Hermitian Jacobi Forms of Higher Degree

by

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DISSERTATION ABSTRACT

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We develop the theory of Hermitian Jacobi forms in degree $n > 1$. This builds on the work of Klaus Haverkamp in [Hav95] who developed this theory in degree $n = 1$. Haverkamp in turn generalized a monograph of Eichler and Zagier, [EZ85]. Hermitian Jacobi forms are holomorphic functions which appear in certain infinite series expansions (Fourier Jacobi expansions) of Hermitian modular forms. In this work we give a definition of Hermitian Jacobi forms in degree $n > 1$, give their relationship to more classical Hermitian modular forms and construct a useful tool for studying Hermitian Jacobi forms, the theta expansion. This theta expansion allows us to relate our forms to classical modular forms via the Eichler-Zagier map and thereby bound the dimension of our space of forms. We then go on to apply the developed theory to prove some non-vanishing results on the Fourier coefficients of Hermitian modular forms.

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CHAPTER 1

INTRODUCTION

We develop core aspects of the theory of Hermitian Jacobi forms in degree $n > 1$. This builds on the work of Klaus Haverkamp in [Hav95] who developed this theory in degree $n = 1$. Haverkamp was in turn generalizing a monograph of Eichler and Zagier [EZ85]. Hermitian Jacobi forms are holomorphic functions which appear in certain infinite series expansions (Fourier Jacobi expansions) of Hermitian modular forms. In this work we give a definition of Hermitian Jacobi forms in a higher-dimensional setting, give their relationship to more classical Hermitian modular forms and construct a useful tool for studying Hermitian Jacobi forms, the theta expansion. This theta expansion allows us to relate our forms to classical modular forms via the Eichler--Zagier map and thereby bound the dimension of our space of forms. We then apply the developed theory to prove some non-vanishing results on the Fourier coefficients of Hermitian modular forms.

1.1 Motivation

Broadly speaking, the goal of this paper is add an additional tool to studying a class of functions known as automorphic forms. These functions can generally be thought of as holomorphic functions on some complex manifold, sometimes identified with the moduli space of some family of geometric objects, that satisfy symmetry properties with respect to a linear algebraic group that acts on the domain manifold. Auotomorphic forms have broad applications in number theory, for example the application of classical modular forms in Andrew Wiles' solution to Fermat's Last Theorem. Today many mathematicians study automorphic forms as objects of interest in their own right or to help shed light on a very broad family of conjectures in the field known as the Langlands Program.

The study of automorphic forms can be divided into different subfields based on the group that governs the symmetry of these forms. Classically this group was $\mathrm{SL}_2(\mathbb{Z})$ with which we associate classical modular forms. In his 1939 paper [Sie39] Carl Ludwig Siegel introduced a higher dimensional analogue known as Siegel modular forms which transform with respect to the group $\mathrm{Sp}_n(\mathbb{Z})$. The class of automorphic forms that we are most interested in here are those known as Hermitian modular forms introduced by Hel Braun in three papers, [Bra49], [Bra51a], and [Bra51b] . We give a formal definition and some introduction to the theory in chapter 3.
In [EZ85], Eichler and Zagier formally introduced Jacobi forms, a class of $\mathbb{C}$-valued functions on $\mathbb{H} \times \mathbb{C}$ that satisfied some symmetry conditions with respect to the group $\mathrm{SL}_2(\mathbb{Z})$ as well as some holomorphy conditions. Two important examples include Jacobi's theta function and the Weierstrass $\wp$ function which parametrizes the complex points of an associated elliptic curve. Jacobi forms also appear as the coefficients in a certain series expansion of Siegel modular forms known as the Fourier Jacobi expansion. In this way Jacobi forms have proven to be a useful tool in studying Siegel modular forms.
In [Hav95] Haverkamp gives a generalization of the theory of Jacobi forms to the setting of Hermitian modular forms, though only in degree 2. This paper is the most direct logical predecessor to the current work. Even with this limit on the degree, there are applications of this work, principally to the study of the Fourier coefficients of Hermitian modular forms of degree 2, as in [AD19]. Our goal in this work is to generalize Haverkamp's work to an arbitrary degree n. As an initial motivation we hoped to extend the results of [AD19] and generalize similar results of [BD22]. As often happens, we found an insurmountable difficulty that is discussed in the final chapter.

1.2 Description of results

Our primary result is the following. Let K be a quadratic imaginary field with ring of integers $\mathcal{O}$.

Theorem 1.1. *Let ϕ be an Hermitian Jacobi form of weight k, degree $n \geq 1$ and invertible index T. Then ϕ has a theta expansion of the form*

$$\phi(\tau, w, z) = \sum_{s \in \left(\mathcal{O}^{\#}\right)^n / T\mathcal{O}^n} h_s(\tau)\theta_{T,s}(\tau, w, z) \tag{1.1}$$

where each h_s is a classical modular form.

Here $w, z \in \mathbb{C}^n$ and $\tau \in \mathbb{H}$ the complex upper-half plane. We define $\theta_{T,s}$ in Equation (4.7). This is Theorem 5.5 in the main text. In Chapter 2 we give new definitions of Hermitian Jacobi forms in degree $n > 1$. These directly generalize those of Haverkamp. These definitions are chosen so that in Chapter 3 we can prove the following proposition.

Proposition 1.2. *Let F be an Hermitian modular form. Then F has a Fourier Jacobi expansion*

$$F\left(\begin{bmatrix} \tau & w \\ z & T \end{bmatrix}\right) = \sum_{T \in \Lambda_n(\mathcal{O})} \phi_T(\tau, w, z) e(TZ) \tag{1.2}$$

where each ϕ_T is an Hermitian Jacobi form.

Here

$$\Lambda_n(\mathcal{O}_K) := \left\{ A \in M_n(\mathbb{C}) | A = \overline{A}^t, a_{i,i} \in \mathbb{Z}, a_{i,j} \in \mathcal{O}^{\#} \right\} \tag{1.3}$$

and $e(TZ)$ is shorthand for $e^{2\pi i \mathrm{tr}(TZ)}$. This is Proposition 4.1 in the main text. We give an introduction to the theory of Hermitian modular forms in Chapter 3 before this proposition.

Chapter 4 is the technical heart of the paper. In this chapter we first introduce our theta functions and explain the existence of the expansion given in Theorem 5.5. The bulk of this chapter is establishing Proposition 4.8, a transformation law for these theta functions under an action of a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The fact that each h_s in the theta expansion is a modular form will eventually follow from this transformation law. In chapter 5 we begin by translating the transformation law for the theta functions into transformations of the coefficients. After proving the theorem above we move onto generalizing the Eichler--Zagier map and the twists there-of to our setting. These functions, which take Hermitian Jacobi forms to classical modular forms, are a central tool in [AD19], Anamby and Das's paper on non-vanishing fundamental Fourier coefficients. In chapter 6 we introduce some of the theory of vector valued Hermitian modular forms in order to give some novel applications of our work to the theory of Hermitian modular forms. In this chapter we prove the following result.

Proposition 1.3. *Let F be a non-zero, possibly vector valued, Hermitian modular form of degree $n \geq 2$. Then F has infinitely many non-zero Fourier Jacobi coefficients with non-singular index.*

The above result essentially tells us that each Hermitian modular form gives rise to many Hermitian Jacobi forms. Using this and the theta expansion we prove the proposition below.

Proposition 1.4. *Let F be a non-zero Hermitian modular form of weight k and degree $n \geq 2$. Then F has infinitely many non-zero Fourier coefficients with index*

$A = \begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}$ *such that*

$$\det(A) \leq C_{k,n-1,D} \det(T)^2 \tag{1.4}$$

where

$$C_{k,n,D} = \frac{(k-n)D^{3+2\lfloor \frac{n}{2} \rfloor}}{12} \tag{1.5}$$

and D is $|\Delta_K|$, the discriminant of the number field K.

In the final section we explain the relationship between this work and the existence of non-zero fundamental Fourier coefficients. As stated in the motivation the initial goal of this work was to generalize [BD22, Theorem 1.1] of Böcherer and Das, stated below, to the Hermitian setting.

Theorem 1.5 (Theorem 1.1 of [BD22])**.** *Let F be a non-zero vector valued Siegel modular form of weight ρ and degree n. Suppose further that $k(\rho) - \frac{n}{2} \geq \varrho(n)$. When n is even, assume that F is cuspidal. Then there exists infinitely many $GL_n(\mathbb{Z})$ inequivalent matrices $T \in \Lambda_n^+$ such that d_T is odd and square free, and $a_F(T) \neq 0$.*

Anamby and Das have proven a similar result, [AD19, Theorem 1], for Hermitian modular forms when the degree $n = 2$ and heavily leveraged the theory of Hermitian Jacobi forms in this setting. While we were not able to generalize the approaches of either of these two papers here, we do outline the general strategy and where the road blocks are to generalizing these approaches.

CHAPTER 2

HERMITIAN JACOBI FORMS

2.1 The Hermitian Jacobi Group

The goal of this chapters is to give the definition of Hermitian Jacobi forms in degree greater than or equal to 1. Let $n \geq 1$ be a integer and K be a quadratic imaginary field with ring of integer $\mathcal{O}$. Let $D = |\Delta_K|$ be the absolute value of the discriminant of K. Hermitian Jacobi forms of degree n are functions $\phi : \mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ which satisfy some transformation properties and admit a Fourier expansion. Before we can define Hermitian Jacobi forms, we need to define a group action on $\mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n$. Let

$$U(n, n) := \left\{ A \in M_{2n}(\mathcal{O}) : \overline{A}^t J_n A = J_n \right\} \tag{2.1}$$

where $J_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$. The group $U(n, n)$ is the unitary group of degree n which governs the transformations of Hermitian modular forms of degree n. We define a right action of $U(1, 1)$ on $\mathbb{C}^n \times \mathbb{C}^n$ via

$$[\lambda, \mu] \cdot A := [\lambda, \mu]\overline{A} = [\overline{a_{11}}\lambda + \overline{a_{21}}\mu, \overline{a_{12}}\lambda + \overline{a_{22}}\mu] \tag{2.2}$$

for any $\lambda, \mu \in \mathbb{C}^n \times \mathbb{C}^n$ and $A = [a_{ij}] \in U(1, 1)$. This action gives rise to the following definition.

Definition 2.1. Define multiplication in $\Gamma_1 := U(1, 1) \ltimes \mathbb{C}^{2n}$ via

$$(A, X_1) \cdot (A, X_2) := (A_1 A_2, X_1 \cdot A_2 + X_2). \tag{2.3}$$

This group will be that which governs the transformation law of our Hermitian Jacobi forms.

Let $(\tau, w, z) \in \mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n$. Throughout the paper we consider w to be a row vector and z to be a column vector. In order to define our group action of Γ_1 on $\mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n$ we define actions of the two constituent groups. First for $[\lambda, \mu] \in \mathbb{C}^n \times \mathbb{C}^n$, where both λ and μ are row vectors, we define

$$[\lambda, \mu] \cdot (\tau, w, z) := (\tau, w + \lambda\tau + \mu, z + \overline{\lambda}^t \tau + \overline{\mu}^t). \tag{2.4}$$

Here we write A^t for the transpose of a matrix. Often in related literature the convention is ${}^tA$. Next we consider the action of $U(1,1)$ on (τ, w, z). We also note that

$$U(1,1) = \{\epsilon M : \epsilon \in \mathcal{O}^\times, M \in \mathrm{SL}_2(\mathbb{Z})\}. \tag{2.5}$$

For $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ define $M\tau = \frac{a\tau+b}{c\tau+d}$ and $j(M,\tau) = c\tau + d$. We define an action of $U(1,1)$ on $\mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n$ by

$$\epsilon M \cdot (\tau, w, z) := \left(M\tau, \frac{\epsilon w}{j(M,\tau)}, \frac{\bar{\epsilon} z}{j(M,\tau)} \right). \tag{2.6}$$

We then define a group action of Γ_1 on $\mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n$ via

$$(A, [\lambda, \mu]) \cdot (\tau, w, z) := A \cdot ([\lambda, \mu] \cdot (\tau, w, z)). \tag{2.7}$$

Remark 2.2. One can check that this is a well-defined group action by first computing that, for $A \in U(1,1)$ and $[\lambda, \mu] \in \mathbb{C}^n \times \mathbb{C}^n$,

$$[\lambda, \mu] \cdot (A(\tau, w, z)) = A([\lambda, \mu] \cdot A) \cdot (\tau, w, z). \tag{2.8}$$

With the group actions defined above we can define associated slash operators. Let $\phi : \mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be a function. For $\epsilon M \in U(1,1)$, $\lambda, \mu \in \mathbb{C}^n$, an integer k and $T \in \Lambda_n(\mathcal{O})$ define

$$\phi|_{T,k}\epsilon M(\tau, w, z) := \epsilon^{-k}(j(M,\tau))^{-k} e\left(-cwTz/j(M,\tau)\right) \phi\left(\epsilon M \cdot (\tau, w, z)\right) \tag{2.9}$$

$$\phi|_T(\lambda, \mu)(\tau, w, z) := e(\lambda T\overline{\lambda}^t \tau + wT\overline{\lambda}^t + \lambda Tz)\phi\left(\tau, w + \lambda\tau + \mu, z + \overline{\lambda}^t\tau + \overline{\mu}^t\right). \tag{2.10}$$

2.2 Hermitian Jacobi Forms

In the previous section we introduced the transformation group for our Hermitian Jacobi forms. The other condition on our forms is the existence of a Fourier expansion of a certain form. In order to give this condition explicitly we introduce a couple indexing sets for these expansions. Let $\mathcal{O}^{\#} := \frac{i}{\sqrt{D}}\mathcal{O}$ be the different of our ring of integers.

Remark 2.3. The set $\mathcal{O}^{\#}$ is the dual lattice of $\mathcal{O}$ with respect to the trace pairing on K. In particular, for each $\alpha \in \mathcal{O}^{\#}$ and $\beta \in \mathcal{O}$ we have

$$2\mathrm{Re}[\alpha\beta] \in \mathbb{Z}. \tag{2.11}$$

Define

$$\Lambda_n(\mathcal{O}) := \left\{ A \in M_n(\mathbb{C}) | A = \overline{A}^t, a_{i,i} \in \mathbb{Z}, a_{i,j} \in \mathcal{O}^{\#} \right\} \tag{2.12}$$

and

$$\Lambda_n^+(\mathcal{O}) := \left\{ A \in \Lambda_n(\mathcal{O}) : A \text{ is positive definite } \right\}. \tag{2.13}$$

Finally define our indexing set, for a matrix $T \in \Lambda_n(\mathcal{O})$,

$$\mathcal{S}_T := \left\{ (m, r) \in \mathbb{Z} \times \left(\mathcal{O}^{\#}\right)^n : \begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix} \in \Lambda_{n+1}^+(\mathcal{O}) \right\}. \tag{2.14}$$

Now we can define an Hermitian Jacobi form.

Definition 2.4. Let k be an integer and $T \in \Lambda_n(\mathcal{O})$. A holomorphic function $\phi : \mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is an Hermitian Jacobi form of weight k, degree n, and index T if

1. ϕ satisfies the following transformation laws

$$\phi|_{T,k} A(\tau, w, z) = \phi(\tau, w, z) \text{ for all } A \in U(1, 1) \tag{2.15}$$

$$\phi|_T [\lambda, \mu](\tau, w, z) = \phi(\tau, w, z) \text{ for all } [\lambda, \mu] \in \mathcal{O}^n \times \mathcal{O}^n \tag{2.16}$$

2. ϕ has a Fourier expansion

$$\phi(\tau, w, z) = \sum_{(m,r)\in\mathcal{S}_T} \alpha(m,r)e(m\tau + wr + \overline{r}^t z). \tag{2.17}$$

We call this space $\mathcal{J}^n_{T,k}(\mathcal{O})$ or just $\mathcal{J}^n_{T,k}$ if $\mathcal{O}$ is clear from context.

These transformations given in condition (1) above can be rephrased as a single requirement using the group Γ_1 though we find it easier to think of these two transformations separately.

CHAPTER 3

HERMITIAN MODULAR FORMS

3.1 Definition and Notation

Hermitian modular forms are a higher dimensional analogue of classical modular forms, similar to Seigel modular forms. Whereas Seigel modular forms transform with respect to the symplectic group, Hermitian modular forms transform with respect to the unitary group $U(n,n)$ for some integer $n \geq 1$. These objects were first introduced in a sequence of three papers by Hel Braun [Bra49], [Bra51a] and [Bra51b]. Since their definition, these forms have grown to be an important class of automorphic forms with a theory that parallels many of the developments in the theory of Seigel modular forms. In this chapter we give an overview of this theory and explain the relationship between Hermitian modular forms and Hermitian Jacobi forms.

First define the Hermitian upper-half space of degree n by

$$\mathcal{H}_n := \{Z \in M_n(\mathbb{C}) : (Z - \overline{Z}^t)/2i > 0\}. \tag{3.1}$$

The group $U(n,n)$ has an action on $\mathcal{H}_n$ defined by

$$M\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \tag{3.2}$$

for $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in U(n,n)$ and $Z \in \mathcal{H}_n$. Now we define Hermitian modular forms.

Definition 3.1. Let k be an integer. A function $F : \mathcal{H}_n \to \mathbb{C}$ is a Hermitian modular form of weight k, and degree n for the group $U(n,n)$ if it is holomorphic and, for any $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in U(n,n)$ and $Z \in \mathcal{H}_n$, we have

$$F(Z) = F|_k M(Z) := \det(CZ+D)^{-k} F(M\langle Z\rangle). \tag{3.3}$$

Here A, B, C and D are $n \times n$ block matrices. When $n = 1$ we require that F be holomorphic at the cusps of $\mathrm{SL}_2(\mathbb{Z})$ as well. Concretely this means that we require that $F(Z)$ is bounded as $\mathrm{Im}(z)$ tends to infinity. We denote the space of such functions by $\mathcal{M}_k^n(\mathcal{O})$.

For a matrix B define $e(B) = e^{2\pi i \mathrm{tr} B}$. Then a Hermitian modular form F has a Fourier expansion

$$F(Z) = \sum_{A \in \Lambda_n(\mathcal{O})} a(F,A) e(AZ). \tag{3.4}$$

If instead the above sum is over $\Lambda_n^+(\mathcal{O})$ we say that F is a cusp form.

3.2 Relationship with Hermitian Jacobi Forms

We now explain the relationship between Hermitian Jacobi forms and Hermitian modular forms. The basic principle is that the Fourier series can be rearranged in order to give F as a sum of Hermitian Jacobi forms multiplied by an exponential term. We first need a lemma in order to verify the transformation law of these coefficients.

Lemma 3.2. *Let $\mu, \lambda \in \mathcal{O}^n$. Then for any $T \in \Lambda_n(\mathcal{O})$ we have that*

$$\lambda T \overline{\mu}^t + \mu T \overline{\lambda}^t \in \mathbb{Z}. \tag{3.5}$$

Proof. We have, using the formula for the trace of a matrix product,

$$\mathrm{tr}\left(\overline{\mu}^t T \lambda + \overline{\lambda}^t T \mu\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} \overline{\mu_j} \lambda_i + t_{ji} \overline{\lambda_j} \mu_i. \tag{3.6}$$

The terms for which $i = j$ are in $\mathbb{Z}$ since

$$t_{ii} \in \mathbb{Z} \text{ and } \left(\overline{\mu_j}\lambda_i + \overline{\lambda_j}\mu_i\right) \in \mathcal{O} \cap \mathbb{R} = \mathbb{Z}. \tag{3.7}$$

Then we have

$$\operatorname{tr}(\overline{\mu}^t T\lambda + \overline{\lambda}^t T\mu) \equiv \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} 2\mathrm{Re}[t_{ij}\overline{\mu_i}\lambda_j] \pmod{\mathbb{Z}} \tag{3.8}$$

Since $t_{ij} \in \mathcal{O}^\#$ and $\overline{\mu_i}\lambda_j \in \mathcal{O}$, Remark 2.3 implies each term in the above sum is an integer. Thus $\operatorname{tr}(\overline{\mu}^t T\lambda + \overline{\lambda}^t T\mu)$ is as well. □

With this we can prove the that the Fourier Jacobi coefficients of an Hermitian modular form are Hermitian Jacobi forms.

