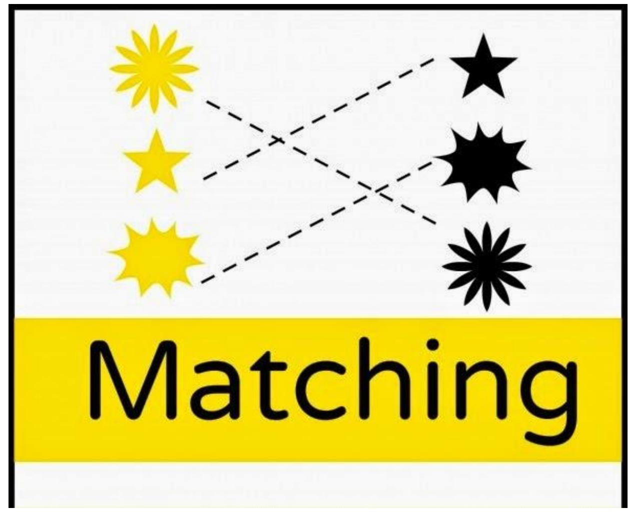
Lesson #8: Why Matching Matters

Lesson #8 Learning Objectives: Upon successfully completing this lesson and the associated homework, students will be able to:

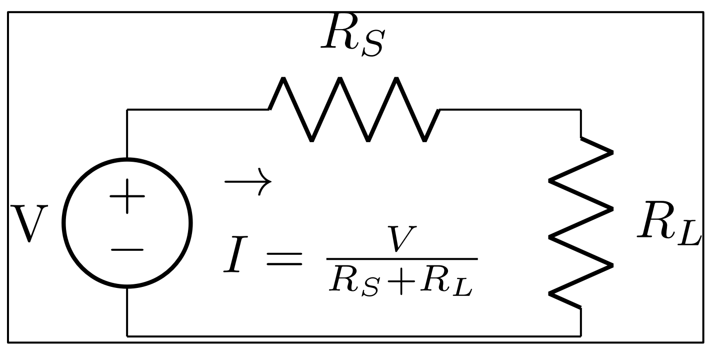
1. Recall the maximum power transfer theorem.
2. Illustrate the maximum power transfer theorem for sinusoidal sources.
3. Calculate the power absorbed by a load resistor given a source impedance and load impedance.
4. Design an appropriate matching network to maximize the power absorbed by the load.



8.1. Maximum Power Transfer Theorem

In general, when we are sending out signals along a transmission line or cable, our goal is to maximize the power that we receive at the other end. In our case, we are going to be connecting our wireless boards to an antenna. We would like to make sure as much of that power as possible is absorbed by the antenna.

For now, we can model an antenna as a voltage source with a series impedance which we connect to our load. Current is obtained from Ohm's law. For a DC system, the power dissipated in the resistors is



$$P_S = \frac{V_S^2}{R_S} \qquad\qquad P_L = \frac{V_L^2}{R_L}$$

Do you remember what the load resistance needs to be to ensure that maximum power is transferred to the load? Can you prove this?

There are two ways to prove this.

From a DC perspective, we remember that the power dissipated in the load is maximized when RL is equal to Rs. Is this the same when our voltage source is sinusoidal?

Since we are dealing with high frequencies, the voltage generated by our antenna voltage source is $V(t) = v_0 \cos(\omega t)$. The instantaneous power dissipated into the load is the product of the voltage across the load and the current flowing through the load[1]. We can rewrite this only in terms of voltage as $P = I \cdot V = \left(\frac{V}{R}\right) V = \frac{V^2}{R}$. We could do the same with current and end up with Joule heating, based on discoveries from James Joule[2], which states $P = I^2 R$.

For a periodic signal, the average power dissipated in the load is the sum of the total power over a period divided by the length in time of the period.

$$P_{av} = \frac{1}{T} \int_{-T/2}^{T/2} \frac{v_s \cos^2(\omega t)}{R_L} \, \mathrm{dt}$$

which after some math…

…will result in

$$P_S = \frac{V_S^2}{2R_S} \qquad P_L = \frac{V_L^2}{2R_L}$$

(Note the factor of 2)

[1] I've just restated Ohm's Law

[2] Joule was a brewer. That's the whole fact.

and these can be rewritten as

$$P_S = \frac{I^2 R_S}{2} \qquad\qquad P_L = \frac{I^2 R_L}{2}$$

and after substitution in the expression for current in the mesh (its on the other page) we find that

$$\begin{aligned} P_L &= \frac{R_L V^2}{2\left(R_S + R_L\right)^2} \\ &= \frac{1}{R_s} \frac{\left(R_L / R_s\right) V^2}{2\left(1 + \left(R_L / R_s\right)\right)^2} \end{aligned}$$

where we've rearranged the power to be a function of the expression (RL / Rs). For now, plot Pl * Rs in matlab as a function of Rl/Rs to determine the ratio for maximum power transfer.

