rocket associated with the completed stage is usually ejected, and the next stage begins. This process continues until the final stage is reached. Chapter 7 of our textbook (*Classical Mechanics: 5th Edition, Kibble and Berkshire*) deals briefly with the mechanics of rockets, and contains problems involving two-stage and three-stage rockets. I noticed that for multi-stage rockets, it would be difficult to calculate an analytical expression for the position or velocity of the rocket as a function of time, especially when taking into account the variable forces of gravity and drag, and that it would be more suitable to do a numerical compuation.

The goal of this project is to simulate a multistage rocket, by first deriving an equation of motion for the motion of a multistage rocket that takes into account forces of gravity and drag, and then using numerical integration to obtain values for the position and velocity of the rocket as functions of time. The example rocket simulated is the Saturn V rocket, a three-stage rocket which was used as part of the Apollo Program to launch a spacecraft to the moon in 1969. Information about Saturn V can be found at: `https://en.wikipedia.org/wiki/Saturn_V#Saturn_V_vehicles_and_launches`.

The implementation of this project was done using the Python programming language, while making use of the numpy, scipy, and matplotlib libraries, which allow for easy numerical computations, integration, and plotting. An object-oriented implementation was used with a class "Rocket" which contains the necessary information for a multistage rocket, as well as methods that can be called on the Rocket class for simulating and plotting the motion. Although the program I wrote simulates the motion of the Saturn V rocket, it could be modified by a different user to simulate the motion of another rocket (including a rocket with a different number of stages), by declaring a new Rocket object with different parameters and calling the same methods.

2 Deriving an Equation of Motion

The equation of motion for a rocket can be derived using conservation of momentum. During each stage the rocket "ejects" a small mass dm downwards at velocity u relative to the rocket, which is traveling at upwards at velocity v. This pushes the rocket upwards. The rocket continues doing this for until all the fuel (or "wet mass") for that stage is burned through. Then the rocket ejects the "dry mass" (mass remaining after fuel is burned through) of that stage of the rocket and begins burning fuel for the next stage. A complete description of the rocket involves the wet and dry mass of each stage, the ejection velocity u for each stage, and the time taken to burn through the fuel of each stage.

Over a small time interval, dt, the momentum before and after can be written, relative to a stationary observer. The momentum before is:

$$p_i = Mv$$

Where M is the mass of the rocket and v is the upward velocity of the rocket. After ejecting a small mass dm at downwards velocity u relative to the rocket, the momentum is:

$$p_f = (M - dm)(v + dv) + dm(v - u)$$

$$= Mv - vdm + Mdv - dmdv + vdm - udm$$

$$= Mv + Mdv - dmdv - udm$$

Neglecting the second-order infinitesimal term, $dmdv$, the change in momentum is:

$$p_f - p_i = Mdv - udm$$

Now, using the equation:

$$dp = F\,dt$$

Taking into account the external forces on the system of gravity and drag:

$$dp = Mdv - udm = (F_{grav} + F_{drag})dt$$

Where F_{grav} depends on height and F_{drag} depends on velocity and height.

$$F_{grav} = (GMM_E)/(R_E + h)^2$$

$$F_{drag} = \frac{1}{2}\rho v^2 AC_d$$

In these equations, $G = 6.67 \times 10^{-11}\ m^3kg^{-1}s^{-2}$, M is the mass of the rocket, M_E is the mass of the Earth $= 5.97 \times 10^{24}$ kg, R_E is the radius of the earth $= 6.38 \times 10^6$ m, and h is the height above the surface of the Earth. v is the upward velocity of the rocket, A is the cross-sectional area of the rocket (for the Saturn V rocket, the diameters of each stage of the rocket are given- I used the largest cross-sectional area of the remaining stages as the value for A at each stage), C_d is the drag coefficient (I used a value of 0.2, following the example in this paper: `https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-07-dynamics-fall-2009/lecture-notes/MIT16_07F09_Lec14.pdf`). ρ is the air density, and can be approximated as:

$$\rho(h) = \rho_0 e^{-h/H}$$

Where ρ_0 is the air density at sea level $= 1.225 kg/m^3$, $H = 8000m$ is the so-called "scale height" of the atmosphere, and h is the height above the surface of the Earth.

