Fiche résumé Meca Flu :

Modèle continu du fluide :

$v(x,t)$, $\rho(x,t)$ → variables locales

Champs Eulérien -

$p(x,t)$ — pression

$v(x,t)$ — vitesse

Propriétés fondamentales des fluides :

- Densité ρ — $kg.m^{-3}$

Diffusive transports in fluids

$$\mathbf{\Phi}_T = -k_T \nabla T$$

Implicit assumptions

- *Local* & *instantaneous* transport: $\mathbf{\Phi}_T(\mathbf{x}, t)$ only depends on the gradient at $\mathbf{x}$ and t
- $\nabla T = 0 \Longrightarrow \mathbf{\Phi}_T = 0$: diffusive transport is tied to *inhomogeneities*
- *Homogeneous* and *isotropic* diffusion: k_T is the same everywhere and in all directions
- Molecular diffusive transport is not affected by the fluid motion ($\neq$ convection)

- Pression p — $N.m^{-2}$ (Pa)

- Tension de surface γ — $N.m^{-1}$

For most common liquids in the air, $\gamma \approx 10-50\,mN.m^{-1}$ — Water $\gamma \approx 72\,mN.m^{-1}$, Ethanol $\gamma \approx 22\,mN.m^{-1}$

- Viscosité η — Pa.s ($kg.m^{-1}.s^{-1}$)

Air $\eta \approx 10^{-5}$ Pa.s

Water $\eta \approx 10^{-3}$ Pa.s → viscosité dynamique

Blood $\eta \approx 10^{-3}$-10^{-2} Pa.s

Honey $\eta \approx$ 2-10 Pa.s

$\nu = \dfrac{\eta}{\rho}$ ($m^2.s^{-1}$)

Nombre de Reynolds — $Re = \dfrac{\rho U L}{\eta} = \dfrac{UL}{\nu}$

Reynolds number

- Convective transport of momentum $\sim \rho U^2$
- Diffusive transport of momentum $\sim \dfrac{\eta U}{L}$

$$\frac{\text{Convection}}{\text{Viscous diffusion}} \sim \frac{\rho U L}{\eta} = \frac{UL}{\nu}$$

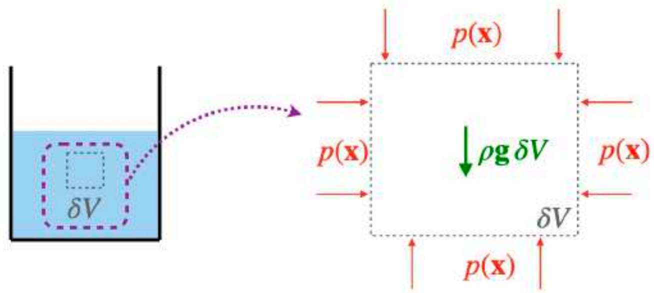
$$\boxed{Re = \frac{\rho U L}{\eta} = \frac{UL}{\nu}}$$

- Re ≪ 1: viscous forces are dominant
- Re ≫ 1: convective (inertial) transport is dominant

The **Reynolds number**, Re, measures the relative importance of **convection** and **viscous diffusion** in **momentum** exchanges

Forces statiques :

Hydrostatics



$\nabla p = \rho g$

$p(x,y,z) = p_0(x,y) - gz$

Forces applied on δV

- Gravity (weight): $\delta \mathbf{F}_g = \int_{\delta V} \rho \mathbf{g}\, dV$
- Pressure forces: $\delta \mathbf{F}_p = \oint_{\delta S} [-p(\mathbf{x})\mathbf{n}(\mathbf{x})] dS$

Gradient theorem: $\int_V \nabla f\, dV = \oint_{\partial V} f \mathbf{n}\, dS$

(Note: despite the volume integral, it only *looks* like a volume force)

⇒ À l'équilibre, les pressions s'égalisent de part et d'autres de l'interface

Energie de surface $\mathcal{E} = \gamma S$

↳ longueur capillaire carac. - $l_c = \sqrt{\frac{\gamma}{\rho g}}$

Loi de Young - Laplace :

Pressure difference at the interface

κ: *Mean* curvature of the interface

Young-Laplace law $\Delta p = p_1 - p_2 = 2\gamma\kappa$ $\quad 2\kappa = \frac{1}{R'} + \frac{1}{R''} = \nabla \cdot \mathbf{n}$

Conditions de non-glissement :

No-slip condition

$\mathbf{u} = \mathbf{U}$ (Solid boundary)

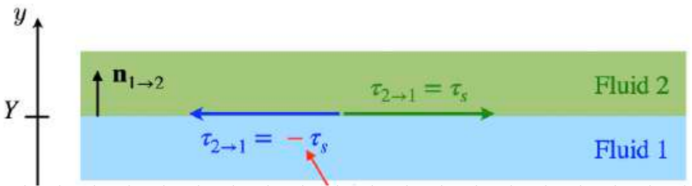
$\mathbf{u}_1 = \mathbf{u}_2$ (Fluid-fluid-interface)

Forces de cisaillement dans un écoulement visqueux:

$$\boxed{\tau_s = \eta \frac{\partial u}{\partial y}\bigg|_{y=0}}$$

ou $\tau_{2\to1} = \eta\, \underline{n_{1\to2}}\, \nabla u$

$\tau_{1\to2} = \eta\, \underline{n_{2\to1}}\, \nabla u$



Dynamic viscosity

$\eta = \frac{1}{3}\rho l^* v^*$

- Units: Pa.s
- Measure of the viscous force

Kinematic viscosity

$\nu = \frac{\eta}{\rho} = \frac{l^* v^*}{3}$

- Units: m^2s^{-1} (diffusivity)
- Measure of momentum diffusion

Ecoulements parallèles :

La direction est fixe et uniforme et l'amplitude (vitesse) est invariante selon cette direction.

$u = u(y, t)\, \underline{e_x}$

Equation du mouvement :

Equation of 1D fluid motion $\boxed{\rho \frac{\partial u}{\partial t}\mathbf{e}_x = \rho \mathbf{g} - \nabla p + \eta \frac{\partial^2 u}{\partial y^2}\mathbf{e}_x}$

Conditions aux limites (!) - Important

Boundary conditions

Rigid wall ← 2 possible types of conditions → Free surface

$\mathbf{u}(y_{wall}, t) = \mathbf{U}_{wall}(t)$

The viscous fluid adheres to the boundary

$\mathbf{f}_v = \eta \frac{\partial u}{\partial y}\bigg|_{y_{surf}} \mathbf{e}_x = 0$

No viscous stress applied on the free surface

(note: it is possible to include here a forcing at the surface: e.g. wind at the surface of the ocean)

Remark: these conditions apply *at each instant* independently.

Résumé écoulements parallèles

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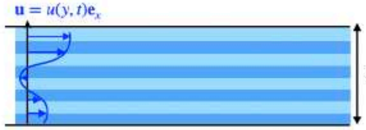
Parallel fluid motion (summary)

$\mathbf{u} = u(y,t)\mathbf{e}_x$



$$\rho\frac{\partial u}{\partial t}\mathbf{e}_x = \rho\mathbf{g} - \nabla p + \eta\frac{\partial^2 u}{\partial y^2}\mathbf{e}_x$$

$$u(y_{\text{wall}}, t) = U_{\text{wall}}(t) \quad \text{or} \quad \eta\frac{\partial u}{\partial y}(y_{\text{surf}}, t) = \tau_s$$

(y_{wall} and y_{surf} designate $y = 0$ and/or $y = h$ depending on the nature of the boundary).

Origin of fluid motion and associated scaling(*), U, for the fluid velocity

- Boundary velocity U_{wall}: $u = O(U_{\text{wall}})$
- Horizontal stress τ_s at a free surface: $u = O(\tau_s h/\eta)$ $\quad \tau_s = \eta\frac{\partial u}{\partial y}(y_{\text{surf}}, t) \sim \eta\frac{U}{h}$
- Imposed (streamwise) pressure gradient P_x: $u = O(P_x h^2/\eta)$ $\quad \eta\frac{U}{h^2} \sim \eta\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} \sim P_x$
- Volume forcing (e.g. $\mathbf{g} \parallel \mathbf{e}_x$): $u = \rho g h^2/\eta$ $\quad \eta\frac{U}{h^2} \sim \eta\frac{\partial^2 u}{\partial y^2} = \rho g \sim \rho g$

(*) Scaling: dimensional dependence of U with the other parameters of the problem

Exo plus classiques

Steady case: Couette-Poiseuille flow

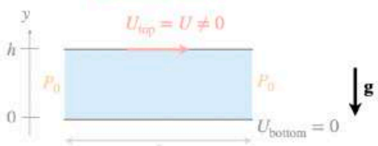
$$\rho\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{=0}\mathbf{e}_x = \rho\mathbf{g} - \nabla p + \eta\frac{d^2u}{dy^2}\mathbf{e}_x \longrightarrow \begin{cases}\frac{\partial p}{\partial x} = \eta\frac{d^2u}{dy^2} \longrightarrow \frac{dp}{dx}(x) = \eta\frac{d^2u}{dy^2}(y) = \text{constant} \\ \frac{\partial p}{\partial y} = -\rho g \longrightarrow p(x,y) = p_0(x) - \rho g y\end{cases}$$

Note: $\mathbf{g}$ is $\perp$ to the flow direction here
$\longrightarrow$ $\mathbf{g} \parallel \mathbf{u}$ is a straightforward generalization

$\perp$ Hydrostatic dependence
$\Longrightarrow$ no influence on the flow

$\boxed{u''(y) = \text{constant} \longrightarrow \textbf{parabolic} \text{ flow profile}}$

Couette flow

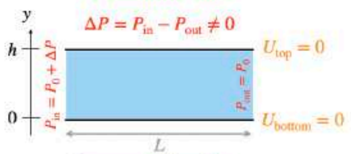


Boundary-driven flow

$u(0) = 0, \quad u(h) = U$
$p(x,y) = p(x+L, y)$

(See details on next slide — not shown)

Poiseuille flow



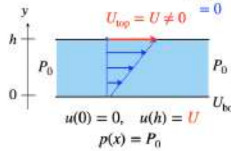
Pressure-driven flow

$u(0) = 0, \quad u(h) = 0$
$p(x,y) = p(x+L, y) + \Delta P$
$\longrightarrow p(x) = P_0 - \frac{x\Delta p}{L}$

NON PROJETÉ

Steady case: Couette flow

$$\underbrace{\cancel{\frac{\partial p}{\partial x}}}_{=0} = \eta\frac{d^2u}{dy^2} \longrightarrow \eta\frac{d^2u}{dy^2} = 0$$



$u(0) = 0, \quad u(h) = U$
$p(x) = P_0$

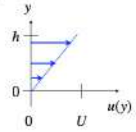
$\eta\frac{d^2u}{dy^2} = 0 \longrightarrow u(y) = ay + b$

$u(0) = 0 \longrightarrow b = 0 \qquad u(y) = ay$

$u(h) = U \longrightarrow a = \frac{U}{h}$

Linear flow profile $\quad \boxed{u(y) = \frac{Uy}{h}}$

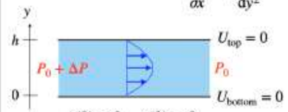
Uniform viscous stress $\quad \boxed{\tau(y) = \frac{\eta U}{h}} = \eta\frac{\partial u}{\partial y}$



In pure Couette flows, the **viscous stress** $\tau(y) > 0$ **is uniform**
$\longrightarrow$ fluid layers all exert the same (downstream) force on the fluid layer beneath them.

