UNIT I
INTRODUCTION TO ALGORITM DESIGN

Algorithms

- A finite set of instructions or logic, written in order, to accomplish a certain predefined task.

- Algorithm is not the complete code or program.

- It is just the core logic (solution) of a problem.

- Can be expressed either as an informal high level description as pseudo code or using a flowchart.

Characteristics of an Algorithm

- **Input**

  An algorithm should have 0 or more well defined inputs.
- **Output**

  An algorithm should have 1 or more well defined outputs
- **Unambiguous**

  Algorithm should be clear and unambiguous.
- **Finiteness**

  Algorithms must terminate after a finite no. of steps.
- **Feasibility**

  Should be feasible with the available resources.
- **Independent**

  An algorithm should have step-by-step directions which should be independent of any programming code.

Pseudo code

- It is one of the methods that could be used to represent an algorithm.
- It is not written in a specific syntax
- Cannot be executed
- Can be read and understood by programmers who are familiar with different programming languages.
- Transformation from pseudo code to the corresponding program code easier.
- Pseudo code allows to include control structures such as WHILE, IF-THEN-ELSE, REPEAT-UNTIL, FOR, and CASE, which are available in many high level languages.

Difference between Algorithm and Pseudocode

Algorithm	Pseudo code
A finite set of instructions or logic, written in order, to accomplish a certain predefined task.	a generic way of describing an algorithm without using any specific programming language-related notations.
It is just the core logic (solution) of a problem	It is an outline of a program, written in a form which can easily be converted into real programming statements.
Easy to understand the logic of a problem	Can be read and understood by programmers who are familiar with different programming languages.
Can be expressed either as an informal high level description as pseudo code or using a flowchart.	It is not written in a specific syntax. It allows to include control structures such as WHILE, IF-THEN-ELSE, REPEAT-UNTIL, FOR, and CASE, which are present in many high level languages.

UNIT I
INTRODUCTION TO ALGORITM DESIGN

Why to design an algorithm?

- General approaches to the construction of efficient solutions to problems
  - They provide templates suited for solving a broad range of diverse problems.
  - They can be translated into common control and data structures provided by most high-level languages.
  - The temporal and spatial requirements of the algorithms which result can be precisely analyzed.

Algorithm Design Approaches

Based on the architecture

- Top Down Approach
- Bottom up Approach

Algorithm Design Techniques

1. Brute Force

- To solve a problem based on the problem's statement and definitions of the concepts involved.
- Easiest approach to apply
- Useful for solving small – size instances of a problem.

- Some examples of brute force algorithms are:
  - Computing a^n *($a > 0$, n a nonnegative integer) by multiplying $a*a*\ldots*a$*
  - Computing *n!*
  - Selection sort, Bubble sort
  - Sequential search

2. **Divide-and-Conquer & Decrease-and-Conquer**

Step 1

Split the given instance of the problem into several smaller sub-instances

Step 2

Independently solve each of the sub-instances

Step 3

Combine the sub-instance solutions.

- With the divide-and-conquer method the size of the **problem instance is reduced by a factor** (e.g. half the input size),
- With the decrease-and-conquer method the size is **reduced by a constant.**

Examples of divide-and-conquer algorithms:

- Computing a^n ($a > 0$, n a nonnegative integer) by recursion
- Binary search in a sorted array (recursion)
- Mergesort algorithm, Quicksort algorithm recursion)
- The algorithm for solving the fake coin problem (recursion)

3. Greedy Algorithms "take what you can get now" strategy

- **At each step the choice must be locally optimal**
- Works well on optimization problems
- **Characteristics**

  1. Greedy-choice property: A global optimum can be arrived at by selecting a local optimum.

  2. Optimal substructure: An optimal solution to the problem contains an optimal solution to sub problems.

- **Examples:**
  - Minimal spanning tree
  - Shortest distance in graphs
  - Greedy algorithm for the Knapsack problem
  - The coin exchange problem
  - Huffman trees for optimal encoding

4.Dynamic Programming

- Finds solutions to subproblems and stores them in memory for later use.
- Characteristics

1. Optimal substructure:

Optimal solution to problem consists of optimal solutions to subproblems

2. Overlapping subproblems:

Few subproblems in total, many recurring instances of each

3. Bottom up approach:

Solve bottom-up, building a table of solved subproblems that are used to solve larger ones.

- **Examples:**
  - Fibonacci numbers computed by iteration.
  - Warshall's algorithm implemented by iterations

5. Backtracking methods

- The method is used for state-space search problems.

What is State-space search problems

- State-space search problems are problems, where the problem representation consists of:
  - initial state
  - goal state(s)
  - a set of intermediate states
  - a set of operators that transform one state into another.
  - a cost function – evaluates the cost of the operations (optional)
  - a utility function – evaluates how close is a given state to the goal state (optional)

- The solving process solution is based on the construction of a state-space tree
- The solution is obtained by searching the tree u
- Examples:
  - DFS problem
  - Maze problems

6. Branch-and-bound

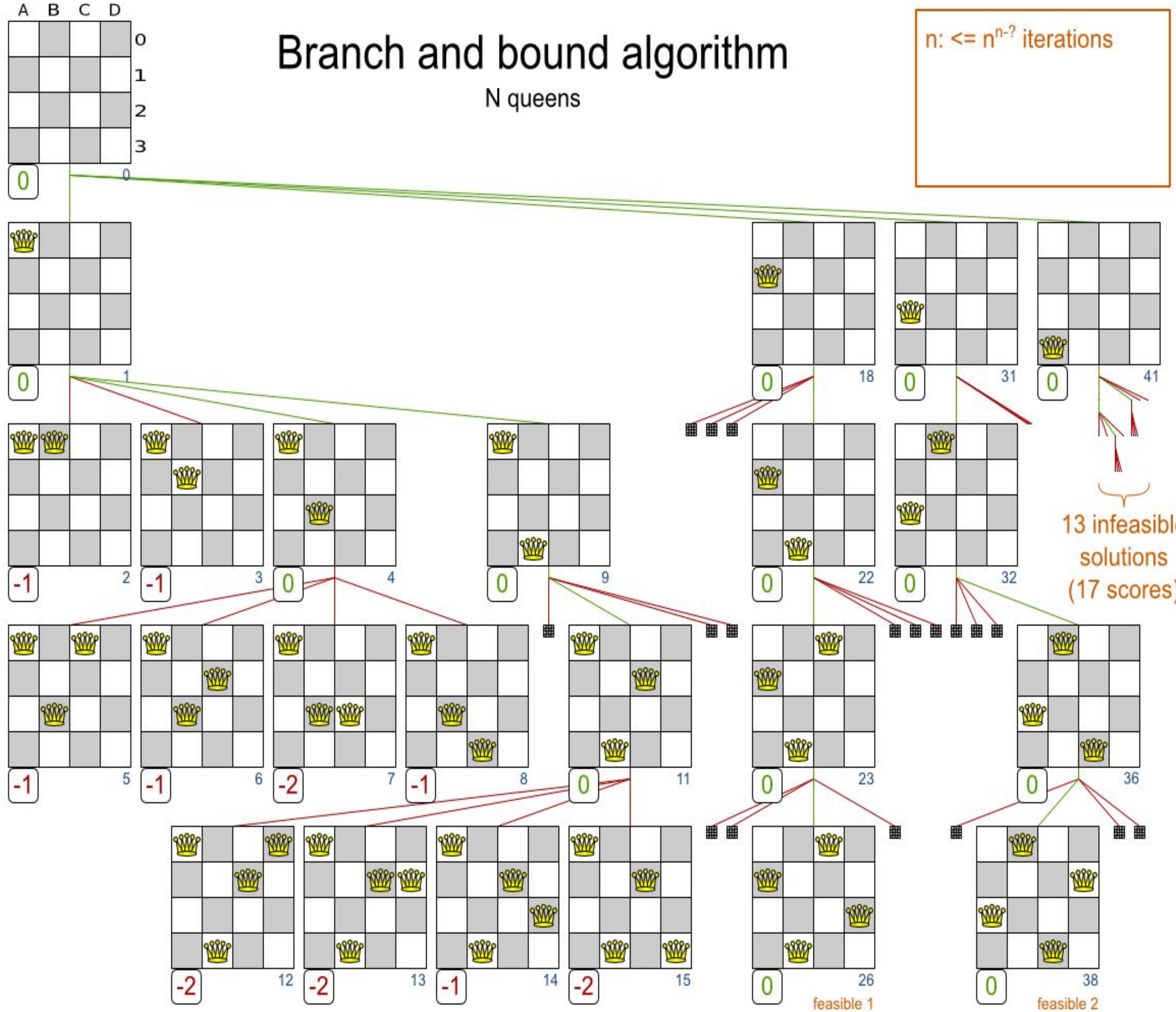
- Branch and bound is used when we can evaluate each node using the cost and utility functions.
- At each step we choose the best node to proceed further.
- Branch-and bound algorithms are implemented using a **priority queue.**
- The state-space tree is built in a **breadth-first manner.**
- **Example:**

