18MAB101T- CALCULUS AND LINEAR ALGEBRA

Unit II - Function of several variables

Dr. E. NANDAKUMAR,
Assistant Professor,
Department of Mathematics,
Kattankulathur-603 203.

In this ppt, we are going to see,

1. Variables
2. Function of several variables
3. Partial derivatives
4. Chain rule
5. Differentiation of Implict functions
6. Total differentiation
7. Total differential
8. Taylor's series

Function of several variables

INTRODUCTION

Definition 1: Independent variable

In a function, the values for the variable which are free to assign is called independent variable.

Definition 2: Dependent variable

In a function, the values for the variable which depends on the value of independent variable is called dependent variable.

Example

$z = x^2 + y^2$

Here x and y are independent variable and z is a dependent variable.

Note: In a function, you have only one dependent variable and the other variables are called independent variable.

Definition 3: Function of several variables

A function which has more than one independent variable is called function of several variables.

Example: $u(x, y, z) = x^2 + y^2 + 2xy - z^2 + xz$

Definition 4: Partial derivative

The derivative of function of several variable with respect to independent variable is called partial derivative and it is denoted by ∂

Example :

$Z = x^3 - y^3 + 3x^2y + 3xy^2$

$\frac{\partial z}{\partial x}$ is called partial derivative with respect to independent variable x

$$\frac{\partial z}{\partial x} = 3x^2 + 6xy + 3y^2$$

In $\frac{\partial z}{\partial x}$, differentiating z with respect to independent variable x and treating the other independent variable as constants.

Example:

Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ for $U = e^x sinycosz$.

Solution:

$$\frac{\partial u}{\partial x} = e^x sinycosz$$

$$\frac{\partial u}{\partial y} = e^x cosycosz$$

$$\frac{\partial u}{\partial z} = -e^x sinysinz$$

Definition: Chain rule

If $z = f(x, y)$ and x and y are function on t then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Example: Find $\frac{dz}{dt}$ where $z = xy^2 + x^2y, x = at^2$ and $y = 2at$

Solution:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = y^2 + 2xy \quad \frac{\partial z}{\partial x} = 2xy + x^2$$

$$\frac{dx}{dt} = 2at \quad \frac{dy}{dt} = 2a$$

$$\frac{dz}{dt} = (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

Substituting $x = at^2$ and $y = 2at$ we get

$$\frac{dz}{dt} = 16a^3t^3 + 10a^3t^4$$

If $u = sin\left(\frac{x}{y}\right),\ \ x = e^t, y = t^2$ Find $\frac{du}{dt}$

Solution:

$$\frac{du}{dt} = \frac{e^t}{t^2} cos\left(\frac{e^t}{t^2}\right)\left(1 - \frac{2}{t}\right)$$

Differentiation of Implict Function

Consider the implict function $f(x, y) = 0$ then $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$.

Example: Find $\dfrac{dy}{dx}$ if $xe^{-y} - 2ye^{x} = 1$

Solution:
Given $f(x,y) = xe^{-y} - 2ye^{x} - 1 = 0$

$$\frac{\partial f}{\partial x} = e^{-y} - 2ye^{x} \quad \frac{\partial f}{\partial y} = e^{-y} - 2ye^{x}$$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{e^{-y} - 2ye^{x}}{-xe^{-y} - 2e^{x}}$$

$$= \frac{e^{-y} - 2ye^{x}}{xe^{-y} + 2e^{x}}$$

Find $\frac{dy}{dx}$ if $(cosx)^y = (siny)^x$

Solution:
$\frac{dy}{dx} = \frac{ytanx + log(siny)}{log(cosx) - xcoty}$.