Steady case: Poiseuille flow

$$\frac{\partial p}{\partial x} = \eta\frac{d^2u}{dy^2} \longrightarrow \eta\frac{d^2u}{dy^2} = \frac{dp}{dx} = -\frac{\Delta P}{L}$$



$u(0) = 0, \quad u(h) = 0$
$p_0(x) = P_0 - \frac{x\Delta P}{L}$

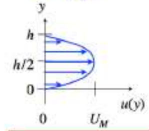
$\eta\frac{d^2u}{dy^2} = -\frac{\Delta P}{L} \longrightarrow u(y) = -\frac{y^2\Delta P}{2\eta L} + ay + b$

$u(0) = 0 \longrightarrow u(y) = -\frac{y^2\Delta P}{2\eta L} + ay$

$u(h) = 0 \longrightarrow a = \frac{h\Delta P}{2\eta L}$

Parabolic flow profile $\quad \boxed{u(y) = \frac{\Delta P}{2\eta L}[y(h-y)]}$

Linear viscous stress $\quad \boxed{\tau(y) = \frac{\Delta P}{2L}[h - 2y]} = \eta\frac{\partial u}{\partial y}$

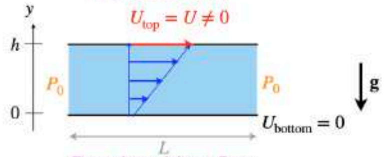


$\boxed{u(y) = \frac{4y(h-y)}{h^2}U_M \qquad U_M = \frac{h^2\Delta P}{8\eta L}}$

$\tau(y > h/2) > 0$ and $\tau(y < h/2) < 0$
central fluid layers **force** outer layers **downstream**

Summary

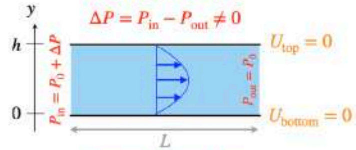
Couette flow



Boundary-driven flow

- **Linear velocity profile:** $u(y) = \frac{Uy}{h}$
- Uniform shear stress: $\tau(y) = \frac{\eta U}{h}$

Poiseuille flow



Pressure-driven flow

- **Parabolic velocity profile:** $u(y) = \frac{4y(h-y)}{h^2}U_M$ with $U_M = \frac{h^2\Delta P}{8\eta L}$
- Linear shear stress: $\tau(y) = \frac{4\eta U_M}{h}\left(1 - \frac{2y}{h}\right)$

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Lecture #2 — Summary

Three types of *internal* forces in a fluid

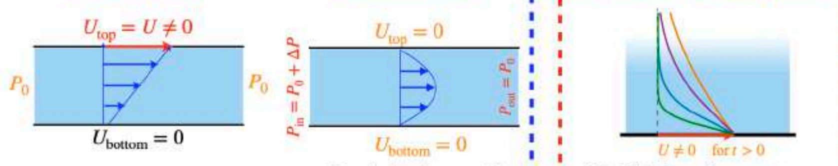
Viscous (dynamic)
- Resistance to **fluid particle deformation**
- *Origin*: molecular diffusion of momentum
- Tangential *or normal* to an interface (see Lect. #4)

Pressure (static)
- Resistance to **compression**
- *Origin*: molecular collisions
- Normal to an interface

Surface tension (static)
- Resistance to **interface formation**
- *Origin*: molecular interaction of non-miscible fluids
- Tangential (and internal) to an interface

STEADY — Parallel (1D) flows $\mathbf{u} = u(y, t)\mathbf{e}_x$ — UNSTEADY

Couette flow — **Poiseuille flow** — **Impulsively-started plate**



Linear flow profile — Parabolic flow profile — 1D diffusion of momentum

Lecture # 3 :

On va souvent chercher à travailler sur des rapports adimensionnés.

Théorème Π :

Buckingham's Pi-theorem

If the problem considered links N **dimensional quantities** and r is the **maximum** number of ***dimensionally-independent*** **quantities** among them, the **number of dimensionless groups** c_i is $N - r$.

What are dimensionally-independent quantities?

n quantities are dimensionally-independent if it is NOT possible to combine $n-1$ of them to obtain a quantity with the same dimension as the remaining one.

Méthode :

Dimensional analysis — Summary

General method

1. Determine the N dimensional quantities to be related and their dimension.
   This step requires intuition and practice: if N is too large, the final result will be of little use. If N is too low, we might miss a critical quantity leading to an irrelevant result.
2. Form the exponent matrix and find its rank r.
3. Construct $N - r$ dimensionless groups — see previous slide.

Details on Step 2

2. Construct dimensional equivalents to the *$N - r$ dependent parameters* by combining powers of the independent parameters.
   $\longrightarrow \quad \rho U^2 R^2 \leftrightarrow D \,, \rho U R \leftrightarrow \eta$

- Use the dimension matrix to find the right power combination (e.g. $\rho^\alpha U^\beta R^\gamma$) of the independent parameters to match the dimension of the dependent parameter of interest (e.g. D) (form/solve a linear system for the required exponents α, β, γ)

	D	R (γ)	U (β)	ρ (α)	η
M	1	0	0	1	1
L	1	1	1	−3	−1
T	−2	0	−1	0	−1

$\longrightarrow \quad 1\times\alpha + 0\times\beta + 0\times\gamma = 1 \quad \longrightarrow \alpha = 1$

$\longrightarrow \quad -3\times\alpha + 1\times\beta + 1\times\gamma = 1$

$\longrightarrow \quad 0\times\alpha + (-1)\times\beta + 0\times\gamma = -2 \quad \longrightarrow \beta = 2$

$\longrightarrow \gamma = 2$

$\longrightarrow$ Indeed $\rho U^2 R^2$ has the same dimension as D !

→ Réfléchir en Ordre de Grandeur /!\

"Méthodes par ODG"

Obtaining scalings - Some « methods »

Method	Example (settling particle)
Identify all the problem's dimensional characteristics/ variables	$\rho, \Delta\rho, g, \eta, a$ (known) U, P (unknown)
A *scaling* will be expressed as product of algebraic *powers* of these variables	$F_v \sim \eta U a$ or $F_g \sim \Delta\rho g a^3$
The scaling of non-uniform fields is their typical magnitude	$\mathbf{u}(\mathbf{x}) \sim U$
If $f(\mathbf{x})$ has a *typical amplitude of variation* F over a *typical distance* L, its gradient ∇f scales as F/L (F may be different from the magnitude of f itself)	$\eta \frac{\partial u}{\partial y} \sim \eta \times \frac{U}{a}$
Scaling of unknown variables may be determined by balancing the scalings of the two dominant effects that determine it.	$F_g \sim F_\eta \longrightarrow \Delta\rho g a^3 \sim \eta U a \Longrightarrow U \sim g a^2 \Delta\rho/\eta$

Nombre adimensionnels :

Relative effect of viscous & convective transports

Reynolds number: $\mathrm{Re} = \frac{\rho U L}{\eta}$: inertia $\longleftrightarrow$ viscosity

Effect of compressibility

Mach number: $\mathrm{M} = \frac{U}{c}$: flow velocity $\longleftrightarrow$ sound wave velocity

BONUS SLIDE

Scalings & Dimensional Analysis

- **What did we learn today?** Different approaches to obtain a simpler approximation/formulation of a fluid mechanics problem involving multiple effects.
- **Dimensional analysis:** consequence of physical dimensions/units (*no physical understanding/mathematical description necessary*)
  Systematic but limited
- **Scaling laws:** based on a physical intuition/understanding of the problem and the identification of the dominant driving/resisting phenomena.
  Potentially far-reaching but requires experience/insight
- **Self-similarity:** particular class of solutions, that depend on reduced/mixed variables; often lead to a system of reduced order (*expected when the system has no externally-imposed time/length scale*)
  Powerful/insightful but limited to very specific situations

Lecture #4

Dans un écoulement parallèle, les lignes de courant et les trajectoires sont confondues

2 approches (Eulérienne et Lagrangienne)

Eulerian framework: $\Theta(\mathbf{x}, t)$ — Temperature at a *fixed* point $\mathbf{x}$

Local derivative $\left(\frac{\partial \Theta}{\partial t}\right)_{\mathbf{x}}$ — Change in temperature observed at a given (**fixed**) *geometric point* $\mathbf{x}$

(NOT the same fluid particle at all times)

Fluid Mechanics approach

Lagrangian framework: $\mathcal{T}_i(t) = \Theta(\mathbf{X}_i(t), t)$ — Temperature of a *moving* fluid particle of trajectory $\mathbf{X}_i(t)$

Material derivative $\frac{d\mathcal{T}_i}{dt}$ — Change in temperature of a given (**moving**) *fluid particle*

(same fluid particle at all times)

This is the quantity needed for applying the laws of physics

(which are only valid for *material* systems)

Formule de la "Material Derivative" :
(A partir de la dérivée locale)

Local derivative $\left(\frac{\partial \Theta}{\partial t}\right)_{\mathbf{x}}$ — Change in temperature at a *fixed* point $\mathbf{x}$

Material derivative $$\frac{D\Theta}{Dt} = \left(\frac{\partial \Theta}{\partial t}\right)_{\mathbf{x}} + \nabla\Theta \cdot \mathbf{u}$$

Change in temperature *following* the moving fluid particle

Local derivative

Convective derivative (associated with the motion of the particle in the non-uniform field)

Tenseurs de gradients :

⇒ Rappels sur la notation d'Einstein.

Notation: $\nabla T = \begin{pmatrix} \frac{\partial T}{\partial x_1} \\ \frac{\partial T}{\partial x_2} \\ \frac{\partial T}{\partial x_3} \end{pmatrix} = \sum_{i=1,2,3} \frac{\partial T}{\partial x_i}\mathbf{e}_i \longrightarrow \boxed{\nabla T = \frac{\partial T}{\partial x_i}\mathbf{e}_i}$

Einstein's convention
Implicit summation over **repeated** *roman* indices
(i.e. $i, j, k, \ldots$ but not $\alpha, \beta, \ldots$)

Albert Einstein
(1879-1955)

Compact index notation

$$\nabla T \equiv \frac{\partial T}{\partial x_i}$$

The reference to the basis $(\mathbf{e}_i)_i$ is left implicit
1 free index ↔ vector

Example: $\forall \mathrm{d}\mathbf{x}, \quad \mathrm{d}T = \nabla T \cdot \mathrm{d}\mathbf{x} \longrightarrow$ in index notation: $\forall \mathrm{d}x_i, \quad \mathrm{d}T = \frac{\partial T}{\partial x_i}\mathrm{d}x_i$

A quick primer on Einstein (index) notation

Rule #1: repeated (also called « dummy ») indices in a product are **summed over** (from 1 to 3)

Rule #2: non-repeated (also called « free ») indices denote **components** of a vector, tensor, etc…

e.g.: $A_{ij}B_{jk} \equiv \sum_j A_{ij}B_{jk} = (\mathbf{AB})_{ik}$ $\qquad u_i\frac{\partial T}{\partial i} \equiv \sum_i u_i \frac{\partial T}{\partial i} = \mathbf{u} \cdot \nabla T$

Scalars:
$a_i b_i \longleftrightarrow \mathbf{a} \cdot \mathbf{b}$
$A_{ij}B_{ji} \longleftrightarrow \mathrm{tr}(\mathbf{A} \cdot \mathbf{B})$
⋮
NO free index

Vectors:
$\frac{\partial f}{\partial x_k} \longleftrightarrow \nabla f,$
$A_{ij}b_j \longleftrightarrow \mathbf{A} \cdot \mathbf{b}$
⋮
1 free index

« Matrices »?
$A_{ij} \longleftrightarrow \mathbf{A}$
$A_{ij}B_{jk} \longleftrightarrow \mathbf{A} \cdot \mathbf{B}$
⋮
2 free indices

Count the **free** indices to determine the *nature* of the type of object you're looking at!

Le gradient dans cette notation :

What type of mathematical object is $\nabla \mathbf{u}$?

$$\forall \mathrm{d}\mathbf{x}, \quad \underbrace{\mathrm{d}\mathbf{u}}_{\text{vector}} = \overbrace{\nabla \mathbf{u}}^{\text{« \cancel{Matrix} »?}} \cdot \underbrace{\mathrm{d}\mathbf{x}}_{\text{vector}}$$

2nd order tensor!

Second-order tensor

$\nabla \mathbf{u}$ is a 2nd order quantity
(two « dimensions »: it « contains » all the $\frac{\partial u_i}{\partial x_j}$ characterizing the changes in u_i with x_j)

$\nabla \mathbf{u}$ is an *intrinsic object* (its definition is **independent** of the chosen basis)
(in contrast with $\frac{\partial u_i}{\partial x_j}$, *its components representation*, which depends on the choice of basis $\mathbf{e}_j$)

Rappels sur les tenseurs d'ordre 2.

A brief introduction to 2nd-order tensors

Properties

- We can perform ALL classical matrix/vector operations with tensors
  Tip: use index notation to familiarise yourself with this in a cartesian basis (⟶ see PC#4)
  e.g. $\mathbf{A} \cdot \mathbf{b} \rightarrow A_{ij}b_j$ is a vector, ${}^t\mathbf{A} \rightarrow A_{ji}$, etc…
- **New product** (tensor product)**:** If **a** and **b** are two **vectors** (components a_i and b_j), $\mathbf{a} \otimes \mathbf{b}$ is the **2nd-order tensor** of components $a_i b_j$.
  $\mathbf{a} \otimes \mathbf{b} \neq \mathbf{b} \otimes \mathbf{a} \longleftrightarrow a_i b_j \neq a_j b_i$
- **Be careful:** for second order tensors, **order matters !**
  e.g. if $\mathbf{A} = \mathbf{a} \otimes \mathbf{b},\ A_{ij} = a_i b_j \leftrightarrow \mathbf{A} = \mathbf{a} \otimes \mathbf{b}$ and $A_{ji} = a_j b_i \leftrightarrow {}^t\mathbf{A} = \mathbf{b} \otimes \mathbf{a} \neq \mathbf{A}$
- **Basis of tensors:** If $(\mathbf{e}_j)_{1 \leq j \leq 3}$ is a base of vectors (in index notation, $\mathbf{a} = a_i \mathbf{e}_i$), the corresponding basis for 2nd-order tensors is noted $(\mathbf{e}_i \otimes \mathbf{e}_j)_{1 \leq i,j \leq 3}$
  (in index notation, $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$)

Properties

- **Classical *contraction*** (« scalar product »)**:** If **A** is a 2nd-order tensor and **b** a vector, $\mathbf{A} \cdot \mathbf{b}$ is the classical « matrix-vector » product represented by $A_{ij}b_j$ in index notation
  (check indices!)
- **New contraction:** double-contraction $\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A} \cdot \mathbf{B})$
  (in index notation, $\mathbf{A} : \mathbf{B} = A_{ij}B_{ji}$ — sum over two different indices i and j)
- **Two important tensor notations**:
  Identity tensor $\mathbf{I} \rightarrow \delta_{ij}$ ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, e.g. $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$)
  Permutation tensor $\boldsymbol{\epsilon} \rightarrow \epsilon_{ijk}$ $\qquad$ Levi-Civita symbol: $\epsilon_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) = (1,2,3) \text{ or circ . perm .} \\ -1 \text{ if } (i,j,k) = (3,2,1) \text{ or circ . perm .} \\ 0 \text{ otherwise} \end{cases}$

Note: ϵ is a **third-order object** (3 free indices!)