Proposition 3.3. *Let $F : \mathcal{H}_{n+1} \to \mathbb{C}$ be a Hermitian modular form of weight k. Let F have Fourier expansion*

$$F(Z) = \sum_{A\in\Lambda_{n+1}(\mathcal{O})} a(F, A)e(AZ).$$

For $T \in \Lambda_n(\mathcal{O})$ define

$$\phi_T(\tau, w, z) := \sum_{(m,r)\in\mathcal{S}_T} a\left(F, \begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right) e(a\tau + wr + \overline{r}^t z).$$

Then

$$F\left(\begin{bmatrix} \tau & w \\ z & Z \end{bmatrix}\right) = \sum_{T\in\Lambda_n(\mathcal{O})} \phi_T(\tau, w, z)e(TZ) \tag{3.9}$$

and each ϕ_T is an Hermitian Jacobi form of degree n, weight k and index T.

Proof. First we address the holomorphy concerns of ϕ_T. The Fourier series of F converges absolutely on compact subsets of the domain $\mathcal{H}_{n+1}$. Hence the sub-sum

$$e(TZ)\phi_T(\tau, z, w) = \sum_{(m,r)\in\mathcal{S}_T} a\left(F, \begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right) e\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\begin{bmatrix} \tau & w \\ z & Z \end{bmatrix}\right) \tag{3.10}$$

will converge absolutely on compact subsets of $\mathcal{H}_n$ and thus so too will the sum defining ϕ_T. In particular we see that ϕ_T is a well-defined holomorphic function on $\mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n$ with the proper Fourier expansion.

In order to prove that ϕ_T satisfies the necessary transformation laws we first need Equation (3.9). If

$$\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix} \in \Lambda_{n+1}(\mathcal{O}) \tag{3.11}$$

then $T \in \Lambda_n(\mathcal{O})$. Observe that

$$e(TZ)e(m\tau + wr + \overline{r}^t z) = e\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\begin{bmatrix} \tau & w \\ z & Z \end{bmatrix}\right). \tag{3.12}$$

so that Equation (3.9) immediately follows from the definition of ϕ_T.

It remains to check that ϕ_T satisfies the proper transformation law. Let $\epsilon \in \mathcal{O}^\times$ and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Note that

$$\mathcal{M} := \begin{bmatrix} \epsilon a & 0 & \epsilon b & 0 \\ 0 & I_n & 0 & 0 \\ \epsilon c & 0 & \epsilon d & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix} \in U(n+1, n+1). \tag{3.13}$$

Since F is an Hermitian modular form of weight k, we have

$$F|_k \mathcal{M} = F. \tag{3.14}$$

If follows from direct computation that

$$\mathcal{M}\left\langle \begin{bmatrix} \tau & w \\ z & Z \end{bmatrix} \right\rangle = \begin{bmatrix} M\tau & \frac{\epsilon w}{j(M,\tau)} \\ \frac{\overline{\epsilon} z}{j(M,\tau)} & Z - \frac{czw}{(c\tau+d)} \end{bmatrix}. \tag{3.15}$$

Expanding Equation (3.14) using Equation (3.9) and comparing the coefficients of $e(TZ)$ gives Equation (2.15) for ϕ_T.

Next consider our other transformation requirement. Let $\lambda, \mu \in \mathcal{O}^n$. Again, we consider these as row vectors. To get Equation (2.16) for ϕ_T we use the matrix

$$\mathcal{N} := \begin{bmatrix} 1 & 0 & 0 & \mu \\ \overline{\lambda}^t & I_n & \overline{\mu}^t & \overline{\lambda}^t \mu \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & I_n \end{bmatrix}. \tag{3.16}$$

We have

$$\mathcal{N}\left\langle \begin{bmatrix} \tau & w \\ z & Z \end{bmatrix} \right\rangle = \begin{bmatrix} \tau & \lambda\tau + w + \mu \\ \overline{\lambda}^t \tau + \overline{\mu}^t + z & Z + \overline{\lambda}^t w + z\lambda + \tau\overline{\lambda}^t \lambda + (\overline{\lambda}^t \mu + \overline{\mu}^t \lambda) \end{bmatrix}. \tag{3.17}$$

If we expand Equation (3.14) using Equation (3.9) and compare the coefficients we almost get the desired equality. On the left-hand side we have an extra term of

$$e(\lambda T \overline{\mu}^t + \mu T \overline{\lambda}^t) = 1$$

by Lemma 3.2. Equation (2.16) follows. $\square$

The expansion above is known as the Fourier Jacobi expansion of an Hermitian modular form. By studying the Fourier expansion of Hermitian Jacobi forms, we can get information about Hermitian modular forms, similar to the Siegel setting, where the theory of Jacobi forms sheds light on Siegel modular forms.

CHAPTER 4

THETA EXPANSIONS

This chapter is the most technically dense. We introduce our theta functions, relate them to Hermitian Jacobi forms and study their transformations. We'll almost exclusively be studying Hermitian Jacobi forms of positive definite index as non-singularity of the index is required for the definition of the theta functions. Proposition 6.11 tells us this requirement is not too restrictive.

4.1 Definitions and Existence

Throughout this section let ϕ be a Hermitian Jacobi form of degree n, weight k and index $T \in \Lambda_n^+(\mathcal{O})$ with Fourier expansion

$$\phi(\tau, w, z) = \sum_{(m,r)\in\mathcal{S}_T} \alpha(m,r)e(m\tau + wr + \overline{r}^t z). \tag{4.1}$$

Since $T \in \Lambda_n^+(\mathcal{O})$ we see that T is necessarily non-singular. We introduce some notation that will be helpful for this section.

Define, for $A \in \Lambda_n(\mathcal{O})$

$$d(A) := \begin{cases} |\det\left(i\sqrt{D}A\right)| & \text{if } n \text{ is even} \\ \left|\frac{-i}{\sqrt{D}}\det\left(i\sqrt{D}A\right)\right| & \text{if } n \text{ is odd} \end{cases}. \tag{4.2}$$

This quantity is called the content of the matrix A. For $n \geq 1$ and $D \in \mathbb{R}$ let

$$E_n(D) = \begin{cases} D & \text{if n is odd} \\ 1 & \text{if n is even} \end{cases}. \tag{4.3}$$

This is a correction term to $d(A)$ that is necessary to relate the content of an $n \times n$ matrix to the content of an $n+1 \times n+1$ matrix. Finally let

$$I_{T,s} := \left\{N \in \mathbb{Z}_{>0} : \frac{N}{E_n(D)d(T)} + \overline{s}^t T^{-1} s \in \mathbb{Z}\right\}. \tag{4.4}$$

This will be the indexing set of our theta coefficients. The main result of this sections is

Proposition 4.1. *Let ϕ be as above. For $s \in \left(\mathcal{O}^{\#}\right)^n$ and an integer $N > 0$ define*

$$\alpha_{s+T\mathcal{O}^n}(N) = a(m, r) \tag{4.5}$$

if $r \equiv s \pmod{T\mathcal{O}^n}$ and $N = d\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right)$. Define for $\tau \in \mathbb{H}$ and $w, z \in \mathbb{C}^n$,

$$h_s(\tau) := \sum_{N \in I_{T,s}} \alpha_{s+T\mathcal{O}^n}(N) e\left(\frac{N}{E_n(D)d(T)}\tau\right) \tag{4.6}$$

$$\theta_{T,s}(\tau, w, z) := \sum_{\substack{r \in \left(\mathcal{O}^{\#}\right)^n \\ r \equiv s \pmod{T\mathcal{O}^n}}} e\left(\overline{r}^t T^{-1} r\tau + wr + \overline{r}^t z\right). \tag{4.7}$$

Then we have

$$\phi(\tau, w, z) = \sum_{s \in \left(\mathcal{O}^{\#}\right)^n / T\mathcal{O}^n} h_s(\tau)\theta_{T,s}(\tau, w, z) \tag{4.8}$$

We will see that the quantity $E_n(D)d(T)$ will be the level of the theta coefficients h_s of our Hermitian Jacobi form ϕ. In order to prove Proposition 4.1 we need 4 preliminary results. We first argue that our theta functions are well-defined holomorphic functions.

Proposition 4.2. *Let $s \in (\mathcal{O}^{\#})^n$. Then*

$$\sum_{\substack{r \in (\mathcal{O}^{\#})^n \\ r \equiv s \pmod{T\mathcal{O}^n}}} e\left(\overline{r}^t T^{-1} r\tau + wr + \overline{r}^t z\right) \tag{4.9}$$

is absolutely and uniformly convergent on compact subsets of $\mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n$.

Proof. Let $K \subset \mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n$ be compact.

First we find $\omega_1, \omega_2, \omega_3 > 0$ such that

$$\left|e\left(\overline{r}^t T^{-1} r\tau\right)\right| \leq \exp\left(-\omega_1 ||r||^2\right) \tag{4.10}$$

$$|e(wr)| \leq \exp(\omega_2 ||r||) \tag{4.11}$$

$$\left|e\left(\overline{r}^t z\right)\right| \leq \exp(\omega_3 ||r||) \tag{4.12}$$

for all $(\tau, w, z) \in K$ and $r \in (\mathcal{O}^{\#})^n$ with $||r||$ sufficiently large. If $d > 0$ is the minimal eigenvalue of T^{-1} then

$$|\overline{r}^t T^{-1} r| \geq d||r||^2. \tag{4.13}$$

This follows from the fact that, because T^{-1} is a positive definite, Hermitian matrix, T^{-1} is orthogonally diagonalizable.

Since $K \subset \mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n$ is compact we can choose some $v_0 > 0$ so that $\mathrm{Im}(\tau) > v_0$ for all $(\tau, w, z) \in K$. We have

$$\left|e\left(\overline{r}^t T^{-1} r\tau\right)\right| \leq \exp\left(-2\pi d v_0 ||r||^2\right). \tag{4.14}$$

Let $\omega_1 = 2\pi d v_0$. Inequalities 4.11 and 4.12 just follow from the Cauchy--Schwarz inequality and the fact that w and z are such that $(\tau, w, z) \in K$ for some τ.

These together give

$$\left|e\left(\overline{r}^t T^{-1} r\tau + wr + \overline{r}^t z\right)\right| \leq \exp\left(-\omega_1 ||r||^2 + (\omega_2 + \omega_3)||r||\right) \tag{4.15}$$

for r with $||r||$ sufficiently large. If $0 \leq \omega_4 < \omega_1$ then for $||r||$ sufficiently large we have

$$-\omega_4 ||r||^2 > -\omega_1 ||r||^2 + (\omega_2 + \omega_3)||r||. \tag{4.16}$$

Let $M_r = \exp(-\omega_4 ||r||^2)$. The for all $(\tau, w, z) \in \mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n$ and $\left(r \in \mathcal{O}^{\#}\right)^n$ with $||r||$ sufficiently large we have

$$\left|e\left(\overline{r}^t T^{-1} r\tau + wr + \overline{r}^t z\right)\right| \leq M_r. \tag{4.17}$$

By the Weierstrass M-test it suffices to show that

$$\sum_{\substack{r \in (\mathcal{O}^{\#})^n \\ r \equiv s \pmod{T\mathcal{O}^n}}} M_r < \infty. \tag{4.18}$$

Let $\Lambda = \{r \in (\mathcal{O}^{\#})^n : r \equiv s \pmod{T\mathcal{O}^n}\}$. We see that Λ is a shift of a lattice in $\mathbb{C}^n$ so that, for

$$\Lambda_N = \{\lambda \in \Lambda : (N-1) \leq ||\lambda|| \leq N\} \tag{4.19}$$

we have $|\Lambda_N| \leq CN^{2n}$ for some fixed constant C and N sufficiently large. After splitting sum (4.18) into a sum over N and a sum over Λ_N we see convergence follows from the ratio test. $\square$

In order to ensure that the coefficients $\alpha_{s+T\mathcal{O}^n}(N)$ are well-defined, we need the following lemma.

Lemma 4.3. *Let r, m, s and p be such that*

$$\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}, \begin{bmatrix} p & \overline{s}^t \\ s & T \end{bmatrix} \in \Lambda_{n+1}(\mathcal{O}). \tag{4.20}$$

If $s \equiv r \pmod{T\mathcal{O}^n}$ and

$$\det\left(\begin{bmatrix} m & r \\ \overline{r}^t & T \end{bmatrix}\right) = \det\left(\begin{bmatrix} p & s \\ \overline{s}^t & T \end{bmatrix}\right)$$

then $a(m,r) = a(p,s)$

Proof. There are two useful equations for this proof. Applying Equation (2.16) with $\mu = 0$ and comparing coefficients of $e[m\tau + wr + \overline{r}^t]$ in the Fourier expansion of ϕ gives

$$a(m,r) = a(m + \lambda T\overline{\lambda}^t + r\lambda + \overline{r}^t\overline{\lambda}^t, r + T\overline{\lambda}^t) \tag{4.21}$$

for each $\lambda \in \mathcal{O}^n$.

Since T is non-singular we can write

$$\det\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right) = \det(T)(m - \overline{r}^t T^{-1} r) \tag{4.22}$$

If

$$\det(T)(m - rT^{-1}\overline{r}^t) = \det(T)(p - sT^{-1}\overline{s}^t) \tag{4.23}$$

then

$$m - rT^{-1}\overline{r}^t = p - sT^{-1}\overline{s}^t. \tag{4.24}$$

Let $\overline{\lambda}^t = T^{-1}(s-r) \in \mathcal{O}^n$. We then compute

$$\overline{r}^t\overline{\lambda}^t + \lambda r + \lambda T\overline{\lambda}^t = \overline{s}^t T^{-1} s - \overline{r}^t T^{-1} r. \tag{4.25}$$

Thus

$$a(m,r) = a(m + sT^{-1}\overline{s}^t - rT^{-1}\overline{r}^t, s) = a(p,s)$$

as desired.

$\square$

We need two more lemmas in order to prove Proposition 4.1. Both are of a similar flavor and are used in comparing the Fourier expansion of ϕ with the theta expansion.

Lemma 4.4. *Let $M \in \Lambda_n^+(\mathcal{O})$. Then $d(M) \in \mathbb{Z}_{>0}$.*

Proof. First consider the case when n is even. Then

$$d(M) = \left|\det(i\sqrt{D}M)\right|. \tag{4.26}$$

We have $\det\left(i\sqrt{D}M\right) \in \mathcal{O}$ because $i\sqrt{D}M \in M_n(\mathcal{O})$. Since $M \in \Lambda_n^+(\mathcal{O})$ we have $M = \overline{M}^t$. Thus

$$\det\left(i\sqrt{D}M\right) = \overline{\det(-i\sqrt{D}M)} = (-1)^n\overline{\det(i\sqrt{D}M)} = \overline{\det(i\sqrt{D}M)}. \tag{4.27}$$

Thus $\det\left(i\sqrt{D}M\right) \in \mathbb{R} \cap \mathcal{O} = \mathbb{Z}$ and since M is positive definite $\det(i\sqrt{D}M) \neq 0$ and $d(M) \in \mathbb{Z}_{>0}$

Now consider when n is odd. In this case we have

$$d(M) = \left|\frac{-i}{\sqrt{D}}\det(i\sqrt{D}M)\right|. \tag{4.28}$$

As in the preceding argument, $\frac{-i}{\sqrt{D}}\det(i\sqrt{D}M) \in \mathbb{R}$. We argue that $\frac{-i}{\sqrt{D}}\det(i\sqrt{D}M) \in \mathcal{O}$. Let $A = i\sqrt{D}M$. The important characteristics of A for the following are that $A = -\overline{A}^t$, $a_{ij} \in \mathcal{O}$ and $a_{ii} \in i\sqrt{D}\mathcal{O}$. Let S_n denote the symmetric group on n letters. We have

$$\det(A) = \sum_{\sigma \in S_n}(-1)^\sigma \prod a_{i,\sigma(i)}. \tag{4.29}$$

Write $S_n = P \coprod Q$ where $P = \{\sigma \in S_n : \sigma = \sigma^{-1}\}$ and $Q = S_n \backslash P$. Then

$$\det(A) = \sum_{\sigma \in P}(-1)^\sigma \prod a_{i,\sigma(i)} + \sum_{\sigma \in Q}(-1)^\sigma \prod a_{i,\sigma(i)}. \tag{4.30}$$

For $\sigma \in Q$ we have $\sigma \neq \sigma^{-1}$ so we can pair summands of index σ with summands of index σ^{-1}. We have

$$(-1)^\sigma \prod a_{i,\sigma(i)} = (-1)^{\sigma^{-1}} \prod a_{i,\sigma(i)} = -(-1)^{\sigma^{-1}} \prod \overline{a_{\sigma(i),i}} \tag{4.31}$$

$$-(-1)^{\sigma^{-1}} \prod \overline{a_{i,\sigma^{-1}(i)}}. \tag{4.32}$$

The terms in the sum over Q after being grouped in this way will be of the form $2i\mathrm{Im}(a)$ for some $a \in \mathcal{O}$. We have

$$\frac{-i}{\sqrt{D}}2i\mathrm{Im}(a) = 2\mathrm{Re}\left(\frac{-i}{\sqrt{D}}a\right) \in \mathbb{Z}. \tag{4.33}$$

Now consider the sum over P. If σ is written as a product of disjoint cycles then the order of σ will be the length of the largest cycle. We see that σ must be a product of disjoint transpositions because σ has order two. Because n is odd σ must have a fixed point. If $\sigma(i) = i$ then $a_{i,\sigma(i)} = a_{i,i} \in i\sqrt{D}\mathcal{O}$. Hence

$$\det(A) = \sum_{\sigma \in P} (-1)^\sigma \prod a_{i,\sigma(i)} \in i\sqrt{D}\mathcal{O} \tag{4.34}$$

so $\frac{-i}{\sqrt{D}} \det(i\sqrt{D}M) \in \mathcal{O} \cap \mathbb{R} = \mathbb{Z}$. Thus $d(M) \in \mathbb{Z}_{>0}$ when n is odd. $\square$

Lemma 4.5. *Let $s \in (\mathcal{O}^{\#})^n$. Then*

$$E_n(D)d(T)\overline{s}^t T^{-1} s \in \mathbb{Z}. \tag{4.35}$$

Proof. First consider when n is odd. In this case we have

$$E_n(D) = D \tag{4.36}$$

$$d(T) = \pm \frac{i}{\sqrt{D}} \det(i\sqrt{D}T). \tag{4.37}$$

We see

$$d(T)T^{-1} = \pm \det(i\sqrt{D}T)(i\sqrt{D}T^{-1}) = \operatorname{Adj}(i\sqrt{D}T) \in M_n(\mathcal{O}). \tag{4.38}$$

Then

$$E_n(D)d(T)\overline{s}T^{-1}s = \pm \overline{\left(i\sqrt{D}s\right)}^t \operatorname{Adj}(i\sqrt{D}T)(i\sqrt{D}s) \in \mathcal{O}. \tag{4.39}$$

We also have $E_n(D)d(T)\overline{s}^t T^{-1}s \in \mathbb{R}$ because T is Hermitian so $E_n(D)d(T)\overline{s}^t T^{-1}s \in \mathbb{Z}$ as desired.

