Sumedh

Eigen values and eigen vectors of a given sq. Matrix

- let A be any given non-zero square matrix
- if There exist a scalar (λ) and non-zero column matrix (X) such that $AX = \lambda X$ then the scalar (λ) is called an eigen value or characteristic value of latent value of the Matrix (A) and (X) is called the corresponding eigen vector or characteristic vector of A

eg $\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- **Determination of λ & X**

$$\because AX = \lambda X$$
$$AX - \lambda IX = 0$$
$$(A - \lambda I)X = 0$$

$$\because X \neq 0$$

$$\therefore |A - \lambda I| = 0$$

This eq. which determines λ is called the characteristic eq. of A. The corresponding eigen vector (X) can now be determined by considering the homogenous eq. $|A - \lambda I|(X) = 0$

- **Method 1**

eg. Find the characteristic eq of matrix

$$A = \begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

The characteristic eq. $|A - \lambda I| = 0$

$$\left[\begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right] = 0$$

$$\begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} - \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = 0$$

$$= \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} - (-2) \begin{vmatrix} 1 & 1 \\ 1 & -1-\lambda \end{vmatrix} + 3 \begin{vmatrix} 1 & 1-\lambda \\ 1 & 3 \end{vmatrix}$$

$$= (2-\lambda)\left[(1-\lambda)(-1-\lambda) - 3\right] + 2\left[(-1-\lambda) - 1\right] + 3\left[3 - (1-\lambda)\right]$$

$$= (2-\lambda)[\lambda^2 - 4] + 2(-2-\lambda) + 3(2+\lambda) = 0$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

$$= \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

Method -2

We can use the following formula

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = Sum of elements of leading diagonal

S_2 = Sum of Minors of elements on leading diagonal

S_3 = $|A|$

eg $A = \begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$

$S_1 = 2 + 1 - 1 = 2$

$S_2 = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$

$= (-1-3) + (-2-3) + (2-(-2))$

$S_2 = -4 - 5 + 4 = -5$

$S_3 = |A| = 2(-1-3) - (-2)(-1-1) + 3(3-1)$

$= 2(-4) + 2(-2) + 3(2)$

$= -6$

Now $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\therefore \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

Hence, we get the same eq.

$\boxed{\text{Note : All the roots of } \lambda \text{ are distinct}}$

- To find eigen values and eigen vectors of the above matrix

$$A = \begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & -3 & -1 \end{vmatrix}$$

Char eq. $= \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

Step 1 : we find the first root by trail & error
Synthetic div

for eg $\lambda = 1$

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & +6 \\ \times & +0 & 1 & -1 & -6 \\ \hline & \underbrace{1 \quad -1 \quad 6}_{\text{Quad eq.}} & & & 0 \end{array}$$

i.e satisfies

$$\lambda^2 - \lambda - 6 = 0$$

$$\lambda(\lambda-3)+2(\lambda-3) = 0$$

$$(\lambda-3)(\lambda+2) = 0$$

$$\therefore \lambda = 1, -2, 3$$

$\hookrightarrow$ eigen value

- To find corresponding eigen vector

$$[A - \lambda I][X] = 0$$

$$\left[\begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right] \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

put $\lambda = 1$

$$\begin{vmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= \begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ x_1 + 0x_2 + x_3 &= 0 \\ x_1 + 3x_2 - 2x_3 &= 0 \end{aligned}$$

[choose two diff eq.]

by cross multiplication

$$\frac{x_1}{\begin{vmatrix} -2 & 3 \\ 0 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix}}$$

$$\frac{x_1}{-2} = \frac{\cancel{-}x_2}{\cancel{-}2} = \frac{x_3}{2}$$

$$X = \begin{vmatrix} -2 \\ 2 \\ 2 \end{vmatrix} = \begin{vmatrix} -1 \\ 1 \\ 1 \end{vmatrix}$$

for $\lambda = -2$

$$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{aligned} 4x_1 - 2x_2 + 3x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \end{aligned}$$

[choose two diff eq]

by cross mult.

$$\frac{x_1}{\begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -2 \\ 1 & 3 \end{vmatrix}}$$

$$\frac{x_1}{-11} = -\frac{x_2}{1} = \frac{x_3}{14}$$

$$X = \begin{vmatrix} -11 \\ -1 \\ 14 \end{vmatrix}$$

$\lambda = 3$

$$\begin{vmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{aligned} -x_1 - 2x_2 + 3x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + 3x_2 - 4x_3 &= 0 \end{aligned}$$

$$\begin{matrix} -1 & -2 & 3 \\ 1 & -2 & 1 \end{matrix}$$

$$\frac{x_1}{4} = \frac{\cancel{-}x_2}{\cancel{-}4} = \frac{x_3}{4}$$

$$X = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$$

Q.1] Find the eigen values & Vector

$$\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

Soln $\quad |A - \lambda I|\,|X| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 18$