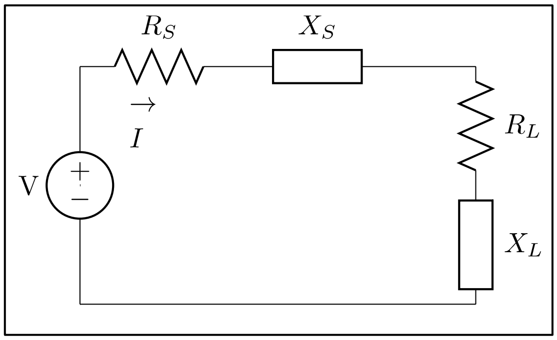
```
clear all;
close all;
rlrs = linspace(.1, 4, 500);

V = 1;
Plxrs = (rlrs) * V^2 ./ (2 * (1 + rlrs).^2);

figure(1);clf;
h = plot(rlrs, Plxrs);
set(h, 'linewidth', 2);

x1 = ???;
line([x1, x1], [0, .16]);
axis([0, 4, 0, .16]);
set(gca, 'fontsize', 14);
xlabel('R_L / R_s');
ylabel('P_L * R_s');
grid on;
```

If you change V, the magnitude of the voltage source, does this change the result?

8.2. Let's make things complex …

When we add in reactance and must consider a source with reactance and a load with reactance, things get a bit different. The current flowing through the circuit model is now written as

$$I = \frac{V}{\left(R_s + jX_s\right) + \left(R_L + jX_L\right)}$$



Where the source (antenna) and load are both complex with real and imaginary components. As a resistor does not care about the phase of the current[3], its dissipated power is only sensitive to the size of the current flowing.

$$P_L = \frac{|I|^2 R_L}{2}$$

and after plugging in our expression for I, we see

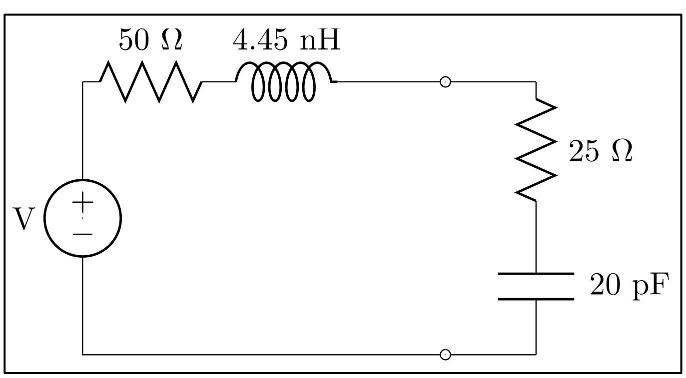
$$\begin{aligned} P_L &= \frac{V^2 R_L}{2|(R_s + jX_s) + (R_L + jX_L)|^2} \\ &= \frac{V^2 R_L}{2|Z_s + Z_L|^2} \end{aligned}$$

The largest power dissipation in the load (what we want) will happen when the denominator is small. For fixed values of the real source and load resistances, the denominator is maximized when the imaginary part is zero. To force the imaginary part of the denominator to zero, the reactances of the source and load need to be equal in magnitude but opposite in sign. This is the definition of the complex conjugate. So the condition for maximum power transfer is also called conjugate matching – transforming the load impedance to the complex conjugate of the source.

Modeling an antenna as a voltage source with a source impedance, we would find that the average power delivered to the load is

$$\begin{aligned} P_{Av} &= \frac{I_{ant}^2 R_{load}}{2} \\ &= \frac{V_{ant}^2}{8R_{ant}} \end{aligned}$$

Consider the following circuit. In each case, determine what the power delivered to the load is. The source impedance is 50 + j25. Frequencies are at 915 MHz. Assume V, the magnitude of the periodic sinusoid, is 1.



A) What is the power delivered to the load?

B) If you transform the impedance to be the same as the source, what would be the power deliverd to the load?

C) If the load is transformed to the conjugate match of the antenna, what is the power deliverd to the load?

[3] Nor the meaning of life, or which Star Wars movie is best

Take aways:

1) In DC systems, maximum power is transferred when the source impedance is equal to the load impedance.
2) For systems with complex loads, the maximum power transfer is achieved when the source impedance is the conjugate of the effective load impedance.
3) Reading a reflection coefficient can tell us approximately how well matched we will be.