Example: Curl of a vector $\boldsymbol{\omega} = \nabla \times \mathbf{u} \longrightarrow \begin{pmatrix} \omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix} \longrightarrow \omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}$ (Check indices!)

Gradient de la vitesse (Notation)

Velocity gradient - matrix representation in a *specific* basis $(\mathbf{e}_j)_j$

$$\nabla\mathbf{u} = \frac{\partial u_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j$$

$\mathbf{e}_i \otimes \mathbf{e}_j$ is represented by the matrix with 0 everywhere except a 1 positioned on the i-th line and j–th column.

$$\nabla\mathbf{u} \equiv \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \boxed{\frac{\partial u_2}{\partial x_3}} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \begin{array}{l} \longrightarrow \text{Line vector } \nabla u_i \\ \text{Line } i=2 \\ \end{array}$$

Line vector ∇u_i (gradient of a specific component)

Line $i = 2$

The (i, j) component of $\nabla\mathbf{u}$ is the derivative of u_i along $\mathbf{e}_j$

Column $j = 3$

Column vector $\dfrac{\partial \mathbf{u}}{\partial x_j}$ (derivative of the full velocity vector in *one* direction)

On peut désormais noter l'accélération des particules fluides

Acceleration of fluid particles

Coming back to where we started from.... we can now define the **acceleration** of fluid particles

Acceleration = rate of change of velocity *for a given fluid particle*

$\longrightarrow$ **Acceleration a is the material derivative of u**

In intrinsic (tensor) form, $\mathbf{a} = \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \dfrac{\partial \mathbf{u}}{\partial t} + \nabla\mathbf{u}\cdot\mathbf{u}$

Contraction on index j

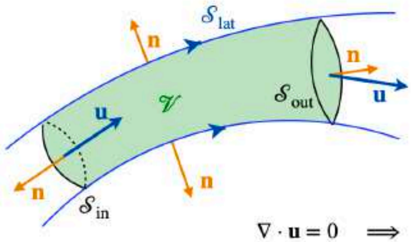
For each component, $a_i = \dfrac{\mathrm{D}u_i}{\mathrm{D}t} = \dfrac{\partial u_i}{\partial t} + \nabla u_i \cdot \mathbf{u}$ $\qquad a_i = \dfrac{\partial u_i}{\partial t} + \dfrac{\partial u_i}{\partial x_j}u_j$

Local acceleration **Convective acceleration** (change due to velocity **gradients**)

Steady flows $\iff \dfrac{\partial \mathbf{u}}{\partial t} = 0$

Conservation du volume pour un fluide incompressible

Incompressible flows: stream tubes



Streamtube
Volume enclosed by the streamlines issued from a closed curve

Incompressibility

$$\nabla\cdot\mathbf{u} = 0 \implies \int_{\mathcal{V}} \nabla\cdot\mathbf{u}\,\mathrm{d}V = 0 = \oint_{\mathcal{S}} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S$$

On streamlines $\quad \mathbf{u}\cdot\mathbf{n} = 0$ on $\mathcal{S}_{\text{lat}} \implies \underbrace{\int_{\mathcal{S}_{\text{in}}} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S}_{=-\mathcal{Q}_{\text{in}}} + \underbrace{\int_{\mathcal{S}_{\text{out}}} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S}_{=\mathcal{Q}_{\text{out}}} = 0$

$$\mathcal{Q}_{\text{in}} = \mathcal{Q}_{\text{out}}$$

> For **incompressible** flows, the **volume flux** of a **streamtube** is **independent** of the cross-section considered $\quad \mathcal{Q} = \displaystyle\int_{\mathcal{S}} \mathbf{u}\cdot\mathbf{n}\;\;\mathrm{d}S$

Rotation du fluide

Rotation of fluid particles: vorticity

$\mathbf{W} = \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^T) \implies \exists\,\mathbf{\Omega}$ (rotation rate of fluid particles) such that $\forall \mathbf{x}$, $\mathbf{W}\cdot\mathbf{x} = \mathbf{\Omega}\times\mathbf{x}$.

$$\mathbf{W} = \frac{1}{2}\begin{pmatrix} 0 & -\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} & 0 & -\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \\ -\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) & \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$

with $\mathbf{\Omega} = \dfrac{\nabla\times\mathbf{u}}{2}$

Vorticity

The **vorticity** field is the **curl** of the velocity: $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ $\quad$ in index notation $\omega_i = \epsilon_{ijk}\dfrac{\partial u_j}{\partial x_k}$

The **vorticity** $\boldsymbol{\omega} = 2\mathbf{\Omega}$ is by definition **twice** the *local* rotation rate of fluid particles

By definition, $\nabla\cdot\boldsymbol{\omega} = 0 \longrightarrow$ it is *solenoidal*

> The **circulation** of a vortex tube is **independent** of the cross section and closed tour used to define it (1st Helmholtz law)

En somme, le gradient de la vitesse contient toute l'information nécessaire sur le fluide

Local decomposition of a fluid flow

Any fluid flow can be decomposed locally (Taylor expansion) as

$$\mathbf{u}(\mathbf{r}+\delta\mathbf{r}) \simeq \mathbf{u}(\mathbf{r}) + \nabla\mathbf{u}\cdot\delta\mathbf{r} + O(|\delta\mathbf{r}|^2)$$

Change of *small* fluid particles

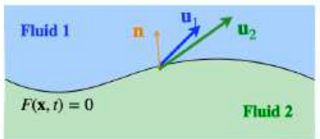
Volume | Shape | Orientation

$$\mathbf{u}(\mathbf{r}+\delta\mathbf{r}) \simeq \mathbf{u}(\mathbf{r}) + \left[\frac{\nabla\cdot\mathbf{u}}{3}\mathbf{I} + \frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right) - \frac{\nabla\cdot\mathbf{u}}{3}\mathbf{I} + \frac{1}{2}\left(\nabla\mathbf{u}-\nabla\mathbf{u}^T\right)\right]\cdot\mathbf{r} + \dots$$

General flow — Rigid translation — Isotropic expansion — Volume-preserving strain $\mathbf{d}-\frac{\mathrm{tr}(\mathbf{d})}{3}\mathbf{I}$ — Rotation

⇒ La composante normale la vitesse du fluide est toujours continue (csq de la cons de la masse)

⇒ Condition limite à une interface en mvmt



Surface parameterization

$\mathbf{x}\in\Sigma \iff F(\mathbf{x},t)=0$

Normal to Σ: $\mathbf{n} = \dfrac{\nabla F}{|\nabla F|}$

Fluids 1 & 2 are non-miscible: no mass flux through $\Sigma \Longrightarrow \Sigma$ is a material surface.

$\mathbf{x}\in\Sigma \iff \mathbf{x}$ is a *material* point $\longrightarrow F=0$ following a moving surface point

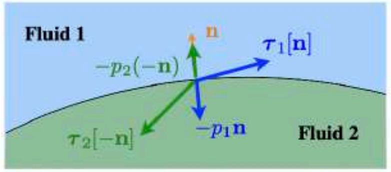
for $\mathbf{x}\in\Sigma$, $\quad \dfrac{DF}{Dt} = \dfrac{\partial F}{\partial t} + \mathbf{u}\cdot\nabla F = 0 \qquad \dfrac{DF}{Dt}$: *material derivative*

$$\boxed{\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \mathbf{u}_1\cdot\nabla F = \frac{\partial F}{\partial t} + \mathbf{u}_2\cdot\nabla F = 0}$$

$\nabla F\cdot\mathbf{u}_1 = \nabla F\cdot\mathbf{u}_2 \iff \mathbf{n}\cdot\mathbf{u}_1 = \mathbf{n}\cdot\mathbf{u}_2$ (impermeability)

This equation relates the velocity of the fluids $(\mathbf{u}_1, \mathbf{u}_2)$ to the position of the interface, $F(\mathbf{x},t)=0$

Résumé des conditions limites :

	Fluid-solid interface	Fluid-fluid
Kinematic (velocity)	$\mathbf{u}(\mathbf{x},t) = \mathbf{V}(\mathbf{x},t)$ (viscous) $\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t) = \mathbf{V}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)$ (non-viscous)	$\mathbf{u}_1(\mathbf{x},t) = \mathbf{u}_2(\mathbf{x},t)$ (viscous) $\mathbf{u}_1(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t) = \mathbf{u}_2(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)$ (non-viscous)
Dynamic (force)	N/A	 $\boldsymbol{\sigma}_1(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t) = \boldsymbol{\sigma}_2(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)$

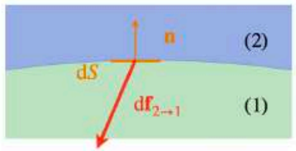
Tenseur "force" :

Stress tensor $\boldsymbol{\sigma}$

Definition

The stress tensor field $\boldsymbol{\sigma}(\mathbf{x})$ is the 2nd order tensor such that the force applied on an oriented surface $\mathbf{dS}$ within the fluid domain is equal to

$$\mathbf{df} = \boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}\,dS$$



$\mathbf{T}(\mathbf{x},\mathbf{n}) = \boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}$ is the **stress force**:
- applied at $\mathbf{x}$
- on the particle **from** which $\mathbf{n}$ is pointing (here: fluid 1)
- by the fluid particle **to** which $\mathbf{n}$ is pointing (here: fluid 2)

Properties

- $\sigma_{ij} = \mathbf{e}_i\cdot(\boldsymbol{\sigma}\cdot\mathbf{e}_j)$ is the component *along* $\mathbf{e}_i$ of the stress applied on a *surface of oriented normal* $\mathbf{e}_j$
- $\boldsymbol{\sigma}$ is symmetric: $\boldsymbol{\sigma} = {}^t\boldsymbol{\sigma}$ $(\sigma_{ij} = \sigma_{ji})$

Viscous stress and Newtonian model

Pressure & viscous contributions

$$\boldsymbol{\sigma} = -p\mathbf{I} + \cancel{\boldsymbol{\tau}(\mathbf{u})} \longrightarrow \boldsymbol{\tau}(\mathbf{d})$$ (viscosity resists fluid particle deformation)

$-p\mathbf{I}$: Pressure stress

In *static* and *moving* fluids
Resists *changes in volume*
(isotropic)

$\boldsymbol{\tau}$: Viscous stress

In *moving* fluids ONLY!
Resists fluid particle *deformation*

Newtonian fluid
(e.g. water or air)

The viscous stress tensor $\boldsymbol{\tau}$ is a **linear** and **instantaneous** function of the local strain rate $\mathbf{d}$

$$\boldsymbol{\tau} = \kappa\, (\nabla \cdot \mathbf{u})\, \mathbf{I} + 2\,\eta\, \mathbf{d}^{\mathrm{dev}} \qquad \mathbf{d}^{\mathrm{dev}} = \mathbf{d} - \frac{\mathrm{tr}(\mathbf{d})}{3}\mathbf{I}$$ (no change in volume)

κ: **Bulk** viscosity (Response to volume changes)

η: **Shear** viscosity (Response to volume-preserving deformations)

Remember: $\nabla \cdot \mathbf{u} = \mathrm{tr}(\mathbf{d})$

lecture #5

lois de conservation

Conservation laws

All conservation principles apply to *closed* systems - same fluid particles at all t (*material* system).