Now consider when n is even. In this case we have

$$E_n(D) = 1 \tag{4.40}$$

$$d(T) = \pm \det(i\sqrt{D}T). \tag{4.41}$$

Note that the diagonal entries of $\mathrm{Adj}(i\sqrt{D}T)$ are the determinants of form $\det(i\sqrt{D}A)$ for some $A \in \Lambda_{n-1}(\mathcal{O})$. Since $n-1$ is odd we see that these diagonal entries lie in $i\sqrt{D}\mathcal{O}$, as in Equation (4.34). Thus

$$d(T)T^{-1} = \pm i\sqrt{D}\mathrm{Adj}(i\sqrt{D}T) \tag{4.42}$$

is Hermitian with diagonal entries in $D\mathcal{O}$ and off diagonal entries in $i\sqrt{D}\mathcal{O}$. We have

$$sA\overline{s}^t = \sum_{i=1}^{n} a_{ii}|s_i|^2 + \sum_{i<j} 2\mathrm{Re}[a_{ij}s_i\overline{s_j}]. \tag{4.43}$$

Since $a_{ii} \in D\mathcal{O}$ $a_{ii}|s_i|^2 \in \mathbb{Z}$. Since $a_{ij} \in i\sqrt{D}\mathcal{O}$ we have $a_{ij}s_i \in \mathcal{O}$. Thus $2\mathrm{Re}[a_{ij}s_i\overline{s_j}] \in \mathbb{Z}$ because $\overline{s_j} \in \mathcal{O}^{\#}$. Thus $d(T)sT^{-1}\overline{s}^t \in \mathbb{Z}$ as desired. □

We now prove the existence of theta expansions for Hermitian-Jacobi forms.

Proof of Proposition 4.1. We first recall the setting. We have an Hermitian Jacobi form ϕ with Fourier expansion

$$\phi(\tau, w, z) = \sum_{(m,r)\in\mathcal{S}_T} \alpha(m,r)e(m\tau + rw + \overline{r}^t z). \tag{4.44}$$

Furthermore we have for $\tau \in \mathbb{H}$ and $w, z \in \mathbb{C}^n$,

$$h_s(\tau) := \sum_{N\in I_{T,s}} \alpha_{s+T\mathcal{O}^n}(N)e\left(\frac{N}{E_n(D)d(T)}\tau\right) \tag{4.45}$$

$$\theta_{T,s}(\tau, w, z) := \sum_{\substack{r\in\left(\mathcal{O}^{\#}\right)^n \\ r\equiv s \pmod{T\mathcal{O}^n}}} e\left(\overline{r}^t T^{-1} r\tau + wr + \overline{r}^t z\right) \tag{4.46}$$

where

$$I_{s,T} := \left\{N \in \mathbb{Z}_{\geq 0} : \frac{N}{E_n(D)d(T)} + \overline{s}^t T^{-1} s \in \mathbb{Z}\right\} \tag{4.47}$$

and

$$\alpha_{s+T\mathcal{O}^n}(N) = a(m,r) \tag{4.48}$$

if $r \equiv s \pmod{T\mathcal{O}^n}$ and $N = d\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right)$.

First we argue that $N = d\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right)$ for some $\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix} \in \Lambda_{n+1}(\mathcal{O})$ with $r \equiv s \pmod{T\mathcal{O}^n}$ if and only if $N \in I_{T,s}$. First suppose $N = d\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right)$. The fact that $N \in \mathbb{Z}_{>0}$ is the content of Lemma 4.4.

Consider $\frac{N}{E_n(D)d(T)} + \overline{s}^t T^{-1} s$. We have

$$\frac{N}{E_n(D)d(T)} + \overline{s}^t T^{-1} s = m - \overline{r}^t T^{-1} r + \overline{s}^t T^{-1} s \tag{4.49}$$

by Equation (4.22). An application of Lemma 3.2 shows that if $r \equiv s \pmod{T\mathcal{O}^n}$

$$\overline{s}^t T^{-1} s - \overline{r}^t T^{-1} r \in \mathbb{Z}. \tag{4.50}$$

We then see that $N \in I_{T,s}$.

Next suppose $N \in I_{T,s}$ then take $m = m(N,r) := \frac{N}{E_n(D)d(T)} + \overline{r}^t T^{-1} r$ for any $r \equiv s \pmod{T\mathcal{O}^n}$ so that $N = d\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right)$. To see that $\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix} > 0$ use that, for $\begin{bmatrix} x_1 \\ x \end{bmatrix} \in \mathbb{C} \times \mathbb{C}^n$ we have

$$\begin{bmatrix} \overline{x_1} & \overline{x}^t \end{bmatrix} \begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix} \begin{bmatrix} x_1 \\ x \end{bmatrix} = |x_1|^2 \left(\frac{N}{E_n(D)d(T)}\right) + \overline{(x + T^{-1}x_1 r)}^t T(x + T^{-1}x_1 r) > 0 \tag{4.51}$$

as long as $\begin{bmatrix} x_1 \\ x \end{bmatrix} \neq 0$. To get

$$\sum_{s \in (\mathcal{O}^{\#})^n / T\mathcal{O}} \theta_s h_s = \phi(\tau, w, z) \tag{4.52}$$

write out both sums and use the argument above to realize $(s, r, N) \mapsto (m(N, r), r)$ gives a bijection between the two indexing sets. $\square$

We will see that the functions h_s are classical modular forms and, since the Fourier coefficients of h_s are directly related to those of ϕ_T (and hence to F), we can use our knowledge of classical modular forms to gain information about Hermitian modular forms. We explain this process in more detail in Chapter 6.

4.2 The Transformation of Theta Functions

In order to prove that the h_s are modular forms we will prove some transformation laws for our theta functions and then use the transformation law satisfied by our Hermitian Jacobi forms to translate these into transformation laws satisfied by the h_s. We introduce some notation.

We then define a character $\chi_D : \mathbb{Z} \to \mathbb{C}$ by the Kronecker symbol

$$\chi_D(q) := \left(\frac{D}{q}\right). \tag{4.53}$$

This is a character modulo D because D is a fundamental discriminant. Recall the group

$$\Gamma_0(r) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{r} \right\}. \tag{4.54}$$

The primary result of this section is as follows:

Proposition 4.6. *For the theta function $\theta_{T,s}$ and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0\left(E_n(D)d(T)\right)$ we have*

$$\theta_{T,s}|_{T,n}MJ = \frac{1}{\det(-i\sqrt{D}T)}\chi_D(d)^n \sum_{s' \in \left(\mathcal{O}^\#\right)^n/T\mathcal{O}^n} e\left[a\left(2Re[\overline{s}^tT^{-1}s'] + b\overline{s}^tT^{-1}s\right)\right]\theta_{T,s'}. \tag{4.55}$$

We follow Haverkamp [Hav95, Satz 4.5] closely for this proof. The steps are as follows.

1. Introduce auxiliary functions ψ and ϕ and prove these satisfy two useful transformations by computing some related exponential integrals.

2. Relate these two functions to our theta functions using Fourier expansions.

3. Translate the transformation laws of ψ and ϕ into those of the theta functions.

4. Relate a Gauss-like sum that appears to our character χ_D.

First we'll introduce these auxiliary functions and prove two transformation laws for ψ and ϕ.

Definition 4.7. Let $s, \sigma, w, z \in \mathbb{C}^n$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(E_n(D)d(T))$ and $\tau \in \mathbb{H}$. Again, we consider w as a row vector and s, σ and z as column vectors. Define

$$\phi(w, z) = e[wT^{-1}z(a\tau + b)a] \tag{4.56}$$

$$\psi_{s,\sigma}(w, z) = \sum_{r \equiv s \pmod{\mathcal{O}^n}} e\left[\overline{r}^tTr\tau + (wr + \overline{r}^tz)(a\tau + b) + 2\mathrm{Re}[\overline{\sigma}^tr]\right]. \tag{4.57}$$

Similarly to Proposition 4.2 $\psi_{s,\sigma}$ is given by a series that converges absolutely and uniformly on compact subsets of $\mathbb{C}^n \times \mathbb{C}^n$.

Remark 4.8. We assume that $a \neq 0$. If $a = 0$ then $c = \pm 1$ so $E_n(D)d(T) = 1$. We see $E_n(D) = 1$ so that n is even and $\Gamma_0(E_n(D)d(T)) = \mathrm{SL}_2(\mathbb{Z})$. If we prove for any $M \in \mathrm{SL}_2(\mathbb{Z})$ with $a \neq 0$ then we see that for any $M \in \mathrm{SL}_2(\mathbb{Z})$ with $a \neq 0$ we have

$$\theta_{T,s}|_{T,n} MJ = \theta_{T,s}. \tag{4.58}$$

Since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} J^{-1}$ and JJ^{-1} have non-zero (1,1) entry we see that $\theta_{T,s}|_{T,n} A = \theta_{T,s}$ for every $A \in \mathrm{SL}_2(\mathbb{Z})$. In particular we have this equation for MJ when $a = 0$. Thus we can assume $a \neq 0$ from here on out.

Now we prove the transformation laws for ϕ and ψ.

Lemma 4.9. *Let $\alpha \in T\mathcal{O}^n$ and $\beta \in (\mathcal{O}^{\#})^n$. Then we have*

$$\phi(w + \overline{\alpha}^t, z + \alpha)\psi_{s,\sigma}(w + \overline{\alpha}^t, z + \alpha) = e\left[2Re[b\overline{s}^t\alpha - a\overline{\sigma}^t T^{-1}\alpha]\right]\phi(w, z)\psi_{s,\sigma}(w, z) \tag{4.59}$$

and

$$\phi(w + \overline{\beta}^t(JM\tau), z + \beta(JM\tau))\psi(w + \overline{\beta}^t(JM\tau), z + \beta(JM\tau)) = \tag{4.60}$$

$$e\left[-\left(\overline{\beta}^t T^{-1}\beta JM\tau + wT^{-1}\beta + \overline{\beta}^t T^{-1}z + 2Re((d\overline{s}^t\beta - c\overline{\sigma}^t T^{-1}\beta))\right)\right]\phi(w, z)\psi(w, z). \tag{4.61}$$

Proof. First we consider the proof of Equation (4.59). We have

$$\phi(w, z)\psi_{s,\sigma}(w, z) \tag{4.62}$$

$$= \sum_{r \equiv s \pmod{\mathcal{O}^n}} e\left[wT^{-1}z(a\tau + b)a + \overline{r}^t Tr\tau + (wr + \overline{r}^t z)(a\tau + b) + 2\mathrm{Re}[\overline{\sigma}^t r]\right]. \tag{4.63}$$

Note that we can replace r in the above sum by $r + aT^{-1}\alpha$ and it will leave the whole sum unchanged because $aT^{-1}\alpha \in \mathcal{O}^n$. If we add and subtract $b(\overline{\alpha}^t r + \overline{r}^t\alpha + a\overline{\alpha}^t T^{-1}\alpha)$ within our exponential then our rth term will be

$$e\left[wT^{-1}z(a\tau+b)a + \overline{\alpha}^tT^{-1}\alpha(a\tau+b)a + (wT^{-1}\alpha + \overline{\alpha}^tT^{-1}z)(a)(a\tau+b)\right] \tag{4.64}$$

$$\cdot e\left[\overline{r}^tTr\tau + (wr+\overline{r}^tz)(a\tau+b) + \overline{r}^t\alpha(a\tau+b) + \overline{\alpha}^tr(a\tau+b) + 2\mathrm{Re}[\overline{\sigma}^tr]\right] \tag{4.65}$$

$$\cdot e[2\mathrm{Re}[\overline{\sigma}^t(aT^{-1}\alpha) - b\overline{r}^t\alpha]] \cdot e[-ab\overline{\alpha}^tT^{-1}\alpha]. \tag{4.66}$$

Since $\alpha \in T\mathcal{O}^n$ we have $\overline{\alpha}^tT^{-1}\alpha \in \mathbb{Z}$ so this last term is just 1. Since $r \equiv s \pmod{\mathcal{O}^n}$ we have

$$e[2\mathrm{Re}[b\overline{r}^t\alpha]] = e[2\mathrm{Re}[b\overline{s}^t\alpha]]. \tag{4.67}$$

Hence our sum for $\phi \cdot \psi$ has rth term

$$e\left[(w+\overline{\alpha}^t)T^{-1}(z+\alpha)(a\tau+b)a\right] \cdot e\left[\overline{r}^tTr\tau + \left((w+\overline{\alpha}^t)r + \overline{r}^t(z+\alpha)\right)(a\tau+b) + 2\mathrm{Re}\left[\overline{\sigma}^tr\right]\right] \tag{4.68}$$

$$\cdot e\left[2\mathrm{Re}[\overline{\sigma}^t(aT^{-1}\alpha) - b\overline{s}^t\alpha]\right]. \tag{4.69}$$

From this our first transformation law follows.

Now we consider the second of these transformation laws. To prove this we make another change of variables in our sum $\phi(w,z)\psi(w,z)$. Note that

$$d(T)T^{-1} = \pm\det\left(i\sqrt{D}T\right)(i\sqrt{D}T)^{-1} \text{ or } \pm i\sqrt{D}\det\left(i\sqrt{D}T\right)(i\sqrt{D}T)^{-1} \tag{4.70}$$

depending on whether n is odd or even. In any event we see that $E_n(D)d(T)T^{-1}\beta \in \mathcal{O}^n$. Thus $cT^{-1}\beta \in \mathcal{O}^n$. We exchange r for $r - cT^{-1}\beta$.

After this change of variables we proceed by direct comparison of the sums in Expressions 4.60 and 4.61. Since $JM\tau = \frac{-c\tau-d}{a\tau+b}$, it suffices to show

$$(\overline{\beta}^t r + a(\overline{\beta}^t T^{-1}z + wT^{-1}\beta + \overline{\beta}^t T^{-1}\beta(JM\tau)) + \overline{r}^t\beta))(-c\tau - d) \tag{4.71}$$

$$\equiv (-c(\overline{\beta}^t T^{-1}\beta) + \overline{r}^t\beta + \overline{\beta}^t r + awT^{-1}\beta + a\overline{\beta}^t T^{-1}z)(-c\tau) + b(wT^{-1}\beta(-c) + (-c)(\overline{\beta}^t T^{-1}z)) \tag{4.72}$$

$$-d(\overline{r}^t\beta + \overline{\beta}^t r) - \overline{\beta}^t T^{-1}z - wT^{-1}\beta - \overline{\beta}^t T^{-1}\beta JM\tau \pmod{\mathbb{Z}} \tag{4.73}$$

The difference of these two modulo $\mathbb{Z}$ is $dc\overline{\beta}^t T^{-1}\beta$ which, by Lemma 4.5, is an integer.

□

Next our goal is to relate these functions ψ and ϕ to our theta functions. The basic idea is to study the Fourier expansion of the product and compute the Fourier coefficients. From here we can use a change of variables to relate our function ψ to θ.

Lemma 4.10. *Let $s' = bTs - a\sigma$ and $\sigma' = dTs - c\sigma$. Then we have*

$$e\left[\overline{s'}^t T^{-1}s' JM\tau + 2Re[\overline{s'}T^{-1}\sigma']\right]\phi(w,z)\cdot\psi_{s,\sigma}(w,z) \tag{4.74}$$

$$= \gamma(0)\sum_{g\in(\mathcal{O}\#)^n} e\left[\overline{(s'+g)}^t T^{-1}(s'+g)JM\tau + wT^{-1}(s'+g) + \overline{(g+s')}^t T^{-1}z + 2Re[\overline{(s'+g)}^t T^{-1}\sigma'\right] \tag{4.75}$$

for some constant $\gamma(0)$.

Proof. Let

$$F(w,z) := \phi(w,z)\psi_{s,\sigma}(w,z). \tag{4.76}$$

We follow the following steps:

1. Define a new function $\widetilde{F}$ and check the transformation

$$\widetilde{F}(w+\overline{\alpha}^t, z+\alpha) = \widetilde{F}(w,z)$$

for all $\alpha \in T\mathcal{O}^n$.

2. Write out the Fourier expansion of $\widetilde{F}$ and translate this to a series expansion of F.

3. Evaluate $F(w+\overline{\beta}^t JM\tau, z+\beta JM\tau)$ in two ways. First evaluate directly, using step (2).

4. Use the equality of 4.60 and 4.61 $F(w+\overline{\beta}^t JM\tau, z+\beta JM\tau)$ and perform the change of variables $g \mapsto g+\beta$ on the sum.

5. Compare the series in (3) and (4) and plug in $g=0$ to get a formula for the coefficients.

6. Use (5) and the series expansion of F to prove the equality of Expressions 4.90 and 4.91.

Step 1: Let $F(w,z) = \phi(w,z)\psi(w,z)$. Define

$$\widetilde{F}(w,z) = e\left[-(wTs' + \overline{s'}^t Tz)\right] F(w,z). \tag{4.77}$$

The transformation law for $\widetilde{F}$ follows from Equation (4.59).