$S_2 = 5 + 20 + 20 = 45$

$S_3 = 8(5) - (-6)(-10) + 2(6)$

$= 40 - 60 + 20$

$= 0$

$\therefore \quad \lambda^3 - 18\lambda^2 + 45\lambda = 0 \qquad$ 5 45 / 3 9 / 3

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$(\lambda - 15)(\lambda - 3) = 0$

$\lambda = 0, 15, 3$

put $\lambda = 0 \longrightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}$

$$\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 7x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 3x_3 = 0$$

$$\begin{vmatrix} 4 & -3 & 1 \\ -6 & 7 & 4 \end{vmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -3 & 1 \\ 7 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 4 & 1 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -3 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{5} = \frac{\cancel{-}x_2}{\cancel{-}10} = \frac{x_3}{10}$$

$$X = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$\lambda = 3$

$$\begin{vmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{vmatrix}$$

$$5x_1 - 6x_2 + 2x_3$$
$$-6x_1 + 4x_2 - 4x_3$$
$$2x_1 - 4x_2 +$$

$$\begin{vmatrix} 5 & -6 & +2 \\ 2 & -4 & 0 \end{vmatrix}$$

$$\frac{x_1}{8} = \frac{\cancel{-}x_2}{\cancel{-}4} = \frac{x_3}{-8}$$

$$X = \begin{bmatrix} 2 \\ +1 \\ -2 \end{bmatrix}$$

$\lambda = 18$

$$\begin{vmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{vmatrix}$$

$$-7x_1 - 6x_2 + 2x_3$$
$$-6x_1 - 8x_2 - 4x_3$$
$$2x_1 - 4x_1 - 12x_3$$

$$\frac{x_1}{20} = \frac{\cancel{+}x_2}{\cancel{+}20} = \frac{x_3}{-10}$$

$$X = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

Q.2] Find eigen values & vectors

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

Type : 2   Non symmetric matrix having repeated eigen value

eg find eigen values /vectors :

$$\begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix}$$

Soln $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 9$

$S_2 = (12-6) + (8-3) + (6-2)$

$= 6 + 5 + 4$

$= 15$

$S_3 = 2(12-6) + 1(6-8) + 1(6-9)$

$= 12 - 2 - 3$

$= 7$

$\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$

$\lambda^2 - 8\lambda + 7 = 0$

$\lambda^2 - 8\lambda - \lambda +$

$(\lambda-1)(\lambda-7)(\lambda-1)$

7

$-7 \quad 8 \quad -1.$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix}$$

$\lambda = 1, 1, 7$

for $\lambda = 7$

$$\begin{vmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{vmatrix}$$

$$-5x_1 + x_2 + x_3 = 0$$
$$2x_1 - 4x_2 + 2x_3 = 0$$
$$3x_1 + 3x_2 - 3x_3 = 0$$

$$\begin{vmatrix} 2 & -4 & 2 \\ 3 & +3 & -3 \end{vmatrix}$$

$$\frac{x_1}{6} = \frac{x_2}{\cancel{-}12} = \frac{x_3}{18} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

for $\lambda = 1$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = 0$$

$$x_1 + x_2 + x_3 = 0$$
$$2(x_1 + x_2 + x_3) = 0$$
$$3(x_1 + x_2 + x_3) = 0$$

$\therefore \quad x_1 + x_2 + x_3 = 0$ (infinite solⁿ)

assume $x_2 = 0$

$$x_1 + x_3 = 0$$

$$x_1 = -x_3$$

asume $x_3 = K$

$\therefore \quad x_1 = -K$

$\therefore \quad \begin{bmatrix} -K \\ 0 \\ K \end{bmatrix}$ ie $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

again for $\lambda = 0$

$$x_1 + x_2 + x_3 = 0$$

assume $x_1 = 0$

$$x_2 + x_3 = 0$$

$$\begin{bmatrix} 0 \\ -K \\ K \end{bmatrix} \text{ i.e } \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

asume $x_3 = K$

$\therefore$ $x_2 = -K$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$S_1 = 3$

$S_2 = 0 + 1 + 1 = 2$

$S_3 = 0$

$\lambda^3 - 3\lambda^2 + 2\lambda = 0$

$\lambda = 0$

$\lambda(\lambda^2 - 3\lambda + 2)$

2

$-2 \quad -1$

$\lambda(\lambda^2 - 2\lambda - \lambda + 2) = 0$

$(\lambda-2)(\lambda-1)(\lambda-0)$

$\lambda = 2, 1, 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$\lambda = 0$
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$x_1 = 0$
$x_2 + x_3 = 0$
$x_2 + x_3 = 0$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$\frac{x_1}{0} = \frac{-x_2}{} = \frac{x_3}{0}$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Q.1] Find the characteristic roots & vectors of the matrix

$$\begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

Sol^n

$S_1 = -1$

$S_2 = -12 + (-3) + (-6)$
$= -21$

$S_3 = -2(-12) + 2(6) + -3(-3)$
$= 24 + 12 + 9$
$= 45$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & +6 & +45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