  8-puzzle problem.

  N queens problem

Branch and bound algorithm

N queens



Recollect

- **Different design Approaches/ Design Paradigms**
  - **Brute force**
  - **Divide and Conquer** **Unit 2**
  - **Greedy Algorithms**
  - **Dynamic Programmin** **} Unit 3**
  - **Backtracking** **Unit 4**
  - **Branch and Bound** **Unit 5**

UNIT I
INTRODUCTION TO ALGORITM DESIGN

Algorithm Analysis

- An algorithm is said to be efficient and fast,
  - if it takes less time to execute
  - consumes less memory space.
- The performance of an algorithm is measured on the basis of
  - Time Complexity
  - Space Complexity

- **Space Complexity**
  - The amount of memory space required by the algorithm in its life cycle.
  - **A fixed part** For example simple variables & constant used and program size etc.
  - **A variable part** For example dynamic memory allocation, recursion stacks space etc.
  - Space complexity S(P) of any algorithm P is

    S(P) = C + SP(I)

    Where C is the fixed part

    S(I) is the variable part of the algorithm

- **Time Complexity - T(n)**
  - The amount of time required by the algorithm to run to completion.
  - T(n) can be measured as the number of steps, provided each step consumes constant time.

Algorithm analysis

- **The *worst-case complexity*** of the algorithm is the function defined by the maximum number of steps taken on any instance of size *n*.
- **The *best-case complexity*** of the algorithm is the function defined by the minimum number of steps taken on any instance of size *n*.
- Finally, **the *average-case complexity*** of the algorithm is the function defined by the average number of steps taken on any instance of size *n*.

Mathematical Analysis

- **For Non-recursive Algorithms**
  - There are four rules to count the operations:
    - **Rule 1: for loops - the size of the loop times the running time of the body**
      - Find the running time of statements when executed only once
      - Find how many times each statement is executed
    - **Rule 2 : Nested loops**
      - The product of the size of the loops times the running time of the body
    - **Rule 3: Consecutive program fragments**
      - The total running time is the maximum of the running time of the individual fragments
    - **Rule 4: If statement**
      - The running time is the maximum of the running times of if stmt and else stmt.

Rule 1: for loops

for(i = 0; i < n; i++) // i = 0; executed only once: O(1)

// i < n; n + 1 times O(n)

// i++ n times O(n)

// total time of the loop heading:

// O(1) + O(n) + O(n) = O(n)

sum = sum + i; // executed n times, O(n)

The loop heading plus the loop body will give: O(n) + O(n) = O(n).

IF

1. The size of the loop is n (loop variable runs from 0, or some fixed constant, to n)
2. The body has constant running time (no nested loops)

Loop running time is: O(n)

$$C(n) = \sum_{i=0}^{n-1} 1 = (n-1) - 0 + 1 = n$$

Rule 2 : Nested loops

```
sum = 0;
for( i = 0; i < n; i++)
   for( j = 0; j < n; j++)
         sum++;
```

- Applying Rule 1 for the nested loop (the 'j' loop) we get O(n) for the body of the outer loop. The outer loop runs n times, therefore the total time for the nested loops will be O(n) * O(n) = O(n*n) = O(n^2)

Mathematical analysis:

Inner loop:

$$S(i) = \sum_{j=0}^{n-1} 1 = (n-1) - 0 + 1 = n$$

Outer loop:

$$C(n) = \sum_{i=0}^{n-1} S(i) = \sum_{i=0}^{n-1} n = n * \sum_{i=0}^{n-1} 1 = n((n-1) - 0 + 1) = n^2$$

```
for( i = 0; i < n; i++)
  for( j = i; j < n; j++)
      sum++;
```

- Here, the number of the times the inner loop is executed depends on the value of i

i = 0, inner loop runs n times

i = 1, inner loop runs (n-1) times

i = 2, inner loop runs (n-2) times

…

i = n – 2, inner loop runs 2 times

i = n – 1, inner loop runs once.

Thus we get: (1 + 2 + … + n) = n*(n+1)/2 = O(n^2)

Running time is the product of the size of the loops times the running time of the body.

Rule 3: Consecutive program fragment

```
sum = 0;
for( i = 0; i < n; i++)
    sum = sum + i;
sum = 0;
for( i = 0; i < n; i++)
    for( j = 0; j < 2*n; j++)
     sum++;
```

- The first loop runs in O(n) time, the second - O(n^2) time, the maximum is O(n^2)

The total running time is the maximum of the running time of the individual fragments

Rule 4: If statement

```
if C
   S1;
else
   S2;
```

The running time is the maximum of the running times of S1 and S2.

The running time is the maximum of the running times of if stmt and else stmt

Exercise

a.

```
sum = 0;
for( i = 0; i < n; i++)
    for( j = 0; j < n * n; j++)
      sum++;
```

Ans : O(n^3)

b.