Step 2: We know $\widetilde{F}$ is holomorphic on $\mathbb{C}^n \times \mathbb{C}^n$ because both ϕ and ψ are. By Step 1 we know that $\widetilde{F}$ is periodic with respect to the lattice

$$\left\{(\overline{\alpha}^t, \alpha) : \alpha \in T\mathcal{O}^n\right\}. \tag{4.78}$$

Because $\widetilde{F}$ is holomorphic $\widetilde{F}$ will have a Fourier expansion over the dual lattice of the form

$$\widetilde{F}(w,z) = \sum_{g \in T^{-1}(\mathcal{O}\#)^n} \gamma(Tg) e[wg + \overline{g}^t z]. \tag{4.79}$$

Then, after the change of variables $g \mapsto T^{-1}g$, we have

$$F(w,z) = \sum_{g \in (\mathcal{O}\#)^n} \gamma(g) e\left[wT^{-1}(s'+g) + \overline{(s'+g)}^t T^{-1} z\right]. \tag{4.80}$$

Step 3: Let $\beta \in \left(\mathcal{O}\#\right)^n$. Plugging in $(w + \overline{\beta}^t JM\tau, z + \beta JM\tau)$ gives

$$F(w + \overline{\beta}^t JM\tau, z + \beta JM\tau) \tag{4.81}$$

$$= \sum_{g \in \left(\mathcal{O}\#\right)^n} \gamma(g) e\left[2\mathrm{Re}[\overline{(s'+g)}^t T^{-1}\beta] JM\tau\right] e\left[wT^{-1}(s'+g) + \overline{(s'+g)}^t T^{-1} z\right]. \tag{4.82}$$

Step 4: Note

$$d\overline{s}^t \beta - c\overline{\sigma}^t T^{-1}\beta = \overline{\sigma'}^t T^{-1}\beta. \tag{4.83}$$

By Equation (4.59) we have,

$$F(w + \overline{\beta}^t JM\tau, z + \beta JM\tau) = e\left[-\left(\overline{\beta}^t T^{-1}\beta JM\tau + wT^{-1}\beta + \overline{\beta}^t T^{-1} z + 2\mathrm{Re}[\overline{\sigma'}^t T^{-1}\beta]\right)\right] F(w,z). \tag{4.84}$$

Using the series expansion of F and replacing the index g by $g+\beta$ gives

$$F(w + \overline{\beta}^t JM\tau, z + \beta JM\tau) \tag{4.85}$$

$$= \sum_{g \in (\mathcal{O}\#)^n} \gamma(g+\beta) e\left[-\left(\overline{\beta}^t T^{-1}\beta JM\tau + 2\mathrm{Re}[\overline{\sigma'}^t T^{-1}\beta]\right)\right] e\left[wT^{-1}(s'+g) + \overline{(s'+g)}^t T^{-1} z\right]. \tag{4.86}$$

Step 5: Comparing 4.82 and 4.86 we see

$$\sum_{g\in\left(\mathcal{O}\#\right)^n} \gamma(g)e\left[2\mathrm{Re}\left[\overline{(s'+g)}^t T^{-1}\beta\right] JM\tau\right] e\left[wT^{-1}g+\overline{g}^t T^{-1}z\right] \quad (4.87)$$

$$= \sum_{g\in\left(\mathcal{O}\#\right)^n} \gamma(g+\beta)e\left[-\overline{\beta}^t T^{-1}\beta JM\tau - 2\mathrm{Re}[\overline{\sigma'}^t T^{-1}\beta]\right] e\left[wT^{-1}g+\overline{g}^t T^{-1}z\right]. \quad (4.88)$$

If we compare the $g=0$ coefficients then we get

$$\gamma(\beta) = \gamma(0)e\left[\left(\overline{\beta}^t T^{-1}\beta + 2\mathrm{Re}[\overline{s'}^t T^{-1}\beta]\right) JM\tau + 2\mathrm{Re}[\overline{\sigma'}^t T^{-1}\beta]\right]. \quad (4.89)$$

Step 6: If we combine Equation (4.89) with 4.80 we get

$$e\left[\overline{s'}^t T^{-1}s' JM\tau + 2\mathrm{Re}[\overline{s'} T^{-1}\sigma']\right] \phi(w,z)\cdot\psi_{s,\sigma}(w,z) \quad (4.90)$$

$$= \gamma(0) \sum_{g\in\left(\mathcal{O}\#\right)^n} e\left[\overline{(s'+g)}^t T^{-1}(s'+g)JM\tau + wT^{-1}(s'+g) + \overline{(g+s')}^t T^{-1}z + 2\mathrm{Re}[\overline{(s'+g)}^t T^{-1}\sigma'\right] \quad (4.91)$$

where

$$\sigma' = dTs - c\sigma \text{ and } s' = bTs - a\sigma. \quad (4.92)$$

as desired. □

Let

$$\eta = \eta(\tau, s, \sigma) = e\left[-\left(\overline{s'}^t T^{-1}s' JM\tau + 2\mathrm{Re}[\overline{s'}^t T^{-1}\sigma']\right)\right]\gamma(0). \quad (4.93)$$

Lemma 4.10 then becomes

$$\phi(w,z)\cdot\psi_{s,\sigma}(w,z) \quad (4.94)$$

$$= \eta \sum_{g\in\left(\mathcal{O}\#\right)^n} e\left[\overline{(s'+g)}^t T^{-1}(s'+g)JM\tau + wT^{-1}(s'+g) + \overline{(g+s')}^t T^{-1}z + 2\mathrm{Re}[\overline{(s'+g)}^t T^{-1}\sigma'\right] \quad (4.95)$$

In the next lemma we compute the value of η. In the process we'll run into the following Gauss sum.

Definition 4.11. For T and $a \in \mathbb{Z}$ non-zero , define

$$G(T,a) = \sum_{s \in \mathcal{O}^n / a\mathcal{O}^n} e\left[\frac{\overline{s}^t T s}{a}\right].$$

We'll also need the following lemma from Haverkamp's thesis [Hav95]:

Lemma 4.12. *For $c \in \mathbb{R}$ and $r, q \in \mathbb{C}$ with $Im(q) < 0$ we have*

$$\int_{Im(z)=c} e\left[-qz^2 + rz\right] dz = (2iq)^{-1/2} e\left[r^2/4q\right] \tag{4.96}$$

Lemma 4.13. *We have*

$$\eta = \frac{1}{\det\left(-i\sqrt{D}T\right)} \frac{1}{\left(a(a\tau+b)\right)^n} \overline{G(bT,a)} e\left[-bd\overline{s}^t T s - ac\overline{\sigma}^t T^{-1} \sigma + 2adRe[\overline{\sigma}^t T s]\right]. \tag{4.97}$$

Proof. We start by multiplying both sides of equation from 4.10 by

$$e\left[-\left(\overline{s'}^t T^{-1} s J M \tau + w T^{-1} s' + \overline{s'}^t T^{-1} z\right)\right].$$

We write the left hand side as

$$\sum_{q \in \mathcal{O}^n} e[R(q,w,z)]$$

where

$$R(q,w,z) = wT^{-1}z(a\tau+b)(a) + \overline{(s+q)}^t T(s+q)\tau + \left(w(s+q) + \overline{(s+q)}^t z\right)(a\tau+b) \tag{4.98}$$

$$+2\mathrm{Re}[\overline{\sigma}^t(s+q)] - \overline{s'}^t T^{-1} s' J M \tau - w T^{-1} s' - \overline{s'}^t T^{-1} z. \tag{4.99}$$

The right hand side will be

$$\eta \sum_{p \equiv s' \pmod{\left(\mathcal{O}\#\right)^n}} e\left[\left(\overline{p}^t T^{-1} p - \overline{s'}^t T^{-1} s'\right) J M \tau + (w T^{-1}(p-s') + \overline{(}p-s')^t T^{-1} z + 2\mathrm{Re}[\overline{p}^t T^{-1} \sigma']\right].$$

There are two transformation laws that R satisfies: Let $g, h \in \mathbb{C}^n$

1.

$$R(ag+h,w,z) = R(h,w,z)+\left(a^2\overline{g}^tTg+2\mathrm{Re}[\overline{(ag)}^tT(s+h)]\right)\tau a(wg+\overline{g}^tz)(a\tau+b)+2a\mathrm{Re}[\overline{\sigma}^tg].$$

2.

$$R(h,w+\overline{g}^tT,z+Tg) = R(h,w,z)+2a\mathrm{Re}[\overline{\sigma}^tg]+(\overline{g}^tz+wg+\overline{g}^tTg)(a)(a\tau+b)+2a\mathrm{Re}[\overline{g}^tT(s+b)]\tau.$$

Both of these follow from direct computation.

From (1) and (2) above together we see that, for $g,h \in \mathcal{O}^n$

$$R(ag+h,w,z) \equiv R(h,w+\overline{g}^tT,z+Tg) \pmod{\mathbb{Z}}.$$

In order to compute η we set $z = \overline{w}^t$ and integrate both the left and right hand sides over $\mathbb{C}^n/(\mathcal{O}^nT)$. We first consider the right side.

The right side is invariant under translation of w by $\alpha \in \mathcal{O}^nT$. Our integral is

$$\eta \sum_{p\equiv s' \pmod{(\mathcal{O}\#)^n}} I(p) \tag{4.100}$$

where

$$I(p,s') := \int_{\mathbb{C}^n/\mathcal{O}^nT} e\left[(wT^{-1}(p-s')+\overline{(}p-s')^tT^{-1}\overline{w}^t+2\mathrm{Re}[\overline{p}^tT^{-1}\sigma']\right]dw. \tag{4.101}$$

Note that if we substitute $w+\mathfrak{z}$ for w we get

$$I(p,s') = e\left[2\mathrm{Re}[(p-s')T^{-1}\overline{\mathfrak{z}}^t]\right]I(p,s') \tag{4.102}$$

for any $\mathfrak{z} \in \mathbb{C}^n$. Hence $I(p,s') = 0$ unless $p = s'$. After plugging in $p = s'$ our right hand side becomes.

$$\eta e\left[2\mathrm{Re}[\overline{s'}^tT^{-1}\sigma']\right]\mathrm{vol}\left(\mathcal{O}^nT\right). \tag{4.103}$$

We have

$$\mathrm{vol}\left(\mathcal{O}^n T\right) = \det(T)^2 \mathrm{vol}(\mathcal{O}^n) = \det(T)^2 \left(\frac{\sqrt{D}}{2}\right)^n. \tag{4.104}$$

Thus our right hand side is

$$\eta \det(T)^2 \left(\frac{\sqrt{D}}{2}\right)^n e\left[2\mathrm{Re}[\overline{s'}^t T^{-1}\sigma']\right] \tag{4.105}$$

Next we compute the left hand side. We have

$$\int_{\mathbb{C}^n/\mathcal{O}^n T} \sum_{p\in\mathcal{O}^n} e[R(t,w,\overline{w}^t)] = \int_{\mathbb{C}/\mathcal{O}^n T} \sum_{h\in\mathcal{O}^n/a\mathcal{O}} e\left[R(h, w+\overline{g}^t T, \overline{w}^t + Tg)\right] \tag{4.106}$$

$$= \sum_{h\in\mathcal{O}^n/a\mathcal{O}^n} \int_{\mathbb{C}} e[R(h,w,\overline{w}^t)] \tag{4.107}$$

$$= \sum_{h\in\mathcal{O}^n/a\mathcal{O}^n} \int_{\mathbb{C}} e\left[R\left(h, w - \frac{\overline{(h+s)}^t T}{a}, \overline{w}^t - \frac{T(h+s)}{a}\right)\right] dw \tag{4.108}$$

Recall that $a \neq 0$. Now we need to compute this integral. After expanding R and simplifying the resulting expression breaks into two pieces:

$$\int_{\mathbb{C}} e\left[wT^{-1}\overline{w}^t(a)(a\tau+b) - 2\mathrm{Re}[wT^{-1}s']\right] dw \tag{4.109}$$

$$\sum_{h\in\mathcal{O}^n/a\mathcal{O}^n} e\left[2\mathrm{Re}[\overline{\sigma}^t(s+h)] - \overline{s'}^t T^{-1}s'JM\tau + \frac{2}{a}\mathrm{Re}[\overline{s'}(h+s)] - \frac{\overline{(s+h)}^t T(s+h)}{a} b\right]. \tag{4.110}$$

First we'll simplify (4.110). Our sum in (4.110) is

$$e\left[\frac{b}{a}\overline{s}^t T s - \overline{s'}^t T^{-1} s' JM\tau\right] \overline{G(bT,a)}. \tag{4.111}$$

Now we consider (4.109). Since T is Hermitian we can find another Hermitian matrix G such that $G^2 = T^{-1}$. Let $v = wG$ and $u = Gs'$. Our integral is, after making the substitution $w \mapsto wG^{-1}$

$$\det(T) \int_{\mathbb{C}^n} e\left[v\overline{v}^t(a)(a\tau+b) - 2\mathrm{Re}[vu]\right] dv. \tag{4.112}$$

If we break this integral into components and write v and u for v_i and u_i respectively we'll find

$$\left[\int_{\mathbb{C}} e\left[(v)^2(a)(a\tau+b) - 2\mathrm{Re}[vu]\right] dv\right]^n. \tag{4.113}$$

Let $v = x + iy$ and $u = p + iq$ so that our integral split into

$$\int_{\mathbb{R}} e\left[x^2(a)(a\tau+b) - 2xp\right] dx \cdot \int_{\mathbb{R}} e\left[y^2(a)(a\tau+b) - 2yq\right] dy. \tag{4.114}$$

We view $\mathbb{R} = \{z \in \mathbb{C} : \mathrm{Im}(z) = 0\}$ so we can apply Lemma 2.6. (b) from Haverkamp's thesis, Lemma 4.12 in this paper, to each of the above. We get that the above integral is equal to

$$\frac{i}{2(a)(a\tau+b)} e\left[-\frac{p^2+q^2}{a(a\tau+b)}\right]. \tag{4.115}$$

In total we find that our integral in (4.110) is equal to

$$\det(T)\left(\frac{i}{2(a)(a\tau+b)}\right)^n e\left[-\frac{1}{(a)(a\tau+b)}\overline{s'}^t T^{-1} s'\right]. \tag{4.116}$$

Combining 4.109 and 4.110 give the expression for the left hand side as

$$e\left[bd\overline{s}^t Ts - 2cb\mathrm{Re}[\overline{s}^t\sigma] + ac\overline{\sigma}^t T^{-1}\sigma\right] G(bT, a)\det(T)\left(\frac{i}{2a(a\tau+b)}\right)^n. \tag{4.117}$$

Setting expressions (4.116) and (4.105) equal and solving for η gives

$$\eta = \frac{1}{\det(T)}\left(\frac{i}{\sqrt{D}(a)(a\tau+b)}\right)^n \overline{G(bT,a)} e\left[-bd\overline{s}Ts - ac\overline{\sigma}^t T^{-1}\sigma + 2ad\mathrm{Re}[\overline{\sigma}^t Ts]\right]. \tag{4.118}$$

A slight simplification of the above gives the desired result. $\square$

Using this we can get a transformation for our theta function which, after analyzing the sum G will be that given in Proposition 4.8.

Proposition 4.14. *We have, for $M \in \Gamma_0(E_n(D)d(T))$,*

$$\theta_{T,s}|_{T,n}(MJ) = \tag{4.119}$$

$$\frac{1}{\det\left(-i\sqrt{D}T\right)} \cdot \frac{1}{d^n} G(bT,d) \sum_{s' \in \left(\mathcal{O}\#\right)^n / T\mathcal{O}^n} e\left[a\left(2Re[\overline{s}^t T^{-1} s'] + b\overline{s}^t T^{-1} s\right)\right] \cdot \theta_{T,s'}. \tag{4.120}$$

Proof. Plugging in our value for η to Equation (4.10) gives

$$\psi_{T^{-1}s,0}\left(\frac{wT}{a\tau+b}, \frac{Tz}{a\tau+b}\right) = \tag{4.121}$$

$$\frac{1}{\det(-i\sqrt{D}T)} \cdot \frac{1}{a^n}\overline{G(bT,a)} \sum_{s \in \left(\mathcal{O}\#\right)^n / T\mathcal{O}^n} e\left[d\left(2\mathrm{Re}[\overline{s}^t T^{-1} s'] - b\overline{s}^t T^{-1} s\right)\right] \cdot \theta_{m,s'}|_{T,n}[-JM]. \tag{4.122}$$

Recall we have

$$\psi_{s,\sigma}(w,z) := \sum_{r \equiv s \pmod{\mathcal{O}^n}} e\left[\overline{r}^t T r \tau + (wr + \overline{r}^t z)(a\tau+b) + 2\mathrm{Re}[\overline{\sigma}^t r]\right]. \tag{4.123}$$

Using this and the change of variables $r \mapsto T^{-1}r$ gives

$$\psi_{T^{-1}s,0}\left(\frac{wT}{a\tau+b}, \frac{Tz}{a\tau+b}\right) = \theta_{T,s}(\tau,w,z). \tag{4.124}$$

After slashing both sides by $M^{-1}J$ and replacing M^{-1} by M we get, for any $M \in \Gamma_0(E_n(D)d(T))$

$$\theta_{T,s}|_{T,n}(MJ) = \tag{4.125}$$

$$\frac{1}{\det\left(-i\sqrt{D}T\right)} \cdot \frac{1}{d^n}\overline{G(-bT,d)} \sum_{s' \in \left(\mathcal{O}\#\right)^n / T\mathcal{O}^n} e\left[a\left(2\mathrm{Re}[\overline{s}^t T^{-1} s'] + b\overline{s}^t T^{-1} s\right)\right] \cdot \theta_{T,s'}. \tag{4.126}$$

$\square$

Our goal over the next few results is to show that $\frac{G(bT,d)}{d^n}$ is a character modulo $E_n(D)d(T)$ and that it does not depend on b. First we show that it is a group homomorphism from the group $\Gamma_0(E_n(D)d(T))$ to $\mathbb{C}^\times$. In the process we'll need to compute some exponential sums that appear, for which we need the following lemmas.

Lemma 4.15. *Let $A \in M_n(\mathcal{O})$. Then $[\mathcal{O}^n : A\mathcal{O}^n] = Nm(\det(A))$.*

Proof. See [Cla14, Lemma 1.15] for a proof of this claim. Here $R = \mathcal{O}$, $M = T$ and $\Lambda = \mathcal{O}$. $\square$

Lemma 4.16. *We have*

$$\sum_{s\in\left(\mathcal{O}^{\#}\right)^n/T\mathcal{O}^n} e\left[2Re\left[\overline{s}^t T^{-1}s'\right]\right] = \begin{cases} 0 & \text{if } s' \neq 0 \\ Nm(\det(i\sqrt{D}T)) & \text{if } s' = 0. \end{cases} \tag{4.127}$$

Proof. Suppose that $s' \neq 0$. I claim $2\mathrm{Re}(\overline{s_0}T^{-1}s') \notin \mathbb{Z}$ for some s_0. Since we know $s' \notin T\mathcal{O}^n$ we must have $T^{-1}(s') \notin \mathcal{O}^n$. We can find $s_0 \in (\mathcal{O}^{\#})^n$ such that $2\mathrm{Re}[\overline{s_0}^t T^{-1}s'] \notin \mathbb{Z}^n$ because $\mathcal{O}$ is the dual lattice of $\mathcal{O}^{\#}$. With this choice of s_0 in mind we have

$$e\left[2\mathrm{Re}(\overline{s_0}T^{-1}s')\right] \sum_{s\in\left(\mathcal{O}^{\#}\right)^n/T\mathcal{O}^n} e\left[2\mathrm{Re}(\overline{s}T^{-1}s')\right] \tag{4.128}$$

$$= \sum_{s\in\left(\mathcal{O}^{\#}\right)^n/T\mathcal{O}^n} e\left[2\mathrm{Re}(\overline{s'}T^{-1}(s+s_0))\right] \tag{4.129}$$

$$= \sum_{s\in\left(\mathcal{O}^{\#}\right)^n/T\mathcal{O}^n} e\left[2\mathrm{Re}(\overline{s}T^{-1}s')\right]. \tag{4.130}$$

Since $e\left[2\mathrm{Re}(\overline{s_0}T^{-1}s')\right] \neq 1$ we must have that the sum is zero. If $s' = 0$ then the result follows from Lemma 4.15. $\square$

Lemma 4.17. *For $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(E_n(D)d(T))$ let*

$$\chi(M) = \frac{G(bT, d)}{d^n}. \tag{4.131}$$

Then $\chi(MM') = \chi(M)\chi(M')$.

Proof. This result follows by applying Proposition 4.14 to $\theta_{T,0}$. By Proposition 4.14 we have

$$\theta_{T,0}|_{T,n}MJ = \frac{1}{\det(-i\sqrt{D}T)}\chi(M) \sum_{s' \in \left(\mathcal{O}^{\#}\right)^n / T\mathcal{O}^n} \theta_{T,s'}. \tag{4.132}$$

Thus

$$\theta_{T,0}|_{T,n}M = \frac{1}{\det(-i\sqrt{D}T)}\chi(M) \sum_{s'} \theta_{T,s'}|_{T,n}(-J). \tag{4.133}$$

Now again by Proposition 4.14 we have

$$\theta_{T,s'}|_{T,n}(-J) = \frac{1}{\det(-i\sqrt{D}T)}(-1)^n G(-T,-1) \sum_{s''} e\left[-2\mathrm{Re}[s''T^{-1}s']\right]\theta_{T,s''}. \tag{4.134}$$

Direct computation from the definition gives $G(-T,-1) = 1$. In total we have

$$\theta_{T,0}|M = \frac{(-1)^n}{\det(-i\sqrt{D}T)^2}\chi(M) \sum_{s'}\sum_{s''} e\left[-2\mathrm{Re}[\overline{s'}^t T^{-1}s'']\right]\theta_{T,s''}. \tag{4.135}$$

By Lemma 4.16 we have

$$\theta_{T,0}|M = \frac{(-1)^n}{\det(-i\sqrt{D}T)^2}\chi(M)\mathrm{Nm}(\det(i\sqrt{D}T))\theta_{T,0} \tag{4.136}$$

$$= \chi(M)\theta_{T,0}. \tag{4.137}$$

Since $\theta_{T,0} \neq 0$ and the slash operator is multiplicative the desired result follows. $\square$

Next we want to compute $\chi(M)$ in the case that d is coprime to $2Dd(T)$. The idea is to try and diagonalize bT modulo d and then reduce to the one dimensional case, which is solved by Haverkamp.

