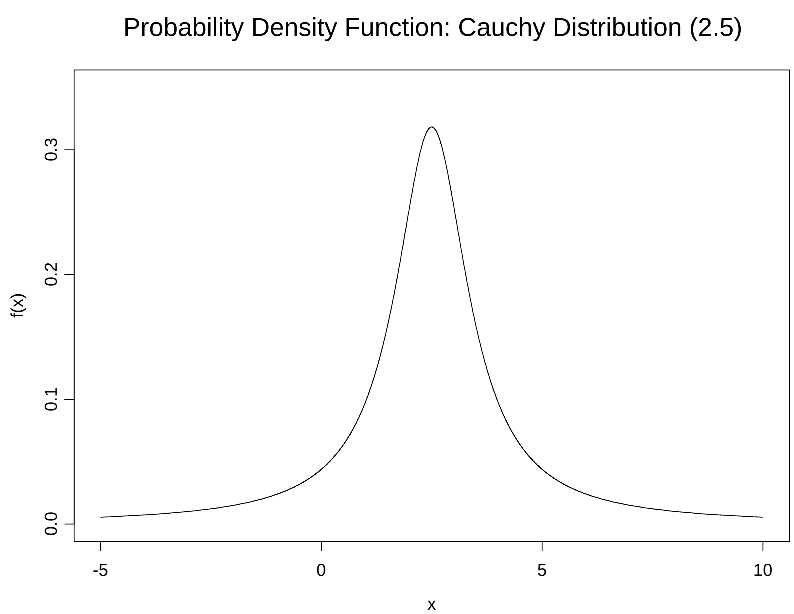
What if There Were No Law of Large Numbers?

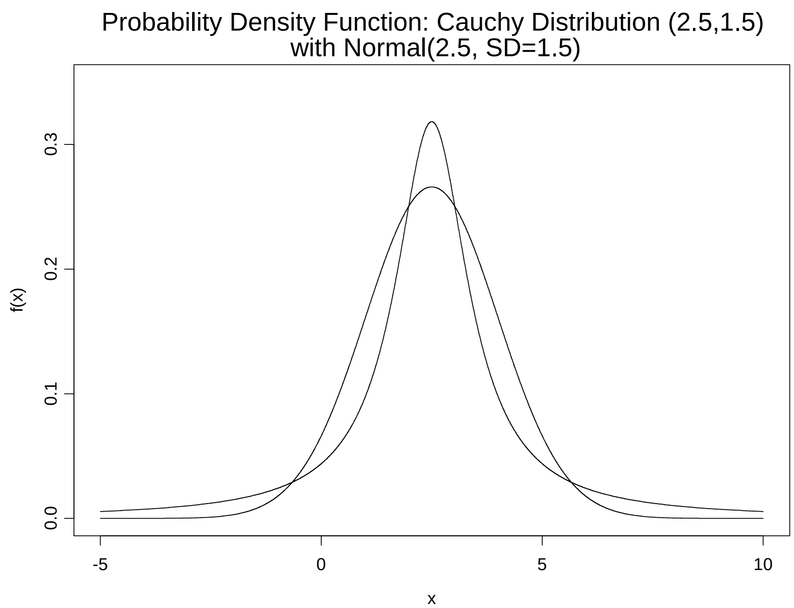
We said that the Law of Large Numbers applies whenever we make independent observations on a random variable *X* that has an expected value.

In those cases the Law guarantees that the sequence of sample means will eventually converge to the expected value of the random variable, E(*X*), or the "mean of the distribution."

But not all distributions have an expected value. Here is an example of one that does not:



This is the density function of a random variable that has a "Cauchy" probability distribution. Here it is again, this time with the normal density with mean 2.5 and standard deviation 1.5 shown for comparison.



The Cauchy probability density function is

$$f(x) = \frac{1}{\pi(1+(x-\theta)^2)} \qquad -\infty < x < \infty$$

The density is centered at θ (θ = 2.5 in the above graphs), and θ is the <u>**median**</u> of the Cauchy distribution.

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If this distribution did have an expected value, that expected value would also equal θ, the center of the distribution. But it does not. In this case (Cauchy distribution) the integral that we use to define the expected value of *X* does not exist.