-15

$-5 \quad 3$

$\lambda = -3 \qquad \lambda^2 - 2\lambda - 15$

$\lambda^2 - 5\lambda + 3\lambda - 15$

$\lambda = -3, -3, 5$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}$$

$\lambda = 5$

$$\begin{vmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{vmatrix}$$

$$\begin{aligned} -7x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 - 4x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 - 5x_3 &= 0 \end{aligned}$$

$$\begin{vmatrix} 1 & -2 & -3 \\ -1 & -2 & -5 \end{vmatrix} \qquad \begin{vmatrix} -7 & 2 & -3 \\ 1 & -2 & -3 \end{vmatrix}$$

$$\frac{x_1}{4} = \frac{\cancel{-}x_2}{\cancel{-}8} = \frac{x_3}{-4} \qquad \frac{x_1}{-12} = \frac{-x_2}{24} = \frac{x_3}{12}$$

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \text{same} \qquad \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$\lambda = \cdot 3$

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{vmatrix}$$

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 0 \end{aligned}$$

$$x_1 + 2x_2 - 3x_3 = 0$$

$$2x_2 - 3x_3 = 0$$

$$x_2 = \frac{3}{2}x_3$$

$$x_3 = t$$

$$\therefore x_2 = \frac{3}{2}t \qquad \begin{pmatrix} 0 \\ t \\ \frac{3}{2}t \end{pmatrix}$$

eg :-

$$\begin{vmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{vmatrix}$$

Soln $S_1 = -3$

$S_2 = +8 - 11 + 6 = 3$

$S_3 = 6(+8) - 6(14) + 5(+7)$
$= 48 - 84 + 35$
$= -1$

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

-1	1	3	+3	1
	0	-1	-2	-1
	1	2	+1	0

$\lambda^2 + 2\lambda + 1 = 0$
$\lambda^2 + \lambda + \lambda + 1 = 0$
$\lambda(\lambda + 1) + 1$

$\lambda = -1, -1, -1$

$$\begin{vmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{vmatrix}$$

$$\begin{vmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{vmatrix}$$

$7x_1 - 6x_2 + 5x_3 = 0$

$x_1 = 0$ $\qquad$ $\cancel{6}x_2 = \cancel{+}\frac{5}{6}x_3$

$$\begin{bmatrix} 0 \\ 5/6t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}$$

Similarly $x_2 = 0$ $\quad$ $x_3 = 0$

$$\begin{bmatrix} t \\ 0 \\ 5/7 \end{bmatrix} \qquad \begin{bmatrix} 6/7 t \\ t \\ 0 \end{bmatrix}$$

Type 3 : Symmetric Matrix having repeated values

let X_1, X_2 be two column matrices, then X_1 & X_2 are orthogonal if $X_1^T X_2 = 0$

eg $X_1 = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$ $\quad$ $X_2 = \begin{vmatrix} 1 \\ -2 \\ 3 \end{vmatrix}$

$$X_1^T = \begin{vmatrix} 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ -2 \\ 3 \end{vmatrix} = 1 - 4 + 3 = 0$$

Note :- When the Matrix A is symmetric, its eigen vectors are orthogonal

Q1] find eigen values & Vectors

$$\begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

eg :- $S_1 = 12$

$S_2 = 8 + 14 + 14$
$= 36$

$S_3 = 6(8) - 2(+1) + 2(-4)$
$= 48 - \cancel{4} - 8$
$= 32$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$\begin{array}{c} 16 \\ \wedge \\ -8 \; -2 \end{array}$$

$$\lambda^2 - 8\lambda - 2\lambda + 16 = 0$$

$$\lambda(\lambda - 8) - 2$$

$$\lambda = +8, +2, +2$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix}$$

$$\begin{vmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{vmatrix}$$

$$-x_1 - x_2 + x_3 = 0$$
$$-2x_1 - 5x_2 - x_1 = 0$$
$$-2x_1 - x_2 - 5x_1 = 0$$

$$\begin{vmatrix} -1 & -1 & 1 \\ -2 & -5 & -1 \end{vmatrix}$$

$$\frac{x_1}{6} = \frac{\cancel{-}x_2}{\cancel{-}3} = \frac{x_3}{3}$$

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{vmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$2x_1 - x_2 + x_3 = 0$$

$x_1 = 0$ $\quad \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\because$ A is symm, $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that

$X_1^T X_3 = 0 \quad X_1^T X_2 = 0$

$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow X_1^T = \begin{vmatrix} 0 & 1 & 1 \end{vmatrix}$

$$X_1^T X_3 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$b + c = 0$$

$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \rightarrow X_1^T = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$

$$X_1^T X_3 = 2a - b + c = 0$$

$$b + c = 0 \quad \text{—(i)}$$
$$2a - b + c = 0 \quad \text{—(ii)}$$

$$\begin{vmatrix} 0 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$\frac{a}{-2} = \frac{-b}{2} = \frac{c}{2}$$