Total differentiation: If $z = f(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are all functions on $'t'$ then,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dt}$$

Example:For $z = f(x_1, x_2, x_3) = {x_1}^2 + {x_2}^2 + {x_3}^2$ where $x_1(t) = t^2$, $x_2(t) = 2t$ and $x_3(t) = 3t^3$ then find $\dfrac{dz}{dt}$

Solution:

$$\frac{\partial f}{\partial x_1} = 2x_1 \quad \frac{\partial f}{\partial x_2} = 2x_2 \quad \frac{\partial f}{\partial x_3} = 2x_3$$
$$\frac{dx_1}{dt} = 2t \quad \frac{dx_2}{dt} = 2 \quad \frac{dx_3}{dt} = 9t^2$$
$$\begin{aligned}\frac{dz}{dt} &= 2(t^2)(2t) + 2(2t)(2) + 2(3t^2)(qt^2)\\ &= 4\mathrm{t}^3 + 8t + 54t^5\\ &= 54\mathrm{t}^5 + 4t^3 + 8t.\end{aligned}$$

Total differential:
If $u = f(x_1, x_2, \cdots, x_n)$ then the total differential of u is given by
$$du = \frac{\partial f}{\partial x_1} \cdot dx_1 + \frac{\partial f}{\partial x_2} \cdot dx_2 + \cdots + \frac{\partial f}{\partial x_n} \cdot dx_n.$$

Example: A metal box without a top has inside dimensions 6ft, 4ft and 2ft. If the metal is 0.1ft thick. Find the approximate volume by using the differential.

Solution: Let x, y, z be the dimensions of a metal box. Then its volume is $V = xyz$ From total differential we have
$$\begin{aligned} dV &= \frac{\partial v}{\partial x} \cdot dx + \frac{\partial V}{\partial y} \cdot dy + \frac{\partial v}{\partial z} \cdot dz \\ &= yzdx + xzdy + xydz \\ &= 8(0.2)+12(0.2)+24(0.1) \\ &= 6.4 \text{ cu.ft} \end{aligned}$$

TAYLOR SERIES

The Taylor series expansions of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b)\right.$$
$$\left.+3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b) + \ldots\right.$$

Where $f_x = \dfrac{\partial f}{\partial x} \quad f_y = \dfrac{\partial f}{\partial y}$

$f_{xx} = \dfrac{\partial^2 f}{\partial x^2} \quad f_{xy} = \dfrac{\partial^2 f}{\partial x \partial y} \quad f_{yy} = \dfrac{\partial^2 f}{\partial y^2}$

$f_{xxx} = \dfrac{\partial^3 f}{\partial x^3} \quad f_{xxy} = \dfrac{\partial^3 f}{\partial x^2 \partial y} \quad f_{xyy} = \dfrac{\partial^3 f}{\partial x \partial y^2}$ and $f_{yyy} = \dfrac{\partial^3 f}{\partial y^3}$ and so on.

Note: If $a = 0$ and $b = 0$ then the Taylor's series is reduce to Macularian's series in two variables

$$f(x,y) = f(0,0) + \left[xf_x(0,0) + yf_y(0,0)\right] +$$
$$\frac{1}{2!}\left(x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right] +$$
$$\frac{1}{3!}\left(x^3 f_{xxx}(0,0) + 3x^2 yf_{xxy}(0,0) + +3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \ldots$$

Problems on Taylor's series

Expand $x^2y + 3y - 2$ in power of $(x - 1)$ and $(y + 2)$ using Taylor series upto terms of third degree.

Solution: The Taylor series expansion of $f(x,y)$ in power of $(x - a)$ and $(y - b)$ is given by

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \ldots$$

Here $a = 1$ and $b = -2$

$f(x,y) = x^2y + 3y - 2 \quad f(1,-2) = -10$

$f_x = 2xy \quad f_x(1,-2) = -4$

$f_y = x^2 + 3 \quad f_y(1,-2) = 4$

$f_{xx} = 2y \quad f_{xx}(1,-2) = -4$

$f_{xy} = 2x \quad f_{xy}(1,-2) = 2$

$f_{yy} = 0 \quad f_{yy}(1,-2) = 0$

$f_{xxx} = 0 \quad f_{xxx}(1,-2) = 0$

$f_{xxy} = 2 \quad f_{xxy}(1,-2) = 2$

$f_{xyy} = 0 \quad f_{xyy}(1,-2) = 0$

$f_{yyy} = 0 \quad f_{yyy}(1,-2) = 0$

Substituting the values we get

$$f(x,y) = -10 + \frac{1}{1!}\left((x-1)(-4) + (y+2)(4)\right)$$

$$+\frac{1}{2!}\left((x-1)^2(-4) + 2(x-1)(y+2)(2)\right] + \frac{1}{3!}\left(3(x-1)^2(y+2)(2)\right] + \ldots$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2) + \ldots$$