1. Mass

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho\, \mathrm{d}\Omega}_{\text{Mass in } \Omega(t)} = 0$$

2. Momentum

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho\mathbf{u}\, \mathrm{d}\Omega}_{\text{Momentum in } \Omega(t)} = \underbrace{\int_{\Omega(t)} \rho\mathbf{f}\, \mathrm{d}\Omega}_{\text{Volume forces (mass density)}} + \underbrace{\int_{\partial\Omega(t)} \boldsymbol{\sigma}\cdot\mathbf{n}\, \mathrm{d}\Sigma}_{\text{Contact forces}}$$

3. Total energy

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho\left(\frac{u^2}{2}+e\right)\mathrm{d}\Omega}_{\text{Total energy (kinetic + internal)}} = \underbrace{\int_{\Omega(t)} \rho\mathbf{f}\cdot\mathbf{u}\, \mathrm{d}\Omega + \int_{\partial\Omega(t)} \mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\, \mathrm{d}\Sigma}_{\text{Rate of work of external forces (volume + contact forces)}} + \underbrace{\int_{\Omega(t)} r\, \mathrm{d}\Omega + \int_{\partial\Omega(t)} -\boldsymbol{\phi}\cdot\mathbf{n}\, \mathrm{d}\Sigma}_{\text{Heat } \textit{received} \text{ (volume sources + heat fluxes } \boldsymbol{\phi})}$$

Common form:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho b\, \mathrm{d}\Omega}_{\text{Content of } \Omega} = \underbrace{\int_{\Omega(t)} \rho\mathbf{f}_b\cdot\mathbf{u}\, \mathrm{d}\Omega}_{\text{Volume sources}} + \underbrace{\int_{\partial\Omega(t)} \boldsymbol{\phi}_b\cdot\mathbf{n}\, \mathrm{d}\Sigma}_{\text{Surface sources}}$$

Théorème de transport de Reynolds :

Reynolds Transport Theorem

For any function $f(\mathbf{r},t)$ and any volume $V(t)$ with boundary velocity $\mathbf{U}(\mathbf{r},t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} f(\mathbf{r},t)\mathrm{d}V = \underbrace{\int_{V(t)} \frac{\partial f}{\partial t}\mathrm{d}V}_{\text{Local changes}} + \underbrace{\int_{\partial V(t)} f(\mathbf{r},t)[\mathbf{U}\cdot\mathbf{n}]\mathrm{d}S}_{\text{Fluxes}}$$

Special case: consider a *material* volume $\Omega(t)$ (i.e. a volume *moving with the fluid*)
$\longrightarrow$ for material volumes, the boundary velocity $\mathbf{U}(\mathbf{r},t)$ is simply the fluid's velocity $\mathbf{u}(\mathbf{r},t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} f(\mathbf{r},t)\mathrm{d}\Omega = \int_{\Omega(t)} \frac{\partial f}{\partial t}\mathrm{d}\Omega + \int_{\partial\Omega(t)} f(\mathbf{r},t)[\mathbf{u}\cdot\mathbf{n}]\mathrm{d}\Sigma$$

lois de conservations sur un volume de contrôle :

Conservation laws on control volumes

Mass

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathrm{d}V = -\int_{\partial V} \rho\Big[\big(\mathbf{u}-\mathbf{U}\big)\cdot\mathbf{n}\Big]\mathrm{d}S$$

Momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u} \mathrm{d}V = -\int_{\partial V} \rho\mathbf{u}\Big[\big(\mathbf{u}-\mathbf{U}\big)\cdot\mathbf{n}\Big]\mathrm{d}S + \int_V \rho\mathbf{f}\,\mathrm{d}V + \int_{\partial V}\boldsymbol{\sigma}\cdot\mathbf{n}\mathrm{d}S$$

Total energy
$e_T = \frac{u^2}{2} + e$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho e_T \mathrm{d}V = -\int_{\partial V} \rho e_T\Big[\big(\mathbf{u}-\mathbf{U}\big)\cdot\mathbf{n}\Big]\mathrm{d}S + \int_V \rho(\mathbf{f}\cdot\mathbf{u}+r)\,\mathrm{d}V + \int_{\partial V}({}^t\boldsymbol{\sigma}\cdot\mathbf{u}-\boldsymbol{\phi})\cdot\mathbf{n}\mathrm{d}S$$

Change in content of V — Convective transport *into* V — Sources

Conclusion: The content (mass, momentum, energy) of $V(t)$ changes in time due to
(i) the *convective* flux *entering* this volume at that time, and
(ii) the volume and surface sources (forces, rate of work, heat transfer).

Note: the convective transport terms vanish if the volume is material ($\mathbf{U} = \mathbf{u}$)
(In that case the volume boundary is impermeable — no fluid comes in/out!

Peut être vu comme un flux aussi (cf slide 5.32)

Summary

5 conservation equations

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) = 0$$

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}$$

$$\rho \frac{De}{Dt} = \rho r - \nabla \cdot \boldsymbol{\phi} - p(\nabla \cdot \mathbf{u}) + {}^t\boldsymbol{\tau} : \nabla \mathbf{u}$$

$\mathbf{u}, T, p, \rho, e$: 7 variables the local state of fluid particles

Using conservation principles alone the evolution of the system is **underdetermined**.
⟶ we need more information/equation: what IS the system considered?

System-dependent

$p(\rho, T), e(\rho, T) \rightarrow$ thermodynamic laws

Ex: perfect gas $p = \rho r T, e = c_V T$

$\boldsymbol{\tau}(\mathbf{u}, \ldots), \boldsymbol{\phi}(T, \ldots) \rightarrow$ **constitutive laws**

⟶ **ex: Newtonian fluid**

Equations de Navier-Stokes :
Énoncées dans le cadre général, cela donne,

General Navier-Stokes equations

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) = 0$$

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{f} - \nabla(p - \lambda \nabla \cdot \mathbf{u}) + \nabla \cdot (2\eta \mathbf{d})$$

$$\rho \frac{De}{Dt} = \rho r - p(\nabla \cdot \mathbf{u}) + K \Delta T + \epsilon$$

$$\epsilon = \kappa(\nabla \cdot \mathbf{u})^2 + \eta(\mathbf{d}^{\text{dev}} : \mathbf{d}^{\text{dev}})$$

⚠ On est dans le cas incompressible le + souvent

A special limit: the incompressible flow

A flow field is *incompressible*, when volume changes of fluid particles are negligible $\longrightarrow \frac{D\rho}{Dt} = 0 \implies \boxed{\nabla \cdot \mathbf{u} = 0}$ Incompressibility condition (a.k.a. *continuity equation*)

Incompressible and homogeneous flows $\iff \rho =$ constant.

Incompressible Navier-Stokes equations $\quad e(\rho, T) = e(T)$

$$\nabla \cdot \mathbf{u} = 0$$

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{f} - \nabla p + \eta \nabla^2 \mathbf{u}$$

Decoupled hydrodynamic problem

$$\rho c_V \frac{DT}{Dt} = \rho r + K \Delta T + \epsilon$$

$$\epsilon = 2\eta\, {}^t\nabla \mathbf{u} : \nabla \mathbf{u}$$

Thermal problem (*forced* transport)

Limites importantes ($Re \gg 1$, $Re \ll 1$)

Some important limits

Re $\gg$ 1 — Inviscid flow (Euler) $\quad \tau_U \sim L/U \ll \tau_\nu \sim L^2/\nu$ (convective transport; momentum diffusion)

$$\mathrm{St}\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \hat{\nabla}\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \frac{\mathbf{e}_z}{\mathrm{Fr}^2} - \hat{\nabla}\hat{p} + \frac{1}{\mathrm{Re}}\hat{\nabla}^2\hat{\mathbf{u}}$$

$$\nabla \cdot \mathbf{u} = 0, \qquad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{u} \cdot \mathbf{u}\right) = \rho \mathbf{g} - \nabla p$$

Euler's equations (See Lect. #8)

Re $\ll$ 1 — Creeping flow (Stokes) $\quad \tau_\nu \sim L^2/\nu \ll \tau_U \sim L/U$ (momentum diffusion; convective transport)

$$\mathrm{St}\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \hat{\nabla}\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \frac{\mathbf{e}_z}{\mathrm{Fr}^2} - \hat{\nabla}\hat{p} + \frac{1}{\mathrm{Re}}\hat{\nabla}^2\hat{\mathbf{u}}$$

$$\cancel{p = \rho U^2 \hat{p}} \longrightarrow p = \frac{\eta U}{L}\hat{p}$$

$$\mathrm{Re}\,\mathrm{St}\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \mathrm{Re}\,\hat{\nabla}\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \frac{\mathrm{Re}\,\mathbf{e}_z}{\mathrm{Fr}^2} - \hat{\nabla}\hat{p} + \hat{\nabla}^2\hat{\mathbf{u}}$$

Why can't we neglect ∇p?
⟶ p is necessary to enforce incompressibility

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla p = \eta \nabla^2 \mathbf{u}$$

Stokes' equations (See Lect. #7)

$\mathrm{Fr} = \frac{U}{\sqrt{gL}} \gg 1$ — Non-buoyant flow $\quad \tau_U \sim L/U \ll \tau_g \sim \sqrt{L/g}$ (convective transport; Buoyant time)

$$\mathrm{St}\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \hat{\nabla}\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \frac{\mathbf{e}_z}{\mathrm{Fr}^2} - \hat{\nabla}\hat{p} + \frac{1}{\mathrm{Re}}\hat{\nabla}^2\hat{\mathbf{u}}$$

The effect of gravity on the fluid *motion* is negligible

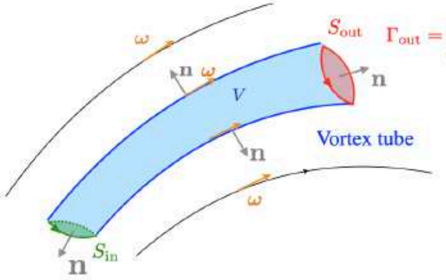
$\mathrm{St} = \frac{\omega L}{U} \ll 1$ — Quasi-steady flow $\quad \tau_U \sim L/U \ll \tau_\omega \sim \omega^{-1}$ (convective transport; intrinsic timescale)

$$\mathrm{St}\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \hat{\nabla}\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \frac{\mathbf{e}_z}{\mathrm{Fr}^2} - \hat{\nabla}\hat{p} + \frac{1}{\mathrm{Re}}\hat{\nabla}^2\hat{\mathbf{u}}$$

History effects in the forcing conditions can be neglected.

Circulation

Circulation $\quad \Gamma = \oint_C \mathbf{u} \cdot \mathrm{d}\mathbf{l} = \int_S \boldsymbol{\omega} \cdot \mathbf{n} \mathrm{d}S$



$\Gamma_{\text{out}} = \oint_{C_{\text{out}}} \mathbf{u} \cdot \mathrm{d}\mathbf{l} = \int_{S_{\text{out}}} \boldsymbol{\omega} \cdot \mathbf{n} \mathrm{d}S$

$\Gamma_{\text{in}} = \oint_{C_{\text{in}}} \mathbf{u} \cdot \mathrm{d}\mathbf{l} = \int_{S_{\text{in}}} \boldsymbol{\omega} \cdot \mathbf{n} \mathrm{d}S$

$\nabla \cdot \boldsymbol{\omega} = 0 \longrightarrow \boxed{\Gamma_{\text{out}} = \Gamma_{\text{in}} = \Gamma}$

The circulation characterises the vortex intensity
(independent of the chosen contour)

Lecture #6 - Ecoulements quasi-parallèles

Dominant dynamics

$\varepsilon \ll 1 \implies |u_z| \ll |u_x|, \left|\frac{\partial u_x}{\partial x}\right| \ll \left|\frac{\partial u_x}{\partial z}\right|$ **Lubrication assumptions**

$\varepsilon^2 \mathrm{Re} \ll 1 \implies |\rho(\nabla \mathbf{u} \cdot \mathbf{u})| \ll |\eta \nabla^2 \mathbf{u}|$

$$\longrightarrow \boxed{\begin{aligned} \frac{\partial p}{\partial x} &= \eta \frac{\partial^2 u_x}{\partial z^2} \\ \frac{\partial p}{\partial z} &= 0 \end{aligned}}$$

~~Parallel flow~~ ? **Quasi-parallel flow!**

Parabolic flow profile in *z*

What is different here from a "true" parallel flow?

- u_x, p depend on x
- boundaries are *slightly inclined*
- non-zero *vertical* flow
- the flow is only *locally* parallel

The problem is *x*-dependent but the flow profile is parabolic *in each cross section*

Propriétés :

Main properties of the flow in elongated channels $\quad \varepsilon = \delta/L \ll 1$

(i) **Spatial derivatives** $\quad \frac{\partial}{\partial x} \sim O\left(\varepsilon \frac{\partial}{\partial z}\right)$

(ii) **Velocity scales** $\quad W = O(\varepsilon U)$

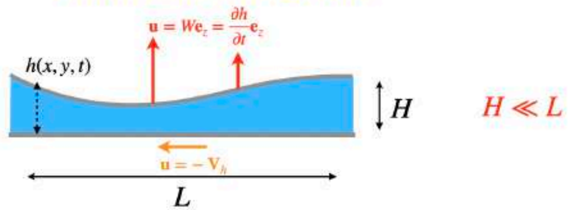
(iii) **Dominant balance** $\quad \frac{\partial p}{\partial x} = \eta \frac{\partial^2 u_x}{\partial z^2}$

(iv) **Pressure scaling** $\quad P_{\text{lub}} = \frac{\eta U}{L}\left(\frac{L}{\delta}\right)^2 = O\left(\frac{P_0}{\varepsilon^2}\right) \gg P_0$

(v) **Dominant forces** $\quad F_0 \ll F_t \ll F_n$

Théorie lubrification :

Canonical framework



$H \ll L$

Two distinct length scales

- L : **longitudinal** dimension /length scale for the **longitudinal** variations of **u**
- H : **cross-flow** dimension / length scale for the **transverse** variations of **u**

We distinguish two different types of boundary conditions

- Relative longitudinal displacement $\longrightarrow$ bottom wall motion $\mathbf{u}|_{z=0} = -\mathbf{V}_h$
- Relative cross-flow displacement $\longrightarrow$ top wall motion $\mathbf{u}|_{z=h} = W\mathbf{e}_z = \frac{\partial h}{\partial t}\mathbf{e}_z$

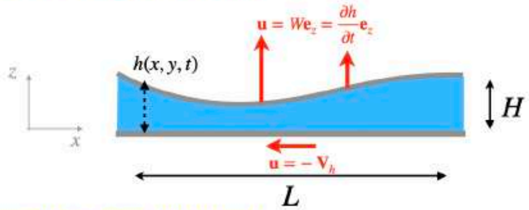
Dans complété lecture #6

Lubrication equations

3D Lubrication equations

(At leading order — see bonus slide)

$$\boxed{\begin{aligned} \nabla_h \cdot \mathbf{u}_h + \frac{\partial u_z}{\partial z} &= 0 \\ \nabla_h p = \eta \frac{\partial^2 \mathbf{u}_h}{\partial z^2} \\ \frac{\partial p}{\partial z} &= 0 \end{aligned}}$$



Same equations as parallel flows!!
But all fields depend on (x, y) — to respect boundary conditions!