So the equation:

$$dp = F\,dt$$

Becomes:

$$Mdv - udm = (\frac{-GMM_E}{(R_E+h)^2} - \frac{1}{2}(\rho_0 e^{-h/H})v^2 AC_d)dt$$

We want the dv term alone on the left-hand-side of the equation. To do this, write dm as $-\frac{dM}{dt}dt$ and subtract from both sides:

$$Mdv = (u\frac{-dM}{dt} + \frac{-GMM_E}{(R_E+h)^2} - \frac{1}{2}(\rho_0 e^{-h/H})v^2 AC_d)dt$$

Dividing by M will finally get dv on the left-hand side of the equation and the right-hand-side can then be integrated to get v(t), which can in turn be integrated to get h(t). Since at each point in time, v and h must be known to calculate drag and gravity, both integrals must be done simultaneously- at each time step, v and h must be computed. Due to the presence of the F_{drag} term, this integral cannot be computed analytically, and must be done numerically.

Dividing by M cancels out the M in the numerator of the F_{grav} (second), and leaves M in the denominators of the first and third terms. The easiest way to to the integral is to explicitly have $\frac{-dM}{dt}$ as a constant term (which it is), and $M(t)$ as a function of time. $\frac{-dM}{dt}$ is known by the total fuel mass of a stage divided by the time of the stage, and $M(t)$ is known since M_i and M_{final} are known and M changes at a constant rate of $\frac{dM}{dt}$.

The final expression to be integrated during each stage is:

$$dv = (\frac{u}{M(t)}\frac{-dM}{dt} - \frac{GM_E}{(R_E+h)^2} - \frac{1}{2M(t)}(\rho_0 e^{-h/H})v^2 AC_d)dt$$

3 Saturn V Rocket Data

The Saturn V rocket is a three-stage rocket: technically, the third-stage is split up into two separate stages, since, during the Apollo Mission, it was stopped once in Earth's orbit, and then resumed later to get the correct lunar trajectory. For simplicity's sake, I implemented the third-stage as a continuous process. The data from the Wikipedia page does not give the ejection velocity, but it does give the thrust, which allows calculating the ejection velocity: $T = u\frac{-dM}{dt}$

Stage 1:

Wet Mass = 2,169,000 kg
Dry Mass = 131,000 kg
Burn Time = 168 s
dM/dt = -12,910 kg/s
Thrust = 33,000 kN
Ejection Velocity = 2556.16 m/s
Diameter = 10 m

Stage 2:

Wet Mass = 444,000 kg
Dry Mass = 36,000 kg
Burn Time = 384 s
dM/dt = -1156 kg/s
Thrust = 5141 kN
Ejection Velocity = 4446.27 m/s
Diameter = 10 m

Stage 3:

Wet Mass = 109,000 kg
Dry Mass = 10,000 kg
Burn Time = 494 s
dM/dt = -220.6 kg/s
Thrust = 1796 kN
Ejection Velocity = 8141.43 m/s
Diameter = 6.604 m

Notes on efficiency: The model above assumes complete fuel efficiency, which is obviously not the reality. I still believe that the model should accurately predict the motion of the rocket, since the thrust data I am using is, to the best of my knowledge, the actual thrust generated by each stage of the rocket. This explains why the ejection velocities seem high- because they may not actually be the ejection velocities, but rather "effective ejection velocities" calculated from the given thrusts and wet mass of each stage.

4 Notes on Implementation

The program 'MultistageRocketSimulator' has two classes defined: Stage, and Rocket. A Stage consists of all the necessary information for a stage: the wet mass, dry mass, ejection velocity, ejection time, and diameter. A Rocket consists of some number of stages. Calling the 'simulate' method on a rocket solves for the position and velocity as functions of time. Calling the 'plot' method plots the velocity and height of the rocket.