Expand $e^x cosy$ in power of x and y as for as the term of the third degree

Solution: $f(x,y) = e^x cosy \quad a = 0$ and $b = 0$

$f(x,y) = e^x cosy \quad f(0,0) = 1$

$f_x = e^x cosy \quad f_x(0,0) = 1$

$f_y = -e^x siny \quad f_y(0,0) = 0$

$f_{xx} = e^x cosy \quad f_{xx}(0,0) = 1$

$f_{xy} = -e^x siny \quad f_{xy}(0,0) = 0$

$f_{yy} = -e^x cosy \quad f_{yy}(0,0) = -1$

$f_{xxx} = e^x cosy \quad f_{xxx}(0,0) = 1$

$f_{xxy} = -e^x siny \quad f_{xxy}(0,0) = 0$

$f_{xyy} = -e^x cosy \quad f_{xyy}(0,0) = -1$

$f_{yyy} = e^x siny \quad f_{yyy}(0,0) = 0$

Substituting these values in the Taylor series we get,

$$f(x,y) = 1 + \frac{x}{1!} + \frac{x^2 - y^2}{2!} + \frac{x^3 - 3xy^2}{3!} + \dots$$

Using Taylor series verify that $cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \dots$

$f(x,y) = cos(x+y) \quad f(0,0) = 1$
$f_x = f_y = -sin(x+y) \Longrightarrow f_x(0,0) = f_y(0,0) = 0$
$f_{xx} = f_{xy} = f_{yy} = -cos(x+y) \Longrightarrow f_{xx}(0,0) = f_{xy}(0,0) = f_{yy}(0,0) = -1$
$f_{xxx} = f_{xxy} = f_{xyy} = f_{yyy} = sin(x+y) \Longrightarrow f_{xxx}(0,0) = f_{xxy}(0,0) =$
$f_{xyy}(0,0) = f_{yyy}(0,0) = 0$
$f_{xxxx} = f_{xxxy} = f_{xxyy} = f_{xyyy} = f_{yyyy} = cos(x+y) \Longrightarrow f_{xxxx}(0,0) =$
$f_{xxxy}(0,0) = f_{xxyy}(0,0) = f_{xyyy}(0,0) = f_{yyyy}(0,0) = 1$
Substituting these values we get

$$cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \dots$$

Problems for practice

1. Using Taylor's series expand $e^x log(1+y)$ upto term of the third degree about (0,0)
2. Find the Taylor series expansion of e^{xy} at (1,1) upto third degree terms.
3. Find the expansions for $cosxsiny$ on powers of x and y upto terms of third degree.

18MAB101T- Calculus And Linear Algebra

Unit II-Functions of Several Variables

Dr. S. SABARINATHAN,
Assistant Professor,
Department of Mathematics,
Kattankulathur-603 203.

Maxima and Minima of Functions two variables

Maximum Value: A function $f(x,y)$ is said to have a maximum value at $x = a, y = b$ if $f(a,b) > f(a+h, b+k)$, for small and independent values of h and k, positive or negative.

Minimum Value: A function $f(x,y)$ is said to have a maximum value at $x = a, y = b$ if $f(a,b) < f(a+h, b+k)$, for small and independent values of h and k, positive or negative.

Extreme Value: $f(a,b)$ is said to be an extremum value of $f(x,y)$ if it is either maximum or minimum.

Working rule to find extreme values (Necessary Conditions)

Step 1: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Step 2: Solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously.