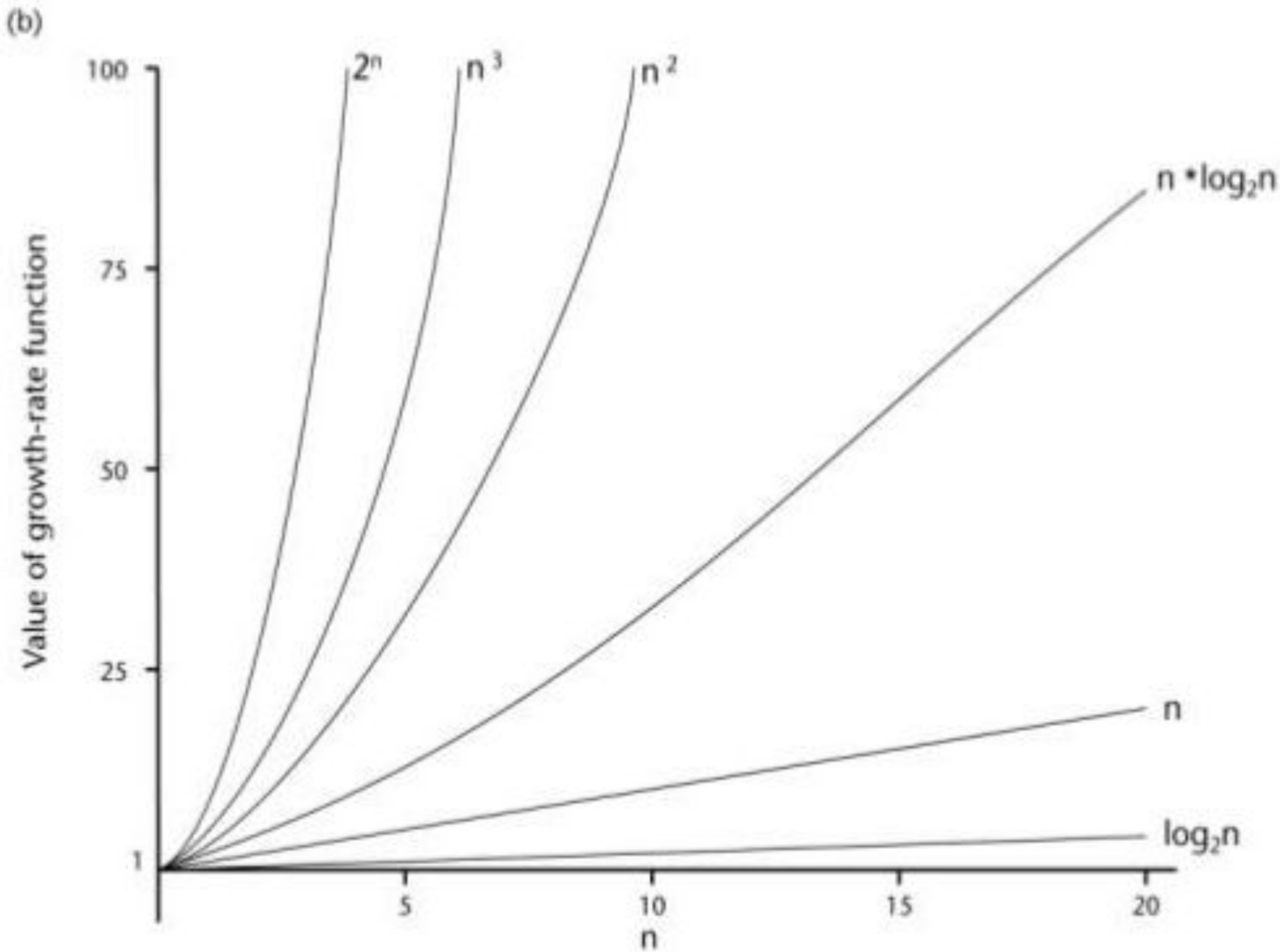
```
sum = 0;
for( i = 0; i < n; i++)
    for( j = 0; j < i; j++)
      sum++;
```

Ans : O(n^2)

c.

Order of Growth Function

	constant	*logarithmic*	*linear*	*N-log-N*	*quadratic*	*cubic*	*exponential*
n	**O(1)**	**O(log n)**	**O(n)**	**O(n log n)**	**O(n^2)**	**O(n^3)**	**O(2^n)**
1	1	1	1	1	1	1	2
2	1	1	2	2	4	8	4
4	1	2	4	8	16	64	16
8	1	3	8	24	64	512	256
16	1	4	16	64	256	4,096	65536
32	1	5	32	160	1,024	32,768	4,294,967,296
64	1	6	64	384	4,069	262,144	1.84×10^{19}


(b)
Value of growth-rate function
100
75
50
25
1
2^n
n^3
n^2
$n * \log_2 n$
n
$\log_2 n$
5
10
15
20
n

Linear Search Analysis

```
linear(a[n], key)
    for( i = 0; i < n; i++)
        if (a[i] == key)
            return i;
    else return -1;
```

- Worst Case : O(n) // Rule 1 for loop explanation
- Average Case :

  If the key is in the first array position: 1 comparison

  If the key is in the second array position: 2 comparisons

  ...

  If the key is in the ith postion : i comparisons

  ...

  So average all these possibilities: $(1+2+3+...+n)/n = [n(n+1)/2] /n = (n+1)/2$ comparisons.

  The average number of comparisons is **$(n+1)/2 = \Theta(n)$.**
- Best Case : O(1)

Binary Search Analysis

```
binarysearch(a[n], key, low, high)
while(low<high)
{
mid = (low+high)/2;
if(a[mid]=key)
   return mid;
elseif (a[mid] > key)
   high=mid-1;
  else
   low=mid+1;
  }
return -1;
```

Worst case analysis: The key is not in the array

Let T(n) be the number of comparisons done in the worst case for an array of size n. For the purposes of analysis, assume n is a power of 2, ie n = 2^k.

Then $T(n) = 2 + T(n/2)$

$= 2 + 2 + T(\frac{n}{2^2})$ // 2nd iteration

$= 2 + 2 + 2 + T(n/2^3)$ // 3rd iteration

...

$= i * 2 + T(n/2^i)$ // ith iteration

... $= k * 2 + T(1)$

Note that k = logn, and that T(1) = 2.

So T(n) = 2logn + 2 = O(logn)

Bubble sort analysis

```
int i, j, temp;
for(i=0; i<n; i++)
{
  for(j=0; j<n-i-1; j++)
  {
    if( a[j] > a[j+1])
    {
      temp = a[j];
      a[j] = a[j+1];
      a[j+1] = temp;
    }
  }
}
```

Worst Case: In Bubble Sort, n-1 comparisons will be done in 1st pass, n-2 in 2nd pass, n-3 in 3rd pass and so on. So the total number of comparisons will be

(n-1)+(n-2)+(n-3)+.....+3+2+1

Sum = n(n-1)/2

Hence the complexity of Bubble Sort is **O(n^2)**.

Best-case Time Complexity will be **O(n)**, it is when the list is already sorted.

Insertion Sorting Analysis

```
int i, j, key;
for(i=1; i<n; i++)
{
  key = a[i];
  j = i-1;
  while(j>=0 && key < a[j])
  {
    a[j+1] = a[j];
    j--;
  }
  a[j+1] = key;
}
```

- Worst Case Time Complexity : O(n^2)
- Best Case Time Complexity : O(n)
- Average Time Complexity : O(n^2)

UNIT I
INTRODUCTION TO ALGORITM DESIGN

Asymptotic Notations

- Main idea of asymptotic analysis
  - To have a measure of efficiency of algorithms
  - That doesn’t depend on machine specific constants,
  - That doesn’t require algorithms to be implemented
  - Time taken by programs to be compared.
- Asymptotic notations
  - Asymptotic notations are mathematical tools to represent time complexity of algorithms for asymptotic analysis.

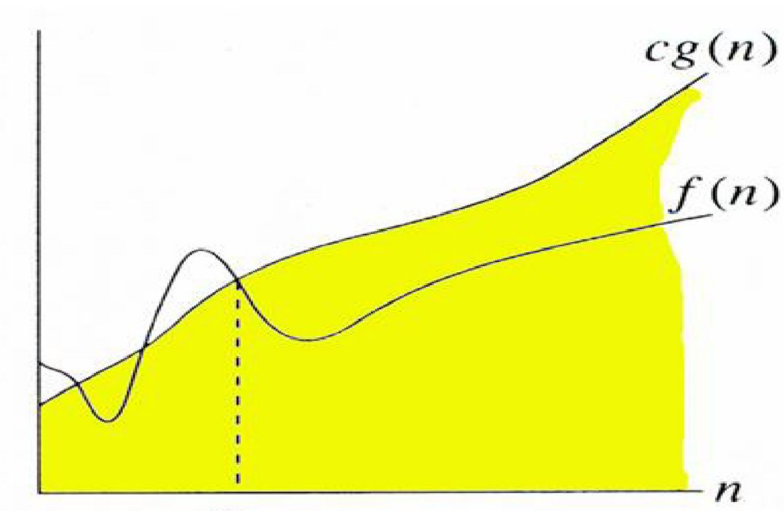
Asymptotic Analysis

- Asymptotic analysis refers to computing the running time of any operation in mathematical units of computation.
  - **O Notation**
  - **Ω Notation**
  - **θ Notation**

- **Big Oh Notation, O**
  - It measures the worst case time complexity or longest amount of time an algorithm can possibly take to complete.