I decided, given that the integral has both v and h terms, that the easiest way to solve it would be as an initial value problem. The equation:

$$dv = (\frac{u}{M(t)}\frac{-dM}{dt} - \frac{GM_E}{(R_E+h)^2} - \frac{1}{2M(t)}(\rho_0 e^{-h/H})v^2 AC_d)dt$$

Becomes:

$$\frac{dv}{dt} = (\frac{u}{M(t)}\frac{-dM}{dt} - \frac{GM_E}{(R_E+h)^2} - \frac{1}{2M(t)}(\rho_0 e^{-h/H})v^2 AC_d)$$

Which can be turned into a system of first order equations by adding the information that the derivative of h(t) is v(t):

$$\frac{dv}{dt} = (\frac{u}{M(t)}\frac{-dM}{dt} - \frac{GM_E}{(R_E+h)^2} - \frac{1}{2M(t)}(\rho_0 e^{-h/H})v^2 AC_d)$$

$$\frac{dh}{dt} = v(t)$$

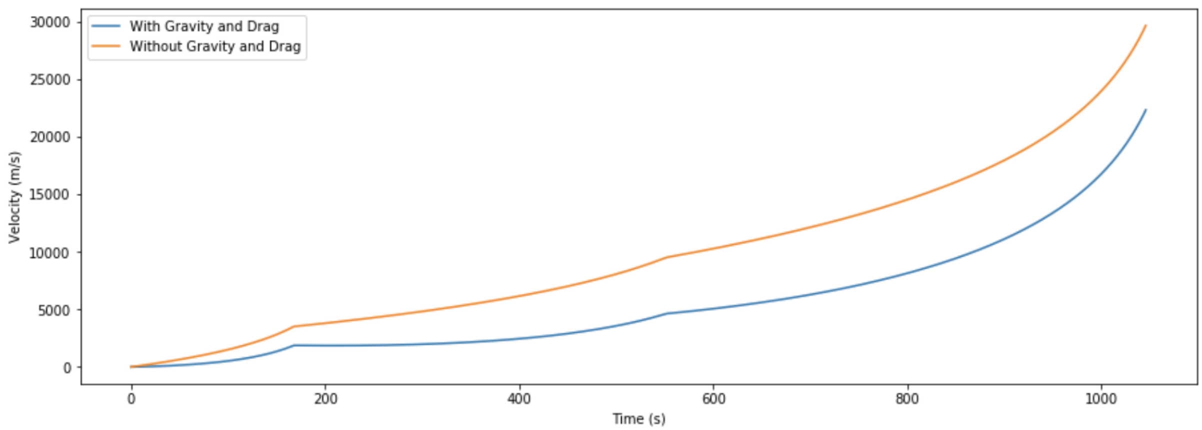
$$v(0) = 0, h(0) = 0$$

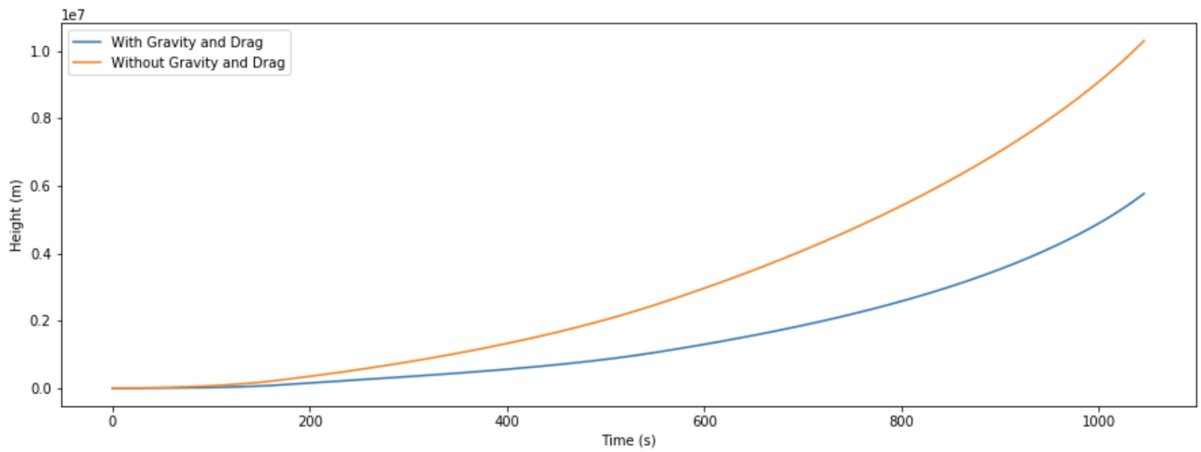
This system can then be integrated using the integrate→ solve initial value problem feature of the scipy library. M(t) and D(t) (max diameter) must also be defined as functions of time.

