Sign Test Exercise

For the problem described in Pagano and Gauvreau, Chapter 11, exercise 1, suppose that we are unwilling to assume that the differences have a normal probability distribution. So instead of the t-test we decide to use a simple "non-parametric" test, the sign test. The sign test is based on the number of positive differences (the number of persons whose LDL cholesterol level on the cornflakes diet is greater than on the oat bran diet).

Pagano, Page 278

	LDL (mmol/l)		
Subject	Corn Flakes	Oat Brand	Sign
1	4.61	3.84	+
2	6.42	5.57	+
3	5.40	5.85	-
4	4.54	4.80	-
5	3.98	3.68	+
6	3.82	2.96	+
7	5.01	4.41	+
8	4.34	3.72	+
9	3.8	3.49	+
10	4.56	3.84	+
11	5.35	5.26	+
12	3.89	3.73	+
13	2.25	1.84	+
14	4.24	4.14	+

Under the hypothesis of no difference between the two diets the probability that a given individual's difference will be positive is 1/2. Under the alternative (LDL cholesterol tends to be greater on the cornflake diet), a positive difference is more likely than a negative one.

(a) Give a reasonable probability model (one with one unknown parameter) for the probability distribution of the number of positive differences in a sample of $n = 14$ subjects.

*The number of positive differences is a random variable with a **binomial probability distribution**. The number of trials is **n = 14** and the success probability, θ, is the probability that for a given subject, the difference*

D = (cholesterol level on cornflake diet)- (level on oat bran diet)

will be positive: $\boldsymbol{\theta = P(D > 0)}$.

(b) In terms of the parameter in your model, state null and alternative hypotheses analogous to those that we used in the original (Pagano) version of this problem.

The general null hypothesis is that diet has no effect on cholesterol level. This would imply that the difference D is just as likely to be negative as positive: $H_0: \theta = 1/2$*. Alternatively we could state the null hypothesis as "LDL level does not tend to be higher on cornflake diet"* $H_0: \theta \leq 1/2$*.*

Either way, the alternative is $H_A: \theta > 1/2$*.*

(c) The sign test rejects the null hypothesis if the number of positive differences observed is greater than or equal to some critical value, k. If we use k = 13 as the critical value, what is our Type I error probability?

Let K be the number of positive differences. Then if we reject the null hypothesis when K is greater than or equal to thirteen our Type I error probability is

$$P(K \geq 13) = \binom{14}{13}(1/2)^{13}(1 - 1/2)^{1} + (1/2)^{14} = 15(1/2)^{14} = 0.0009$$

(d) If we use k = 13 as the critical value, what is the power of the sign test if in fact the probability that the difference is positive is 5/8? ...3/4? ...7/8?

If θ = 5/8 then the power is

$$P_{\theta=5/8}(K \geq 13 \mid \theta = 5/8) = \binom{14}{13}(5/8)^{13}(3/8)^{1} + (5/8)^{14} = 0.013$$

If θ = 6/8 (i.e. θ = 3/4=6/8) then the power is

$$P_{\theta=3/4}(K \geq 13 \mid \theta = 6/8) = \binom{14}{13}(3/4)^{13}(1/4)^{1} + (3/4)^{14} = 0.101$$

If θ = 7/8 then the power is

$$P_{\theta=7/8}(K \geq 13 \mid \theta = 7/8) = \binom{14}{13}(7/8)^{13}(1/8)^{1} + (7/8)^{14} = 0.463$$

(e) If we want a test with α no greater than 0.05, what critical value should we use?

$$P(\ K\ \geq 13 \mid \theta = 1/2\ ) = 0.001$$

$$P(\ K\ \geq 12 \mid \theta = 1/2\ ) = 0.006$$

$$P(\ K\ \geq 11 \mid \theta = 1/2\ ) = 0.029$$

$$P(\ K\ \geq 10 \mid \theta = 1/2\ ) = 0.090$$

Critical value of ***11*** *gives Type I error probability of 0.03. The next smaller critical value, 10, would give an error probability of 0.09, which is too large (greater than 0.05). Of course any value less than 10 would give an even larger error probability.*

(f) If we apply the sign test to the data what is the p-value?

The number of positive differences in the sample is 12, so

$$p\text{-}value = P_{\theta=1/2}(\ K\ \geq 12 \mid \theta = 1/2\ ) = 0.006$$

(already calculated in part (e)).

(g) Suppose the probability distribution of the differences is actually normal with mean μ = 0.3 and σ = 0.4.

(i) What is the power of the sign test (with k=13)?
[Hint: What is P(D > 0)?]

The probability of a positive difference is

$$\theta = P(D > 0) = P\left(\frac{D-0.3}{0.4} > -\frac{0.3}{0.4}\right) = P(Z > -0.75) = 0.773$$

so the power (the probability of 13 or more positive differences) is

$$P(K \geq 13 | \theta = 0.773) = \binom{14}{13}(0.773)^{13}(0.227)^{1} + (0.773)^{14} = 0.140$$

(ii) Calculate the power of the z-test with the same α, 0.001 (i.e. the z-test with critical value 3.090). Which test has greater power, the z-test or the sign test? Can you explain why the sign test has less power?

The power of the z test is

$$P\left(\frac{\sqrt{14}\ \overline{D}}{0.4} > 3.090\right) = P\left(\frac{\sqrt{14}\,(\overline{D}-0.3)}{0.4} > 3.090 - \frac{0.3\sqrt{14}}{0.4}\right)$$

$$= P(\ Z\ > 0.284\ ) = 0.388$$

The sign test is less powerful than the z test because it ignores the magnitudes of the differences, looking only at their signs. Because it ignores this important information, it loses power. This is the price it must pay for freedom from the z-test's assumption of a normal distribution with known variance.