Quasi-steady problem
(time-dependent through the boundary conditions — see bonus slide)

p is constant across the gap like in boundary layers…
(origin: separation of scales)

Valid even for $\mathrm{Re}_L = O(1)$

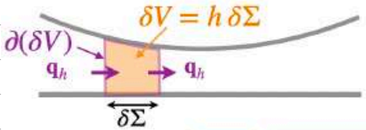
Validity conditions

$\varepsilon \ll 1$	$\varepsilon^2 \mathrm{Re}_L \ll 1$
Thin gap	Inertia not too large
$\varepsilon = \dfrac{H}{L}$	$\mathrm{Re}_L = \dfrac{\rho U L}{\eta}$

In some texts, (including GHP), $\mathrm{Re}_H = \dfrac{\rho U H}{\eta} = \varepsilon \mathrm{Re}_L$ is used $\longrightarrow$ in that case the validity condition writes $\varepsilon \mathrm{Re}_H \ll 1$

Equation de Reynolds

Mass conservation:



$$\frac{\mathrm{d}}{\mathrm{d}t}\overbrace{\int_{\delta V} \rho \mathrm{d}V}^{\approx \rho h \delta\Sigma} = -\int_{\partial(\delta V)} \rho \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S$$

$$\int_{\delta V} \rho \mathrm{d}V \approx \rho h \delta\Sigma \qquad \int_{\partial(\delta V)} \rho \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\partial(\delta\Sigma)}\int_0^h \rho \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}z\,\mathrm{d}S$$

$$\approx \rho\delta\Sigma \times \nabla_h \cdot \overbrace{\int_0^h \mathbf{u}_h \mathrm{d}z}^{=\mathbf{q}_h}$$

Volume change — Net flux out

$$\frac{\partial h}{\partial t} + \nabla_h \cdot \mathbf{q}_h = 0$$

Reynolds equation

$$\boxed{\frac{\partial h}{\partial t} = \nabla_h \cdot \left(\frac{h^3 \nabla_h p}{12\eta} - \frac{Uh}{2}\right)}$$

Two different problems:

- If h is known (e.g. rigid confinement) $\longrightarrow$ **2nd order PDE** for p in *space* (at each t)
  Pressure adjusts (once again) immediately to enforce mass conservation (and incompressibility)
- If p is known (e.g. free surface/thin film) $\longrightarrow$ **2nd** (or 4th) **order PDE** for h in *space AND time*
  Here, p plays the role of the forcing mechanism and h adjusts to mechanical balance

Fluid (lubrication) forces

At leading order $\quad \mathbf{f} = p\mathbf{e}_z - \left(p\nabla_h h + \eta \dfrac{\partial \mathbf{u}_h}{\partial z}\right)$

$O(P\varepsilon^{-2})$, vertical $\qquad O(P\varepsilon^{-1})$, horizontal

Summary

- Fluid stresses have **different scalings in both directions:**
  - $\dfrac{\eta U}{L\varepsilon^2}$ (forces $\perp$ wall) $\gg$ $\dfrac{\eta U}{L\varepsilon}$ (forces $\parallel$ wall)
- **Total** fluid force $\longrightarrow$ local fluid stresses $\times$ area of the lubrication film
  region where $|\nabla_h h| \ll 1$ & $h \ll L$
  E.g.: for a sphere of radius L at a distance δ from the wall, $\mathcal{A} \sim \delta L$
- For reference, fluid forces applied on the outer regions are $O(\eta U L)$ ***(subdominant)***

Film mince ($\neq$ lubrication)

Rigid confinement: fluid motion is **constrained** by the motion of **rigid** walls
(the boundaries' position is KNOWN — up to a rigid body motion)

Thin-film flows: the thin fluid layer has at least one **freely-deformable interface**
(the boundaries' position is UNKNOWN and results from a force balance on the interface)

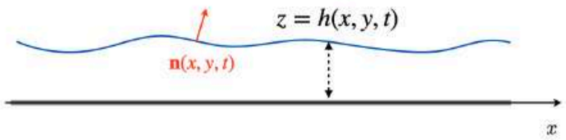
Lire dans complète 6.42.

$\Rightarrow$ cf page suivante pour rappel sur inter-

-face liquide-liquide

Liquid-liquid interfaces (a reminder)

Geometric description



In a thin film, $|\nabla_h h| \sim \varepsilon \ll 1$

Interface position: $z = h(x, y, t) \quad \Longleftrightarrow \quad F(\mathbf{x}, t) = 0 \quad (F = z - h)$

Normal vector:

$$\mathbf{n}(\sim \nabla F) = \frac{\mathbf{e}_z - \nabla_h h}{\sqrt{1 + |\nabla_h h|^2}}$$

for thin films, $|\nabla_h h| \ll 1$ & $\mathbf{n} \approx \mathbf{e}_z - \nabla_h h + O(\varepsilon^2)$

Curvature: $\kappa = \nabla \cdot \mathbf{n}$ (Signed!) for thin films, $\kappa \approx -\nabla_h^2 h$

No-slip condition: $\mathbf{u}^{(1)} = \mathbf{u}^{(2)}$ at $z = h(x, y, t)$ (*unique* definition of $\mathbf{u}$ at the interface!)

Interface evolution:

"The boundary between two non-miscible fluids is a **material** surface"

(Fluid particles at the interface **remain** at the interface) — see Lect. #4

$$\frac{\mathrm{D}F}{\mathrm{D}t} = 0 \text{ at } z = h \text{ with } F = z - h \quad \Longleftrightarrow \quad \frac{\partial h}{\partial t} + \mathbf{u}_h \cdot \nabla_h h = u_z \text{ at } z = h$$

All three terms are of the same order $O(\varepsilon)$ in the lubrication limit.

$(h, u_z, \nabla_h h) = O(\varepsilon), \mathbf{u}_h = O(1)$

Force balance — see Lect. #4

$$[\boldsymbol{\sigma}^{(2)} - \boldsymbol{\sigma}^{(1)}] \cdot \mathbf{n} = \gamma\kappa\mathbf{n} - \nabla_h \gamma \text{ at } z = h(x, y, t)$$

Fluid stresses applied on each side — Stress **within** the interface

For a thin-film ($|\nabla_h h| \ll 1$) in contact with the air ($\boldsymbol{\sigma}^{(2)} = -p_0\mathbf{I}$)

Geometry

$$\mathbf{n} = \mathbf{e}_z - \nabla_h h + O(\varepsilon^2)$$

$$\kappa = -\nabla_h^2 h + O(\varepsilon^2)$$

Dynamic boundary conditions

$$p - p_0 = -\gamma \nabla_h^2 h \quad \text{(Laplace, } \perp\text{)}$$

$$\eta \frac{\partial \mathbf{u}_h}{\partial z} = \nabla_h \gamma \quad \text{(Marangoni, } \parallel\text{)}$$

General thin-film equations

In deriving the lubrication equations, p was the **non-hydrostatic part** of pressure

$\longrightarrow$ To obtain the equation for the *actual* pressure, p needs to be replaced by $\tilde{p} = p + \rho g z$

Momentum conservation across the film ($\mathbf{e}_z$)

g : Gravity

γ : Surface tension

Part #2: $\frac{\partial \tilde{p}}{\partial z} = 0 \quad \Longrightarrow$ For a thin film, $\frac{\partial p}{\partial z} = -\rho g$ & $p(z = h) = p_0 - \kappa\gamma$

$$\boxed{\textbf{Pressure distribution: } p(r, z, t) = \rho g(h - z) + p_0 - \gamma \nabla_h^2 h}$$

For simplicity, we assumed here that $\mathbf{g}$ is normal to the film ($\mathbf{g} = -g\mathbf{e}_z$).

For a vertical film, the effect of gravity would have to be included in the momentum balance **along** the film.

Momentum conservation along the film ($\mathbf{e}_x, \mathbf{e}_y$)

Part #2: $\eta \frac{\partial^2 \mathbf{u}_h}{\partial z^2} = \nabla_h \tilde{p} \quad \Longrightarrow$ For a thin film, $\eta \frac{\partial^2 \mathbf{u}_h}{\partial z^2} = \nabla_h p$

& $\mathbf{u}_h(z = 0) = \mathbf{V}_h$, $\quad \eta \frac{\partial \mathbf{u}_h}{\partial z}(z = h) = \nabla_h \gamma$

γ : Surface tension

$\mathbf{V}_h$: horizontal velocity of the rigid boundary

$$\boxed{\textbf{Flow profile: } \mathbf{u}_h = \frac{z^2 - 2zh}{2\eta} \nabla_h p + \frac{z}{\eta} \nabla_h \gamma + \mathbf{V}_h}$$

General thin-film equations

$$\textbf{Flow profile: } \mathbf{u}_h = \frac{z^2 - 2zh}{2\eta} \nabla_h p + \frac{z}{\eta} \nabla_h \gamma + \mathbf{V}_h$$

$$\textbf{Pressure distribution: } p(r, z, t) = \rho g(h - z) + p_0 - \gamma \nabla_h^2 h$$

Horizontal volume flux:

$$\mathbf{q}_h = \int_0^h \mathbf{u}_h \mathrm{d}z = -\frac{h^3}{3\eta} \nabla_h p + \frac{h^2}{2\eta} \nabla_h \gamma + h\mathbf{V}_h$$

Mass conservation (Reynolds equation)

$$\frac{\partial h}{\partial t} = -\nabla_h \cdot \mathbf{q}_h$$

$$\frac{\partial h}{\partial t} = \frac{1}{3\eta} \nabla_h \cdot (h^3 \nabla_h p) - \frac{1}{2\eta} \nabla_h \cdot (h^2 \nabla_h \gamma) - \mathbf{V}_h \nabla_h h$$

with the horizontal pressure gradient: $\nabla_h p = \rho g \nabla_h h - \gamma \nabla_h (\nabla_h^2 h)$

Lecture #7 - Ecoulements de Stokes

From Navier-Stokes to Stokes equations

Non-dimensional Navier Stokes equations

(Using $\eta U/L$ as a typical pressure scale — see Lect. #5 or bonus slide)

$$\hat\nabla\cdot\hat{\mathbf{u}}=0$$

$$\mathrm{Re}_\omega\frac{\partial\hat{\mathbf{u}}}{\partial\hat t}+\mathrm{Re}\,\hat\nabla\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}=-\hat\nabla\hat p+\hat\nabla^2\hat{\mathbf{u}}
\xrightarrow{\mathrm{Re},\,\mathrm{Re}_\omega\ll 1}
\begin{array}{c}\hat\nabla\cdot\hat{\mathbf{u}}=0\\ \hat\nabla^2\hat{\mathbf{u}}=\hat\nabla\hat p\end{array}
\longrightarrow
\boxed{\begin{array}{c}\nabla\cdot\mathbf{u}=0\\ \eta\nabla^2\mathbf{u}=\nabla p\end{array}}$$

Stokes equations

$$\mathrm{Re}=\frac{\rho UL}{\eta},\quad \mathrm{Re}_\omega=\mathrm{Re}\times\mathrm{St}=\frac{\rho\omega L^2}{\eta}$$

$$\boxed{\mathrm{Re},\,\mathrm{Re}_\omega\ll 1}$$

$\iff$ Inertial effects associated with both parts of the fluid particles' acceleration are negligible

Some important consequences

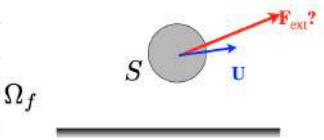
- $\nabla\cdot\boldsymbol{\sigma}=0$ (momentum balance) $\Longrightarrow$ **NO net force on *ANY* fluid volume**
- NO convective acceleration $(\rho\nabla\mathbf{u}\cdot\mathbf{u})$ $\Longrightarrow$ **NO non-linearities**
- NO local time derivative $\left(\rho\dfrac{\partial\mathbf{u}}{\partial t}\right)$ $\Longrightarrow$ **Instantaneous equation** (quasi-static)

(Time dependence is nevertheless maintained by/in the **forcing** boundary conditions)

Propriétés des équations de Stokes

Model problem

What is the force exerted on a rigid spherical particle translating with a velocity $\mathbf{U}$?