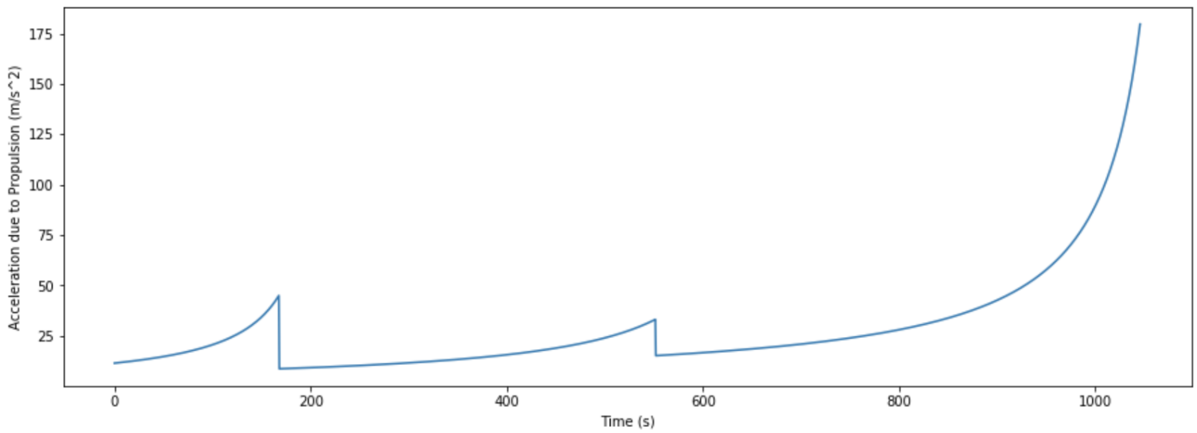
5 Results

I had the program simulate the rocket with and without the effects of gravity and drag, and plot the resulting motion. I also plotted the acceleration due to propulsion, gravity, and drag as functions of time. The results seem to make sense. Gravity has a major impact on slowing down the rocket in the beginning stages when it has not gained much momentum. Drag is basically negligible (reaches a peak of $-0.175m/s^2$, but recedes as the rocket travels into the less dense upper atmosphere).

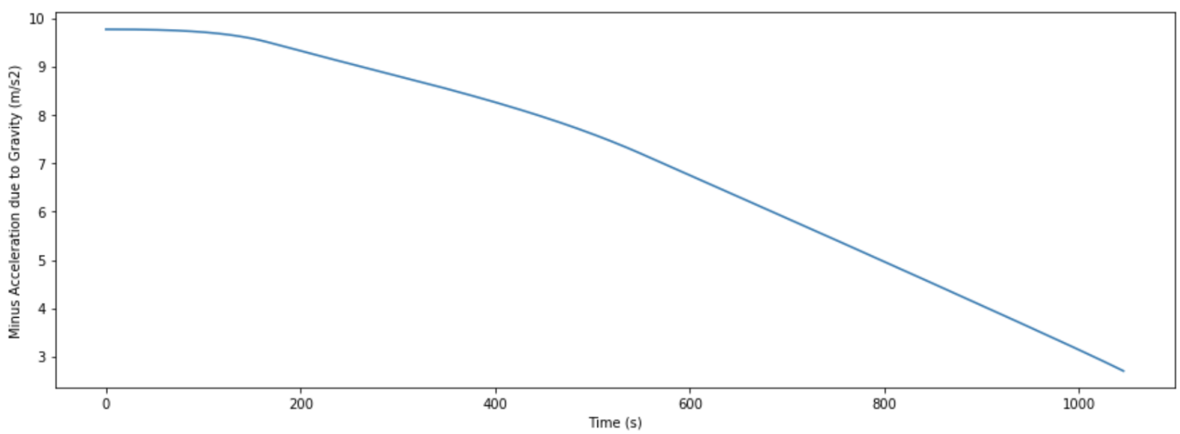
Apologies for the messy image formatting!


