CS 559 Homework 1

Stephen Szemis

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Problem 1:
Suppose you have a set of medical data and you are looking at the relation between being a football player and having a concussion at some point in life. Say event A is that you have had a concussion at some point and event B is that you are a football player. $P(A|B)$ is therefore the probability that you've had a concussion given that you are a football player; while $P(B|A)$ is the probability that you are a football player given that you've had a concussion. We know that football players are more likely to have concussions when compared to the general population, so it's pretty clear that $P(A|B)$ is not the same as $P(B|A)$.

Problem 2:
Show relation between independence and correlation.

1. Given X and Y are two continuous independent random variables we know

$$f(X,Y) = f_x(X)f_y(Y)$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$E(XY) = E(X)E(Y) - E(X)E(Y)$ for independent variables therefore $Cov(X,Y) = E(X)(Y) - E(X)E(Y) = 0$ which show that they are uncorrelated.

2. Suppose $X \sim Uniform[-1,1]$ and $Y = X^2$

First let's show that this is uncorrelated.

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$Cov(X,Y) = E(X * X^2) - E(X)E(X^2) = E(X^3) - E(X)E(X^2)$$

$$E(X) = \frac{1}{2}(1-1) = 0$$

$$E(X^3) = \frac{1-1}{(3+1)(-2)} = 0$$

$$Cov(X,Y) = 0 - 0 * E(X^2) = 0$$

So they are uncorrelated, and they are clearly not independent since X determines Y.

Problem 3:
We start from the simple Discriminant function

$$g_j(\mathbf{x}) = ln(P(\mathbf{x}|\omega_j)) + ln(\omega_j)$$

We need to expand this out. If we want the probability of a series of independent events, we can simply multiply the probability of each "individual" event. So we get.. . .

$$P(\mathbf{x}|\omega_j) = \prod_{i=1}^{d} P(x_i|\omega_j)$$

Now we need to know what $P(x_i|\omega_j)$ is in terms of P_{ij} (as defined in the problem). P_{ij} is the probability of $x_i = 1$ given event ω_j. In order to find the probability of any value of x_i we need to account for both when it could be 0 or 1. This is very similar to the binomial distribution but for a very special case. We can formalize this logic as:

$$P(x_i|\omega_j) = \binom{1}{1}(P_{ij})^{x_i} * (1 - P_{ij})^{1-x_i}$$

Simplified and plugged into the previous product we get.

$$P(\mathbf{x}|\omega_j) = \prod_{i=1}^{d}(P_{ij})^{x_i} * (1 - P_{ij})^{1-x_i}$$

Which, when placed in our Discriminant function gives us.

$$g_j(\mathbf{x}) = ln(\prod_{i=1}^{d}(P_{ij})^{x_i} * (1 - P_{ij})^{1-x_i}) + ln(\omega_j)$$

Thanks to logarithmic properties we can simplify this into.

$$g_j(\mathbf{x}) = \sum_{i=1}^{d} \big(x_i ln(P_{ij}) + (1 - x_i) ln(1 - P_{ij})\big) + ln(P(\omega_j))$$

Which with one more division and some splitting can become our final equation.

$$g_j(\mathbf{x}) = \sum_{i=1}^{d} \frac{(1 - x_i)}{(1 - x_i)} * \big(x_i ln(P_{ij}) + (1 - x_i) ln(1 - P_{ij})\big) + ln(P(\omega_j))$$

$$g_j(\mathbf{x}) = \sum_{i=1}^{d} \big(x_i \frac{ln(P_{ij})}{(1 - x_i)}\big) + \sum_{i=1}^{d} ln(1 - P_{ij}) + ln(P(\omega_j))$$

Problem 4:
We will first list all known information.

$$P(x|\omega_1) = N(4, 1)$$

$$P(x|\omega_2) = N(8, 1)$$

$$P(\omega_2) = \frac{1}{4}$$

$$\lambda = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$$

We can find the prior for ω_1 easily from the other prior.

$$P(\omega_1) = \frac{3}{4}$$

We can write the Normal distribution explicitly.

$$P(x|\omega_1) = \frac{1}{2\pi} e^{-\frac{(x-4)^2}{2}}$$

$$P(x|\omega_1) = \frac{1}{2\pi} e^{-\frac{(x-10)^2}{2}}$$

And we can then calculate the LHS of our decision rule.

$$\frac{P(x|\omega_1)}{P(x|\omega_2)} = \frac{\frac{1}{2\pi} e^{-\frac{(x-4)^2}{2}}}{\frac{1}{2\pi} e^{-\frac{(x-10)^2}{2}}} = \frac{e^{-\frac{(x-4)^2}{2}}}{e^{-\frac{(x-10)^2}{2}}} = e^{-\frac{(x-4)^2}{2} + \frac{(x-10)^2}{2}}$$

$$\frac{P(x|\omega_1)}{P(x|\omega_2)} = e^{\frac{(x-10)^2-(x-4)^2}{2}} = e^{\frac{x^2-20x+100-x^2+8x-16}{2}} = e^{-6x+42}$$