(Ω_f): $\eta\nabla^2\mathbf{u}=\nabla p$, $\nabla\cdot\mathbf{u}=0$

(S): $\mathbf{u}=\mathbf{U}$ (Input)

$\int_S\boldsymbol{\sigma}\cdot\mathbf{n}\,dS=\mathbf{F}_h$ (Output)

1. Linearity

Stokes equations are **linear**.

$$\begin{array}{l}\mathbf{U}_1\to(\mathbf{u}_1,p_1)\to\mathbf{F}_1\\ \mathbf{U}_2\to(\mathbf{u}_2,p_2)\to\mathbf{F}_2\end{array}\longrightarrow \mathbf{U}_1+\lambda\mathbf{U}_2\to(\mathbf{u}_1+\lambda\mathbf{u}_2,p_1+\lambda p_2)\to\mathbf{F}_1+\lambda\mathbf{F}_2$$

You can construct solutions by linear superposition.

2. Unicity

Stokes equations are **well-posed**

There is a **unique** solution for a complete set of boundary conditions.

3. Instantaneous response

No $\dfrac{\partial}{\partial t}$ in the equations

The output ($\mathbf{F}_h$) responds **instantaneously** to the input

(here: forcing boundary condition, $\mathbf{u}=\mathbf{U}$, on S)

$\mathbf{F}_h(t_0)$ depends exclusively on $\mathbf{U}(t_0)$ (and not $t'<t^*$)

THERE ARE NO HISTORY EFFECTS!

Physically: information (momentum) is transmitted *instantaneously* throughout the entire fluid domain by viscous diffusion.

4. Reversibility

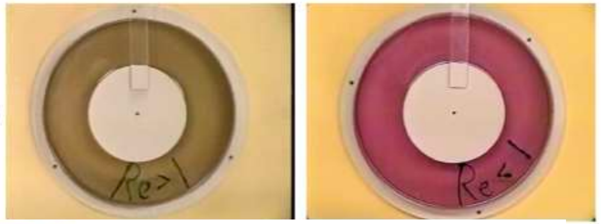
Stokes equations are **reversible**

$\mathbf{U}\to-\mathbf{U}\quad\Rightarrow\quad\mathbf{F}_h\to-\mathbf{F}_h$ in the equations

"Reversing the input reverses the output"

$\mathrm{Re}\gtrsim 1$ vs. $\mathrm{Re}\lesssim 1$

(Movies: G. M. Homsy)



5. Harmonic properties

$$\boxed{\nabla^2\boldsymbol{\omega}=0}\xleftarrow{\nabla\times(A)}\begin{array}{l}\eta\nabla^2\mathbf{u}=\nabla p\quad(A)\\ \nabla\cdot\mathbf{u}=0\quad(B)\end{array}\xrightarrow{\nabla\cdot(A)}\boxed{\nabla^2p=0}$$

Pressure p and vorticity ω are harmonic functions.

They are solutions of Laplace's equation.

What is rr? $\Leftrightarrow\mathbf{r}\otimes\mathbf{r}$

E.g.: Spherical harmonics (see electrostatics!)

$$f(\mathbf{x})=\frac{1}{r}\qquad \nabla f=-\frac{\mathbf{r}}{r^3}\ \text{(source)}\qquad \nabla\nabla f\cdot\mathbf{d}=\left(\frac{3\mathbf{rr}}{r^5}-\frac{\mathbf{I}}{r^3}\right)\cdot\mathbf{d}\ \text{(dipole)}\qquad\cdots$$

Fundamental basis of solutions for p and ω

|| cf slide pour étude détaillée de l'écoulement autour d'une sphère en translation

Calcul en notation d'Einstein des gradients:

Finding gradients, divergence,... using index notation

$p=\eta h(r)\mathbf{r}\cdot\mathbf{U}=\eta h r_i U_i$

$$\longrightarrow\quad\frac{\partial p}{\partial x_i}=\eta U_j\frac{\partial}{\partial x_i}(x_j h)=\eta U_j\left[\delta_{ij}h+\frac{x_ix_jh'}{r}\right]$$

$$\longrightarrow\quad\Delta p=\frac{\partial^2p}{\partial x_i\partial x_i}=\eta U_jx_j\left(\frac{5h'}{r}+r\left[\frac{h'}{r}\right]'\right)$$

$$\mathbf{u}=\big(f(r)\mathbf{I}+g(r)\mathbf{rr}\big)\cdot\mathbf{U}\longrightarrow u_i=fU_i+gx_ix_kU_k$$

$$\nabla\mathbf{u}\ \longrightarrow\ \frac{\partial u_i}{\partial x_j}=\frac{x_jf'}{r}U_i+\left(\frac{x_jg'}{r}x_ix_k+g\delta_{ij}x_k+gx_i\delta_{jk}\right)U_k$$

$$\nabla\cdot\mathbf{u}\longrightarrow\ \frac{\partial u_i}{\partial x_i}=\left(\frac{f'}{r}+rg'+4\right)U_ix_i$$

$$\frac{\partial^2u_i}{\partial x_j^2}=\left(\frac{\delta_{jj}f'}{r}+\frac{x_jx_j}{r}\left[\frac{f'}{r}\right]'\right)U_i+\left(\frac{5x_ix_kg'}{r}+\frac{x_ix_jx_jx_k}{r}\left[\frac{g'}{r}\right]'+2g\delta_{ij}\delta_{jk}+\frac{2x_ix_kg'}{r}\right)U_k$$

$$\nabla^2\mathbf{u}\longrightarrow\frac{\partial^2u_i}{\partial x_j^2}=\left(f''+\frac{2f'}{r}+2g\right)U_i+\left(\frac{6g'}{r}+g''\right)x_ix_kU_k$$

$r_jr_j=r^2$

$\dfrac{\partial x_j}{\partial x_i}=\delta_{ij}$

If α is a function of r only, and $r^2=x_ix_i$, $\dfrac{\partial}{\partial x_i}[\alpha(r)]=\dfrac{\partial r}{\partial x_i}\times\alpha'(r)=\dfrac{x_i\alpha(r)}{r}$

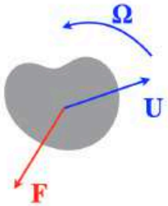
$\delta_{jj}=\mathrm{tr}(\mathbf{I})=3$

The letter used for repeated index can be changed: $x_jy_j=x_ky_k$

Et dans le cadre non sphérique?

General particles - Resistance matrix

There is no analytical solution for non-spherical objects.



$\nabla\cdot\boldsymbol{\sigma}=0, \qquad \nabla\cdot\mathbf{u}=0 \quad \text{in } \Omega_f$

$\mathbf{u}=\mathbf{U}+\boldsymbol{\Omega}\times\mathbf{r} \quad \text{on } S$

Linear problem in $(\mathbf{U},\boldsymbol{\Omega})$

$(\mathbf{u},\boldsymbol{\sigma})$ are linear function of $(\mathbf{U},\boldsymbol{\Omega})$

$$\mathbf{F}=\int_S \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}S, \qquad \mathbf{L}=\int_S \mathbf{r}\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{d}S$$

Linearity: $\begin{pmatrix}\mathbf{F}\\ \mathbf{L}\end{pmatrix}=\mathscr{L}\begin{pmatrix}\mathbf{U}\\ \boldsymbol{\Omega}\end{pmatrix}$ Linear relation between two (6×1) vectors

$$\begin{pmatrix}\mathbf{F}\\ \mathbf{L}\end{pmatrix}=-\mathbf{R}\cdot\begin{pmatrix}\mathbf{U}\\ \boldsymbol{\Omega}\end{pmatrix}=-\eta a\begin{pmatrix}\mathbf{A} & a\mathbf{B}\\ a\mathbf{C} & a^2\mathbf{D}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{U}\\ \boldsymbol{\Omega}\end{pmatrix}$$

Resistance matrix (6 x 6)

Property: $\mathbf{R}$ is symmetric positive definite

Théorème réciprocal de Lorentz:

Theorem

If $(\mathbf{u},\boldsymbol{\sigma})$ and $(\tilde{\mathbf{u}},\tilde{\boldsymbol{\sigma}})$ are two Stokes flow solutions in the **same** fluid domain Ω_f bounded by a surface $\partial\Omega_f$, then

$$\int_{\partial\Omega_f}\tilde{\mathbf{u}}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}S=\int_{\partial\Omega_f}\mathbf{u}\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}\,\mathrm{d}S$$

Effect of a moving fluid on an passive particle

Perturbed flow

$\nabla\cdot\mathbf{u}'=0,\ \nabla\cdot\boldsymbol{\sigma}'=0$

$\mathbf{u}'=\mathbf{U}'+\boldsymbol{\Omega}'\times\mathbf{r}-\mathbf{u}_\infty$ on S

$\mathbf{u}'\to 0$ far from S

$\int_S \boldsymbol{\sigma}'\cdot\mathbf{n}\,\mathrm{d}S=0, \quad \int_S \mathbf{r}\times(\boldsymbol{\sigma}'\cdot\mathbf{n})\,\mathrm{d}S=0$

Dragged particle problem

$\nabla\cdot\tilde{\mathbf{u}}=0,\ \nabla\cdot\tilde{\boldsymbol{\sigma}}=0$

$\tilde{\mathbf{u}}=\tilde{\mathbf{U}}+\tilde{\boldsymbol{\Omega}}\times\mathbf{r}$ on S

$\tilde{\mathbf{u}}\to 0$ far from S

$\int_S \tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}\,\mathrm{d}S=\tilde{\mathbf{F}}, \quad \int_S \mathbf{r}\times(\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n})\,\mathrm{d}S=\tilde{\mathbf{L}}$

Two Stokes flow problems $(\mathbf{u}',\boldsymbol{\sigma}')$ and $(\tilde{\mathbf{u}},\tilde{\boldsymbol{\sigma}})$ outside S with different boundary conditions

Lorentz Reciprocal Theorem $\quad \int_S \mathbf{u}'\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}\,\mathrm{d}S=\int_S \tilde{\mathbf{u}}\cdot\boldsymbol{\sigma}'\cdot\mathbf{n}\,\mathrm{d}S$

dans slide 7.46

After substitution $\quad \boxed{\mathbf{U}'\cdot\tilde{\mathbf{F}}+\boldsymbol{\Omega}'\cdot\tilde{\mathbf{L}}=\int_S \mathbf{u}_\infty\cdot\tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}\,\mathrm{d}S}$

Fin de Amphi 7 pour limites du modèle de Stokes ($\underline{Re}=0$)

Lecture #8 - Limite d'Euler

Nouveau cadre d'étude:

Inviscid flow
There is no source of entropy

$\longrightarrow$ NO viscous dissipation ($\boldsymbol{\tau}=0 \Leftrightarrow \boldsymbol{\sigma}=-p\mathbf{I}$),
$\longrightarrow$ NO heat transfer ($\boldsymbol{\phi}=0$),
$\longrightarrow$ NO heat source r

General inviscid flow equations

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}=-\rho\nabla\cdot\mathbf{u} \qquad \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}=-\nabla p+\rho\mathbf{f}$$

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{u^2}{2}+e\right)=\rho\mathbf{f}\cdot\mathbf{u}-\nabla\cdot(p\mathbf{u})$$

Bernoulli's first theorem (Bernoulli I)

(Pressure changes *along* streamlines)

In an **inviscid steady flow**, the total **charge** $H=e+\frac{p}{\rho}+\frac{u^2}{2}+gz$ is a constant along *each* streamline ($\mathbf{u}\cdot\nabla H=0$).

Dans un écoulement stationnaire, on arrive au 1er théorème de Bernoulli:

Bernoulli's first theorem (Bernoulli I)

(Pressure changes *along* streamlines)

Bernoulli's 1st theorem (steady flows)

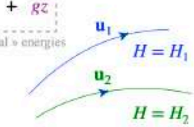
In an inviscid *steady* flow, the total **charge** $H=e+\frac{p}{\rho}+\frac{u^2}{2}+gz$ is a constant along a *given* streamline ($\mathbf{u}\cdot\nabla H=0$).