O(g(n)) = { f(n): there exist positive constants c and n0 such that

0 <= f(n) <= cg(n) for all n >= n0}



$f(n) = O(g(n))$

0 <= f(n) <= cg(n) for all n >= n0}

If f(n) = 3n+2

g(n)=n

Then

3n+2 <= cn

For instance c=4

3n+2<=4n

n>=2

Hence f(n) = O(g(n))

If f(n) = 3n+2 $g(n)=n^2$

Then

$3n+2 <= cn^2$

For instance c=1

n=1,2,3,4,**5,6…**

$3n+2<=n^2$

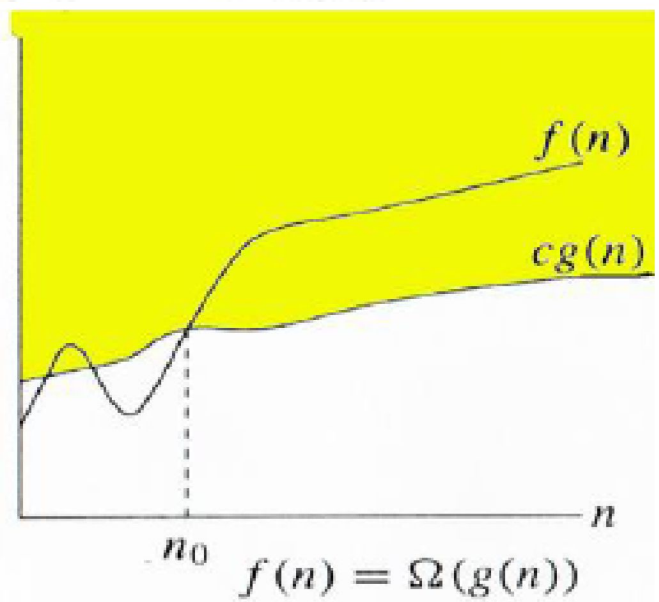
n>=5

Hence f(n) = O(g(n))

- **Ω Notation: (Best Case)**
  - Ω notation provides an asymptotic lower bound

$$\Omega(g(n)) = \{f(n): \text{there exist positive constants } c \text{ and } n0 \text{ such that}$$

$$0 <= cg(n) <= f(n) \text{ for all } n >= n0\}.$$



$$0 <= cg(n) <= f(n)\ for\ all\ n >= n0\}.$$

If f(n) = 3n+2

g(n)=n

Then

f(n)>= cg(n)

For instance c=1

3n+2 >= n

Hence f(n) = omega(g(n))

If f(n) = 3n+2 g(n)=n^2

Then

3n+2 >= cn^2

For instance c=1

n=**1,2,3,4,5,6…**

3n+2 >=n^2

Hence f(n) = omega(n)

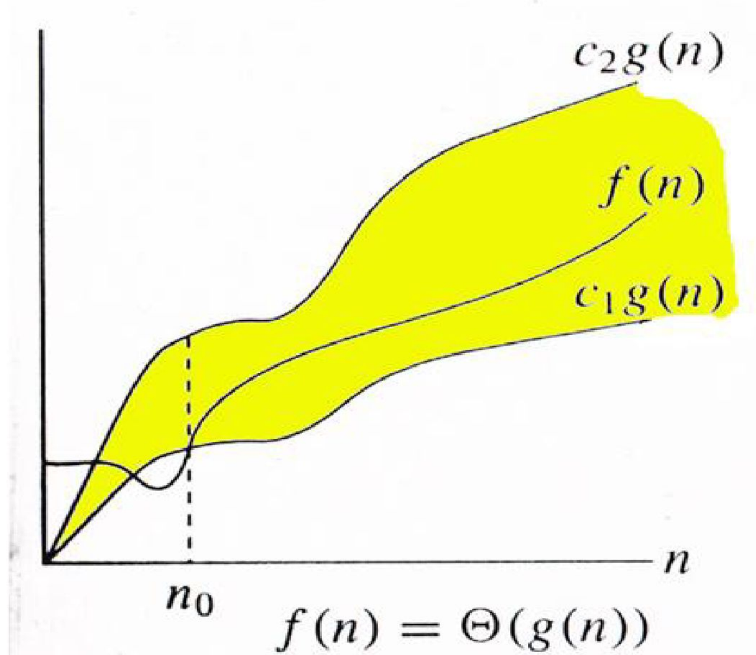
logn

log(logn)

Θ Notation:

$\Theta((g(n)) = \{f(n)$: there exist positive constants c1, c2 and n0 such that

$$0 <= c1*g(n) <= f(n) <= c2*g(n) \text{ for all } n >= n0\}$$



$0 <= c1*g(n) <= f(n) <= c2*g(n)$ for all $n >= n0\}$

If $f(n) = 3n+2$

$g(n)=n$

Then

$c1*g(n) <= f(n) <= c2*g(n)$

$C1,c2>0$ and $N>n_0$

For instance $c2=4$

$f(n)<= c2g(n)$

$3n+2 <= 4n \quad n_{0=1}$

$f(n)>= c1g(n)$

For instance $c1=1$

$3n+2 >= n$

Hence $f(n) = theta(n)$

Order of Growth Function

	constant	*logarithmic*	*linear*	*N-log-N*	*quadratic*	*cubic*	*exponential*
n	O(1)	O(log n)	O(n)	O(n log n)	O(n^2)	O(n^3)	O(2^n)
1	1	1	1	1	1	1	2
2	1	1	2	2	4	8	4
4	1	2	4	8	16	64	16
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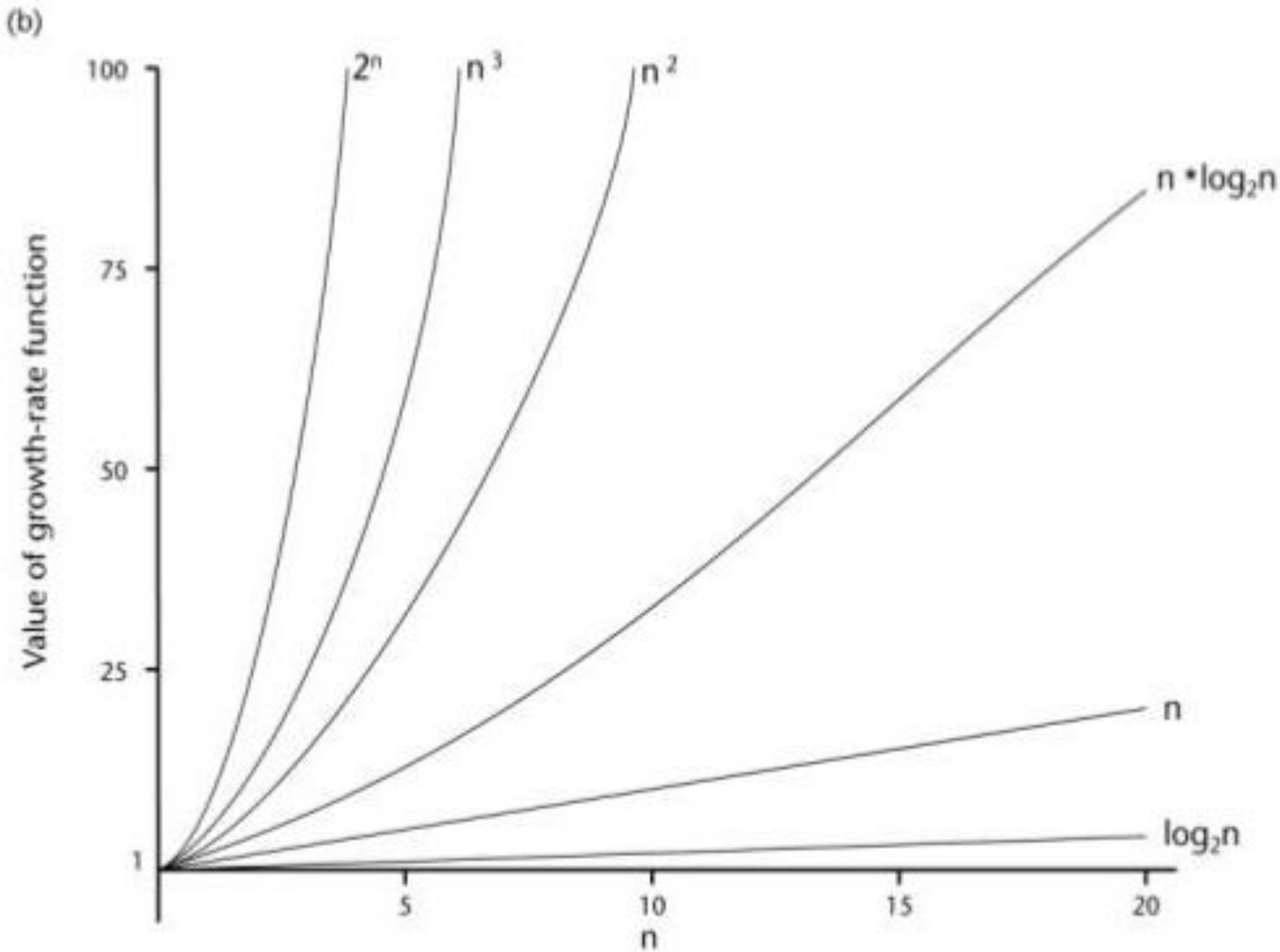