Let the solutions be $(a, b), (c, d), \ldots$

Stationary Points: The point (a, b) at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ are called stationary points of the function $f(x, y)$.

Stationary values: The values of $f(x, y)$ at the stationary points are called stationary values of the function $f(x, y)$.

Note: Every extremum value is a stationary value but a stationary value need not be an extremum.

Notations:

$$p = \frac{\partial f}{\partial x}, q = \frac{\partial f}{\partial y}, r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y} \text{ and } t = \frac{\partial^2 f}{\partial y^2}$$

Sufficient Condition for Maxima and Minima

Let (a,b) be a stationary point. Then if

- $rt - s^2 > 0$ at (a,b) and $r < 0$ $(t < 0)$ then $f(a,b)$ is maximum value.
- $rt - s^2 > 0$ at (a,b) and $r > 0$ $(t > 0)$ then $f(a,b)$ is minimum value.
- $rt - s^2 < 0$ at (a,b) then $f(a,b)$ has neither a maximum nor a minimum value. In this case, the point (a,b) is called a saddle point of the function $f(x,y)$.
- if $rt - s^2 = 0$, then the case is doubtful and hence further investigations are required.

Example 1:

Discuss the maximum and minimum of $x^2 + y^2 + 6x + 12$.

Solution: Let $f(x,y) = x^2 + y^2 + 6x + 12$
Now $p = 2x + 6, q = 2y$, $r = 2$, $s = 0$ and $t = 2$
The stationary points are given by $p = 0, q = 0$

$$\Rightarrow 2x + 6 = 0 \text{ and } 2y = 0$$

$$\Rightarrow x = -3 \text{ and } y = 0$$

$\therefore$ (-3, 0) is the stationary point.

	(-3,0)
r	$2\ (> 0)$
s	0
t	$2\ (> 0)$
$rt - s^2$	$4\ (> 0)$

Hence $f(x,y)$ is minimum when $x = -3$ and $y = 0$.

Example 2:

Examine $f(x,y) = x^3 + y^3 - 3xy$ for maximum and minimum values.

Solution: Let $f(x,y) = x^3 + y^3 - 3xy$
Now $p = 3x^2 - 3y, q = 3y^2 - 3x, r = 6x, s = -3$ and $t = 6y$
The stationary points are given by $p = 0, q = 0$

$$\Rightarrow 3x^2 - 3y = 0 \text{ and } 3y^2 - 3x = 0$$

$$x^2 = y \tag{1}$$

$$\text{and} \quad y^2 = x \tag{2}$$

Substituting (2) in (1), we get $x^2 = \sqrt{x}$

$$\Rightarrow x^4 = x \Rightarrow x(x^3 - 1) = 0$$

$$\Rightarrow x = 0, 1$$

$$\therefore \; y = 0, 1$$

Example 2: (Contd.)

Therefore (0, 0) and (1, 1) are the stationary points.

	(0,0)	(1,1)
$r = 6x$	0	$6\ (> 0)$
$s = -3$	-3	-3
$t = 6y$	0	$6\ (> 0)$
$rt - s^2$	-9 (<0)	27 (>0)

At (0,0) is a saddle point and at (1,1) is a point of minimum value.

$\therefore$ the minimum value of $f(1,1) = 1 + 1 - 3 = -1$.

Example 3:

Find the maximum or minimum value of $\sin x + \sin y + \sin(x+y)$.

Solution: Given $f(x,y) = \sin x + \sin y + \sin(x+y)$
Now $p = \cos x + \cos(x+y), q = \cos y + \cos(x+y)$
$r = -\sin x - \sin(x+y), t = -\sin y - \sin(x+y)$ and $s = -\sin(x+y)$
The stationary points are obtained by equating $p = 0$ and $q = 0$
$\Rightarrow \cos x + \cos(x+y) = 0$ and $\cos y + \cos(x+y) = 0$
$\therefore \cos x = -\cos(x+y)$
$\Rightarrow \cos x = \cos(\pi - (x+y)) \Rightarrow x = \pi - (x+y)$

$$2x + y = \pi \tag{3}$$

Similarly $q = 0$, we get

$$x + 2y = \pi \tag{4}$$

Solving (3) and (4), we get $x = \dfrac{\pi}{3}$ and $y = \dfrac{\pi}{3}$
$\therefore$ the stationary points is $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$.