Lemma 4.18. *We have for d coprime to $2Dd(T)$ and b coprime to d,*

$$G(bT,d)/d^n = (\chi_D(d))^n \tag{4.138}$$

Proof. Recall

$$G(T,a) = \sum_{s \in \mathcal{O}^n/a\mathcal{O}^n} e\left[\frac{\overline{s}^t T s}{a}\right]. \tag{4.139}$$

We first reduce to the case when $d = p^r$. We show that for coprime integers p and q we have

$$G(bT, pq) = G(pT, q)G(qT, p). \tag{4.140}$$

Note that $(x, y) \mapsto px + qy$ gives a bijection from $\mathcal{O}^n/p\mathcal{O}^n \times \mathcal{O}^n/q\mathcal{O}^n$ to $\mathcal{O}^n/pq\mathcal{O}^n$. Thus

$$G(T,pq) = \sum_{s \in \mathcal{O}^n/pq\mathcal{O}^n} e\left[\frac{\overline{s}^t T s}{pq}\right] \tag{4.141}$$

$$= \sum_{x \in \mathcal{O}^n/p} \sum_{y \in \mathcal{O}^n/q} e\left[\frac{\overline{x}^t (qT) x}{p} + \frac{\overline{y}^t (pT) y}{q} + 2\mathrm{Re}[\overline{x}^t T y]\right]. \tag{4.142}$$

The result then follows from the fact that, since $T \in \Lambda_n(\mathcal{O})$ and $x, y \in \mathcal{O}^n$ we have

$$2\mathrm{Re}[\overline{x}^t T y] \in \mathbb{Z} \tag{4.143}$$

by Lemma 3.2. We can then reduce to the case when $d = p^r$ by splitting d into prime factors in this way.

Since d is coprime to D we see that $i\sqrt{D}$ is an invertible element of $\mathcal{O}/d\mathcal{O}$. If T does not have entries in $\mathcal{O}$ then we can replace T by DT by replacing s with $i\sqrt{D}s$ in our sum for G. We assume that T has entries in $\mathcal{O}$.

Next I claim that we can find $P \in \mathrm{GL}_n(\mathcal{O}/d\mathcal{O})$ such that $\overline{P}^t T P$ is diagonal modulo d. We prove the result by induction on the dimension n. The $n = 1$ case is clear. Now suppose that we know that for any $n \times n$ half-integral Hermitian matrix $R \in M_n(\mathcal{O})$ with determinant coprime to p we can find an invertible G such that $\overline{G}^t R G$ is diagonal modulo d. Using our induction hypothesis it suffices to show that T is equivalent to a matrix

$$\begin{bmatrix} \ell & 0 \\ 0 & R \end{bmatrix} \tag{4.144}$$

where R is an $n \times n$ Hermitian matrix.

First we argue that we can assume one of the diagonal entries of T is coprime to p. To do this we construct a vector v such that $\overline{v}^t T v$ is coprime to p. We'll then extend $\{v\}$ to a basis for $\mathcal{O}^n / d\mathcal{O}^n$ to get a matrix P such that $\overline{P}^t T P$ has the first diagonal entry coprime to p. Suppose that all the diagonal entries are divisible by p. I claim that since $\det(T)$ is coprime to p we can find P such that $\overline{P}^t T P$ has an off diagonal entry a_{ij} such that $|a_{ij}|^2$ is coprime to p.

If p is inert we can take $P = I_{n+1}$. This immediately follows from the fact that $\det(T)$ is coprime to p. Suppose that $p = \mathfrak{p}\overline{\mathfrak{p}}$. Let $Q = \{\sigma \in S_n : \sigma = \sigma^{-1}\}$. Then, after pairing the terms corresponding to σ and σ^{-1} we have

$$\det(T) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n t_{i\sigma(i)} = \sum_{\sigma \in Q} (-1)^\sigma \prod_{i=1}^n t_{i\sigma(i)} + \sum_{\sigma \notin Q} (-1)^\sigma 2\mathrm{Re}\left[\prod_{i=1}^n t_{i\sigma(i)}\right]. \tag{4.145}$$

Since $\det(T)$ is coprime to p we must have either $\prod_{i=1}^n t_{i\sigma(i)}$ for $\sigma = \sigma^{-1}$ or $2\mathrm{Re}\left[\prod_{i=1}^n t_{i\sigma(i)}\right]$ for $\sigma \neq \sigma^{-1}$ not lie in p.

First suppose that $\prod_{i=1}^n t_{i\sigma(i)}$ is coprime to p with $\sigma = \sigma^{-1}$. Since t_{ii} is divisible by p by assumption we can find i with $\sigma(i) \neq i$. Then $|t_{i\sigma(i)}|^2$ divides the above product and hence must be coprime to p.

Now suppose that $2\mathrm{Re}\left[\prod_{i=1}^n t_{i\sigma(i)}\right]$ is coprime to p for some $\sigma \neq \sigma^{-1}$. Note that none of the $t_{i\sigma(i)}$ can lie in $p\mathcal{O}$. If for some i $t_{i\sigma(i)} \notin \mathfrak{p}, \overline{\mathfrak{p}}$ then we'll have $|t_{i\sigma(i)}|^2$ coprime to p and we're done. Consider when $t_{i\sigma(i)} \in \mathfrak{p}$ or $\overline{\mathfrak{p}}$ for each i. Since $\sigma \neq \sigma^{-1}$ we can choose i_0 with $\sigma(i_0) \neq \sigma^{-1}(i_0)$. Without loss of generality assume

$t_{i_0\sigma(i_0)} \in \mathfrak{p}$. Then $t_{i_0\sigma^{-1}(i_0)} \in \overline{\mathfrak{p}}$. Now choose P to be the elementary matrix which replaces column $\sigma^{-1}(i_0)$ by column $\sigma^{-1}(i_0)$ added to column $\sigma(i_0)$. Then $\overline{P}^t TP$ will have $(i_0, \sigma^{-1}(i_0))$ entry $t_{i_0,\sigma(i_0)} + t_{i_0,\sigma^{-1}(i_0)}$. Since $t_{i,\sigma(i)} \notin p\mathcal{O} = \mathfrak{p} \cap \overline{\mathfrak{p}}$ we see that $t_{i_0,\sigma(i_0)} + t_{i_0,\sigma^{-1}(i_0)}$ does not lie in $\mathfrak{p}$ or $\overline{\mathfrak{p}}$ and hence has norm coprime to p. Thus in any event we can conjugate by an invertible matrix P and assume that T has an entry t_{ij} with $|t_{ij}|^2$ coprime to p.

Let $v = [v_i]$ be a vector with $v_i = t_{ij}$ and $v_j = 1$ and all other entries zero. Then

$$\overline{v}^t Tv = |t_{ij}|^2 t_{ii} + 2|t_{ij}|^2 + t_{jj} \tag{4.146}$$

which is coprime to p. The set $\{v\} \cup \{e_k\}_{k \neq j}$ will give a basis for $\mathcal{O}^n/p^r\mathcal{O}^n$. Let P be the matrix with columns $v, b_2, \dots b_n$ where $b_2, \dots, b_n$ are the basis vectors chosen above. Then the first entry of $\overline{P}^t TP$ is coprime to p, as it is equal to $\overline{v}^t Tv$. Thus in any event we can assume one of the diagonal entries of T is coprime to p. If one of the other diagonal entries is coprime, say t_{ii} to p we can move it to the front by reordering our basis. Thus we can assume t_{11} is coprime to p. Using this and conjugation by transvections ($t_{i,j}(a) = I_{n+1} + aE_{ij}$ with $i \neq j$) we can eliminate the top row, and since our conjugation preserves the Hermitian property, it will also eliminate the first column as well. To do this note that after conjugation by $t_{1,j}(a)$ our matrix has entry $(1, j)$ entry given by

$$t_{1,j} + at_{1,1}. \tag{4.147}$$

Since $t_{1,1}$ is coprime to p we can choose a so that this is congruent to zero. Since this conjugation does not interfere with entries left of $(1, j)$ we can progressively reduce the entire first row to zero (except the very first entry). Then we can apply the induction hypothesis to get the desired result.

After a change of variables by our matrix P we can assume that T is a diagonal matrix. Note that each of the diagonal entries must be coprime to p. Let m_i be the ith diagonal entry of T. We can break our sum defining G into a product of n sums of the form

$$\sum_{s_i \in \mathcal{O}/d\mathcal{O}} e\left[\frac{m_i N(s_i)}{d}\right]. \tag{4.148}$$

Now as long as d is odd, by [Hav95, Lemma .04], we have

$$\sum_{s_i \in \mathcal{O}/d\mathcal{O}} e\left[\frac{m_i N(s_i)}{d}\right] = \chi_D(d)d. \tag{4.149}$$

From this the result follows. □

Now we prove Proposition 4.8.

Proof of Proposition 4.8. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(E_n(D)d(T))$. By Proposition 4.14 it suffices to show $\chi(M) = \chi_D(d)^n$. This follows immediately from Lemma 4.18 when $(d, 2D) = 1$.Suppose that d is not coprime to $2D$. Let s be the product of all prime factors of $2D$ not dividing d. Note that when the lower right entry of M' is 1 $\chi(M') = 1$. Then

$$\chi(M) = \chi(M)\chi\left(\begin{bmatrix} 1 + sd(T) & s \\ d(T) & 1 \end{bmatrix}\right) = \chi\left(\begin{bmatrix} * & * \\ * & sc + d \end{bmatrix}\right). \tag{4.150}$$

Let p divide $2D$. If p divides d then it doesn't divide c because $ad - bc = 1$ and it doesn't divide s by construction. Hence in this case p doesn't divide $sc + d$. If p doesn't divide d then it must divide s by construction and hence sc. Since p doesn't divide d it doesn't divide $sc + d$. Thus $sc + d$ is coprime to $2D$. We see $\chi(M) = \chi_D(sc + d)^n$. If n is even then this is 1 because χ_D is a real character so $\chi(M) = 1\chi_D(d)^n$. If n is odd then D divides c and so $\chi_D(sc + d)^n = \chi_D(d)^n$. Proposition 4.8 then follows. □

4.3 Linear Independence of Theta Functions

One of the important properties we'd like to know about our theta functions is that they're linearly independent. This allows us to conclude that if we have two theta expansions that are equal, they must have identical theta coefficients, which will become useful in the next chapter.

Proposition 4.19. *We have, for $s, s' \in (\mathcal{O}^{\#})^n$ with $s \not\equiv s' \pmod{T\mathcal{O}^n}$,*

$$\int_{P_\tau^n} \theta_{T,s}(\tau, w, z)\overline{\theta_{T,s'}(\tau, w, z)}dwdz = 0. \tag{4.151}$$

Here

$$P_\tau := \{(\alpha + \beta\omega + \gamma\tau + \delta\omega\tau, \alpha + \beta\overline{\omega} + \gamma\tau + \delta\overline{\omega}\tau) : 0 \leq \alpha, \beta, \gamma, \delta < 1\} \subset \mathbb{C} \times \mathbb{C} \tag{4.152}$$

and $\mathcal{O} = \mathbb{Z} + \omega\mathbb{Z}$.

Proof. Let

$$\theta_{T,s}(\tau, w, z) = \sum_{\substack{\sigma \in (\mathcal{O}^{\#})^n \\ \sigma \equiv s \pmod{T\mathcal{O}^n}}} e\left[\overline{\sigma}^t T^{-1}\sigma\tau + w\sigma + \overline{\sigma}^t z\right] \tag{4.153}$$

$$\theta_{T,s'}(\tau, w, z) = \sum_{\substack{\rho \in (\mathcal{O}^{\#})^n \\ \rho \equiv s' \pmod{T\mathcal{O}^n}}} e\left[\overline{\rho}^t T^{-1}\rho\tau + w\rho + \overline{\rho}^t z\right]. \tag{4.154}$$

After exchanging the order of summation and integration integral 4.151 we get

$$\sum_{\substack{\sigma \equiv s \\ \rho \equiv s'}} e\left[\overline{\sigma}^t T^{-1}\sigma - \overline{\rho}^t T^{-1}\rho\overline{\tau}\right] I \tag{4.155}$$

where

$$I = \int_{P_\tau^n} e\left[w\sigma + \overline{\sigma}^t z - \overline{\rho}^t \overline{w}^t - \overline{z}^t \rho\right] dzdw. \tag{4.156}$$

Consider the substitution $w \mapsto w + \omega z^t$, $z \mapsto w^t + \overline{\omega} z$. If we call this substitution $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$ then the equations below follow from direct computation:

- $\det(H) = D^{n/2}$
- $H^{-1}(P_\tau^n) = \left((\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z})^2\right)^n$.

Then

$$I = D^n \int_{\left((\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})^2\right)^n} e\left(w(2\mathrm{Re}[\sigma]) + 2\mathrm{Re}[\omega\sigma]z - \overline{w}(2\mathrm{Re}[\rho]) - 2\mathrm{Re}[\omega\rho]\overline{z}\right) dwdz. \tag{4.157}$$

If we write $w = x_1 + iy_1$ and $z = x_2 + iy_2$ then we get

$$I = D^n \int e\left[x_1\left(2\mathrm{Re}[\sigma] - 2\mathrm{Re}[\rho]\right)\right] e\left[\left(2\mathrm{Re}[\omega\sigma] - 2\mathrm{Re}[\omega\rho]\right)x_2\right] dx_1 dx_2 e\left[f(y_1, y_2)\right] dy_1 dy_2 \tag{4.158}$$

for some unimportant linear function f. We see, after some potential shifting of the domain, the integrals with respect to x_1 and x_2 are

$$\int_{[0,1]^n} e\left[x_1\left(2\mathrm{Re}[\sigma] - 2\mathrm{Re}[\rho]\right)\right] dx_1 \cdot \int_{[0,1]^n} e\left[\left(2\mathrm{Re}[\omega\sigma] - 2\mathrm{Re}[\omega\rho]\right)x_2\right] dx_2. \tag{4.159}$$

Since $s \not\equiv s' \pmod{T\mathcal{O}^n}$ we must have $\sigma \not\equiv \rho \pmod{T\mathcal{O}^n}$ which means in particular that $\sigma \neq \rho$. We must have either $2\mathrm{Re}[\sigma] \neq 2\mathrm{Re}[\rho]$ or $2\mathrm{Re}[\omega\sigma] \neq 2\mathrm{Re}[\omega\rho]$ since otherwise $\sigma = \rho$. Since $\sigma, \rho \in (\mathcal{O}^{\#})^n$ we must have $2\mathrm{Re}[\sigma], 2\mathrm{Re}[\omega\sigma], 2\mathrm{Re}[\rho],$ and $2\mathrm{Re}[\omega\rho]$ must all lie in $\mathbb{Z}^n$. Thus $I = 0$ as desired and the result follows. $\square$

Corollary 4.20. *The collection $\{\theta_s\}_{s\in(\mathcal{O}^{\#})^n/T\mathcal{O}^n}$ is linearly independent over $\mathbb{C}$.*

Proof. Suppose we have

$$\sum_{s\in(\mathcal{O}^{\#})^n/T\mathcal{O}^n} c_s\theta_s = 0. \tag{4.160}$$

Then for any $s' \in \left(\mathcal{O}^{\#}\right)^n / T\mathcal{O}^n$ we have

$$0 = \sum_s c_s \int_{P_\tau} \theta_s \overline{\theta_{s'}} dwdz = c_{s'} \int_{P_\tau} |\theta_{s'}|^2 dwdz. \tag{4.161}$$

Note that $\theta_s \neq 0$ on P_τ, plug in $(i, 0, 0)$ for example, so that

$$\int_{P_\tau} |\theta_{s'}|^2 dwdz \neq 0 \tag{4.162}$$

and so we must have $c_{s'} = 0$ as desired. $\square$

CHAPTER 5

THE EICHLER--ZAGIER MAP FOR HERMITIAN JACOBI FORMS

5.1 Eichler--Zagier Map

Now that we have theta expansions we can define an Eichler--Zagier map. This will take our Hermitian-Jacobi form and give us a classical modular form. This will be a generalization of a similar map from [EZ85]. We give a brief overview of the original Eichler--Zagier map.

Given a classical Jacobi form of index m ϕ Eichler and Zagier give a series expansion of the form

$$\phi(\tau, z) = \sum_{\mu \pmod{2m}} h_\mu(\tau)\theta_{m,\mu}(\tau, z). \tag{5.1}$$

Each of the functions h_μ is a classical modular form of half integer weight and the map

$$\phi \mapsto \sum_{\mu \pmod{()2m}} h_\mu(4m\tau), \tag{5.2}$$

the original Eichler--Zagier map, gives an isomorphism from the space of Jacobi forms of index m and weight k and a particular subspace of modular forms of weight $k-\frac{1}{2}$. We generalize this map here and explore some of the relationship between our Hermitian Jacobi forms and classical modular forms

First we need to translate the transformation law of our theta function into a transformation law for our coefficients. Throughout let ϕ be an Hermitian Jacobi form of degree n, weight k and non-singular index T with theta expansion

$$\phi = \sum_{s \in (\mathcal{O}^{\#})^n / T\mathcal{O}^n} \theta_{T,s} h_s. \tag{5.3}$$

Let $\Theta = [\theta_s]_{s\in(\mathcal{O}\#)^n/T\mathcal{O}^n}$ and $H = [h_s]_{(\mathcal{O}\#)^n/T\mathcal{O}^n}$ so that $\phi = \Theta^t H$. We can define slash operators component-wise on these vector-valued functions. Using the definitions directly gives

$$\left(\Theta^t H\right)|_{T,k}M = \left(\Theta|_{T,n}M\right)^t H|_{k-n}M \tag{5.4}$$

for any $M \in \mathrm{SL}_2(\mathbb{Z})$. If we translate Proposition 4.8 into a law for Θ we find that if

$$U_T(MJ) = \left[\frac{1}{\det\left(-i\sqrt{D}T\right)}\chi_D(d)^n e\left[a\left(b\overline{s}^t T^{-1}s + 2\mathrm{Re}\left[\overline{s}^t T^{-1}s'\right]\right)\right]\right]_{s,s'} \tag{5.5}$$

then

$$\Theta|_{T,n}MJ = U_T(MJ)\Theta \tag{5.6}$$

for any $M \in \Gamma_0(E_n(D)d(T))$.

We also have

$$\theta_s\Big|_{T,n}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = e\left[\overline{s}^t T^{-1}s\right]\theta_s \tag{5.7}$$

so that

$$U_T\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right) = \left[\delta_{s,s'}e\left[\overline{s}^t T^{-1}s'\right]\right]_{s,s'\in(\mathcal{O}\#)^n/T\mathcal{O}^n}. \tag{5.8}$$

These two computations show that for every $M \in \mathrm{SL}_2(\mathbb{Z})$ there exists some matrix $U_T(M)$ such that $\Theta|_{T,n}M = U_T(M)\Theta$.