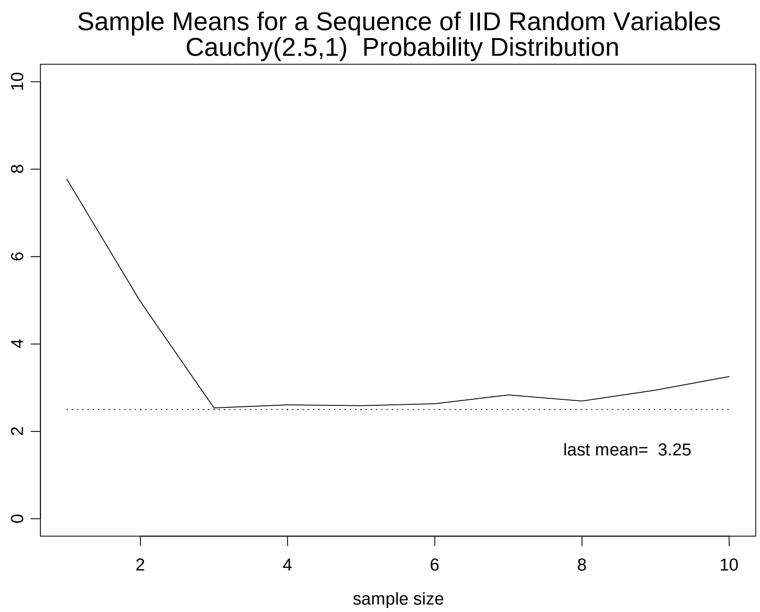
$$E(X) = \int_{-\infty}^{\infty} x \; f(x) \, dx = ???$$

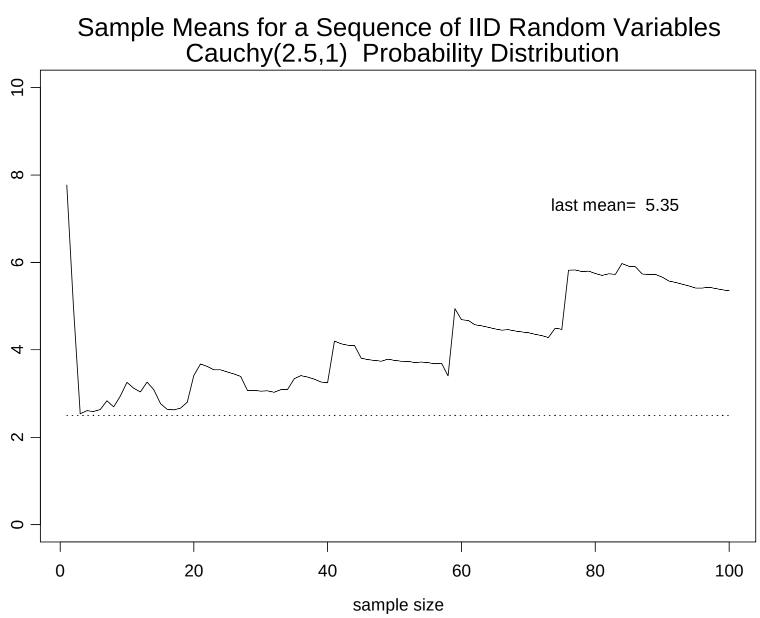
Big deal? So what?

Let's see what happens if we repeat the experiment that we used to demonstrate the Law of Large Numbers in action:

Generate a sequence of independent random variables, all with this Cauchy distribution (centered at 2.5), and look at the sequence of sample means.

Here are the first 10 means:





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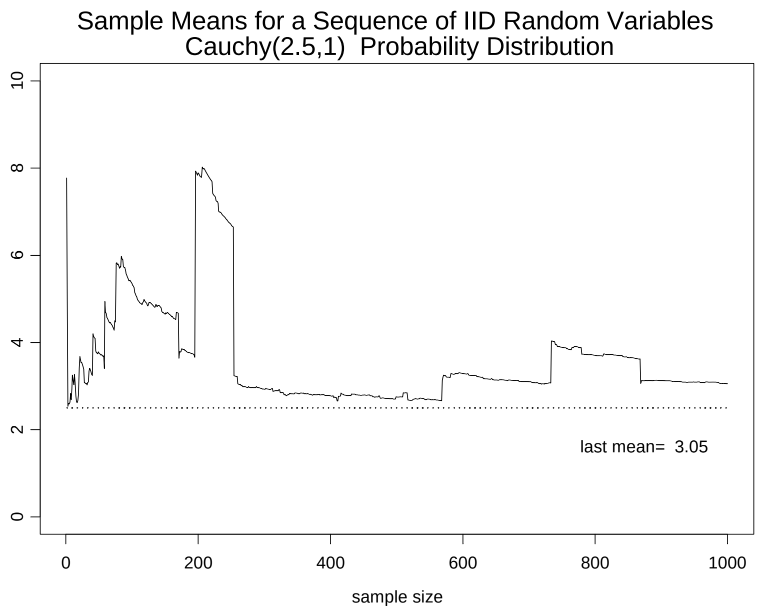
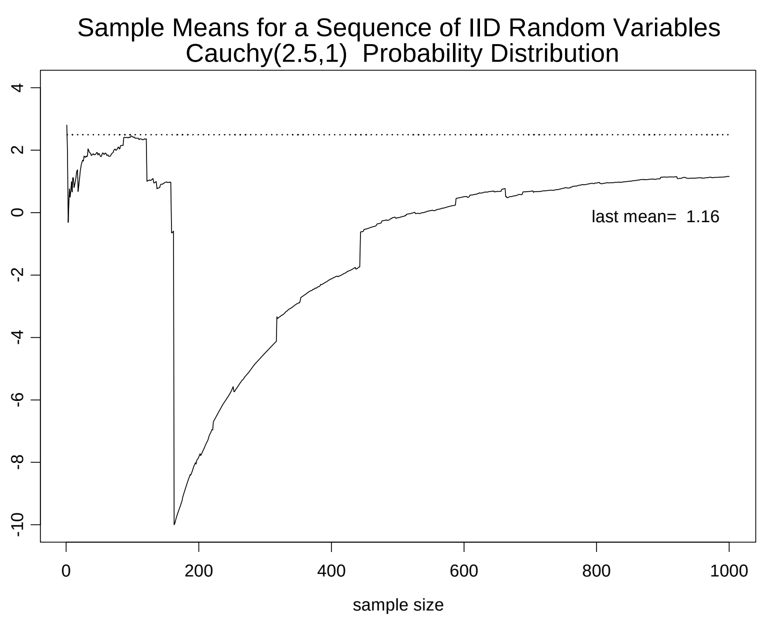
Here is 1000 observations:



Close enough you say? Here is another 1000:



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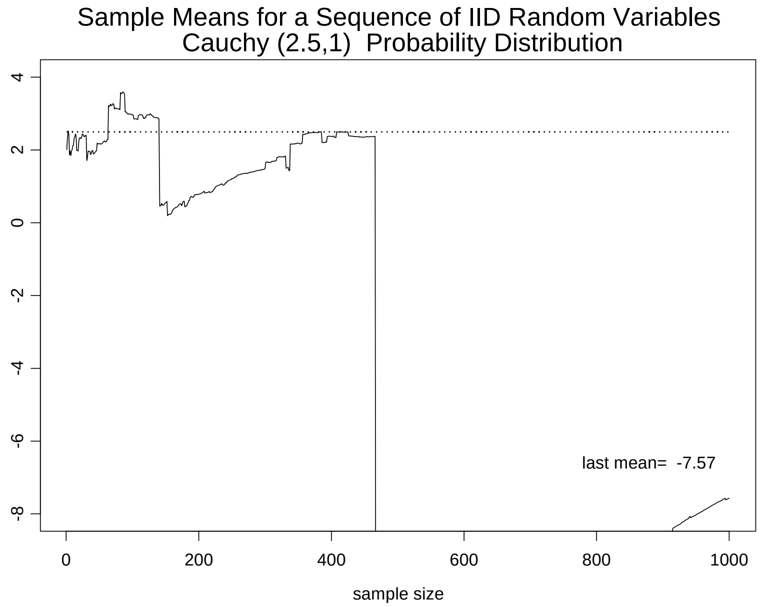
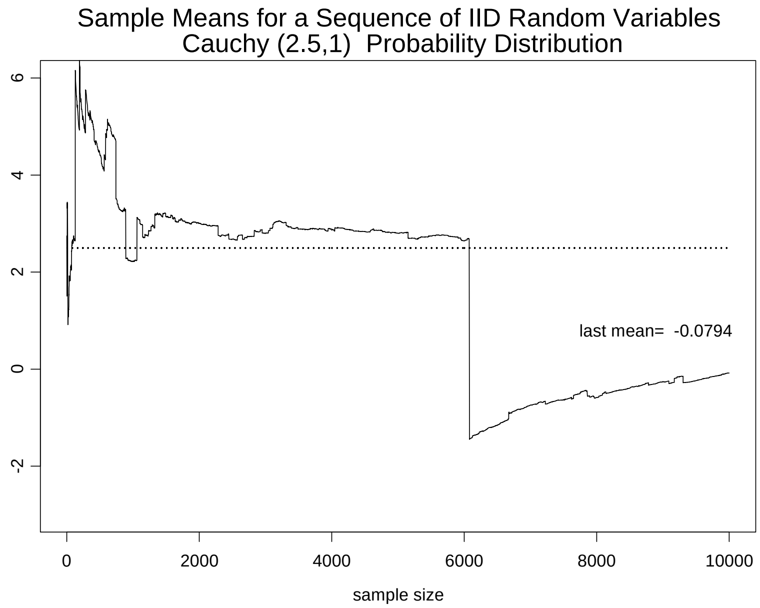
Here is another 1000:



And here is 10,000:



The sequence of sample means never settles down. It does not converge to the value 2.5 (or to any other value).