Example 3: (contd.)

	$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$
$r = -\sin x - \sin(x+y)$	$-\sqrt{3}\ (<0)$
$s = -\sin(x+y)$	$-\frac{\sqrt{3}}{2}$
$t = -\sin y - \sin(x+y)$	$-\sqrt{3}\ (<0)$
$rt - s^2$	$\frac{9}{4}\ (>0)$

$\therefore$ the point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a maximum point.
Hence the maximum value of

$$f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{2}.$$

Example 4:

Find the extreme values of the function
$f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Solution: Given $f(x,y) = x^3 + y^3 - 3x - 12y + 20$
Now $p = 3x^2 - 3, q = 3y^2 - 12,\ r = 6x, s = 0$ and $t = 6y$
The stationary points are obtained by equating $p = 0$ and $q = 0$

$p = 0$	$q = 0$
$\Rightarrow 3x^2 - 3 = 0$ $\Rightarrow x^2 - 1 = 0$ $\Rightarrow x = \pm 1$	$\Rightarrow 3y^2 - 12$ $\Rightarrow y^2 - 4 = 0$ $\Rightarrow y = \pm 2$

$\therefore$ the stationary points are (1,2), (1,-2), (-1,2), (-1,-2).

Example 4: (Contd.)

	(1,2)	(1,-2)	(-1,2)	(-1,-2)
$r = 6x$	6 (> 0)	6 (> 0)	-6 (< 0)	-6 (> 0)
$s = 0$	0	0	0	0
$t = 6y$	12 (> 0)	-12 (< 0)	12 (> 0)	-12 (< 0)
$rt - s^2$	72 (<0)	-72 (<0)	-72 (< 0)	72 (> 0)
	min.	saddle	saddle	max.

Hence the maximum value of $f(-1,-2)$ is 38 and the minimum value of $f(1,2)$ is 2.

Example 4:

Examine for extreme values of $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$.

Solution: Given $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$

Now $p = y - \frac{a^3}{x^2}, q = x - \frac{a^3}{y^2}, r = \frac{2a^3}{x^3}, s = 1$ and $t = \frac{2a^3}{y^3}$

The stationary points are obtained by equating $p = 0$ and $q = 0$

$$\Rightarrow y - \frac{a^3}{x^2} = 0 \qquad \text{and} \tag{5}$$

$$x - \frac{a^3}{y^2} = 0 \tag{6}$$

From (5) $\Rightarrow y = \frac{a^3}{x^2}$

Example 4: (Contd.)

Substituting this value in (6), we get

$$x - \frac{x^4}{a^3} = 0$$

$$\Rightarrow x\left(1 - \frac{x^3}{a^3}\right) = 0$$

$$\Rightarrow x = 0, a.$$

When $x = 0 \Rightarrow y = \infty$ and When $x = a \Rightarrow y = a$
Omit $(0, \infty)$, the stationary point is (a, a).

Example 4: (Contd.)

	(a,a)
$r = \dfrac{2a^3}{x^3}$	$2\ (>0)$
$s = 1$	1
$t = \dfrac{2a^3}{y^3}$	$2\ (>0)$
$rt - s^2$	$3\ (>0)$

$\therefore$ the point (a,a) is a minimum point.
Hence the minimum value of $f(a,a) = 3a^2$.

Example 5:

Examine $f(x,y) = x^3 + y^3 - 3axy$ for maxima and minima.

Solution: Given $f(x,y) = x^3 + y^3 - 3axy$
Now $p = 3x^2 - 3ay, q = 3y^2 - 3ax$, $r = 6x, s = -3a$ and $t = 6y$
The stationary points are obtained by equating $p = 0$ and $q = 0$

$$\Rightarrow 3x^2 - 3ay = 0 \text{ and } 3y^2 - 3ax = 0$$

$$\Rightarrow x^2 = ay \text{ and } y^2 = ax$$

Solving these two equations, we get (0,0) and (a,a).
Therefore the stationary points are (0,0) and (a,a).