Our first goal is to show $U_T(M)$ is always unitary.

Lemma 5.1. *Let $M \in SL_2(\mathbb{Z})$. Then $U_T(M)$ is unitary.*

Proof. Since U_T is multiplicative, it suffices to check that $U_T(J)$ and $U_T\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right)$ are unitary, since these matrices generate $\mathrm{SL}_2(\mathbb{Z})$.

The fact that $U_T\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right)$ is unitary follows from the fact that T is Hermitian. Consider $U_T(J)$. The i,j entry of $\overline{U_T(J)}^t U_T(J)$ is

$$\frac{1}{|\det\left(\sqrt{D}T\right)|^2}\sum_{1\le k\le\left|\left(\mathcal{O}^\#\right)^n/T\mathcal{O}^n\right|} e\left[2\mathrm{Re}(\overline{(s_i-s_j)}T^{-1}s_k)\right]. \tag{5.9}$$

By Lemma 4.15 the diagonal entries are all 1. To show that the off diagonal entries are 0 apply Lemma 4.16. Thus $U_T(J)$ is unitary as desired. □

Proposition 5.2. *We have, for $M \in \Gamma_0(E_n(D)d(T))$,*

$$h_s|_{k-n}MJ = \frac{(\chi_D(d))^n}{\det(i\sqrt{D}T)}\sum_{s'\in\left(\mathcal{O}^\#\right)^n/T\mathcal{O}^n} e\left[-a(b\overline{s}^tT^{-1}s+2Re[\overline{s}^tT^{-1}s'])\right]h_{s'}. \tag{5.10}$$

Proof. Since $\phi|_{T,k}MJ=\phi$ we have

$$\Theta^tH = (\Theta|_{T,n}MJ)^t(H|_{k-n}MJ) = \Theta^t(U_T(MJ))^t(H|_{k-n}MJ). \tag{5.11}$$

By the linear independence of theta functions, Corollary 4.20, we see that

$$(U_T(MJ))^{-t}H = H|_{k-n}MJ. \tag{5.12}$$

By Lemma 5.1 we know $U_T(M)$ is unitary for any $M\in \mathrm{SL}_2(\mathbb{Z})$. Thus Equation (5.12) gives

$$H|_{k-n}MJ = \overline{U_T(MJ)}H \tag{5.13}$$

and translating this to each h_s gives the desired result. □

Lemma 5.3. *If $x\equiv y \pmod{E_n(D)d(T)}$ then we have*

$$\overline{s}^tT^{-1}sx \equiv \overline{s}^tT^{-1}sy \pmod{\mathbb{Z}}. \tag{5.14}$$

Proof. This follows immediately from Lemma 4.5 □

Corollary 5.4. *Let ϕ be an Hermitian Jacobi form of degree n, invertible index T and weight k. Let h_s be a theta coefficient of ϕ. Then $h_s \in S_{k-n}(\Gamma(E_n(D)d(T)))$.*

Proof. First we address holomorphy concerns. We know that h_s is given by a Fourier series that converges absolutely and uniformly on compact subsets of $\mathbb{H}$. This gives us holomorphy on $\mathbb{H}$ and boundedness of h_s at infinity. To get holomorphy at other cusps recall that, for $H = [h_s]$ our vector of theta coefficients we have, for any $M \in \mathrm{SL}_2(\mathbb{Z})$ $H|_{k-n}M = \overline{U_T(M)}^t H$. We then see that $h_s|_{k-n}M$ has a similar Fourier expansion (given as some linear combination of the other theta coefficients) and is thus bounded at infinity as well. Since T is non-singular the Fourier expansions of each h_s has no constant term. Hence each h_s is a cusp form.

The important part is the transformation law. For $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(E_n(D)d(T))$ we have, by Proposition 5.2,

$$h_s|_{k-n}MJ = \frac{\chi_D(d)^n}{\det(i\sqrt{D}T)} \sum_{s'} e\left[-a(b\overline{s}^tT^{-1}s + 2\mathrm{Re}[\overline{s}^tT^{-1}s'])\right] h_{s'}. \tag{5.15}$$

If n is even then $\chi_D(d)^n = 1$ since $\chi_D(d) = \pm 1$. If n is odd then $d \equiv 1 \pmod{D}$ and χ_D is a character modulo D we must have $\chi_D(d) = 1$. By Lemma 5.3 we have

$$e\left[-a\left(b\overline{s}^tT^{-1}s + 2\mathrm{Re}[\overline{s}^tT^{-1}s']\right)\right] = e\left[-2\mathrm{Re}[\overline{s}^tT^{-1}s']\right]. \tag{5.16}$$

Using Proposition 5.2 we see

$$h_s|J = \frac{1}{\det(i\sqrt{D}T)} \sum_{s'} e\left[-2\mathrm{Re}[\overline{s}^tT^{-1}s']\right] h'_s. \tag{5.17}$$

Hence $h_s|MJ = h_s|J$ so $h_s|M = h_s$. $\square$

Theorem 5.5. *Let ϕ be an Hermitian modular form of weight k, degree $n \geq 1$ and non-singular index T. Then ϕ has a theta expansion of the form*

$$\phi(\tau, w, z) = \sum_{s \in \left(\mathcal{O}^\#\right)^n / T\mathcal{O}^n} h_s(\tau)\theta_s(\tau, w, z) \tag{5.18}$$

where each h_s is a classical modular form.

Proof. The existence of this expansion is Proposition 4.1 and the fact that each h_s is a classical modular form is Proposition 5.4. $\square$

Using this we can prove the following corollary.

Corollary 5.6. *The space of Hermitian Jacobi forms of degree n and invertible index T is finite dimensional.*

Proof. We have an injective map of $\mathbb{C}$ vector spaces

$$\mathcal{H} : \mathcal{J}_{T,k}^n \to \bigoplus_{s\in(\mathcal{O}^\#)^n/T\mathcal{O}^n} S_{k-n}(\Gamma(E_n(D)d(T))) \tag{5.19}$$

defined by $\mathcal{H}(\phi) = [h_s]_{s\in(\mathcal{O}^\#)^n/T\mathcal{O}^n}$. We know

$$\dim_{\mathbb{C}}(M_{k-n}(\Gamma(E_n(D)d(T))) = \left\lfloor \frac{(k-n)(E_n(D)d(T))^3 \prod_{p|E_n(D)d(T)}(1-1/p^2)}{12} \right\rfloor \tag{5.20}$$

so we get

$$\begin{aligned}\dim\left(\mathcal{J}_{T,k}^n\right) &\leq \left|(\mathcal{O}^\#)^n/T\mathcal{O}^n\right| \left\lfloor \frac{(k-n)(E_n(D)d(T))^3 \prod_{p|E_n(D)d(T)}(1-1/p^2)}{12} \right\rfloor &(5.21)\\ &= D^n \det(T)^2 \left\lfloor \frac{(k-n)(E_n(D)d(T))^3 \prod_{p|E_n(D)d(T)}(1-1/p^2)}{12} \right\rfloor &(5.22)\end{aligned}$$

$\square$

We define our Eichler--Zagier map and prove it is well-defined.

Definition 5.7. Define $\iota : \mathcal{J}_{T,k}^n \to S_{k-n}(\Gamma_0(E_n(D)d(T)), \chi_D^n)$ by

$$\iota(\phi)(\tau) = h(\tau) := \sum_{s\in\left(\mathcal{O}^\#\right)^n/T\mathcal{O}^n} h_s(E_n(D)d(T)\tau). \tag{5.23}$$

We argue that ι is well-defined.

Proposition 5.8. *If we define ι as above then $\iota(\phi) \in S_{k-n}(\Gamma_0(E_n(D)d(T)), \chi_D^n)$ for $\phi \in \mathcal{J}_{T,k}^n$.*

Proof. Let ϕ be an Hermitian Jacobi form of degree n and invertible index T. Let h_s be a theta coefficient of ϕ and $\begin{bmatrix} a & b \\ E_n(D)d(T)c & d \end{bmatrix} \in \Gamma_0(E_n(D)d(T))$. We have

$$h|_{k-n} \begin{bmatrix} a & b \\ E_n(D)d(T)c & d \end{bmatrix} = \sum_s \left(h_s|_{k-n} \begin{bmatrix} a & E_n(D)d(T)b \\ c & d \end{bmatrix} \right) (E_n(D)d(T)\tau). \quad (5.24)$$

In order to use the above result note that

$$\begin{bmatrix} a & E_n(D)d(T)b \\ c & d \end{bmatrix} = J \begin{bmatrix} -d & c \\ bE_n(D)d(T) & -a \end{bmatrix} J. \quad (5.25)$$

Using this and the transformation for each h_s gives

$$h|_{k-n}M \quad (5.26)$$

$$= \frac{\chi_D(d)^n}{\det(\sqrt{D}T)^2} \sum_s \sum_{s'} \sum_{s''} e\left[-2\mathrm{Re}[\overline{s}^t T^{-1} s'] + dc\overline{s'}^t T^{-1} s' + 2d\mathrm{Re}[\overline{s'}^t T^{-1} s'']\right] h_{s''}. \quad (5.27)$$

We see that, computing the sum over s, this inner sum will be zero unless $s' \equiv 0$, in which case it is $\det(\sqrt{D}T)^2$. Simplifying gives the desired transformation for h. $\square$

The following results give an important subspace of $\mathscr{J}_{k,T}^n$ on which the Eichler--Zagier map is injective, and introduce a family of maps between different spaces of Hermitian Jacobi forms. These maps give us some additional relationships between spaces of Hermitian Jacobi forms and generalize useful constructions in [AD19].

Definition 5.9. Let $k \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{>0}$ and $T \in \Lambda_n^+(\mathcal{O})$. Define a subspace

$$\mathscr{J}_{k,T}^{n,\mathrm{spez}} \subset \mathscr{J}_{k,T}^n \quad (5.28)$$

consisting of those $\phi \in \mathscr{J}_{k,T}^{n}$ whose Fourier coefficients $\alpha(m,r)$ depend only on $det\left(\begin{bmatrix} m & r \\ \overline{r}^t & T \end{bmatrix}\right)$. That is $\phi \in \mathscr{J}_{k,T}^{n,\text{spez}}$ if, whenever $d\left(\begin{bmatrix} m & r \\ \overline{r}^t & T \end{bmatrix}\right) = d\left(\begin{bmatrix} m' & r' \\ \overline{r'}^t & T \end{bmatrix}\right)$, we also have $\alpha(m,r) = \alpha(m',r')$.

Proposition 5.10. *The Eichler--Zagier map is injective on* $\mathscr{J}_{k,T}^{n,spez}$.

Proof. Let

$$\phi = \sum_{s \in (\mathcal{O}^{\#})^n / T\mathcal{O}^n} \theta_{T,s} h_s \in \mathscr{J}_{k,T}^{n,\text{spez}} \tag{5.29}$$

be non-zero. Choose $s \in \left(\mathcal{O}^{\#}\right)^n$ so that $h_s \neq 0$. Recall the Fourier expansion of h_s

$$h_s(\tau) = \sum_{N \in I_{T,s}} \alpha_{s+T\mathcal{O}^n}(N) e\left[\frac{N}{E_n(D) d(T)} \tau\right]. \tag{5.30}$$

Choose N such that h_s has non-zero Nth Fourier coefficient.

Since $\phi \in \mathscr{J}_{k,T}^{n,\text{spez}}$ we have $\alpha_{s+T\mathcal{O}^n}(N) = \alpha_{r+T\mathcal{O}^n}(N)$ for any $r \in (\mathcal{O}^{\#})^n$ and $N \in I_{T,s}$. Denote this common value by $\alpha(N)$. Then we have

$$\iota(\phi) = \sum_{r \in (\mathcal{O}^{\#})^n / T\mathcal{O}^n} \sum_{N \in I_{T,r}} \alpha(N) e\left[\tau\right]. \tag{5.31}$$

From this we see that the Nth Fourier coefficient of $\iota(\phi)$ is

$$\alpha(N) \cdot \left|\left\{r \in \left(\mathcal{O}^{\#}\right)^n : N \in I_{T,r}\right\}\right|. \tag{5.32}$$

Since $\alpha(N) \neq 0$ and $s \in \{r : N \in I_{T,r}\}$ we see that $\iota(\phi)$ has non-zero Nth Fourier coefficient and is thus non-zero. $\square$

In [AD19, Proposition 4.8] Anamby and Das prove that this is the maximal subspace of $\mathcal{J}_{k,T}^{n}$ on which the Eichler Zagier map is injective when $n = 1$ under some additional conditions on T and our quadratic imaginary field K.

Definition 5.11. Let $P \in M_n(\mathcal{O})$ be a matrix with non-zero determinant. Define, for $\phi \in \mathscr{J}^n_{k,T}$ a function $\phi|U_P \in \mathscr{J}^n_{k,PT\overline{P}^t}$ defined by

$$\phi|U_P(\tau, w, z) := \phi(\tau, wP, \overline{P}^t z). \tag{5.33}$$

One can check that $PT\overline{P}^t \in \Lambda_n(\mathcal{O})$ and that $\phi|U_P$ has the correct transformations. The Fourier expansion is discussed in the following proposition.

Proposition 5.12. *Let $\phi \in \mathscr{J}^n_{k,T}$ have Fourier expansion*

$$\phi(\tau, w, z) = \sum_{\substack{m \in \mathbb{Z}, r \in (\mathcal{O}^\#)^n \\ \begin{bmatrix} m & r \\ \overline{r}^t & T \end{bmatrix} \in \Lambda^+_{n+1}(\mathcal{O})}} \alpha(m, r) e(m\tau + wr + \overline{r}^t z). \tag{5.34}$$

Then we have

$$\phi|U_P(\tau, w, z) = \sum_{\substack{m \in \mathbb{Z}, r \in (\mathcal{O}^\#)^n \\ \begin{bmatrix} m & r \\ \overline{r}^t & T \end{bmatrix} \in \Lambda^+_{n+1}(\mathcal{O})}} \beta(m, r) e(m\tau + wr + \overline{r}^t z). \tag{5.35}$$

where

$$\beta(m, r) := \begin{cases} 0 \ \textit{if } r \notin P(\mathcal{O}^\#)^n \\ \alpha(m, r') \ \textit{if } r = Pr' \textit{ for } r' \in \left(\mathcal{O}^\#\right)^n. \end{cases} \tag{5.36}$$

Note that β is well-defined provided $\det(P) \neq 0$.

Proof. The result follows immediately from the definition of $\phi|U_P$. $\square$

Proposition 5.13. *Let $\phi \in \mathscr{J}^{n,spez}_{k,T}$ and $P \in M_n(\mathcal{O})$ be non-singular. Then $\iota(\phi|U_P)$ is non-zero.*

Proof. For this proof we write ι_T and $\iota_{PT\overline{P}^t}$ for the Eichler Zagier maps of index T and $PT\overline{P}^t$ respectively. In order to show $\iota_{PT\overline{P}^t}(\phi|U_P)$ is non-zero we show

$$\iota_{PT\overline{P}^t}(\phi|U_P)(\tau) = \iota_T(\phi)(\tau). \tag{5.37}$$

First we relate the theta coefficients of $\phi|U_P$ to those of ϕ. Consider when $s \notin P(\mathcal{O}^{\#})^n$. In this case the sth theta coefficient of $\phi|U_P$ is zero because for any $N \in I_{PT\overline{P}^t,s}$ we have

$$\alpha_{s+PT\overline{P}^t\mathcal{O}^n}(N) = a_{\phi|U_P}(m,r) = 0 \tag{5.38}$$

as every Fourier coefficient of $\phi|U_P$ with $r \notin P(\mathcal{O}^{\#})^n$ is zero by Equation (5.36). Now suppose $s \in P(\mathcal{O}^{\#})^n$. Let $Ps' = s$, and for $N \in \mathbb{Z}_{\geq 0}$ let $f(N) := \frac{N}{|\det(P)|^2}$. Using Lemma 4.5 directly reveals that f gives a well-defined bijection from $I_{s,PT\overline{P}^t}$ to $I_{s',T}$. From this and Equation (5.36) we see that $h_s(\tau) = h_{s'}(\tau)$. Thus we see that

$$\iota_{PT\overline{P}^t}(\phi|U_P) = \iota_T(\phi)\left(|\det(P)|^2\tau\right) \neq 0. \tag{5.39}$$

$\square$

5.2 Twists of the Eichler--Zagier Map

In this section we'll define twists of the Eichler--Zagier map. These slight variations of the Eichler--Zagier map can be non-zero on a given input even when our original Eichler--Zagier map is zero on that input. Though we are unable to generalize their result here, Anamby and Das in [AD19, Proposition 3.2], were able to show that for any non-zero form $\phi \in \mathcal{J}_{k,p}$, there exists some character such that the twist of the Eichler--Zagier map by that character sends ϕ to something non-zero. We know introduce these twists in our setting. Let

$$f = \begin{cases} E_n(D)d(T) & \text{if } D \text{ is odd} \\ E_n(D)d(T)/2 & \text{if } D \text{ is even} \end{cases} \tag{5.40}$$

Let $\psi : \mathbb{Z} \to \mathbb{C}$ be a character modulo $2fE_n(D)d(T)$ and $g : (\mathcal{O}^{\#})^n/T\mathcal{O}^n \to \mathbb{C}$ be such that, for any $d \in \mathbb{Z}$ and $s \in (\mathcal{O}^{\#})^n/T\mathcal{O}^n$ we have

$$g(ds) = \psi(d)g(s). \tag{5.41}$$

Define the twist of the Eichler--Zagier map by g to be

$$\iota_g(\phi) = \sum_{s\in\left(\mathcal{O}^{\#}\right)^n/T\mathcal{O}^n} \overline{g(s)}h_s(E_n(D)d(T)\tau). \tag{5.42}$$

One can show, in similar fashion to our proof for ι, that $\iota_g(\phi) \in S_{k-n}(\Gamma_0(2fE_n(D)d(T), \chi_D\overline{\psi})$. See [Hav95, Proposition 5.8]. We now introduce a group G and give an action of this group on $\mathcal{J}$. This action gives $\mathcal{J}$ the structure of a G-representation and hence allows us to decompose the space into eigenspaces. We then see that these eigenspaces interact nicely with our twisted Eichler--Zagier maps.

Definition 5.14. Let

$$G := \left\{\mu + E_n(D)d(T) \in \mathcal{O}/E_n(D)d(T)\mathcal{O} : N(\mu) \equiv 1 \pmod{E_n(D)d(T)}\right\}. \tag{5.43}$$

Then G is a multiplicative group.

Proposition 5.15. *Define, for* $\mu \in G$,

$$W_\mu : \mathcal{J}^n_{T,k}(\mathcal{O}) \to \mathcal{J}^n_{T,k}(\mathcal{O}) \tag{5.44}$$

$$\sum \theta_s h_s \mapsto \sum \theta_s h_{\mu s}. \tag{5.45}$$

Then $\{W_\mu\}$ *is a commuting family of diagonalizable maps.*

Proof. First we check that W_μ gives a well-defined map. We verify the follow three conditions:

1. $W_\mu(\phi)$ transforms properly with respect to $\epsilon M \in U(1,1)$.

2. $W_\mu(\phi)$ transforms properly with respect to $[\lambda, \mu] \in \mathcal{O}^n \times \mathcal{O}^n$.