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

eg :- find eigen valus / vec of the Matrix

$$\begin{vmatrix} 2 & -1 & +1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$S_1 = 6$

$$S_2 = 3 + 3 + 3$$
$$= 9$$

$$S_3 = 2(3) - 1(+\cancel{2}1) + (-1)$$
$$= 6 - \cancel{3} - 1$$
$$= +\cancel{3}4$$

$$\lambda^2 - 6\lambda^2 + 9\lambda - 4 = 0$$

1	1	−6	9	−4
	0	1	−5	4
	1	−6	4	0

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda^2 - 4\lambda - \lambda + 4$$

$$(\lambda - 4) - 1$$

$$(\lambda - 4)(\lambda - 1) = 0$$

$$\lambda = 4, 1, 1$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

$\lambda = 4$ $$\begin{vmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{vmatrix}$$

$$-2x_1 - x_2 + x_3 = 0$$

$$\begin{vmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \end{vmatrix}$$

$$\frac{x_1}{3} = \frac{-x_2}{3} = \frac{x_3}{3}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{vmatrix} 1 & -1 & 1 \\ -2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$x_1 = 0$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$a - b + c = 0$$
$$+b + c = 0$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{1} : \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

Type 4: 2x2 Matrix

eg $\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix}$

$[A - \lambda I] = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix}$

$4 - \lambda - 4\lambda + \lambda^2 - 10 = 0$

$\lambda^2 - 5\lambda - 6 = 0$

$\lambda^2 - 6\lambda + \lambda - 6$

$(\lambda - 6)(\lambda + 1) = 0$

$\lambda = +6, -1$

$\lambda = 6 \quad \begin{vmatrix} -5 & 2 \\ 5 & -2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = 0$

- Diagonalisation of a matrix by orthogonal transformation

let A be a symm. mat., Form a matrix N whoose columns are normalised eigen vectors of A, $\because$ N is Orthogonal we use $N^{-1} = N^T$

Then $D = N^T A N$ which is called orthogonal reduct^n or ortho. transformat^n

Note :- $\because$ A is symm. ortho. transformat^n is possible only for real symm. matrix

Note : If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then the normalised vector of X is :-

$$\begin{bmatrix} \dfrac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3}} \\ \\ \dfrac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3}} \\ \\ \dfrac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3}} \end{bmatrix}$$

Working rule for diagonalisation

S-1 : To find the characteristic eq.

S-2 : To solve the Char. eq.

S-3 : Find eigen vectors

S-4 : If the eigen vectors are orthogonal then form a normalised modal matrix N

S-5 : Find N^T

S-6 : Calc. AN

S-7 : Calc. $D = N^T A N$

Q.1] Diagonalise the matrix $A = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}$
by means of ortho. trans.

Soln

$S_1 = 9$

$S_2 = 8 + 8 + 8 = 24$

$S_3 = 3(8) + 1(-4) + 1(-4)$

$= 16$

$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -9 & 24 & -16 \\ & 0 & 1 & -8 & 16 \\ \hline & 1 & -8 & +16 & 0 \end{array}$$

$\lambda^2 - 8\lambda + 16 = 0$

$\lambda^2 - 4\lambda - 4\lambda + 16 = 0$

$(\lambda - 4)(\lambda - 4)(\lambda - 1) = 0$

$\lambda = 4, 4, 1$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$X_1 = \frac{x_1}{-3} = \frac{+x_2}{+3} = \frac{x_2}{3}$$

$$\boxed{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

$$\bar{X_1} = \begin{vmatrix} \frac{-1}{\sqrt{3}} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{vmatrix}$$

$\lambda = 4$

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{vmatrix}$$

$$x_1 - x_2 - x_3 = 0$$

$$x_1 = 0 \qquad x_2 = -x_3$$

$$\boxed{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

$$\overline{X_2} = \begin{pmatrix} 0/\sqrt{2} \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$-a + b + c = 0$$
$$-b + c = 0$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$\frac{x_1}{2} = \frac{\cancel{-}x_2}{\cancel{-}1} = \frac{x_3}{1} \quad \boxed{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}$$

$$\overline{X_3} = \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$$\therefore N = \begin{pmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

$$N^T = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

$$AN = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} -3/\sqrt{3} + 1/\sqrt{3} + 1/\sqrt{3} & 0 + 1/\sqrt{2} - 1/\sqrt{2} & 6\sqrt{6} + 2/\sqrt{6} \\ -1/\sqrt{3} + 3/\sqrt{3} - 1/\sqrt{3} & 0 + 3/\sqrt{2} + 1/\sqrt{2} & 5\sqrt{6} - 1/\sqrt{6} \\ -1/\sqrt{3} - 1/\sqrt{3} + 3/\sqrt{3} & 0 - 1/\sqrt{2} - 3/\sqrt{2} & 2/\sqrt{6} - 1/\sqrt{6} + 3/\sqrt{6} \end{pmatrix}$$

$$AN = \begin{pmatrix} -1/\sqrt{3} & 0 & 8/\sqrt{6} \\ 1/\sqrt{3} & 4/\sqrt{2} & 4/\sqrt{6} \\ 1/\sqrt{3} & -4/\sqrt{2} & 4/\sqrt{6} \end{pmatrix}$$