Example 5: (Contd.)

	(0,0)	(a, a)
$r = 6x$	0	$6a$
$s = -3a$	-3a	-3a
$t = 6y$	0	$6a$
$rt - s^2$	-9 (<0)	$27a^2$ (>0)

Hence the point (a, a) is a minimum if $a > 0$ and (a, a) is a maximum if $a < 0$.

Example 6:

Find the extreme values of $f(x,y) = x^3y^2(1-x-y)$.

Solution: Given $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

Now $p = 3x^2y^2 - 4x^3y^2 - 3x^2y^2 = x^2y^2(3-4x-3y)$,

$q = 2x^3y - 2x^4y - 3x^3y^2 = x^3y(2-2x-3y)$,

$r = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1-2x-y)$,

$s = 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6-8x-9y)$ and
$t = 2x^3 - 2x^4 - 6x^3y = x^3(2-2x-6y)$

The stationary points are obtained by equating $p = 0$ and $q = 0$

$$\Rightarrow x^2y^2(3-4x-3y) = 0 \text{ and } x^3y(2-2x-3y) = 0$$

$$\Rightarrow x = 0, y = 0, 4x+3y = 3 \text{ and } x = 0, y = 0, 2x+3y = 2$$

Example 6: (Contd.)

$$\Rightarrow 4x + 3y = 3 \qquad \text{and} \tag{7}$$

$$2x + 3y = 2 \tag{8}$$

Solving these two equations, we get $\left(\frac{1}{2}, \frac{1}{3}\right)$.

Put $x = 0$ in (7), we get $y = 1$

Put $y = 0$ in (7), we get $x = \frac{3}{4}$

Put $x = 0$ in (8), we get $y = \frac{2}{3}$

Put $y = 0$ in (8), we get $x = 1$

$\therefore$ the stationary points are (0,0), $\left(\frac{1}{2}, \frac{1}{3}\right)$, (0,1), $\left(0, \frac{2}{3}\right)$, $\left(\frac{3}{4}, 0\right)$ and (1,0).

Example 6: (Contd.)

	$(0,0)$	$\left(\frac{1}{2},\frac{1}{3}\right)$	$(0,1)$	$\left(0,\frac{2}{3}\right)$	$\left(\frac{3}{4},0\right)$	$(1,0)$
$r = 6xy^2(1-2x-y)$	0	$-\frac{1}{9}\ (<0)$	0	0	0	0
$s = x^2y(6-8x-9y)$	0	$-\frac{1}{12}$	0	0	0	0
$t = x^3(2-2x-6y)$	0	$-\frac{1}{8}\ (<0)$	0	0	$\frac{27}{128}$	0
$rt-s^2$	0	$\frac{1}{144}\ (>0)$	0	0	0	0
	inco.	Max.	incon.	inco.	inco.	inco.

Therefore $\left(\frac{1}{2},\frac{1}{3}\right)$ is a maximum point.

Hence the maximum value of

$$f\left(\frac{1}{2},\frac{1}{3}\right) = \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(1-\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{432}.$$

Lagrange's Method of Undetermined Multipliers

This method is to find the maximum or minimum value of a function of three or more variables, given the constraints.
Let

$$u = f(x, y, z) \tag{9}$$

be a function of three variables which is to be tested for maximum or minimum value, subject to the condition (constraint)

$$g(x, y, z) = 0 \tag{10}$$

By total differentials, we have

$$du = \frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial y}.dy + \frac{\partial f}{\partial z}.dz \quad \text{by} \quad (9) \tag{11}$$

$$0 = \frac{\partial g}{\partial x}.dx + \frac{\partial g}{\partial y}.dy + \frac{\partial g}{\partial z}.dz \quad \text{by} \quad (10) \tag{12}$$