(b)
Value of growth-rate function
100
75
50
25
1
5
10
15
20
n
2ⁿ
n ³
n ²
n *log₂n
n
log₂n

UNIT I
INTRODUCTION TO ALGORITHM DESIGN

Recursion recall

- Functions calls itself
- consists of one or more base cases and one or more recursive cases.

$$f(n) = \begin{cases} 0 & if\ n = 0 \\ f(n-1) + 1 & otherwise \end{cases}$$

$$\begin{aligned} & & f(0) &= 0 \\ f(n) &= f(n-1) + 1 & for\ all\ n &> 0 \end{aligned}$$

Mathematical Analysis - Induction

- Consider a recursive algorithm to compute the maximum element in an array of integers.
- You may assume the existence of a function "max(a,b)" that returns the maximum of two integers a and b .

```
Function FIND − ARRAY − MAX (A , n)
if (n = 1) then
        return (A[1])
else
        return (max (A[n] , FIND − ARRAY − MAX (A , n − 1)))
end if
```

Solution

- Let p(n) stand for the proposition that Algorithm finds and returns the maximum integer in the locations A[1] through A[n]. Accordingly, we have to show that $(\forall n)$ p(n) is true.
- **BASIS:** When there is only one element in the array , i.e., n=1, then this element is clearly the maximum element and it is returned on Line 2. We thus see that p(1) is true.
- **INDUCTIVE STEP:** Assume that Algorithm finds and returns the maximum element, when there are exactly k elements in A.
- Now consider the case in which there are k + 1 elements in A. Since $(k + 1) > 1$, Line 4 will be executed.
- From the inductive hypothesis, we know that the maximum elements in A[1] through A[k] is returned. Now the maximum element in A is either A[k+1] or the maximum element in A[1] through A[k] (say r). Thus, returning the maximum of A[k+1] and r clearly gives the maximum element in A, thereby proving that $p(k) \rightarrow p(k+1)$.
- By applying the principle of mathematical induction, we can conclude that $(\forall n)$ p(n) is true, i.e., Algorithm is correct.

- Find the exact solution to the recurrence relation using mathematical induction method

$$T(1) = 0$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n, \qquad n \geq 2$$

- Solution is T(n)=nlogn

Solution

- **Basis:** At n = 1, both the closed form and the recurrence relation agree (0=0) and so the basis is true.
- **Inductive step:** Assume that T(r) = rlogr for all 1≤ r ≤ k . Now consider T(k+1). As per the recurrence relation, we have,

$$T(k+1) = 2T(\frac{k+1}{2}) + (k+1);\ since\ (k+1) \geq 2$$

$$= 2\left(\frac{(k+1)}{2} log \frac{k+1}{2}\right) + (k+1) as\ per\ the\ inductive\ hypothesis;\ since \frac{k+1}{2} < k$$

$$= (k+1)[log(k+1) - log\,2] + (k+1)$$

$$= (k+1)log(k+1) - (k+1) + (k+1)$$

$$= (k+1)\,log(k+1)$$

- We can therefore apply the principle of mathematical induction to conclude that the exact solution to the given recurrence is nlogn.

UNIT I
INTRODUCTION TO ALGORITM DESIGN

Substitution method

- Guess the solution.
- Use induction to find the constants and show that the solution works.

Example

$$T(n) = \begin{cases} 1 & \text{if } n = 1\,, \\ 2T(n/2) + n & \text{if } n > 1\,. \end{cases}$$

1. *Guess:* $T(n) = n\lg n + n$.

   [Here, we have a recurrence with
   tion, rather than asymptotic notation, and the solution is al
   asymptotic. Weíll have to check boundary conditions a

2. *Induction:*

   Basis: $n = 1 \Rightarrow n\lg n + n = 1 = T(n)$

   Inductive step: Inductive hypothesis is that $T(k) = k\lg k + k$ for all $k < n$. Weíll use this inductive hypothesis for $T(n/2)$.

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + n \\ &= 2\left(\frac{n}{2}\lg\frac{n}{2} + \frac{n}{2}\right) + n \qquad \text{(by inductive hypothesis)} \\ &= n\lg\frac{n}{2} + n + n \\ &= n(\lg n - \lg 2) + n + n \\ &= n\lg n - n + n + n \\ &= n\lg n + n\,. \end{aligned}$$

■

Generally, we use asymptotic notation:

- We would write $T(n) = 2T(n/2) + \Theta(n)$.
- We assume $T(n) = O(1)$ for sufficiently small n.
- We express the solution by asymptotic notation: $T(n) = \Theta(n \lg n)$.
- We donít worry about boundary cases, nor do we show base cases in the substitution proof.
    - $T(n)$ is always constant for any constant n.
    - Since we are ultimately interested in an asymptotic solution to a recurrence, it will always be possible to choose base cases that work.
    - When we want an asymptotic solution to a recurrence, we donít worry about the base cases in our proofs.
    - When we want an exact solution, then we have to deal with base cases.

For the substitution method:

- Name the constant in the additive term.
- Show the upper (O) and lower (Ω) bounds separately. Might need to use different constants for each.

Another Example

- How to prove by the substitution method that if T(n)=T(n−1)+Θ(n)then T(n)=Θ(n^2)

We guess that $T(n) \leq O(n^2)$.

$$
\begin{aligned}
T(n) &\leq c_1(n-1)^2 + \Theta(n) \\
&\leq c_1(n-1)^2 + c_0 n \\
&\leq c_1(n^2 - 2n + 1) + c_0 n \\
&\leq c_1 n^2 - (2c_1 - c_0)n + c_1 \\
&\leq c_1 n^2 \text{for } n_0 \geq 1 \text{ and } c_0 > c_1
\end{aligned}
$$

Thus $T(n) \in O(n^2)$. Similarly, we can prove that $T(n) \in \Omega(n^2)$. Consequently, $T(n) \in \Theta(n^2)$.