3. $W_\mu(\phi)$ to has an appropriate Fourier expansion.

By equations 5.13 and 5.8 we have

$$h_{\mu s}|_{k-n} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = e\left[-\overline{\mu s}^t T^{-1} \mu s\right] h_{\mu s}. \tag{5.46}$$

After computing $\theta_s|_{T,n}\epsilon I$ directly from the definition and using the relationship between the transformation for Θ and H, we can get that

$$h_{\mu s}|_{k-n}\epsilon I = \epsilon h_{\epsilon^{-1}\mu s}. \tag{5.47}$$

Finally we have

$$h_{\mu s}|_{k-n}J = \frac{1}{\det(i\sqrt{D}T)} \sum_{s' \in (\mathcal{O}^\#)^n / T\mathcal{O}^n} e\left[-2\mathrm{Re}[\mathrm{Nm}(\mu)\overline{s}T^{-1}s']\right] h_{\mu s'}. \tag{5.48}$$

One can quickly check that the conditions on μ give

$$e\left[-2\mathrm{Re}[\mathrm{Nm}(\mu)\overline{s}T^{-1}s']\right] h_{\mu s'} = e\left[-2\mathrm{Re}[\overline{s}T^{-1}s']\right] h_{\mu s'}. \tag{5.49}$$

Hence, since $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \epsilon I, J \right\}$ generate $U(1,1)$, we have

$$H^\mu|\epsilon M = \overline{U_T(\epsilon M)} H^\mu. \tag{5.50}$$

This, together with the transformation law for Θ, gives the transformation law for $W_\mu(\phi)$.

One can compute directly that $\theta_s|[\mu,\lambda] = \theta_s$. This will give the desired transformation for $W_\mu(\phi)$.

We move onto the Fourier expansion of $W_\mu(\phi)$. Writing this out naively gives

$$\sum_{(s,r,N)\in S_1} \alpha_{\mu s+T\mathcal{O}^n}(N)e\left[\left(\frac{N}{E_n(D)d(T)}+\overline{r}^tT^{-1}r\right)\tau + rw + \overline{r}^tz\right]. \tag{5.51}$$

Here

$$S_1 = \left\{(s,r,N) \in (\mathcal{O}^\#)^n/T\mathcal{O}^n \times (\mathcal{O}^\#)^n \times \mathbb{Z}_{\geq 0} : r \equiv s \pmod{T\mathcal{O}^n} \text{ and } N \in I_{T,s}\right\}. \tag{5.52}$$

If we define $m(N,r) = \frac{N}{E_n(D)d(T)}+\overline{r}^tT^{-1}r$ then we have $f(s,N,r) := (m(N,r),r)$ gives a well-defined bijection between S_1 and S_2 where

$$S_2 = \left\{(m,r) : \begin{bmatrix} m & \overline{r}^t \\ r & T\end{bmatrix} \in \Lambda^+_{n+1}(\mathcal{O})\right\}. \tag{5.53}$$

This is essentially the same as the argument that gives the existence of theta expansions given in Proposition 4.1. From this the Fourier expansion will follow. It is clear that this family of operators commutes so it remains to show that that each operator is diagonalizable. By Proposition 5.6 we know that $\mathcal{J}^n_{T,k}$ is a finite dimensional space. The fact that each W_μ is diagonalizable just follows from the fact that G is a finite group and $\mu \mapsto W_\mu$ gives a representation of G on $\mathcal{J}^n_{T,k}$. $\square$

Corollary 5.16. *If, for a character $\eta : G \to \mathbb{C}$ we define*

$$\mathcal{J}^{n,\eta}_{k,T} := \left\{\phi \in \mathcal{J}^n_{T,k} : W_\mu(\phi) = \eta(\mu)\phi\right\}. \tag{5.54}$$

then we have

$$\mathcal{J}^n_{T,k} = \bigoplus_{n,\eta} \mathcal{J}^{n,\eta}_{T,k}. \tag{5.55}$$

We now can study how how twists act on these spaces.

Proposition 5.17. *Let $g : (\mathcal{O}^{\#})^n/T\mathcal{O}^n \to \mathbb{C}$ be such that $g(\mu s) = \eta(\mu)g(s)$ for $\mu \in G$. Recall that we have a map $\iota_g : \mathcal{J}^n_{T,k} \to S_{k-n}(\Gamma_0(2fE_n(D)d(T), \chi_D\overline{\eta})$. For a character $\eta' \neq \overline{\eta}$ we have $\iota_g(W_{\mu'}(h)) = 0$.*

Proof. We have

$$\iota_g(\phi) = \sum_{s \in \left(\mathcal{O}^{\#}\right)^n/T\mathcal{O}^n} g(s)h_s(E_n(D)d(T)\tau) = \sum_s g(\mu s)h_{\mu s}(E_n(D)d(T)\tau). \tag{5.56}$$

Since $W_\mu(\phi) = \eta(\mu)\phi$ we have, by the linear independence of theta functions, $h_{\mu s} = \eta(\mu)h_s$. We then have

$$\iota_g(\phi) = \eta(\mu)\eta'(\mu)\sum_s g(s)h_s(E_n(D)d(T)\tau) = \eta(mu)\eta'(\mu)\iota_g(\phi). \tag{5.57}$$

If we choose μ such that $\overline{\eta(\mu)} \neq \eta'(\mu)$ then we see $\iota_g(\phi) = 0$ as desired. ∎

In the next section we explore applications of the main results presented in Chapters 4 and 5.

CHAPTER 6

NON-VANISHING FOURIER COEFFICIENTS

6.1 Vector-valued Hermitian Modular Forms

The main goal of this chapter is to prove that an Hermitian modular form has infinitely many non-zero Fourier Jacobi coefficients. We give an argument by induction on the degree of the Hermitian modular form. The base case follows from the fact that classical modular forms have infinitely many non-zero Fourier coefficients. In order to use the induction hypothesis we need to work with vector-valued Hermitian modular forms. We start by introducing the theory of those forms here. For a reference on the basic theory see [FM15].

Definition 6.1. Fix a quadratic imaginary field K, positive integer $n \geq 1$, a vector space V and a representation $\rho : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V)$. Let $U(n,n) = U(n,n)(\mathcal{O})$. A vector-valued Hermitian modular form of weight ρ and degree n is a holomorphic function $F : \mathbb{H}_n \to V$ such that

$$F(MZ) = \rho\left(CZ + D, \overline{CZ}^t + \overline{D}\right)^{-1} F(Z) \tag{6.1}$$

for any $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in U(n,n)$.

We'll be studying those with polynomial representations.

Definition 6.2. A representation $\rho : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V)$ is polynomial if there is a basis of V such that the coordinate functions $\rho_{ij} : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}$ are polynomial in the entries of the input matrix. Similarly we can define the notion of a polynomial representation on $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

In order to use our induction hypothesis we need to link F to an Hermitian modular form of smaller degree. Below we give the definition of this related, smaller degree, Hermitian modular form. This follows a similar construction given by Böcherer and Das in [BD22, Section 3.1.]. Note that the reduction in degree entails moving to vector-valued Hermtian modular forms, even when starting with a scalar-valued Hermitian modular form.

Definition 6.3. Let $F : \mathbb{H}_{n+1} \to V$ be an Hermitian modular form. For $A \in \mathbb{H}_{n+1}$ write $A = \begin{bmatrix} \tau & w \\ z & Z \end{bmatrix}$ with $w \in \mathbb{C}^n$ a row vector, $z \in \mathbb{C}^n$ a column vector and $Z \in \mathbb{H}_n$.

For fixed τ, Z the map $(w, z) \mapsto F\left(\begin{bmatrix} \tau & w \\ z & Z \end{bmatrix}\right)$ is a holomorphic function on an open set containing $(0, 0)$. Hence we can take a Taylor expansion to get

$$F\left(\begin{bmatrix} \tau & w \\ z & Z \end{bmatrix}\right) = \sum_{\lambda,\lambda'} F_{\lambda,\lambda'}(\tau, Z) w^{\lambda} z^{\lambda'} \tag{6.2}$$

for some holomorphic functions $F_{\lambda,\lambda'}$. Here λ and λ' are multi-indices. For two such multi-indices let $\nu(\lambda, \lambda')$ be the degree, that is the sum of the entries in both tuples. Choose the multi-index (λ, λ') of the smallest degree such that $F_{\lambda,\lambda'} \neq 0$. If this degree is ν_0 then define

$$F^0(\tau, Z) := \sum_{\substack{(\lambda,\lambda') \\ \nu(\lambda,\lambda')=\nu_0}} F_{\lambda,\lambda'}(\tau, Z) x_2^{\lambda_2} \cdots x_n^{\lambda_n} y_2^{\lambda'_2} \cdots y_n^{\lambda'_n}. \tag{6.3}$$

We view F^0 as a function from $\mathbb{H} \times \mathbb{H}_n \to \mathbb{C}[x_2, \ldots, x_n, y_2, \ldots, y_n]_{\nu_0}$. Here the subscript indicates the polynomials are homogeneous of degree ν_0.

Our goal is to show that, for fixed τ, $F^0(\tau, Z)$ is a vector valued Hermitian modular form of degree n. Then we'll show that the Fourier coefficients of F^0 are the Fourier-Jacobi coefficients of F. F^0 will have infinitely many non-zero Fourier coefficients by induction and we'll be able to conclude that F^0 has infinitely many non-zero Fourier-Jacobi coefficients as desired.

The main difficulty in showing that F^0 is a vector valued Hermitian modular form of degree n is showing that F^0 satisfies the proper transformation law. The following lemma will be necessary. We first introduce some useful notation. Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in U(n,n)$ and $Z \in \mathbb{H}_n$. Then we define

$$\hat{g} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{bmatrix} \tag{6.4}$$

and

$$\mu_g(Z) := CZ + D \text{ and } \lambda_g(Z) := \overline{C}Z^t + \overline{D}. \tag{6.5}$$

Lemma 6.4. *Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in U(n,n)$. We have*

$$\hat{g} \begin{bmatrix} \tau & w \\ z & Z \end{bmatrix} = \begin{bmatrix} \tau - w(CZ+D)^{-1}Cz & w(CZ+D)^{-1} \\ \lambda_g(Z)^{-t}z & gZ \end{bmatrix} \tag{6.6}$$

Proof. Direct computation gives

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \tau & w \\ z & Z \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 \\ -(\mu_g(Z))^{-1}Cz & (\mu_g(Z))^{-1}. \end{bmatrix} \tag{6.7}$$

One can then compute

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \tau & w \\ z & Z \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \right) \left(\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \tau & w \\ z & Z \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} \tau - w(\mu_g(Z))^{-1}cz & -w(\mu_g(Z))^{-1} \\ Az - (gZ)Cz & gZ \end{bmatrix}$$

To prove the claim it suffices to prove that

$$(\lambda_g(Z))^{-t} = A - (gZ)C. \tag{6.8}$$

We know A, B, C and D satisfy

$$\overline{D}^t A - \overline{B}^t C = I_n \tag{6.9}$$

$$\overline{A}^t C - \overline{C}^t A = 0 \tag{6.10}$$

$$\overline{B}D - \overline{D}B = 0. \tag{6.11}$$

because $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in U(n,n)$. From these equations it follows that

$$\lambda_g(Z)^t A = Z\overline{A}^t C + I_n + \overline{B}^t C$$

$$\lambda_g(Z)^t (AZ + B) = (Z\overline{A}^t + \overline{B}^t)\mu_g(Z)$$

from which Equation (6.8) follows. □

Definition 6.5. Let $F : \mathbb{H}_{n+1} \to V$ be a Hermitian modular form of weight

$$\rho : \mathrm{GL}_{n+1}(\mathbb{C}) \times \mathrm{GL}_{n+1}(\mathbb{C}) \to \mathrm{GL}(V). \tag{6.12}$$

Define a representation

$$\rho^0 : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V \otimes \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]_{v_0}) \tag{6.13}$$

by

$$\rho^0(\alpha, \beta) \cdot v \otimes f(\vec{x}, \vec{y}) := (\rho(\hat{\alpha}, \hat{\beta}) \cdot v) \otimes f(x\alpha, \beta^T y). \tag{6.14}$$

Proposition 6.6. *For fixed τ,*

$$F^0(\tau, Z) : \mathbb{H}_n \to V \otimes \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]_{v_0} \tag{6.15}$$

is a Hermitian modular form of weight ρ^0.

Proof. The fact that F^0 is holomorphic follows immediately from the holomorphy of each $F_{\lambda,\lambda'}$ which in turn follows from the holomorphy of F. Next we consider the transformation law. Let $\mathcal{Z} = \begin{bmatrix} \tau & w \\ z & Z \end{bmatrix} \in \mathbb{H}_{n+1}$. Lemma 6.4 implies

$$F|_\rho \hat{g} = \rho(\mu_{\hat{g}}(\mathcal{Z}), \lambda_{\hat{g}}(\mathcal{Z}))^{-1} \cdot \sum_{\lambda,\lambda'} F_{\lambda,\lambda'}(\tau - w(\mu_g(Z))^{-1}Cz, gZ)((\lambda_g(Z))^{-t}z)^{\lambda'}(w(\mu_g(Z))^{-1})^{\lambda}. \tag{6.16}$$

If we consider the Taylor expansion of $F_{\lambda,\lambda'}$ we see

$$F|\rho\hat{g} = \rho(\mu_{\hat{g}}(\mathcal{Z}), \lambda_{\hat{g}}(\mathcal{Z}))^{-1} \cdot \sum_{\substack{\lambda,\lambda' \\ \nu(\lambda,\lambda')=\nu_0}} F_{\lambda,\lambda'}(\tau, gZ)((\lambda_g(Z))^{-t}z)^{\lambda'}(w(\mu_g(Z))^{-1})^{\lambda} + \text{h.o.t.} \tag{6.17}$$

It follows that

$$F^0(\tau, Z) = \rho^0(CZ + D, \overline{C}Z^T + \overline{D})^{-1} F^0(\tau, gZ). \tag{6.18}$$

□

6.2 Vector-Valued Hermitian Jacobi Forms

With vector-valued Hermitian modular forms we can construct vector-valued Hermitian Jacobi forms similarly to the scalar case. Rather than give a general theory of such forms we do the bare minimum to relate these forms to the scalar setting so that we can use the previously developed theory there.

Definition 6.7. Let V be a finite dimensional complex vector space and $\rho : \mathrm{GL}_{n+1}(\mathbb{C})\times \mathrm{GL}_{n+1}(\mathbb{C}) \to \mathrm{GL}(V)$ be a representation. A vector-valued Hermitian Jacobi form of degree n, index T and weight ρ is a holomorphic function $\varphi : \mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n \to V$ such that

1. For all $M = \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(1,1)$

$$\varphi(\tau, w, z) = \rho\left(\begin{bmatrix} \epsilon c\tau + \epsilon d & \epsilon cw \\ 0 & I_{n-1} \end{bmatrix}, \begin{bmatrix} \overline{\epsilon} c\tau + \overline{\epsilon} d & \overline{\epsilon} cz^t \\ 0 & I_{n-1} \end{bmatrix}\right)^{-1} e\left(-cwTz/j(M,\tau)\right)\varphi\left(\epsilon M \cdot (\tau, w, z)\right) \tag{6.19}$$

2. For all $\lambda, \mu \in \mathcal{O}_K^n$

$$\varphi(\tau, w, z) = \rho\left(\begin{bmatrix} 1 & -\lambda \\ 0 & I_{n-1} \end{bmatrix}, \begin{bmatrix} 1 & -\overline{\lambda} \\ 0 & I_{n-1} \end{bmatrix}\right)^{-1} e_T(\overline{\lambda}^t \lambda\tau + z\lambda + \overline{\lambda}^T w)\varphi(\tau, w + \lambda\tau + \mu, z + \overline{\mu}^t + \overline{\lambda}^t \tau). \tag{6.20}$$

3. φ has a Fourier expansion of the form

$$\varphi(\tau, w, z) := \sum_{(m,r)\in\mathcal{S}_T} \alpha(m, r)e(m\tau + wr + \overline{r}z) \tag{6.21}$$

where $\alpha(m, r) \in V$.

Proposition 6.8. *Let $F : \mathbb{H}_{n+1} \to V$ be a vector valued Hermitian modular form of weight ρ. Let F have Fourier expansion*

$$F(Z) = \sum_{A\in\Lambda_{n+1}(\mathcal{O})} a(F, A)e(AZ). \tag{6.22}$$

For $T \in \Lambda_n(\mathcal{O})$ define

$$\phi_T(\tau, w, z) := \sum_{\substack{m \in \mathbb{Z}, r \in (\mathcal{O}^{\#})^n \\ \begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix} \in \Lambda^+_{n+1}(\mathcal{O})}} a\left(F, \begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right) e(a\tau + wr + \overline{r}^t z). \tag{6.23}$$

Then

$$F\left(\begin{bmatrix} \tau & w \\ z & Z \end{bmatrix}\right) = \sum_{T \in \Lambda_n(\mathcal{O})} \varphi_T(\tau, w, z) e(TZ) \tag{6.24}$$

and each φ_T is a vector valued Hermitian Jacobi form of degree $n-1$, weight ρ and index T.

Proof. See the proof of Proposition 3.3. This follows in a nearly identical fashion.

□

Our next goal is to show that if φ is a vector valued Hermitian Jacobi form then it has a component which is a scalar valued Hermitian Jacobi form. This will allow us to use our already developed theory in this setting.

Proposition 6.9. *Let φ be a non-zero Hermitian Jacobi form of degree n, index T and weight $\rho : GL_{n+1}(\mathbb{C}) \times GL_{n+1}(\mathbb{C}) \to GL(V)$. Suppose that ρ is a polynomial representation. Then there exists a basis for V and a component of φ with respect to this basis that is a vector-valued Hermitian Jacobi form with co-domain $\mathbb{C}$.*

Proof. First recall the Lie-Kolchin theorem: For a connected solvable linear algebraic group G and a representation $\rho : G \to \mathrm{GL}(V)$, the image $\rho(G)$ is simultaneously triangularizable. Let $B_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ be the subgroup of upper triangular matrices. Recall that $B_n(\mathbb{C})$ is both solvable and connected. Thus $B_n(\mathbb{C}) \times B_n(\mathbb{C})$ is as well and so we can apply the Lie-Kolchin theorem to get a basis $\{v_i\}_{i=1}^m$for V such that, whenever A, B are upper triangular $\rho(A, B)$ is upper triangular in this basis.

Let φ_i be φ composed with the ith coordinate function from V to $\mathbb{C}$ with respect to this basis and let r be the largest i such that φ_i is non-zero. I claim that

$$\varphi_r : \mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \tag{6.25}$$

is a scalar valued Hermitian Jacobi form. First we determine the weight. Let $\rho_{r,r}(A, B)$ be the (r, r) entry of $\rho(A, B)$. I claim that the maps

$$f_1 : z \mapsto \rho_{r,r}\left(\begin{bmatrix} z & 0 \\ 0 & I_{n-1} \end{bmatrix}, I_n\right)$$
$$f_2 : z \mapsto \rho_{r,r}\left(I_n, \begin{bmatrix} z & 0 \\ 0 & I_{n-1} \end{bmatrix}\right)$$

are polynomial homomorphisms from $\mathbb{C}$ to $\mathbb{C}$. If a representation is polynomial in one choice of basis, then it is polynomial in any choice of basis. Hence $f_1, f_2 \in \mathbb{C}[z]$. When $\rho_{r,r}$ is restricted to upper-triangular matrices it is a homomorphism because on the ring of upper-triangular matrices the (r, r)th coordinate map is a homomorphism. Thus f_1 and f_2 are polynomial homomorphisms. The only such maps are of the form $z \mapsto \lambda^k$. Let $f_i(z) = z^{k_i}$. Define a one-dimensional representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ by

$$(A, B) \mapsto \det(A)^{k_1} \times \det(B)^{k_2}. \tag{6.26}$$

I claim this is the weight of φ_r.