Doing the same for RHS of our decision rule, we get.

$$\frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}}\frac{P(\omega_2)}{P(\omega_1)} = \frac{1-0}{3-0}\frac{1}{3} = \frac{1}{9}$$

So we have the full inequality.

$$e^{-6x+42} > \frac{1}{9}$$

We can take the natural log of both sides, since it is an monotonically increasing function.

$$-6x + 42 > ln(\frac{1}{9})$$
$$-6x + 42 > -2.197$$
$$x < 7.366$$

Problem 5:

1. The normal decision rule is the following

$$\text{Choose } \omega_i \text{ if } P(\omega_i|x) \geq P(\omega_j|x), \forall j$$

   This is known. However, we need to add an addition rule deciding whether rejection is more optimal when compared to a normal classification. In essence, we need to compare the Risk of rejection versus the Risk of classification. The two cases condition risk is as follows.

$$R(\alpha_{c+1}|x) = \sum_{j=1}^{c} \lambda_r P(\omega_j|x) = \lambda_r$$

$$R(\alpha_i|x) = \sum_{i \neq j} \lambda_s P(\omega_j|x) = \lambda_s(1 - P(\omega_j|x))$$

   We create our inequality.

$$R(\alpha_{c+1}|x) \geq R(\alpha_i|x)$$

And solve.

$$\lambda_r \geq \lambda_s(1 - P(\omega_j|x))$$
$$\frac{\lambda_r}{\lambda_s} \geq 1 - P(\omega_j|x)$$
$$P(\omega_j|x) \geq 1 - \frac{\lambda_r}{\lambda_s}$$

So our decision rule can be written as

$$\text{Choose } \omega_i \text{ if } P(\omega_i|x) \geq P(\omega_j|x), \forall j$$
$$\text{and } P(\omega_j|x) \geq 1 - \frac{\lambda_r}{\lambda_s}$$

2. If $\lambda_r = 0$ then our second condition can never be met, since the RHS of our inequality is 1, and the LHS is a probability (i.e. less then 1). This means that we will always choose to reject, since there is no cost to rejecting.

3. If $\lambda_r > \lambda_s$ then the ratio is greater then 1, meaning the RHS of our inequality is less then 0. Which means the cost of rejection is so high as to never be worth it. It is always more optimal to classify something in that case.

Problem 6:
Maximum Likelihood Estimation

1.

$$p(x, \eta) = h(x) \exp\left\{\eta^T T(x) - A(\eta)\right\}$$

Because $p(x, \eta)$ is a probability density function, we know that the integral of it is always equal to 1 (thank you for this hint professor!). So, using that we have.

$$\int h(x) \exp\left\{\eta^T T(x) - A(\eta)\right\} \mathrm{d}x = 1$$
$$\exp\{-A(\eta)\} \int h(x) \exp\left\{\eta^T T(x)\right\} \mathrm{d}x = 1$$
$$\exp\{A(\eta)\} = \int h(x) \exp\left\{\eta^T T(x)\right\} \mathrm{d}x$$
$$A(\eta) = \ln \int h(x) \exp\left\{\eta^T T(x)\right\} \mathrm{d}x$$

2.

$$\frac{\partial}{\partial \eta} A(\eta) = \frac{1}{\int h(x) \exp\left\{\eta^T T(x)\right\} \mathrm{d}x} * \frac{\partial}{\partial \eta} \int h(x) \exp\left\{\eta^T T(x)\right\} \mathrm{d}x$$

$$= \frac{1}{\int h(x) \exp\left\{\eta^T T(x)\right\} \mathrm{d}x} * \int h(x) T(x) \exp\left\{\eta^T T(x)\right\} \mathrm{d}x$$

$$= \frac{1}{\exp\left\{A(\eta)\right\}} * \int h(x) T(x) \exp\left\{\eta^T T(x)\right\} \mathrm{d}x$$

$$= \int h(x) T(x) \exp\left\{\eta^T T(x) - A(x)\right\}$$

$$= \int p(x, \eta) T(x) \mathrm{d}x$$

$$= E[T(x)]$$

3.