Video Link

https://www.youtube.com/watch?v=Zhhh9qpAVN0

UNIT I
INTRODUCTION TO ALGORITM DESIGN

Recurrence relation - recursion

- solving recurrences
  - expanding the recurrence into a tree
  - summing the cost at each level

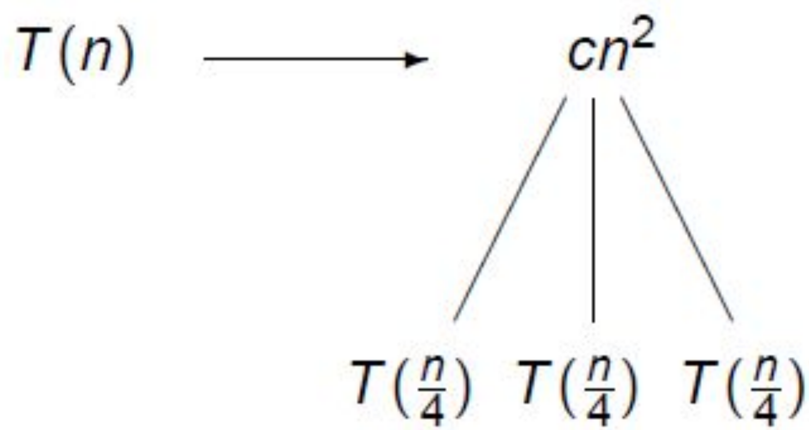
Example

Consider the recurrence relation

$$T(n) = 3T(n/4) + cn^2 \quad \text{for some constant } c.$$

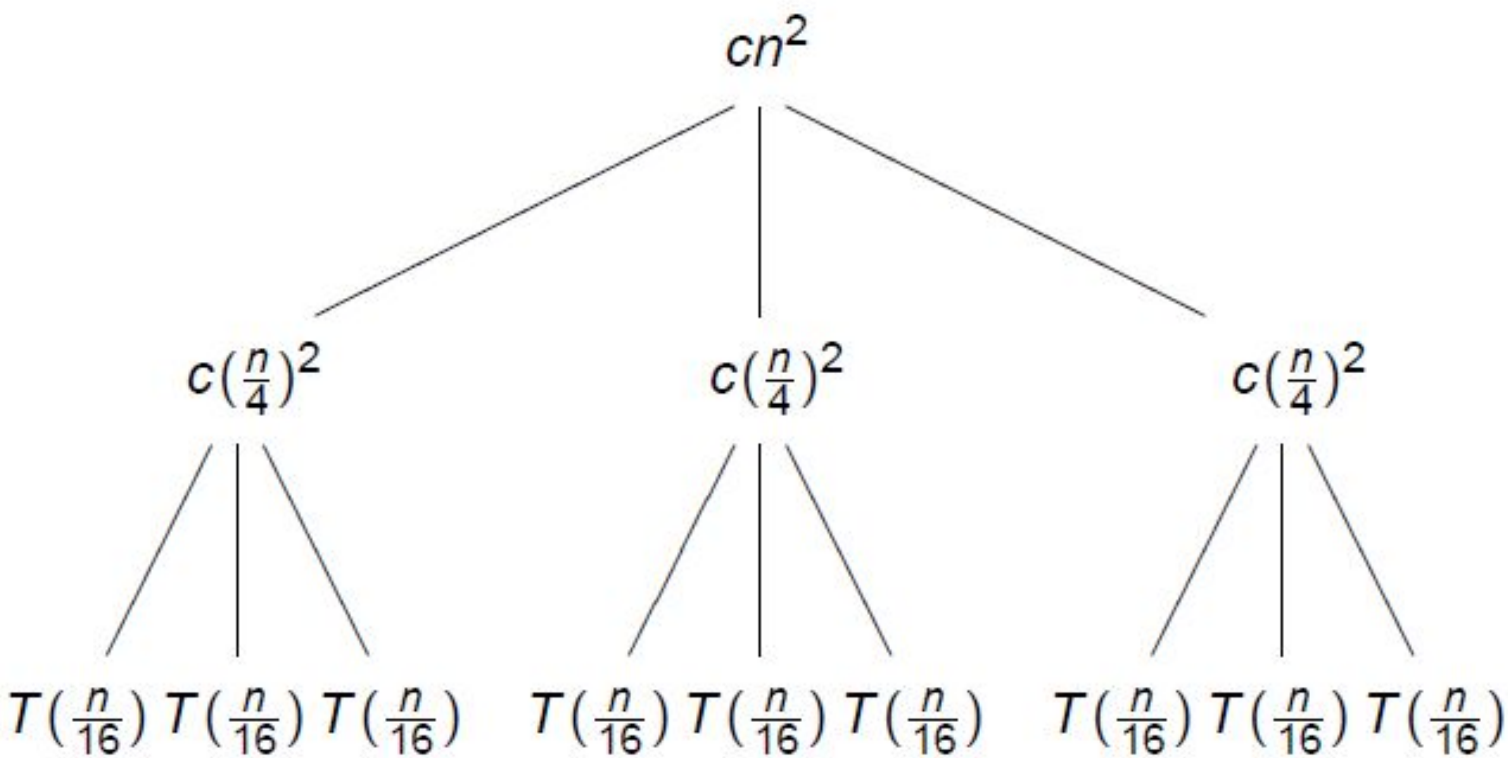
We assume that n is an exact power of 4.

In the recursion-tree method we expand $T(n)$ into a tree:



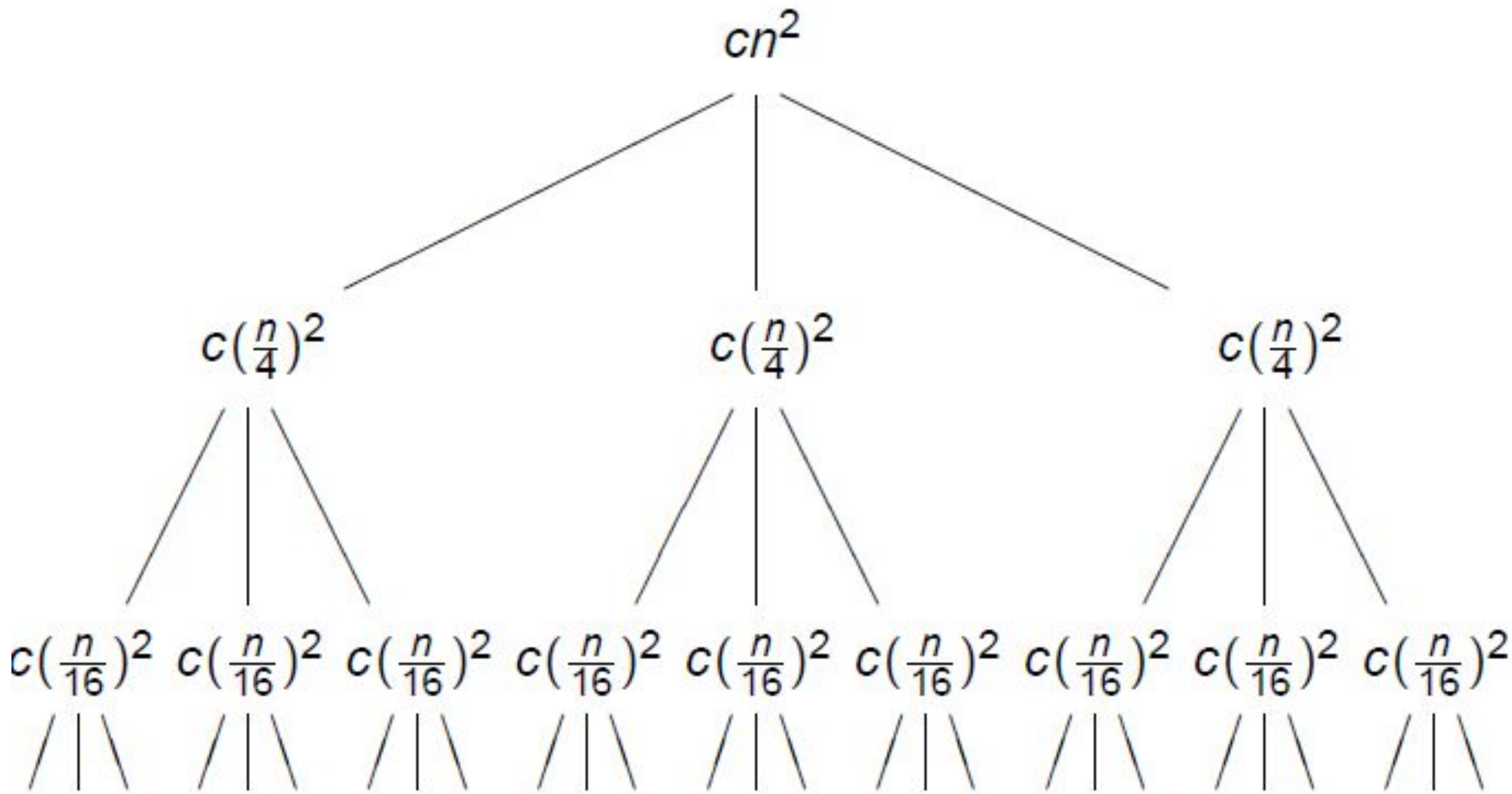
Expand T(n/4)