$$N^T AN = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} & 0 & 8/\sqrt{6} \\ 1/\sqrt{3} & 4/\sqrt{2} & 4/\sqrt{6} \\ 1/\sqrt{3} & -4/\sqrt{2} & 4/\sqrt{6} \end{pmatrix}$$

$$N^T A N = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} & 0 & 8/\sqrt{6} \\ 1/\sqrt{3} & 4/\sqrt{2} & 4/\sqrt{6} \\ 1/\sqrt{3} & -4/\sqrt{2} & 4/\sqrt{6} \end{pmatrix}$$

$$\begin{pmatrix} 1/3 + 1/3 + 1/3 & 0 + \cancel{4/\sqrt{3}.\sqrt{2}} - \cancel{(\;)} & -\cancel{8/\sqrt{18} + 4/\sqrt{18}} + 4/\sqrt{18} \\ \cancel{0 + 1/\sqrt{6}} - 1/6 & 0 + 4/\sqrt{4} + 4/\sqrt{4} & 0 + \cancel{4/\sqrt{12}} - \cancel{4/\sqrt{12}} \\ 0 & 0 & 16/\sqrt{36} + 4/\sqrt{36} + 4/\sqrt{36} \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

- Quadratic form

A homogeneous polynomial of second degree in any no. of variables is called a quad. form

eg $x_1^2 + 7x_2^2 + 3x_2^2 - 3x_1x_2 + 4x_2x_3 - 10x_1x_3$

The matric corresponding the Quad form :-

$$\begin{vmatrix} \text{coeff of } x_1^2 & \frac{1}{2}\text{coeff of } x_1x_2 & \frac{1}{2}\text{coeff of } x_1x_3 \\ \frac{1}{2}\text{coeff of } x_2x_1 & \text{coeff of } x_2^2 & \frac{1}{2}\text{coeff of } x_2x_3 \\ \frac{1}{2}\text{coeff of } x_3x_1 & \frac{1}{2}\text{coeff of } x_3x_2 & \text{coeff of } x_3^2 \end{vmatrix}$$

eg :- $2x_1^2 - 2x_2^2 + 4x_3^3 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & +3 \\ -3 & +3 & 4 \end{bmatrix}$$

eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 9 \\ 3 & 9 & 3 \end{bmatrix}$ $x_1^2 + 4x_2^2 + 3x_3^2 + 4x_1x_2 + 6x_1x_3 + 18x_2x_3$

eg $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -4 \end{bmatrix} \rightarrow \boxed{3x_1^2 + 9x_2^2 - 4x_3^2}$

Canonical form

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1) Index :- The number of +ve terms in canonical form is called as index of the form and denoted by p.

2) Signature :- The diff^n b/w +ve terms p and negative terms $r-p$ where r is rank of the matrix in canonical form is called as signature of the quadratic form.

$\therefore$ signature $= p-(r-p)$

$= p-r+p$

$= \boxed{2p-r}$ //

3) Nature of quadratic form:-

$Q = X^T A X$ in N variables is said to be

1) positive definite :- If all the eigen values of A are +ve numbers.

1, 2, 3

2) negative definite :- If all the eigen values of A are -ve numbers.

-3, -2, -1

3) positive semi-definite :- If all the eigen values of $A \geq 0$ and atleast one eigen value is 0.

0, 8, 15

4) negative semi-definite :- If all the eigen values of $A \leq 0$ and atleast one eigen value is 0.

0, -2, -3

5) Indefinite in all other cases or If A has both +ve and -ve eigen values.

-1, 0, 2

- Reduction of Quad form to Cannonical form using diagonal transformation

Step -1) $Q = X^T AX$

2) Find eigen values $\lambda_1, \lambda_2, \lambda_3$

3) Eigen vectors X_1, X_2, X_3 of A (symmetric)

4) We normalize eigen vectors and called them as $\bar{X_1}, \bar{X_2}, \bar{X_3}$ and form a matrix $N = [\bar{X_1} \ \bar{X_2} \ \bar{X_3}]$

5) $$N^T AN = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

and using transformation

$X = NY$ (Assume)

$$Q = X^T AX$$
$$Q = (NY)^T A (NY)$$
$$= Y^T (N^T AN) Y$$
$$= [y_1 \ y_2 \ y_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

Thus, using the transformation $X = NY$, we have reduced Q to A sum of squares in which coefficient of square terms are eigen values of A since N is orthogonal

$X \to NY$ is called orthogonal transformation.