Lagrange's Method of Undetermined Multipliers

The conditions for $f(x,y,z)$ to have a maximum point or a minimum point is $du = 0$. Therefore (11), we get

$$\frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial y}.dy + \frac{\partial f}{\partial z}.dz = 0 \tag{13}$$

Multiply (12) by λ, we get

$$\lambda\frac{\partial g}{\partial x}.dx + \lambda\frac{\partial g}{\partial y}.dy + \lambda\frac{\partial g}{\partial z}.dz = 0 \tag{14}$$

Adding (13) and (14), we get

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z}\right)dz = 0$$

Here λ is called the Lagrange multiplier.

Lagrange's Method of Undetermined Multipliers

Now we shall choose λ such that

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0$$

Solving the above equations along with the given relation, we get the values of x, y, z and λ.
These values give finally the required maximum or minimum value of the function $f(x, y, z)$.

Working Rule

To find the maximum and minimum values of $f(x,y,z)$ where x,y,z are subject to the constraint $g(x,y,z)$
We define a Function

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

- Find $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$
- Set $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$ and then solve we get x,y,z.

Example 1:

A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions in order that the total surface area is minimum.

Solution: Given

$$g(x,y,z) = xyz - 32 = 0 \tag{15}$$

Let x, y, z be the dimension of rectangular box open at the top.
Total surface area (S): $f(x,y,z) = xy + 2xz + 2yz$
We define the function

$$F(x,y,z) = xy + 2xz + 2yz + \lambda(xyz - 32)$$

At the critical points, we have

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \Rightarrow y + 2z + \lambda yz = 0 \tag{16}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \Rightarrow x + 2z + \lambda xz = 0 \tag{17}$$

Example 1: (Contd.)

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0 \Rightarrow 2x + 2y + \lambda xy = 0 \tag{18}$$

$(16) \times x - (17) \times y \Rightarrow 2(zx - zy) = 0 \Rightarrow z \neq 0, x - y = 0$

$$\Rightarrow x = y \tag{19}$$

$$\Rightarrow y^2 - 2yz = 0 \text{ by } (19)$$

$$\Rightarrow y(y - 2z) = 0 \Rightarrow y \neq 0, y - 2z = 0$$

$$\Rightarrow z = \frac{y}{2} \tag{20}$$

Using (19) and (20) in (16) we get $x = 4$.
$\therefore \quad y = 4, z = 2.$
Hence the dimensions are 4cm, 4cm and 2cm.

Example 2:

Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: The given ellipsoid is

$$g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \tag{21}$$

Let $2x, 2y, 2z$ be the dimensions of the required parallelopiped.
The volume of the parallelopiped (V) : $f(x,y,z)=8xyz$
We define the function

$$F(x,y,z)=8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

At the critical points, we have

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=0 \Rightarrow 8yz+\lambda\left(\frac{2x}{a^2}\right)=0 \tag{22}$$

Example 2: (Contd.)

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \Rightarrow 8xz + \lambda \left(\frac{2y}{b^2} \right) = 0 \tag{23}$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0 \Rightarrow 8xy + \lambda \left(\frac{2z}{c^2} \right) = 0 \tag{24}$$

$(22) \times x + (23) \times y + (24) \times z$, we get

$$24xyz + 2\lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = 0$$

$$\Rightarrow 2\lambda = -24xyz \text{ by } (21)$$

$$\lambda = -12xyz \tag{25}$$

Using (25) in (24), we get

$$8xy + (-12xyz) \left(\frac{2z}{c^2} \right) = 0$$

Example 2: (Contd.)

$$\Rightarrow 8xy\left(1-\frac{3z^2}{c^2}\right)=0$$

$$\Rightarrow \frac{3z^2}{c^2}=1$$

$$\Rightarrow z=\frac{c}{\sqrt{3}}, \text{ since } x\neq 0, y\neq 0$$

Similarly $y=\dfrac{b}{\sqrt{3}}, c=\dfrac{a}{\sqrt{3}}$

Hence the volume of rectangular parallelopiped is $V=\dfrac{8abc}{3\sqrt{3}}$ units.