Applying $T(n) = 3T(n/4) + cn^2$ to $T(n/4)$ leads to $T(n/4) = 3T(n/16) + c(n/4)^2$, expanding the leaves:



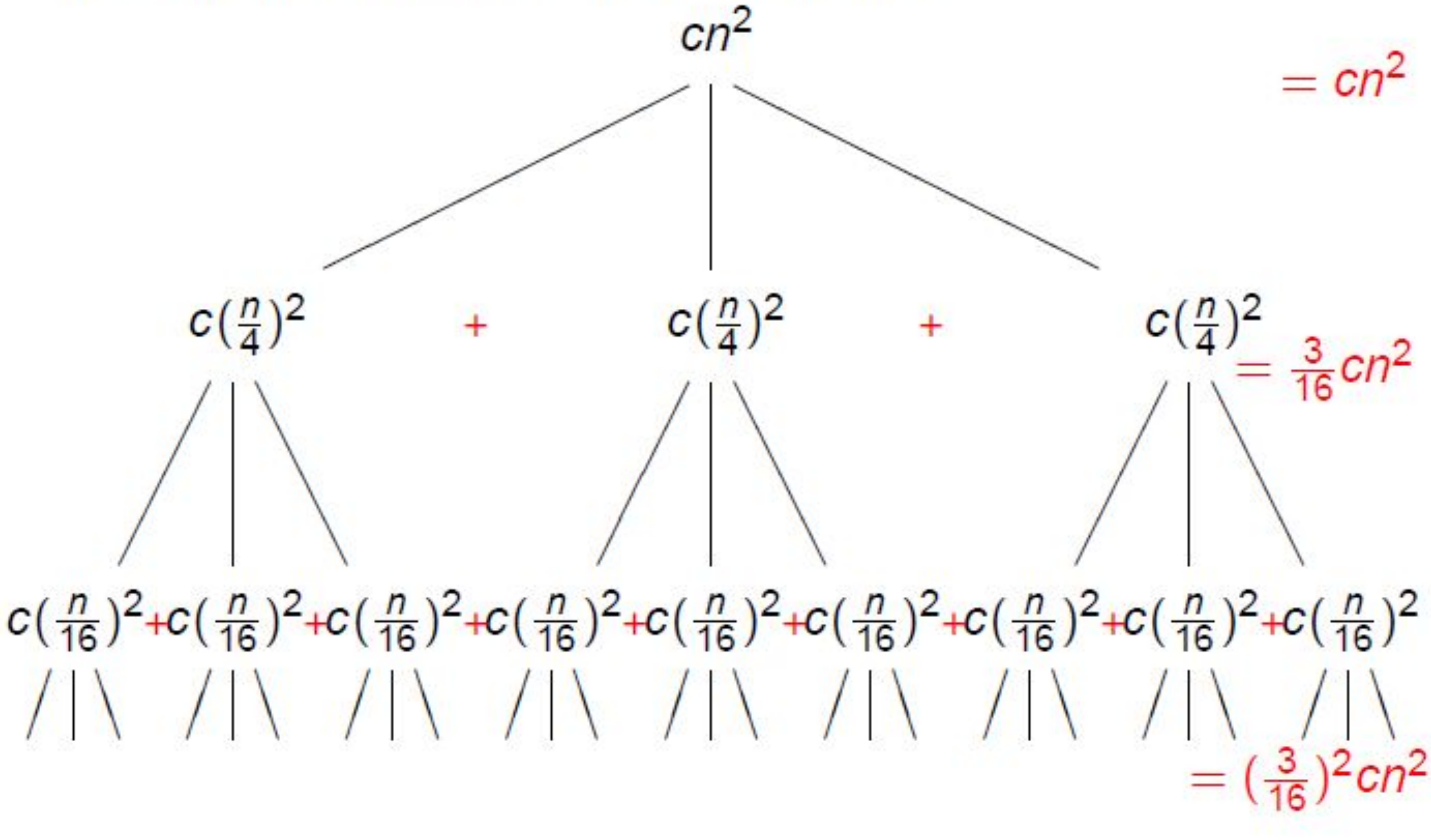
Expand T(n/16)

Applying $T(n) = 3T(n/4) + cn^2$ to $T(n/16)$ leads to $T(n/16) = 3T(n/64) + c(n/16)^2$, expanding the leaves:



Summing the cost at each level

We sum the cost at each level of the tree:



Adding up the costs

$$
\begin{aligned}
T(n) &= cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots \\
&= cn^2 \left(1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \cdots\right)
\end{aligned}
$$

The $\cdots$ disappear if $n = 16$,
or the tree has depth at least 2 if $n \geq 16 = 4^2$.

For $n = 4^k$, $k = \log_4(n)$, we have:

$$
T(n) = cn^2 \sum_{i=0}^{\log_4(n)} \left(\frac{3}{16}\right)^i.
$$

Applying the geometric sum

Applying

$$S_n = \sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

to

$$T(n) = cn^2 \sum_{i=0}^{\log_4(n)} \left(\frac{3}{16}\right)^i$$

with $r = \frac{3}{16}$ leads to

$$T(n) = cn^2 \frac{\left(\frac{3}{16}\right)^{\log_4(n)+1} - 1}{\frac{3}{16} - 1}.$$

Polishing the result we get

Instead of $T(n) \leq dn^2$ for some constant d, we have

$$T(n) = cn^2 \frac{\left(\frac{3}{16}\right)^{\log_4(n)+1} - 1}{\frac{3}{16} - 1}.$$

Recall

$$T(n) = cn^2 \sum_{i=0}^{\log_4(n)} \left(\frac{3}{16}\right)^i.$$

To remove the $\log_4(n)$ factor, we consider

$$\begin{aligned} T(n) &\leq cn^2 \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i \\ &= cn^2 \frac{-1}{\frac{3}{16} - 1} \leq dn^2, \text{ for some constant } d. \end{aligned}$$