Q] $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1$
Convert to diagonal cannonical form & find value, rank, nature, index & signature

Soln

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$S_1 = 11$
$S_2 = 14 + 8 + 14 = 36$
$S_3 = 3(14) - (-1+3) + (1-5)$
$= 42 - 2 - 4$
$= 36$

$$\lambda^3 - 11\lambda + 36\lambda - 36 = 0$$

$\lambda = 2$	1	-11	36	-36
	0	2	-18	36
	1	-9	+18	0

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda^2 - 6\lambda - 3\lambda + 18 = 0$$

$$\lambda = 6, 3, 2$$

$$\begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + 3x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned} \qquad \begin{vmatrix} 1 & -1 & 1 \\ -1 & +3 & -1 \end{vmatrix}$$

$$\frac{x_1}{-2} = \frac{-x_2}{0} = \frac{x_3}{2} \qquad \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$\frac{-2}{\sqrt{8}}, \; 0, \; \frac{2}{\sqrt{8}}$$

$\lambda = 6$ $$\begin{vmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{vmatrix}$$

$$\begin{vmatrix} -3 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\frac{x_1}{-2} = \frac{\cancel{+}x_2}{\cancel{+}4} = \frac{x_3}{+2} \qquad \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad \frac{-1}{\sqrt{6}}, \; \frac{2}{\sqrt{6}}, \; \frac{1}{\sqrt{6}}$$

$\lambda = 3$ $$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad -\frac{1}{\sqrt{3}}, \; -\frac{1}{\sqrt{3}}, \; \frac{1}{\sqrt{3}}$$

ortho

$$N = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$D = N^T A N$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Rank = No. of non-zero rows in D

$= 2y_1^2 + 3y_2^2 + 6y_3^2$

Index $(p) = 3$

Sign $= 2p - r$
$= 2(3) - 3$
$= 3$

Nature +ve (definite eigen values)

Q.1] $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$

- Determination of Nature of Quad form without reducing them to cannonical form

let A be the sq. matrix of the quad form

$$A = \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_1 = |a_{11}|$$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

① The quad form is +ve Definite if $D_1, D_2\ D_3$ are +ve

② The quad form is −ve Definite if $D_1, D_3, \ldots$ are −ve and $D_2, D_4, \ldots$ are all +ve

$\rightarrow$ $(-1)^n D_n > 0$ for all n

③ The quad form is +ve semi-definite if $D_1, D_2, \ldots \geq 0$ and atleast one $D_i = 0$ $(i = 1, 2, \ldots)$

④ The quad form is −ve semi-definite if

$(-1)^n D_n \geq 0$ and atleast one $D_i = 0$ $(i = 1, 2, \ldots)$

⑤ In all other cases, it's indefinite.

eg :- determine the nature of foll. quad. form without reducing them to cannonical form.

i) $6x^2 + 3y^2 + 14z^2 + 4yz + 18xz + 4xy$

Soln

$$Q = \begin{vmatrix} x & y & z \end{vmatrix} \begin{vmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix}$$

$D_1 = 6$

$D_2 = 14$

$D_3 = 6\left(\overset{4}{38}\right) + 2(-10) + 9(-23)$

$= 228 - 20 - 207$

$= 1$

Hence +ve definite.

ii) $x_1^2 + 2x_2^2$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$D_1 = 1$
$D_2 = 2$
$D_3 = 0$

Hence +ve semi-definite

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$$\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix}$$

$D_1 = 1$

$D_2 = 1$

$D_3 = 1(5) + 1(-4) - 1(3) = -2$

Hence indefinite.

HW i) $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$

ii) $2x_1x_2 + 2x_2x_3 - 2x_1x_3$

iii) $2x^2 + 3y^2 + 2z^2 + 2xy$

- Cayley - Hamilton Theorem

Every sq. matrix satisfies its own characteristic eq.

eg:- Verify Cayley Ham. thm and find Inverse of the Matrix

$$\begin{vmatrix} 2 & 1 \\ 1 & -5 \end{vmatrix}$$

Soln $(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & -5-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 11 = 0$$

from C.H. Thm

$A^2 + 3A - 11 = 0$

$$\begin{vmatrix} 2 & 1 \\ 1 & -5 \end{vmatrix} \begin{vmatrix} 2 & 1 \\ 1 & -5 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 1 & -5 \end{vmatrix} - 11 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 5 & -3 \\ -3 & 26 \end{vmatrix} + \begin{vmatrix} 6 & 3 \\ 3 & -15 \end{vmatrix} + \begin{vmatrix} -11 & 0 \\ 0 & -11 \end{vmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ Verified

Now,

$$A^2 + 3A - 11I = 0$$

Pre - multiply by A^{-1}

$$A^{-1} A^2 + 3A^{-}A - 11A^{-1} = 0$$

$$A + 3I - 11A^{-1} = 0$$

$$A^{-1} = \frac{1}{11}[A + 3I]$$

$$= \frac{1}{11}\begin{bmatrix} 5 & 1 \\ 1 & -2 \end{bmatrix}$$

Q.] Verify Cayley Hamilton thm for the Matrix

$$\begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

Soln $S_1 = 5$

$S_2 = 3 + 1 + 5 = 9$

$S_3 = 3 + 2(0) - 2(+2) = 1$

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

$$A^3 - 5A^2 + 9A - I = 0$$

$$A^2 = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\begin{matrix} 1-2+0 & 2+6+4 & -2+0-2 \\ -1-3 \; 0 & -2+9 \; 0 & 2 \; 0 \; 0 \\ 0+2 \; 0 & 0-6-2 & 0 \; 0 \; 1 \end{matrix}$$

$$A^2 = \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix}$$

$$A^3 \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\begin{vmatrix} -1-12 \; 0 & -2+36+8 & +2-4 \\ -4-7 \; 0 & -8+21-4 & +8 \; 0+2 \\ 2 \; 8 \; 0 & 4-24-2 & -4 \; 0 \; 1 \end{vmatrix}$$

$$A^3 \begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & -10 \\ 10 & -22 & -3 \end{pmatrix}$$

$$A^3 - 5A^2 + 9A - I$$

$$\begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & -10 \\ 10 & -22 & -3 \end{pmatrix} - 5\begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} + 9\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} = 0$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A^{-1} pre mul.