A few important remarks

(i) **Total charge $\leftrightarrow$ sum of energies** : $H=\underbrace{e}_{\text{Internal}}+\underbrace{\frac{u^2}{2}}_{\text{Kinetic}}+\underbrace{\frac{p}{\rho}}_{\text{Pressure}}+\underbrace{gz}_{\text{Gravitational}}$ « Potential » energies

(ii) H is constant along **each streamline** *independently*

(iii) Bernoulli I is valid for **steady** flows **only**



(iv) Bernoulli I is valid for both **irrotational** ($\boldsymbol{\omega}=0$) and **rotational** flows (with vorticity)

(v) Bernoulli I is also valid for **compressible** flows or any **conservative** force $\mathbf{f}=-\nabla\phi$
(In that case, gz should be replaced by ϕ, the potential associated with the external force)

(vi) Bernoulli I for a general steady flow is equivalent to the conservation of total energy
(see GHP §5.3 & bonus slide)

En négligeant la gravité, on arrive à :

Stagnation pressure (incompressible flows)

If gravity is neglected, $\boxed{\text{Bernoulli I} \iff p + \frac{1}{2}\rho u^2 = p_{\text{stag}} = \text{constant along } \textbf{\textit{each}} \text{ streamline.}}$

Stagnation (or total) **pressure** $p_{\text{stag}} = \underbrace{p}_{\text{Static pressure}} + \underbrace{\frac{1}{2}\rho u^2}_{\text{Dynamic pressure } p_d}$

Stagnation point $p = p_{\text{stag}} = p_\infty + \frac{1}{2}\rho u_\infty^2$

Conclusions

(i) The **stagnation pressure** is the pressure reached at a **stagnation point** ($u = 0$).

(ii) Along a streamline, p is **larger** where the flow **slows** down.

Loi de Biot-Savart.

Biot-Savart law

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \iff \mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi}\iiint \frac{(\mathbf{x}-\mathbf{x}') \times \boldsymbol{\omega}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} dV + \mathbf{u}_{\text{irr.}}$$

Any irrotational flow ($\nabla \times \mathbf{u}_{\text{irr.}} = 0$)

- This provides the flow induced by **vorticity sources** (i.e. vortices, regions with $\boldsymbol{\omega} \neq 0$)

Vorticity transport/sources (see Lect. #5)

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\boldsymbol{\omega}\cdot\mathbf{u}}_{\text{Transport}} = \underbrace{\nabla\mathbf{u}\cdot\boldsymbol{\omega}}_{\text{Stretching/tilting}} + \cancel{\underbrace{\nu\nabla^2\boldsymbol{\omega}}_{\text{Viscous diffusion}}} + \underbrace{\frac{\nabla\rho \times \nabla p}{\rho^2}}_{\text{Baroclinic torque}}$$

INVISCID

if all external volume forces are conservative !

Barotropic fluid

A barotropic flow is a flow for which $\nabla p \times \nabla \rho = 0 \iff \rho = \rho(p)$ (and not $\rho = \rho(p, T)$)

Examples of barotropic flows $\rho(p)$:

- Any incompressible homogenous fluid ρ = constant.
- A perfect gas at constant temperature

$$\boxed{\textbf{For a barotropic inviscid fluid, } \frac{D\boldsymbol{\omega}}{Dt} = \nabla\mathbf{u}\cdot\boldsymbol{\omega}}$$

✓ Vorticity is transported *like a material vector* (see Lect. #4)

✓ In an *inviscid barotropic* flow, there is **no** source of vorticity!

i.e. no vorticity is created/removed: if there was none initially (irrotational), it stays that way!

=> Théorème de Lagrange (Écoulements poten-tiels)

Lagrange's theorem

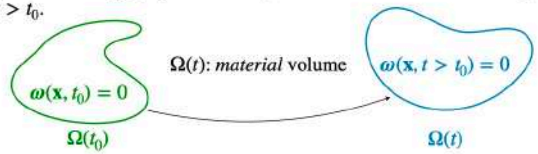
Lagrange's theorem

An **inviscid incompressible** flow that is **irrotational at a given time** t_0 remains **irrotational at later times** $t > t_0$.

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\boldsymbol{\omega}\cdot\mathbf{u}}_{\text{Transport}} = \underbrace{\nabla\mathbf{u}\cdot\boldsymbol{\omega}}_{\text{Stretching/tilting}}$$

There are NO vorticity sources $\longrightarrow$ *existing* vorticity is only transported and distorted.

If initially, $\boldsymbol{\omega} = 0$ in $\Omega(t_0)$ (e.g. fluid at rest), then $\boldsymbol{\omega}$ is also zero in the transported volume $\Omega(t)$ for $t > t_0$.



This result is applicable to any subpart Ω_1 of or the complete flow domain Ω_f !

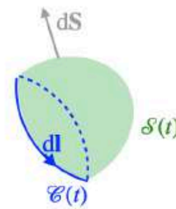
This theorem is essential for the study of potential flows: it guarantees their existence !

The theorem also applies to more general barotropic flows $\rho = \rho(p)$

Théorème de Kelvin :

Kelvin's theorem

Circulation $\Gamma(t) = \oint_{\mathscr{C}(t)} \mathbf{u}\cdot d\mathbf{l} = \iint_{\mathscr{S}(t)} \boldsymbol{\omega}\cdot d\mathbf{S}$



$\mathscr{C}$: **material contour**

(it follows the fluid in its motion)

Kelvin's theorem

In an inviscid **incompressible** flow, the **circulation** $\Gamma(t)$ around a **material contour** $\mathscr{C}(t)$ remains **constant** at all times:

$$\frac{d\Gamma(t)}{dt} = 0$$

The theorem also applies to more general barotropic flows $\rho = \rho(p)$

Ecoulements irrotationnels :

By definition, an irrotational flow has *no* vorticity: $\boldsymbol{\omega} = \nabla \times \mathbf{u} = 0$

$\Longrightarrow$ $\mathbf{u}$ derives from a scalar *potential*: $\mathbf{u} = \nabla \varphi$

cf slide 8.40 pour "méthode des images"

Second théorème de Bernoulli:

Bernoulli's 2nd theorem for irrotational flows (incompressible)

In an **inviscid irrotational** and ***incompressible*** flow,

$$\frac{\partial \varphi}{\partial t} + \frac{u^2}{2} + \frac{p}{\rho} + gz = F(t) \text{ with } \mathbf{u} = \nabla \varphi,$$

is uniform at each instant in the entire domain.

Distinction à faire entre les 2 thm de Bernoulli :

Bernoulli's theorems (I vs. II)

Bernoulli I (steady flows)	**Bernoulli II** (potential flows)
Conditions of applicability: Inviscid flow, Steady flow	**Conditions of applicability:** Inviscid flow, Irrotational (potential) flow
Conserved quantity: $H = \frac{u^2}{2} + \frac{p}{\rho} + \phi$	**Conserved quantity:** $F(t) = \frac{\partial \varphi}{\partial t} + \frac{u^2}{2} + \frac{p}{\rho} + \phi$
Where? On each streamline (independently!)	**Where?** Everywhere

⚠ Make sure to use the correct version (and to explain why!)

d'Alembert's paradox

There is **no net force** on a cylinder in **steady irrotational flows** (neither drag nor lift!)

cf amphi 8.50 pour ex d'applica°

Kutta-Joukowski Theorem

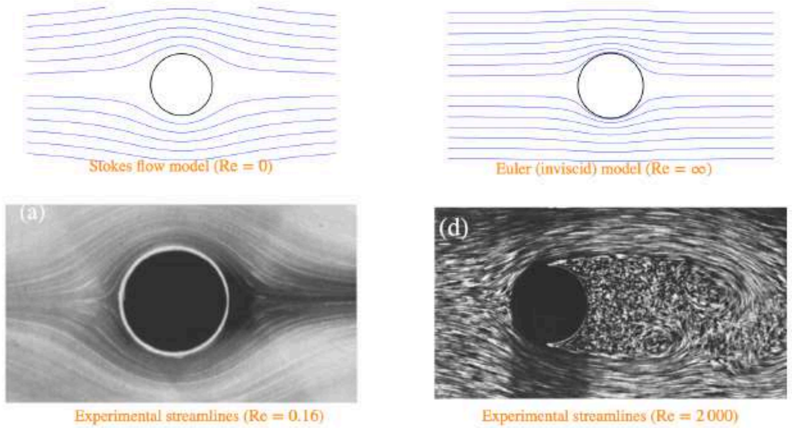
For any body shape in a (steady) 2D inviscid & irrotational flow

$$\mathbf{F} = \rho \mathbf{U} \times \boldsymbol{\Gamma}, \qquad \boldsymbol{\Gamma} = \int_{\mathcal{S}} \mathbf{n} \times \mathbf{u} \, dl$$

Lecture #9 - Advanced Topics

Flow over a bluff body at various Re

Re $\ll$ 1 | Re $\gg$ 1



Good agreement ! | **Not so good…**

La limite d'Euler n'est pas toujours représentative

Equations couche-limite.

Boundary Layer Equations

Boundary Layer Equations ($y = O(\delta_L)$)

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$\rho\left(u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \eta\frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

Et les conditions limites?

It all depends on the scale you are looking at…

⚠CORRECTED

Boundary Layer Equations (valid if $y \ll L$)

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$\rho\left(u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \eta\frac{\partial^2 u_x}{\partial y^2}$$

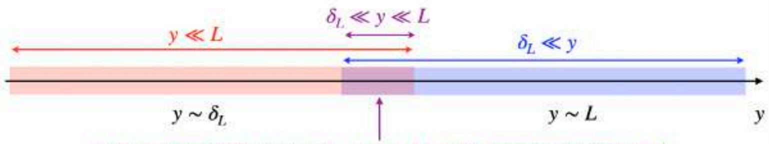
$$\frac{\partial p}{\partial y} = 0$$

Euler Equations (valid if $y \gg \delta_L$),

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$\rho\left(u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x}$$

$$\rho\left(u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y}$$



In the **intermediate** region ($\delta_L \ll y \ll L$), **both approaches** are **valid**:
They describe the **same function** ⟶ they must **match**!

See bonus slide for more details

For $\delta_L \ll y \ll L$, the inviscid and Boundary Layer solutions are **both** valid
$\Longrightarrow$ they **match!**

Matching condition

Boundary layer solution "far from the wall"

$$u_x^{BL}(x, y \gg \delta_L) \sim u_x^e(x, y \ll L)$$

$$p^{BL}(x, y \gg \delta_L) \sim p^e(x, y \ll L)$$

Inviscid solution "at the wall"

Cas Blasius.

Boundary layer over a flat plate (Blasius)

Canonical case: uniform flow over a semi-infinite plate (No intrinsic length scale)

Inviscid (trivial) solution: $\mathbf{u}_e(\mathbf{x}) = U\mathbf{e}_x$ & $p_e(\mathbf{x}) = p_0$ (uniform flow)

Boundary layer equations

$$\frac{\partial p}{\partial y} = 0 \quad \Longrightarrow \quad p = p_e = p_0$$

$$u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = \underbrace{\cancel{-\frac{1}{\rho}\frac{\partial p}{\partial x}}}_{=0} + \nu\frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$
$$u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = \nu\frac{\partial^2 u_x}{\partial y^2}$$
$u_x = u_y = 0$ @ $y = 0$ (No-slip wall)
$u_x \to U$ @ $y \to \infty$ (Matching)
$u_x = U$ @ $x = 0$ (Edge)

2D incompressible flow

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$
$$u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = \nu\frac{\partial^2 u_x}{\partial y^2}$$
$u_x = u_y = 0$ @ $y = 0$ (No-slip wall)
$u_x \to U$ @ $y \to \infty$ (Matching)
$u_x = U$ @ $x = 0$ (Edge)

$$\xrightarrow{u_x = \frac{\partial \psi}{\partial y},\ u_y = -\frac{\partial \psi}{\partial x}}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3}$$
$\psi = \frac{\partial \psi}{\partial y} = 0$ @ $y = 0$ (No-slip wall)
$\psi \to Uy$ @ $y \to \infty$ (Matching)
$\frac{\partial \psi}{\partial y} = U$ @ $x = 0$ (Edge)

Self-similar solution

Why do we suspect a self-similar solution?