Verify the guess by applying substitution method

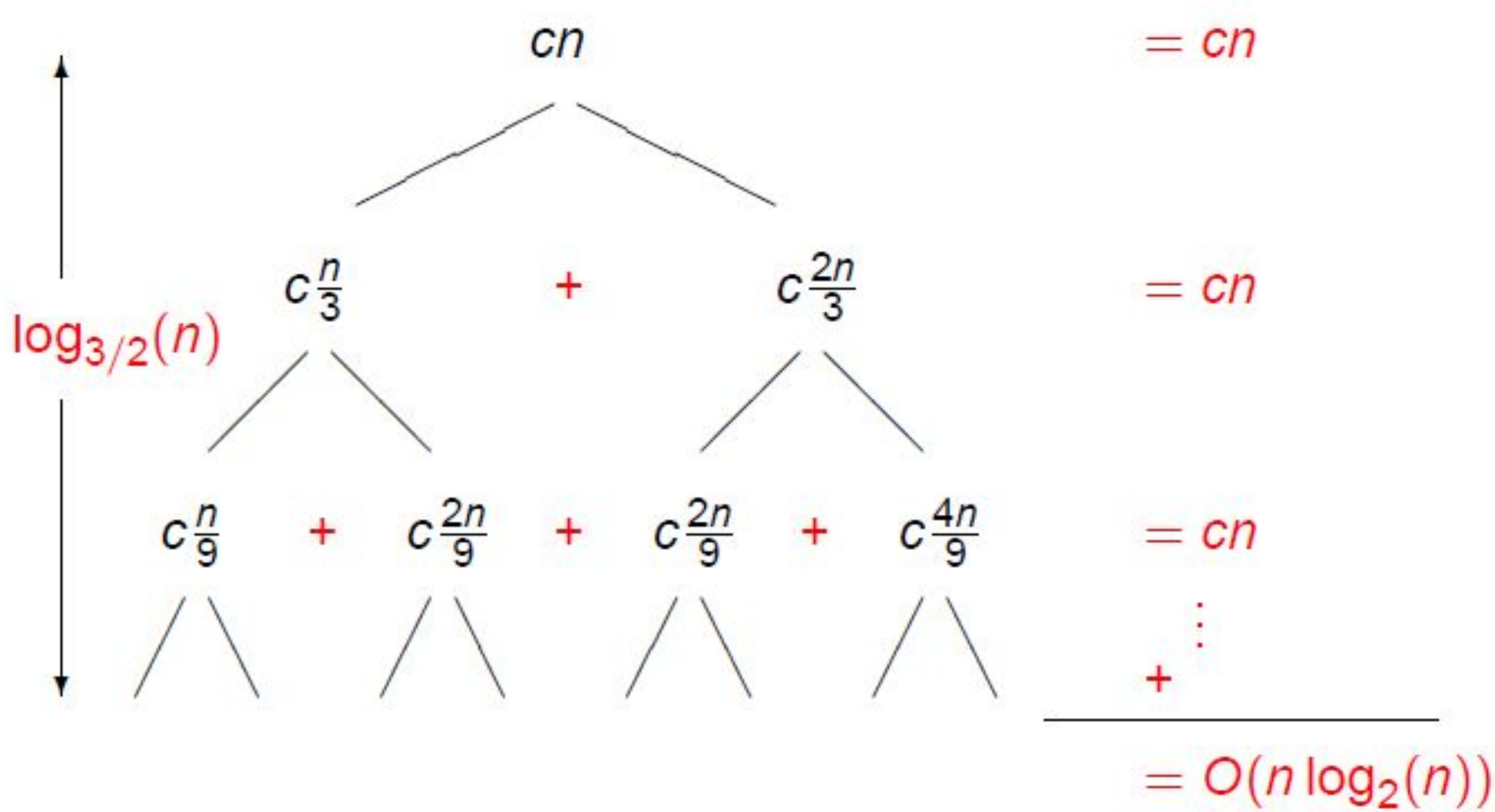
Let us see if $T(n) \le dn^2$ is good for $T(n) = 3T(n/4) + cn^2$.

Applying the substitution method:

$$
\begin{aligned}
T(n) &= 3T(n/4) + cn^2 \\
&\le 3d\left(\frac{n}{4}\right)^2 + cn^2 \\
&= \left(\frac{3}{16}d + c\right)n^2 \\
&= \frac{3}{16}\left(d + \frac{16}{3}c\right)n^2 \\
&\le \frac{3}{16}(2d)n^2, \quad \text{if } d \ge \frac{16}{3}c \\
&\le dn^2
\end{aligned}
$$

Lets see another example

Consider $T(n) = T(n/3) + T(2n/3) + cn$.



Lets practice

- Consider $T(n) = 3T(n/2) + n$. *Use a recursion tree to* derive a guess for an asymptotic upper bound for $T(n)$ and verify the guess with the substitution method.
- $T(n) = T(n/2) + n^2$.
- $T(n) = 2T(n - 1) + 1$.

UNIT I
INTRODUCTION TO ALGORITM DESIGN

Master's theorem method

- Master Method is a direct way to get the solution. The master method works only for following type of recurrences or for recurrences that can be transformed to following type.

$T(n) = aT(n/b) + f(n)$ where $a >= 1$ and $b > 1$ and $f(n)$ is an asymptotically positive function.

- There are following three cases:

1. If $f(n) < O(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If $f(n) > \Omega(n^{\log_b a})$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$.

Solve the problems

- $T(n) = 3T(n/2) + n^2$
- $T(n) = 7T(n/2) + n^2$
- $T(n) = 4T(n/2) + n^2$
- $T(n) = 3T(n/4) + n \lg n$

$T(n) = 3T(n/2) + n^2$

$T(n) = aT(n/b) + f(n)$ where $a >= 1$ and $b > 1$

$a = 3$ $b = 2$ $f(n) = n^2$

1. If $f(n) < O(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If $f(n) > \Omega(n^{\log_b a})$, and f(n) satisfies the regularity condition, then $T(n) = \Theta(f(n))$.

Step 1: Calculate $n^{\log_b a} = n^{\log_2 3} = n^{1.58}$

Step 2: Compare with f(n)

Since $f(n) > n^{\log_b a}$

i.e. $n^2 > n^{1.58}$

Step 3: Case 3 is satisfied hence complexity is given as

$T(n) = \Theta(f(n)) = \Theta(n^2)$

$T(n) = 7T(n/2) + n^2$

$T(n) = aT(n/b) + f(n)$ where $a >= 1$ and $b > 1$

$a = 7$ $b = 2$ $f(n) = n^2$

1. If $f(n) < O(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If $f(n) > \Omega(n^{\log_b a})$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$.

Step 1: Calculate $n^{\log_b a} = n^{\log_2 7} = n^{2.80}$

Step 2: Compare with $f(n)$

since $f(n) < n^{\log_b a}$

Step 3: Case 1 is satisfied hence complexity is given as

$T(n) = \Theta(n^{\log_b a}) = \Theta(n^{2.8})$

$T(n) = 4T(n/2) + n^2$

T(n) = aT(n/b) + f(n) where a >= 1 and b > 1

$a = 4$ $b = 2$ $f(n) = n^2$

1. If $f(n) < O(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If $f(n) > \Omega(n^{\log_b a})$, and f(n) satisfies the regularity condition, then $T(n) = \Theta(f(n))$.

Step 1: Calculate $n^{\log_b a} = n^{\log_2 4} = n^2$

Step 2: Compare with f(n) // Since $f(n) = n^{\log_b a} * \log^0 n$

Step 3: Case 2 is satisfied hence complexity is given as

$$\mathbf{T(n) = \Theta(f(n)\log n) = \Theta(n^2 \log n)}$$