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$A^{-1} = A^2 - 5A + 9I$$

$$= \begin{vmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{vmatrix} - 5 \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{vmatrix} -5 & -10 & +10 \\ +5 & -15 & 0 \\ 0 & +10 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & +5 \end{vmatrix}$$

A pre mul

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$A^4 = 5A^3 - 9A^2 + A$$

$$\begin{vmatrix} -65 & 210 & -10 \\ -55 & 45 & 50 \\ 50 & -110 & -15 \end{vmatrix} +$$

Q.7 $\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$ $\begin{vmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{vmatrix}$

Soln

$S_1 = 6$

$S_2 = 3 + 3 + 3 =$

$S_3 = 2(3) - 1(4) + (4)$

$= 6$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 6 = 0$$

$$A^3 - 6\lambda^2 + 9A - 6I = 0$$

$$A^2 = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 4+1-1 & -2-2 & -1 & 2 \; 1 \; 2 \\ -2 & -2+1 & 1 \; 4 \; 1 & -1 \; -2 \; -2 \\ -2 & +1 \; -2 & 1 \; -2 \; -2 & -1 \; 1 \; 4 \end{vmatrix}$$

$$A^2 = \begin{vmatrix} 4 & -5 & 5 \\ -3 & 6 & -5 \\ -3 & -3 & 4 \end{vmatrix}$$

$$A^3 = \begin{vmatrix} 4 & -5 & 5 \\ -3 & 6 & -5 \\ -3 & -3 & 4 \end{vmatrix} \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 8 \; 5 \; -5 & -4 \; -10 \; -5 & 4 \; 5 \; 10 \\ -6 \; -6 \; 5 & 3 \; 12 \; 5 & -3 \; -6 \; -10 \\ -6 \; 3 \; -4 & 3 \; -6 \; -4 & -3 \; 3 \; 8 \end{vmatrix}$$

$$A^3 = \begin{vmatrix} 8 & -19 & 19 \\ -7 & 20 & -18 \\ -7 & -7 & 8 \end{vmatrix}$$

$A^3 - 6A^2 + 11A - 6I = 0$

$$\begin{vmatrix} 8 & -19 & 19 \\ -7 & 20 & -18 \\ -7 & -7 & 8 \end{vmatrix} - 6\begin{vmatrix} 4 & -5 & 5 \\ -3 & 6 & -5 \\ -3 & -3 & 4 \end{vmatrix} + 11\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} - \begin{vmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

A^{-1} pre mul

$$A^2 - 6A + 11I = 6A^{-1}$$

$$\frac{1}{6}\left(\begin{vmatrix} 4 & -5 & 5 \\ -3 & 6 & -5 \\ -3 & -3 & 4 \end{vmatrix} - 6\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{vmatrix} \right) = A^{-1}$$

$$\frac{1}{6}\begin{vmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ 3 & 3 & 3 \end{vmatrix} = A^{-1}$$

A pre mul

$$A^4 = 6A^3 - 11A^2 + 6A$$

$$= 6\begin{vmatrix} 8 & -19 & 19 \\ -7 & 20 & -18 \\ -7 & -7 & 8 \end{vmatrix} - 11\begin{vmatrix} 4 & -5 & 5 \\ -3 & 6 & -5 \\ -3 & -3 & 4 \end{vmatrix} + 6\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix}$$

$$A^4 = \begin{vmatrix} 56 & -65 & 175 \\ -15 & 66 & -59 \\ -15 & -15 & 16 \end{vmatrix}$$

Q.] $\begin{vmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{vmatrix}$

Soln

$S_1 = 6$

$S_2 = -11 + 14 + 8 = 11$

$S_3 = 8(-11) - 8(-10) - 2(-7)$

$= -88 + 80 + 14$

$= 6$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6I = 0$

$A^3 - 6A^2 + 11A - 6I = 0$

$$A^2 = \begin{vmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{vmatrix} \begin{vmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 64 - 32 - 6 & -64 + 24 + 8 & -16 + 16 - 2 \\ 32 - 12 - 6 & -32 + 9 + 8 & -8 + 6 - 2 \\ 24 - 16 \;\; 3 & -24 + 12 - 4 & -6 + 8 \;\; 1 \end{vmatrix}$$

$$A^2 = \begin{vmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{vmatrix}$$