The problem has **no intrinsic length scale!** (Semi-infinite plate of zero thickness)

$$\boxed{\begin{gathered}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\nu\frac{\partial^3\psi}{\partial y^3}\\ \psi=0,\quad \frac{\partial\psi}{\partial y}=0 \text{ @ } y=0\\ \psi\to Uy \text{ @} y\to\infty\\ \frac{\partial\psi}{\partial y}=U \text{ @ } x=0\end{gathered}}$$

$\psi = a\hat{\psi}$, $x = b\hat{x}$, $y = c\hat{y}$, $\nu = d\hat{\nu}$, $U = e\hat{U}$

$$\longrightarrow\ \boxed{\begin{gathered}\frac{a^2}{bc^2}\left(\frac{\partial\hat\psi}{\partial\hat y}\frac{\partial^2\hat\psi}{\partial\hat x\partial\hat y}-\frac{\partial\hat\psi}{\partial\hat x}\frac{\partial^2\hat\psi}{\partial\hat y^2}\right)=\frac{ad}{c^3}\nu\frac{\partial^3\hat\psi}{\partial\hat y^3}\\ a\hat\psi=0,\quad \frac{a}{c}\frac{\partial\hat\psi}{\partial\hat y}=0 \text{ @ } c\hat y=0\\ a\hat\psi\to ce\hat U\hat y \text{ @ } \hat y\to\infty\\ \frac{a}{b}\frac{\partial\hat\psi}{\partial\hat y}=e\hat U \text{ @ } \hat x=0\end{gathered}}$$

The problem is left unchanged if $\frac{ac}{bd}=\frac{a}{be}=1 \Leftrightarrow \frac{a}{\sqrt{bed}}=\frac{c}{\sqrt{bd/e}}=1$ ⚠CORRECTED

⟶ Search for $\psi = U\delta(x)f(\xi)$ with $\xi=\frac{y}{\delta(x)}$ and $\delta(x)=\sqrt{\frac{\nu x}{U}}$

Reminder: $\frac{a}{\sqrt{bed}}=\frac{c}{\sqrt{bd/e}}=1$ suggests that the corresponding combinations $\psi/\sqrt{\nu x U}$ and $\xi = y/\sqrt{\nu x/U}$ are the ones to appear in the self-similar solution

BONUS SLIDE

Self-similar solution

We search for $\psi = U\delta(x)f(\xi)$ with $\xi=\frac{y}{\delta(x)}$ and $\delta(x)=\sqrt{\frac{\nu x}{U}}$

(See also GHP §10.4.1)

After simplification:

$$f'''=-\underbrace{\frac{U\delta(x)\delta'(x)}{\nu}}_{\text{Function of } x}ff''$$

Self-similarity?
The equation should depend only on ξ

$$\boxed{\frac{\delta\delta' U}{\nu}=\text{dimensionless \& constant}} \longrightarrow \delta(x)=\sqrt{\frac{\nu x}{U}}$$

Boundary layer thickness

$$\boxed{f'''=-\frac{ff''}{2}}$$

Boundary conditions:

$\psi=0,\ \frac{\partial\psi}{\partial y}=0 \text{ @ } y=0 \longrightarrow f(0)=0$ $\quad y=0 \leftrightarrow \xi=0$

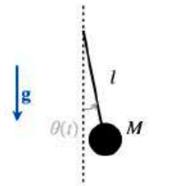
$\psi\to Uy \text{ @ } y\to\infty \longrightarrow f'(\infty)=1$ $\quad y=\infty \leftrightarrow \xi=\infty$

$\frac{\partial\psi}{\partial y}=U \text{ @ } x=0 \longrightarrow f'(\infty)=1$ $\quad x=0 \leftrightarrow \xi=\infty$

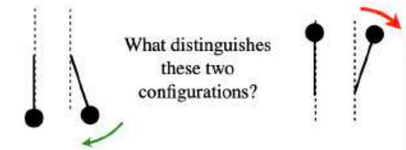
Lecture #10 - Instabilités et singularités

What do we mean by (in)stability?

Classical example: the pendulum

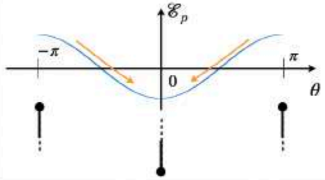


Two possible equilibria



What distinguishes these two configurations?

Potential energy
$\mathscr{E}_p(t)=Mgz(t)=Mgl[1-\cos\theta(t)]$



⟶ Stable:
- Small perturbations do not grow in time
- $\mathscr{E}_p$ is *minimum*

⟶ **Unstable:**
- Small perturbations grow (exponentially) in time
- $\mathscr{E}_p$ is *maximum*

Equilibrium: *position* (*state*) of the system that does not change in time (i.e. *stationary* solution of the evolution equations)

Rayleigh-Taylor instability

Initial state: flat interface, heavier fluid on top

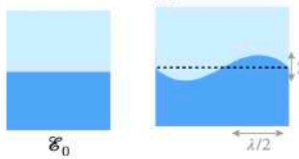
Equilibrium configuration

What mechanisms oppose/slow down the overturning?

Viscosity: dissipation of the released potential energy through viscous stresses
Inertia: reduces the acceleration of the fluid

Slow down the perturbation growth

Surface tension: **opposes** the deformation of the flat interface (**cost** in surface energy)



$\Delta\mathscr{E}=\mathscr{E}-\mathscr{E}_0=\Delta\mathscr{E}_g+\Delta\mathscr{E}_\gamma$

$\Delta\mathscr{E}_g\sim-(\rho_1-\rho_0)g\lambda a^2$ (released potential energy)

$\Delta\mathscr{E}_\gamma\sim\gamma\lambda\left(\frac{a^2}{\lambda^2}\right)$ (surface gained)

Instability: $\Delta\mathscr{E}\lesssim 0$ (*net* energy release)
$\Longleftrightarrow\ \lambda\gtrsim l_c$ with $l_c=\sqrt{\frac{\gamma}{g\Delta\rho}}$
⚠ TYPO

The capillary length l_c is a cut-off scale:
Longer waves are unstable
Shorter waves are stable

Méthode d'analyse linéaire des instabilités

BONUS SLIDE

Linear stability analysis - Method

Equations of motion:

$\longrightarrow$ Navier-Stokes (or Euler, Stokes, …)

e.g. incompressible & homogenous fluid

$$(S):\quad \nabla\cdot\mathbf{u}=0,\qquad \rho\left(\frac{\partial \mathbf{u}}{\partial t}+\nabla\mathbf{u}\cdot\mathbf{u}\right)=-\nabla p+\nabla^2\mathbf{u},\ \text{and boundary conditions…}$$

1. Base state:

$\longrightarrow (\mathbf{u},p)=(\bar{\mathbf{u}}(\mathbf{x}),\bar{p}(\mathbf{x}))$ (steady flow)

- State of reference whose *stability* is investigated
  (possible generalization to analyse the stability of *unsteady* flows)
- $(\bar{\mathbf{u}},\bar{p})$ is a solution of (S):
  $(\bar{S}):\quad \nabla\cdot\bar{\mathbf{u}}=0,\qquad \rho\nabla\bar{\mathbf{u}}\cdot\bar{\mathbf{u}}=-\nabla\bar{p}+\nabla^2\bar{\mathbf{u}}$, and boundary conditions…

2. Perturbation around the base state:

$\longrightarrow [\mathbf{u},p]=[\bar{\mathbf{u}}(\mathbf{x}),\bar{p}(\mathbf{x})]+[\mathbf{u}'(\mathbf{x},t),p'(\mathbf{x},t)]$

Linear stability:

The perturbation is small compared to the base state: $|\mathbf{u}'|\ll|\bar{\mathbf{u}}|$, $|p'|\ll|\bar{p}|$

BONUS SLIDE

Linear stability analysis

3. Perturbation equations:

$[\mathbf{u},p]=[\bar{\mathbf{u}}(\mathbf{x}),\bar{p}(\mathbf{x})]+[\mathbf{u}'(\mathbf{x},t),p'(\mathbf{x},t)]$ with $[\mathbf{u}'(\mathbf{x},t),p'(\mathbf{x},t)]\sim\varepsilon\,[\bar{\mathbf{u}}(\mathbf{x}),\bar{p}(\mathbf{x})]$

$\longrightarrow$ expand the original system keeping only terms up to $O(\varepsilon)$

$$(S)\quad \begin{cases}\nabla\cdot\mathbf{u}=0\\[4pt] \dfrac{\partial\mathbf{u}}{\partial t}+\nabla\mathbf{u}\cdot\mathbf{u}=-\dfrac{\nabla p}{\rho_0}\end{cases}\ \xrightarrow[+[\mathbf{u}'(\mathbf{x},t),p'(\mathbf{x},t)]]{[\mathbf{u},p]=[\bar{\mathbf{u}}(\mathbf{x}),\bar{p}(\mathbf{x})]}\ \begin{cases}\cancel{\nabla\cdot\bar{\mathbf{u}}}+\nabla\cdot\mathbf{u}'=0\\[4pt] \cancel{\dfrac{\partial\bar{\mathbf{u}}}{\partial t}}+\dfrac{\partial\mathbf{u}'}{\partial t}+\cancel{\nabla\bar{\mathbf{u}}\cdot\bar{\mathbf{u}}}+\nabla\mathbf{u}'\cdot\bar{\mathbf{u}}\\ \quad+\nabla\bar{\mathbf{u}}\cdot\mathbf{u}'+\nabla\mathbf{u}'\cdot\mathbf{u}'=-\cancel{\dfrac{\nabla\bar{p}}{\rho_0}}-\dfrac{\nabla p'}{\rho_0}\end{cases}$$

Base state equation

Higher order $O(\varepsilon^2)$

Linearized equations

$$(S')\quad \begin{cases}\nabla\cdot\mathbf{u}'=0\\[4pt] \dfrac{\partial\mathbf{u}'}{\partial t}+\nabla\mathbf{u}'\cdot\bar{\mathbf{u}}+\nabla\bar{\mathbf{u}}\cdot\mathbf{u}'=-\dfrac{\nabla p'}{\rho_0}\end{cases}$$

All terms are $O(\varepsilon)$ and **linear** functions of one of the perturbed fields

(Do the same thing on the boundary conditions)

Linear stability analysis

4. Temporal stability analysis & Normal modes

For illustration, we assume (i) that the **full flow** is **two-dimensional** (x,z) and (ii) that the **base state** depends **only on** z

Linearized equations

$$(S')\quad \begin{cases}\nabla\cdot\mathbf{u}'=0\\[4pt] \dfrac{\partial\mathbf{u}'}{\partial t}+\nabla\mathbf{u}'\cdot\bar{\mathbf{u}}+\nabla\bar{\mathbf{u}}\cdot\mathbf{u}'=-\dfrac{\nabla p'}{\rho_0}\end{cases}$$

+ linearized boundary conditions

- **All** terms are **linear** functions of $(\mathbf{u}',p')$
- **Invariant** problem w.r.t the origin of (x,t)
  (e.g. $t\to t+\tau$ does not change anything)

$\longrightarrow$ Search for perturbations of the form $[\mathbf{u}'(\mathbf{x},t),p'(\mathbf{x},t)]=\mathrm{e}^{\sigma t+\mathrm{i}kx}\,[\mathbf{U}(z,k),P(z,k)]$

For a given **wavenumber** k, finding the acceptable **mode shapes** $[\mathbf{U}(z,k),P(z,k)]$ and corresponding **growth rate** σ results in an **eigenvalue problem**

$\sigma(k)$: dispersion relation

- If $\mathrm{Re}(\sigma)>0$: exponential **growth** $\longrightarrow$ **unstable mode**
- If $\mathrm{Re}(\sigma)<0$: exponential **decay** $\longrightarrow$ **stable mode**
- If $\mathrm{Im}(\sigma)\neq 0$: **oscillatory mode**

Finding the eigenvalue problem

4. Temporal stability analysis & Normal modes

Linearized equations

$$(S')\quad \begin{cases}\nabla\cdot\mathbf{u}'=0\\[4pt] \dfrac{\partial\mathbf{u}'}{\partial t}+\nabla\mathbf{u}'\cdot\bar{\mathbf{u}}+\nabla\bar{\mathbf{u}}\cdot\mathbf{u}'=-\dfrac{\nabla p'}{\rho_0}\end{cases}$$

+ linearized boundary conditions

$[\mathbf{u}'(\mathbf{x},t),p'(\mathbf{x},t)]=\mathrm{e}^{\sigma t+\mathrm{i}kx}\,[\mathbf{U}(z,k),P(z,k)]$

Derivatives of $\mathbf{u}'$ or p' with respect to t (resp. x) amount to multiplications of $\mathbf{U}$ or P by σ (resp. $\mathrm{i}k$)

$$\nabla\cdot\mathbf{u}'\longrightarrow\left[\mathrm{i}kU_x+\frac{\mathrm{d}U_z}{\mathrm{d}z}\right]\mathrm{e}^{\sigma t+\mathrm{i}kx}$$

$$\frac{\partial\mathbf{u}'}{\partial t}\longrightarrow\sigma\,\mathbf{U}\,\mathrm{e}^{\sigma t+\mathrm{i}kx}$$

$$\nabla\mathbf{u}'\cdot\bar{\mathbf{u}}\longrightarrow\left[\mathrm{i}k\bar{u}_x\mathbf{U}+\bar{u}_z\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}z}\right]\mathrm{e}^{\sigma t+\mathrm{i}kx}$$

$$\nabla\bar{\mathbf{u}}\cdot\mathbf{u}'\longrightarrow\left[\mathrm{i}kU_x\bar{\mathbf{u}}+U_z\frac{\mathrm{d}\bar{\mathbf{u}}}{\mathrm{d}z}\right]\mathrm{e}^{\sigma t+\mathrm{i}kx}$$

$$\nabla p'\longrightarrow\mathrm{i}kP\,\mathrm{e}^{\sigma t+\mathrm{i}kx}$$

$$\mathrm{i}kU_x+\frac{\mathrm{d}U_z}{\mathrm{d}z}=0$$

$$\sigma\,\mathbf{U}+\mathrm{i}k\bar{u}_x\mathbf{U}+\bar{u}_z\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}z}+\mathrm{i}kU_x\bar{\mathbf{u}}+U_z\frac{\mathrm{d}\bar{\mathbf{u}}}{\mathrm{d}z}=-\frac{\mathrm{i}kP}{\rho_0}$$

+ boundary conditions on $\mathbf{U},P$

For a given **wavenumber** k, this can be seen as an eigenvalue problem for σ (eigenvalue) and $[\mathbf{U},P]$ (eigenvector)