Let $M = \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(1, 1)$. First we consider transformation (1). We have

$$\left(\begin{bmatrix} \epsilon c\tau + \epsilon d & \epsilon c w \\ 0 & I_{n-1} \end{bmatrix}, \begin{bmatrix} \overline{\epsilon} c\tau + \overline{\epsilon} d & \overline{\epsilon} c w^t \\ 0 & I_{n-1} \end{bmatrix}\right) \tag{6.27}$$

$$= \left(\begin{bmatrix} 1 & \epsilon c w \\ 0 & I_{n-1} \end{bmatrix}, \begin{bmatrix} 1 & \overline{\epsilon} c w^t \\ 0 & I_{n-1} \end{bmatrix} \right) \left(\begin{bmatrix} \epsilon c\tau + \epsilon d & 0 \\ 0 & I_{n-1} \end{bmatrix}, \begin{bmatrix} \overline{\epsilon} c\tau + \overline{\epsilon} d & 0 \\ 0 & I_{n-1} \end{bmatrix} \right) \tag{6.28}$$

If (A, B) are unitriangular, that is they are upper triangular with 1's along the diagonal, then so is $\rho(A, B)$ since the subgroup of unitriangular matrices is the derived subgroup of the Borel subgroup. Thus $\rho \left(\begin{bmatrix} 1 & \epsilon c w \\ 0 & I_{n-1} \end{bmatrix}, \begin{bmatrix} 1 & \overline{\epsilon} c w^t \\ 0 & I_{n-1} \end{bmatrix} \right)$ will be unitriangular and will leave the rth component of a vector unchanged. We have then, using the transformation law for φ,

$$\begin{aligned} \varphi_r(\tau, w, z) &= \rho_{r,r} \left(\begin{bmatrix} \epsilon c\tau + \epsilon d & 0 \\ 0 & I_{n-1} \end{bmatrix}, \begin{bmatrix} \overline{\epsilon} c\tau + \overline{\epsilon} d & 0 \\ 0 & I_{n-1} \end{bmatrix} \right)^{-1} e\left(-cwTz/j(M, \tau)\right) \varphi_r\left(\epsilon M \cdot (\tau, w, z)\right) \\ &= (c\tau + d)^{-k_1 - k_2} \epsilon^{k_2 - k_1} e\left(-cwTz/j(M, \tau)\right) \varphi_r\left(\epsilon M \cdot (\tau, w, z)\right) \end{aligned}$$

as desired.

Next we check the other desired transformation. We have, for φ

$$\varphi(\tau, w, z) = \rho \left(\begin{bmatrix} 1 & -\lambda \\ 0 & I_{n-1} \end{bmatrix}, \begin{bmatrix} 1 & -\overline{\lambda} \\ 0 & I_{n-1} \end{bmatrix} \right)^{-1} e\left[\lambda T \overline{\lambda}^t \tau + wT\overline{\lambda}^t + \lambda T z\right] \tag{6.29}$$

$$\cdot \varphi(\tau, w + \lambda\tau + \mu, z + \overline{\lambda}^t \tau + \overline{\mu}). \tag{6.30}$$

Since φ takes uni-triangular to uni-triangular we see, for $\lambda \in \mathcal{O}^n$

$$\rho \left(\begin{bmatrix} 1 & -\lambda \\ 0 & I_{n-1} \end{bmatrix}, \begin{bmatrix} 1 & -\overline{\lambda} \\ 0 & I_{n-1} \end{bmatrix} \right) \tag{6.31}$$

is uni-triangular. A comparison of the rth entries on either side and the fact that φ_r is the last non-zero entry in φ gives the desired result.

The existence of the proper Fourier series for φ_r follows directly from the Fourier series for φ and the definition of φ_r. $\square$

Remark 6.10. A vector-valued Hermitian Jacobi form of scalar weight, as in the result above, and a Hermitian Jacobi form are not quite the same. The only difference being transformation with respect to the matrices $\epsilon M \in U(1,1)$ when $\epsilon \neq \pm 1$. That said, the important theory we have developed, i.e. that of theta expansions and theta coefficients, will still exist for vector-valued Hermitian Jacobi forms of scalar weight.

6.3 Non-zero Fourier Coefficients

Now that we've linked vector-valued Hermitian Jacobi forms to the scalar setting we can prove a few results on the non-zero Fourier coefficients of Hermitian modular forms.

Proposition 6.11. *Let F be a non-zero vector valued Hermitian modular form of degree $n \geq 2$. Then F has infinitely many non-zero Fourier Jacobi coefficients with non-singular index.*

Proof. We prove the result by induction on the degree n. First consider when $n = 2$. Let F^0 be as in Definition 6.3 and let V be the codomain of F^0 and fix τ_0 so that $F^0(\tau_0, Z)$ is non-zero as a function of Z. By Proposition 6.6 we know F^0 is an Hermitian modular form. Analogously to Proposition 6.9 we can choose a basis for V such that F^0 has a non-zero component, say F^0_r that is a scalar-valued modular form. Because F^0_r has infinitely many Fourier coefficients with non-zero index so too will F^0. If F has Fourier Jacobi expansion

$$\sum_{n \geq 0} \varphi_n(\tau, w, z) e(nZ) \tag{6.32}$$

then

$$a_{F^0}(n) = c(\tau_0) \sum_{\nu(\lambda,\lambda')=\nu_0} \frac{\partial}{\partial w^{\lambda}} \frac{\partial}{\partial z^{\lambda'}} \varphi_n(\tau_0, w, z)|_{w,z=0} \tag{6.33}$$

for some constant $c(\tau_0)$. Hence, for each $n > 0$ with $a_{F^0}(n) \neq 0$ we must also have $\varphi_n \neq 0$. Then F has infinitely many non-zero Fourier-Jacobi coefficients with non-singular index because F^0 has infinitely many non-zero Fourier coefficients $a_{F^0}(n)$ with $n > 0$.

Now suppose we know the result for vector valued Hermitian forms of degree n and let F have degree $n+1$. Let F^0 be as in Definition 6.3 and again call the codomain V. By hypothesis F^0 has infinitely many non-zero Fourier Jacobi coefficients of non-singular index. Let φ be one such coefficient say of index T. By Proposition 6.9 we can choose a basis of V and a component of φ with respect to this basis such that $\varphi_r : \mathbb{H} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is a vector valued Hermitian Jacobi form of scalar weight. As in Remark 6.10 φ_r will have a theta expansion. Choose a non-zero theta coefficient h_s. By Corollary 5.4 h_s will be a classical modular form and hence will have infinitely many non-zero Fourier coefficients of non-zero index. Each such coefficient of h_s will give rise to a non-zero Fourier coefficient of φ_r with non-singular index. Recall if $\alpha_{s+T\mathcal{O}^n}(N) \neq 0$ then φ_r has a Fourier coefficient $\alpha(m,r) \neq 0$ with

$$d\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right) = N. \tag{6.34}$$

This Fourier coefficient of non-singular index for φ_r will guarantee one for F^0. An argument identical to the one given at the end of the preceding paragraph will show that for each Fourier coefficient of h_s with non-zero index, we get a different non-zero Fourier Jacobi coefficient of F with non-singular index. Thus, since h_s has infinitely many non-zero Fourier coefficients with non-zero index, F has infinitely many non-zero Fourier Jacobi coefficients with non-singular index. $\square$

Next we'll look at the actual Fourier coefficients of F. This result reduces to the Sturm bound for classical modular forms though in dimensions ≥ 2 it is not nearly as restrictive as that result. We no longer need consider vector valued Hermitian Jacobi forms. First recall the Sturm bound which states the following: Let $f = \sum a_n q^n$ be a classical modular form of weight k and level N. If $a_n = 0$ for all $n \leq \lfloor \frac{km}{12} \rfloor$ then $f = 0$. Here

$$m = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right). \tag{6.35}$$

We now state our generalization of this result.

Definition 6.12. Define, for $k \geq 1, n \geq 1$ and $D \in \mathbb{R}$,

$$C_{k,n,D} = \frac{(k-n)D^{3+2\lfloor \frac{n}{2} \rfloor}}{12}. \tag{6.36}$$

Proposition 6.13. *Let ϕ be a non-zero Hermitian Jacobi form of non-singular index T, weight k and degree n. Suppose that ϕ has Fourier expansion*

$$\sum_{(m,r) \in \mathcal{S}_T} \alpha(m,r) e\left[m\tau + wr + \overline{r}^t z\right]. \tag{6.37}$$

Then there exists $(m,r) \in \mathcal{S}_T$ with $\alpha(m,r) \neq 0$ and

$$\det\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right) \leq C_{k,n,D} \det(T)^2. \tag{6.38}$$

Proof. We start with the theta expansion

$$\phi = \sum_{s \in \left(\mathcal{O}^{\#}\right)^n / T\mathcal{O}^n} \theta_s h_s. \tag{6.39}$$

Since $\phi \neq 0$ we can find $s \in \left(\mathcal{O}^{\#}\right)^n / T\mathcal{O}^n$ such that $h_s \neq 0$. By Corollary 5.4 we have $h_s \in S_{k-n}\left(\Gamma(E_n(D)d(T))\right)$. The Sturm bound for classical modular forms implies that we can find a non-zero Fourier coefficient of h_s, say $\alpha_{s+T\mathcal{O}^n}(N)$ with

$$N < \frac{(k-n)(E_n(D)d(T))^3}{12} \prod_{p | E_n(D)d(T)} \left(1 - \frac{1}{p^2}\right) \leq \frac{(k-n)(E_n(D)d(T))^3}{12}. \tag{6.40}$$

We see that ϕ will have a non-zero Fourier coefficient with index (m, r) such that $d\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right) = N$ and $r \equiv s \pmod{T\mathcal{O}^n}$ by definition of h_s. From this the result follows. $\square$

Proposition 6.14. *Let F be an Hermitian modular form of weight k and degree $n \geq 2$ with non-zero Tth Fourier Jacobi coefficient for some non-singular T. Then F has a non-zero Fourier coefficient with index $A \in \Lambda_n(\mathcal{O})$ such that*

$$A = \begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix} \tag{6.41}$$

and

$$\det(A) \leq C_{k,n-1,D} \det(T)^2 \tag{6.42}$$

Proof. Let ϕ_T be the non-zero Tth Fourier Jacobi coefficient of F. Then ϕ_T is a non-zero Hermitian Jacobi form of weight k, index T and degree $n-1$. By Proposition 6.13 ϕ_T has a non-zero Fourier coefficient $\alpha(m, r)$ with

$$\det\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right) \leq C_{k,n-1,D} \det(T)^2. \tag{6.43}$$

Since

$$a\left(\begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}\right) = \alpha(m, r) \neq 0 \tag{6.44}$$

by definition of φ_T (see Proposition 3.3.) we see that the proposition is satisfied for $A = \begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}$. $\square$

.

Corollary 6.15. *Let F be a non-zero Hermitian modular form of weight k and degree $n \geq 2$. Then F has infinitely many non-zero Fourier coefficients with index* $A = \begin{bmatrix} m & \overline{r}^t \\ r & T \end{bmatrix}$ *such that*

$$\det(A) \leq C_{k,n-1,D} \det(T)^2 \tag{6.45}$$

Proof. By Proposition 6.11 we know F has infinitely many non-zero Fourier Jacobi coefficients whose index is a non-singular matrix. For each such coefficient we will get a non-zero Fourier coefficient with index A satisfying the conditions of the proposition. □

In this chapter we've seen a few consequences the developed theory has on the theory of Hermitian modular forms. As stated in the introduction original intent of this work was to generalize some results on non-vanishing Fourier coefficients of Hermitian modular forms to higher degree. In the next chapter we discuss some of the issues in generalizing their results to this setting.

CHAPTER 7

FUNDAMENTAL FOURIER COEFFICIENTS

7.1 Difficulties and Roadblocks

The goal of this final chapter is to explain several difficulties in using this theory to generalize the papers of Böcherer and Das [BD22] and Anamby and Das [AD19]. Though the approaches of Böcherer and Das are superficially distinct, the fundamental issue to generalizing both is the same. First we outline the general approach of Böcherer and Das. The main theorem of their paper is as follows:

Theorem 7.1. *Let F be a non-zero vector valued Seigel modular form of weight ρ and degree n. Suppose further that $k(\rho) - \frac{n}{2} \geq \varrho(n)$. When n is even, assume that F is cuspidal. Then there exists infinitely many $GL_n(\mathbb{Z})$ inequivalent matrices $T \in \Lambda_n^+$ such that $d(T)$ is odd and square free, and $a_F(T) \neq 0$.*

Note here two matrices in $T, T' \in \Lambda_n^+$ are inequivalent over $\mathrm{GL}_n(\mathbb{Z})$ if $\overline{A}^t TA \neq T'$ for any $A \in \mathrm{GL}_n(\mathbb{Z})$. The overarching strategy is a proof by induction on the degree n of the form. The base case of $n = 1$, which is the above statement for classical modular forms, is proven in [AD19].

The first step in the induction is to construct a non-zero Fourier Jacobi coefficient φ_T for which T has an odd, square-free discriminant. This follows from the construction of vector valued form F^0, very similar to that given in Definition 6.3, and the induction hypothesis. From this φ_T they derive infinitely many non-zero Fourier coefficients $a_F(A)$ such that d_A is odd and square free. Böcherer and Das then prove, in analogue to 6.9, that φ_T has a non-zero component which is a scalar-valued Jacobi form, say $\varphi_T^{(r)}$.

The final step before dealing with classical modular forms is to construct a non-zero theta coefficient of this scalar-valued Jacobi, say h_μ, such that μ is primitive. Here $\mu \in \mathbb{Z}^{n-1}/2T\mathbb{Z}^{n-1}$ is primitive if

$$\frac{\overline{\mu}}{2}^t T^{-1} \frac{\mu}{2} \tag{7.1}$$

has the largest possible denominator. This step is accomplished in [BD22, Proposition 3.5] and is the step that I was unable to generalize to my setting.

The existence of such a primitive h_μ together with the base case and the relationship between the Fourier coefficients of F and those of h_μ prove the result.

We say a few words about generalizing this approach to the Hermitian setting. As illustrated in Chapter 6 we can generalize these results and constructions until we need the existence of a primitive h_μ. In what follows we give a definition for primitivity in our setting and explain the particular difficulty we faced in proving Proposition 3.5. from Böcherer and Das' paper in this setting.

Definition 7.2. We say $\mu \in \left(\mathcal{O}^{\#}\right)^n$ is primitive with respect to $T \in \Lambda_n^+(\mathcal{O})$ if

$$\frac{1}{i\sqrt{D}} \left(i\sqrt{D}T\right)^{-1} [i\sqrt{D}\mu]$$

has denominator exactly $d(T)T\sqrt{D}$ if n is odd and $d(T)$ if n is even. Similarly $\mu \in (\mathcal{O})^n$ is primitive if

$$\frac{1}{i\sqrt{D}} \left(i\sqrt{D}T\right)^{-1} [\mu]$$

has denominator exactly $id(T)\sqrt{D}$ if n is odd and $d(T)$ if n is even. This notion of primitive descends to $\mathcal{O}^n/i\sqrt{D}\mathcal{O}^n$ which can be seen by computing

$$\frac{1}{i\sqrt{D}}(i\sqrt{D}T)^{-1}[\mu + i\sqrt{D}Tq] \tag{7.2}$$

$$= \frac{1}{i\sqrt{D}} \left(i\sqrt{D}T\right)^{-1} [\mu] + \frac{1}{i\sqrt{D}} \left(\overline{\mu}^t q - \overline{q}^t \mu - \overline{q}^t (i\sqrt{D}T) q\right). \tag{7.3}$$

The existence of a non-zero primitive theta coefficient in Böcherer and Das boils down to proving that the matrix

$$\left[e\left(\frac{\mu\nu}{p}\right)\right]_{\substack{\mu\in\mathbb{Z}/p\mathbb{Z}^x \\ \nu\in\mathbb{Z}/p\mathbb{Z},\ \nu^2=\nu_0^2}} \tag{7.4}$$

has maximal rank for some fixed $\nu_0 \in \mathbb{Z}/p\mathbb{Z}$. Such a matrix either has 2 columns or one column depending on whether $\nu_0 \equiv 0$ or not and so demonstrating that this matrix has maximal rank is relatively straightforward.
In the Hermitian setting the analogous matrix looks like

$$\left[e\left(\frac{(\overline{\mu}\nu+\overline{\nu}\mu)}{p}\right)\right]_{\substack{\mu\in\mathcal{O}/p\mathcal{O}^\times \\ \nu\in\mathcal{O}/p\mathcal{O},\ |\nu|^2\equiv|\nu_0|^2 \pmod p}} \tag{7.5}$$

and in particular the number of columns is equal to the number of solutions to $|\nu|^2 \equiv |\nu_0|^2$ in $\mathcal{O}/p\mathcal{O}$, which could be as high as $2p-1$ if p is split in $\mathcal{O}$. The matrices that appear here can fail to have maximal rank and hence are not sufficient to prove that φ_T has a non-zero primitive theta coefficient. Without such a coefficient we are still able to lift a Fourier coefficient of h_μ to that of F, we just can no longer guarantee that the index of this Fourier coefficient of F will be primitive. In [AD19] Anamby and Das give the following result:

Theorem 7.3. *Let F be an Hermitian cusp form of degree 2. Then $a(F,T) \neq 0$ for infinitely many matrices T such that $D\det(T)$ is of the form $p_K^\alpha n$ where n is square-free with $(n, p_K) = 1$ and $0 \le \alpha \le 2$ if $D \neq 8$ and $0 \le \alpha \le 2$ if $D \neq -8$ and $0 \le \alpha \le 3$ if $D = -8$.*

Our goal would be to generalize this result beyond degree 2. To prove this result Anamby and Das essentially leverage the Eichler--Zagier map and the twists there-of to relate the Fourier coefficients of a non-zero Fourier Jacobi coefficient to those of a classical modular form. By conjugating by a matrix in $\mathrm{GL}_2(\mathcal{O})$, they are able to guarantee the existence of a non-zero Fourier Jacobi coefficient of prime index. The central result of their work is that, for forms of prime index, either the Eichler--Zagier map or a twist of this map, is injective. This allows them to get a non-zero classical cusp form of a certain index and level with Fourier coefficients equal to those of the Hermitian Jacobi form. Then, having reduced to the case of classical modular forms, they prove a non-vanishing result in this setting.

The roadblock in trying to generalize Anamby and Das is essentially the same. The first is getting a Fourier Jacobi of prime determinant. Anamby and Das use a result specific to the setting of $n = 2$, though this can possibly be circumvented by using an induction argument. The second, and more fundamental, is trying to generalize the non-vanishing of either the Eichler--Zagier map or a twist on a given form of prime index.

To prove this result Anamby and Das make the following argument. If each twist sends a form to zero, then all the "primitive" theta coefficients must be zero. They then, just like in Böcherer and Das, look at $h_s|J$, get a family of sums that must be zero and from this construct a matrix they argue must be of maximal rank and arrive at a contradiction with the fact that all the "primitive" theta coefficients are zero. Thus we run into the same issue.

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