$$l(\eta) = \sum_{k=1}^{n} ln(p(x_k|\eta))$$

$$= \sum_{k=1}^{n} ln(h(x_k)) + \eta T(x_k) - A(\eta)$$

$$= \sum_{k=1}^{n} ln(h(x_k)) + \eta \sum_{k=1}^{n} T(x_k) - (n * A(\eta))$$

Take derivative

$$= 0 + \sum_{k=1}^{n} T(x_k) - n * E[T(x)] = 0$$

$$A'(\eta) = E[T(x)] = \frac{1}{n} \sum_{k=1}^{n} T(x_k)$$

Problem 7:
Logistic Regression

1. It's a derivative of the log likelihood function.

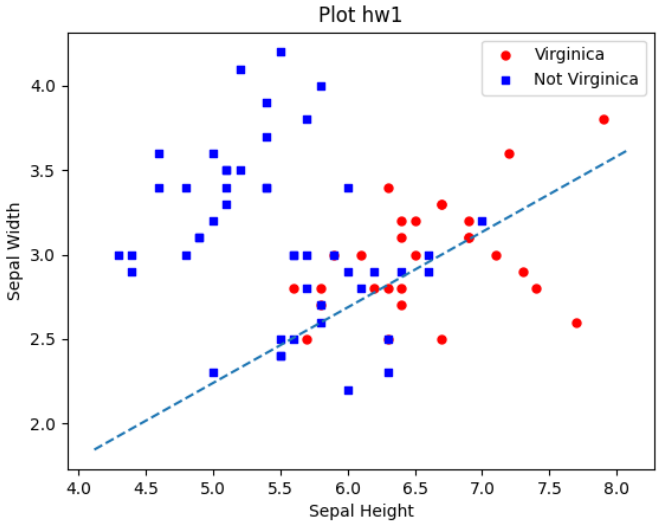
2. First the code

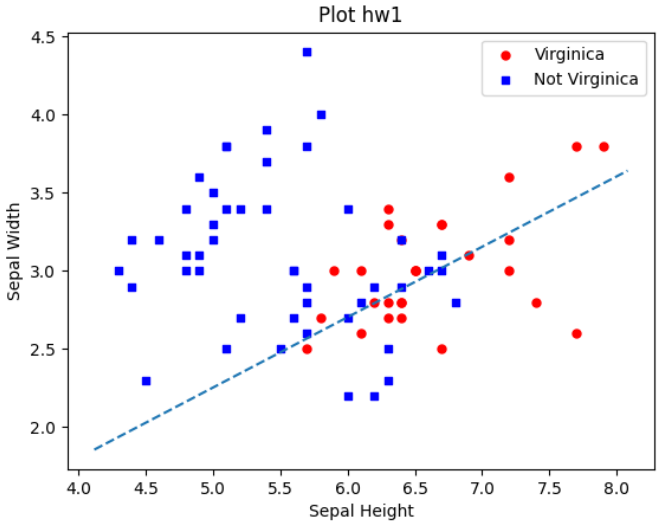
```
from sklearn.datasets import load_iris
from sklearn.model_selection import train_test_split
import numpy as np
import matplotlib.pyplot as plt

# I pledge my honor that I have abided by the Steven's honor system. - Step

# Constants
test_size = 0.5
n = 8000 # number of iterations
training_rate = 0.001

# Load data
iris = load_iris()

# Gradient of loss (binary cross-entropy) is
# vectorized is quiet simple.
# This was very confusing to derive...
def derivative(X, Y, theta):
    # Our current prediction of class labels
    # Based on P(Y=1) = 1 / 1 + e^{-X * theta}
    pred = 1 / (1 + np.exp(-np.dot(X, theta)))
    temp = np.subtract(pred.T, Y)
    return np.dot(X.T, temp.T)

# Takes a default set of weights and uses gradient decent to find
# optimal configuration.
# Params:
# Y -> training targets
# X -> training data
# max_iter -> max iterations
# rate -> training rate
```

```
# (option) theta -> initial weights, defaults to [1] * dimesions
def train(X, Y, max_iter, rate, theta):
    curr_iter = 0
    while (curr_iter < max_iter):
        z = (rate * derivative(X, Y, theta))
        theta = theta - z
        curr_iter += 1
    return theta

# Ignore all but sepal length and width
features = iris['data'][ : , : 2]

# 1 for virginica, 0 otherwise
labels = iris['target']
labels[labels < 2] = 0
labels[labels == 2] = 1

# for n in range(1000, 11000, 1000):

X_train, X_test, Y_train, Y_test = train_test_split(features, labels, test_si
initial_theta = np.array([[1], [1]])
model = train(np.array(X_train), np.array([Y_train]), n, training_rate, initi

# Sigmoid
result = 1 / (1 + np.exp(-np.dot(X_test, model)))

result[result >= 0.5] = 1
result[result <= 0.5] = 0
accuracy = 1 - (np.count_nonzero(result.T - Y_test.T) / len(result))

# Prepare graph
blue = []
red = []
for i in range(len(result)):
    if Y_test[i] == 1:
        red.append(X_test[i])
    else:
        blue.append(X_test[i])
```

```
fig = plt.figure()
ax = fig.add_subplot(111)

# Graph our true values
ax.scatter([i[0] for i in red], [j[1] for j in red], s=25, c='r', marker='o',
ax.scatter([i[0] for i in blue], [j[1] for j in blue], s=25, c='b', marker='s

# Plot the decision boundary
slope = -model[0] / model[1]
axes = plt.gca()
x_vals = np.array(axes.get_xlim())
y_vals = slope * x_vals
plt.plot(x_vals, y_vals, '--')

# Actually plot it
plt.legend(loc='upper right');
ax.set_title("Plot hw1")
ax.set_xlabel('Sepal Height')
ax.set_ylabel('Sepal Width')
plt.show()
```

And here are the graphs, note that the dashed line marks where our decision boundary is.





From my testing we find an accuracy around 68%.