With Gravity and Drag
Without Gravity and Drag
Velocity (m/s)
Time (s)

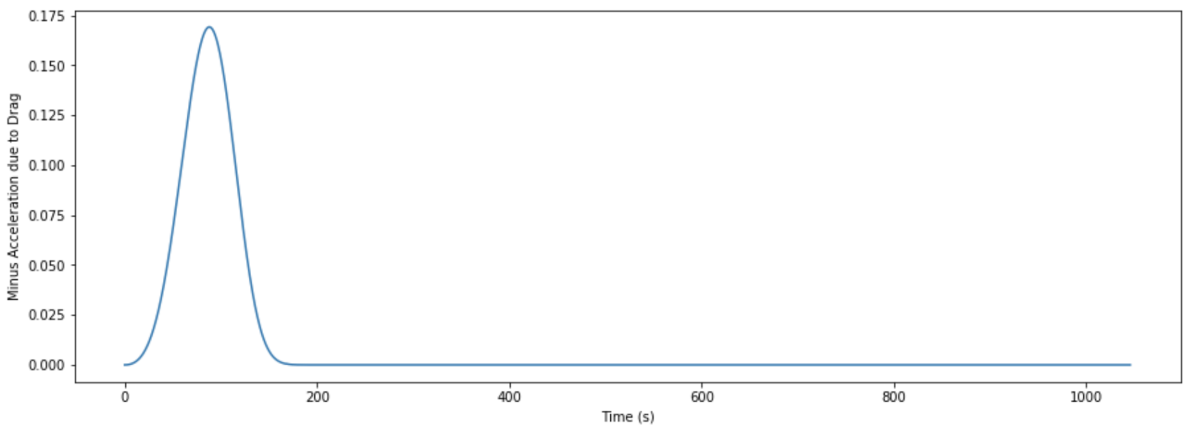

1e7
With Gravity and Drag
Without Gravity and Drag
Height (m)
Time (s)



Acceleration due to Propulsion (m/s^2)
175
150
125
100
75
50
25
0
200
400
600
800
1000
Time (s)



Minus Acceleration due to Gravity (m/s2)
10
9
8
7
6
5
4
3
0
200
400
600
800
1000
Time (s)



6 Code

The code for this project can be viewed at `https://github.com/RyanKierulf/MultistageRocketSimulator`.

7 Conclusion

The results show the expected motion for a multistage rocket. The height of the rocket increases at an increasing rate, and is exiting the atmosphere at the end of the rocket's final stage. The velocity also increases at an increasing rate at each stage, but takes time to build momentum. It makes sense that the velocity should increase at an increasing rate during each stage, since although the propulsion is constant, the mass of the rocket decreases as it burns fuel. By also simulating the motion of the rocket while neglecting gravity and drag, one can see that the motion has the same shape, but achieves greater height and velocity, as would be expected. Finally, plotting the accelerations due to propulsion, gravity, and drag as functions of time also yield expected results: the acceleration due to propulsion looks very close to the derivative of the velocity graph, the acceleration due to gravity starts at a value of 9.81 and steadily decreases, and the acceleration due to drag increases initially as the rocket travels faster and faster, before decreasing as the air in the upper atmosphere becomes less dense.

8 References

1. https://web.mit.edu/16.unified/www/SPRING/propulsion/notes/node103.html
2. https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-07-dynamics-fall-2009/lecture-notes/MIT16_07F09_Lec14.pdf
3. https://en.wikipedia.org/wiki/Saturn_V#Saturn_V_vehicles_and_launches