$$A^3 = \begin{vmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{vmatrix} \begin{vmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 208 - 128 - 6 & -208 + 96 + 8 & -52 + 64 - 2 \\ 112 + 60 - 12 & -112 - 45 + 16 & -28 - 30 - 4 \\ 88 - 64 \;\; 9 & -88 + 48 - 12 & -22 + 32 \;\; 3 \end{vmatrix}$$

$$A^3 = \begin{vmatrix} 74 & -104 & 10 \\ 160 & -141 & -62 \\ 33 & -52 & 13 \end{vmatrix}$$

$$A^3 - 6A^2 + 11A - 6I = 0$$

$$\begin{vmatrix} 74 & -104 & 10 \\ 160 & -141 & -62 \\ 33 & -52 & 13 \end{vmatrix} - 6 \begin{vmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{vmatrix} + 11 \begin{vmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{vmatrix} - 6 \begin{vmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 0$$

Q.] $\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix}$

$S_1 = 6$

$S_2 = 3 + 3 + 3 = 9$

$S_3 = 2(3) - 1$

Q.] Find Char eq. of $A = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$ and hence

find the Matrix $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$S_1 = 5$

$S_2 = 2 + 3 + 2 = 7$

$S_3 = 2(2) + 1(0) + 1(-1) = 3$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3I = 0$$

We know, $A^3 - 5A^2 + 7A - 3I = 0$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$A^5(\cancel{A^3 - 5A^2 + 7A - 3}^{\,0}) + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$A^4 - 5A^3 + 8A^2 - 2A + I$$

$$A(A^3 - 5A^2 + 8A - 2I) + I$$

$$A(A^3 - 5A^2 + 7A - 3I + 8A - 2I - 7A + 3I) + I$$

$$A(\cancel{A^3 - 5A^2 + 7A - 3I}^{\,0} + A + I) + I$$

$$A^2 + A + I$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Q.] If $A = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$ find $A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + I$

Soln $A = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix}$

$$3 - \lambda - 3\lambda + \lambda^2 - 8$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$A^2 - 4A - 5I = 0 \quad \text{(By CHT)}$$

long division

$$A^3 + 9A^2 + 35A + 187$$

$A^2 - 4A - 5I$) $\cancel{A^5} + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$

$\cancel{A^5} - 4A^4 - 5A^3$
(−) (+) (−)

$9A^4 - A^3 + 2A$
$9A^4 - 36A^3 - 45A^2$
(−) (+) (+)

$35A^3 + 47A^2 - 4A$
$35A^3 - 140A^2 - 175A$

$\underline{919A + 942I}$

$$A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$$

$$= \cancel{(A^2 - 4A + 5I)(A^3 + 9A^2 + 35A + 187)} \quad 0$$

$$+ (919A + 942I)$$

$$= 919 \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} + 942 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1861 & 3676 \\ 1838 & 3699 \end{vmatrix}$$

- **Properties of Eigen Values**

① Every sq. matrix and its Transpose have the same eigen values

② If $\lambda_1, \lambda_2 \ldots\ldots \lambda_n$ are the eigen value of mat (A) then $1/\lambda_1 + 1/\lambda_2 + \ldots.\ 1/\lambda_n$ are the eigen values of mat (A^{-1}) where non of λ^{-1} are 0

③ If $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ are the eigen values of mat (A) then

④ $\lambda_1, \lambda_2, \ldots.\ \lambda_n$ are the eigen values of mat (A) then $K\lambda_1, K\lambda_2, \ldots\ K\lambda_n$ are eigen values of mat (KA)

⑤ The eigen values of a real symmetric mat. are all real.

⑥ Eigen Values of a triangular Matrix are just the diagonal elements of the matrix

⑦ 0 is an eigen value of mat (A) if + only if the matrix is singular

⑧ If A and B are orthogonal mat then their product is also orthogonal

Note : Sum of the eigen values of the matrix is equal sum of leading diagonal elements of mat = Trace of mat.

Product of the eigen values of mat. = $|A|$

Q.1] find sum + product of eigen values of the matrix

$$\begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

Sum = -1

Product = $-2(-12) + 2(+6) + 3(+3)$

$= +24 + 12 + 9$

$= 45$

Q.2] two of the eigen values of A = $\begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$ are 3 + 6 find A^{-1}

$3+6+x = Tr = 11$

$x = 2$

$A^{-1} = 1/3 , 1/6 , 1/2$

Q.3] find eigen values of A^3 if $A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{vmatrix}$

Soln by prop 6

$A = 1, 2, 3$

$A^3 = 1^3, 2^3, 3^2 = 1, 8, 24$

Q.4] 2 is a eigen value of A $\begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$

$x + y = 0$

$x = -y$

$x \; z y \qquad -y^2 z =$

$2(-4) - 2(2) + 2(2)$

$-8 - 4 + 4$

-8

$y^2 = 4 \qquad y = 2$

$x = -2$

Q.5] if A $\begin{vmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{vmatrix}$ ar 2, 2, 3

find A^{-1} A^2

$1/2 \; 1/2 \; 1/3 \qquad 4 \; 4 \; 9$