※ In this case, increasing the sample size does nothing because we cannot learn about E(X) by watching the sequence of sample means.

Does the fact that the sequence of means will never settle down and eventually leads us to θ mean that we cannot learn about the value of this parameter (θ) by making more and more independent observations on the Cauchy distribution?

No. We must look for a different way to estimate θ because the sample mean doesn't work.

Here's an idea -- the parameter we want to estimate is the median of the distribution. How about estimating it by the sample median? If instead of the sample means we look at the sequence of sample medians, everything is OK.

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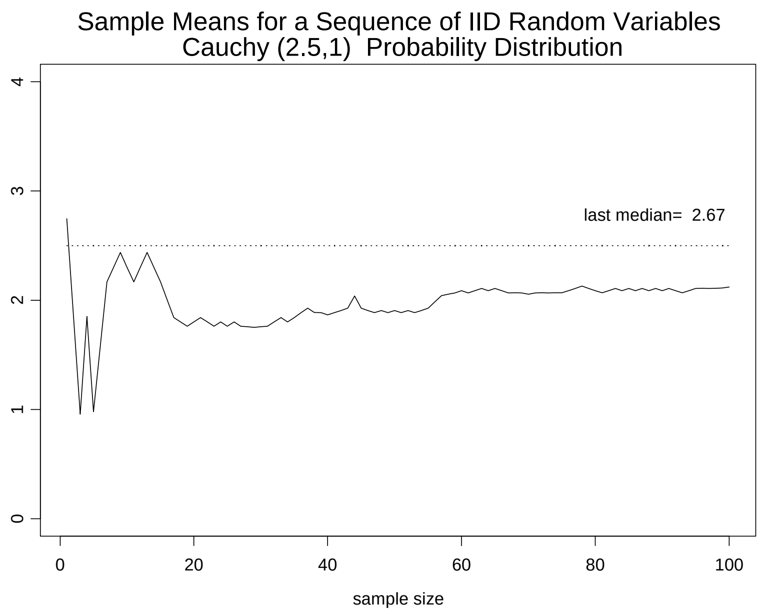
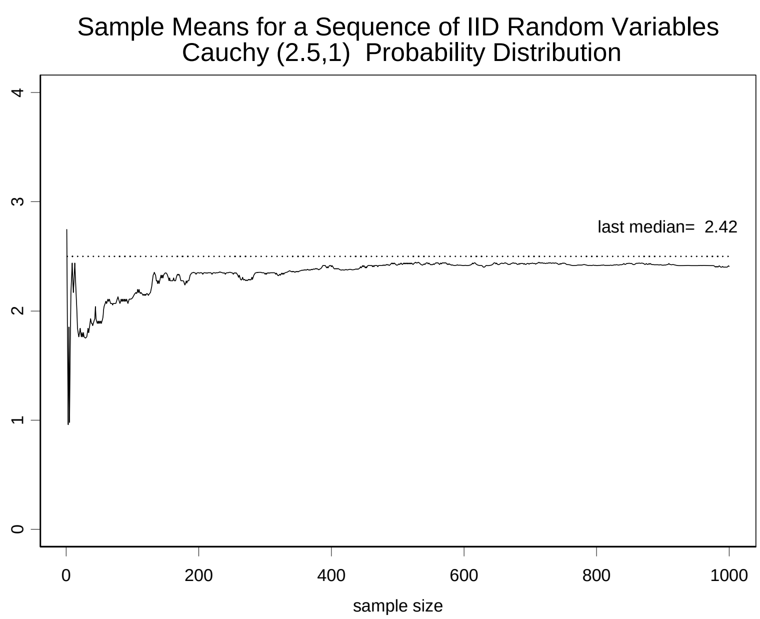
Here is the first 100 sample medians for the previous sequence of 10,000 Cauchy(2.5,1) random variables.



We only need 1,000 observations to see that the Law is again working:



What if there was no Law?

The explanation for why the sequence of sample <u>medians</u> goes to the <u>median</u> of the distribution is also found in the Law of Large Numbers.