Example 3:

Find the dimensions of the rectangular box, open at the top of maximum capacity whose surface is 432 sq.cm.

Solution: Let x, y, z be the dimensions of the rectangular bx, open at the top.
Given its surface area

$$g(x,y,x) = xy + 2yz + 2zx - 432 = 0 \tag{26}$$

The volume is $(V) : f(x,y,z) = xyz$
We define the function

$$F(x,y,z) = xyz + \lambda\,(xy + 2yz + 2zx - 432)$$

At the critical points, we get

$$yz + \lambda(y + 2z) = 0 \tag{27}$$

$$xz + \lambda(x + 2z) = 0 \tag{28}$$

Example 3: (Contd.)

$$xy + \lambda(2y + 2x) = 0 \tag{29}$$

$(27) \times x - (28) \times y \Rightarrow 2\lambda z(x - y) = 0$

$$\Rightarrow x = y \text{ since } \lambda \neq 0, z \neq 0 \tag{30}$$

$(28) \times x - (29) \times z \Rightarrow \lambda y(x - 2z) = 0$

$$\Rightarrow z = \frac{x}{2} \text{ since } \lambda \neq 0, y \neq 0 \tag{31}$$

Using (30) and (31) in (26), we get $x = 12$.
$\therefore \quad y = 12, z = 6$.
Hence the dimensions of the rectangular box are 12 cm, 12 cm and 6 cm.

Example 4:

Find the maximum and minimum distance of the point (3,4,12) from the sphere $x^2 + y^2 + z^2 = 1$.

Solution: Let (x, y, z) be any point on the sphere.
Given

$$g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \tag{32}$$

Distance of the point (x, y, z) from $(3, 4, 12)$ is given by
$d = \sqrt{(x-3)^2 + (y-4)^2 + (z-1)^2}$
$f(x, y, z) =$ square of the distance from the point $(3, 4, 12)$ to the sphere

$$\text{i.e.} \quad f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-1)^2$$

We define the function

$$F(x, y, z) = (x-3)^2 + (y-4)^2 + (z-1)^2 + \lambda\left(x^2 + y^2 + z^2 - 1\right)$$

Example 4: (Contd.)

At the critical points, we have

$$2(x-3)+2\lambda x=0 \tag{33}$$

$$2(y-4)+2\lambda y=0 \tag{34}$$

$$2(z-12)+2\lambda z=0 \tag{35}$$

From (33), (34) and (35), we get

$$x=\frac{3}{1+\lambda},\ y=\frac{4}{1+\lambda}, z=\frac{12}{1+\lambda} \tag{36}$$

Using (36) in (32), we get

$$(1+\lambda)^2=169 \Rightarrow 1+\lambda=\pm 13 \tag{37}$$

Example 4: (Contd.)

Using (37) in (36), we get

$$\left(\frac{3}{13},\frac{4}{13},\frac{12}{13}\right) \text{ and } \left(\frac{-3}{13},\frac{-4}{13},\frac{-12}{13}\right)$$

Therefore the distance are

$$\sqrt{\left(3-\frac{3}{13}\right)^2+\left(4-\frac{4}{13}\right)^2+\left(12-\frac{12}{13}\right)^2}=12 \text{ and}$$

$$\sqrt{\left(3+\frac{3}{13}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2}=14$$

Hence the maximum distance is 14 and the minimum distance is 12.

Example 5:

If $\frac{3}{x}+\frac{4}{y}+\frac{5}{z}=6$ find the values of x, y, z which make $x+y+z$ is minimum.

Solution:

Given

$$g(x,y,z)=\frac{3}{x}+\frac{4}{y}+\frac{5}{z}-6=0 \tag{38}$$

The required function is $f(x,y,z)=x+y+z$

We define the function

$$F(x,y,z)=x+y+z+\lambda\left(\frac{3}{x}+\frac{4}{y}+\frac{5}{z}-6\right)$$

At the critical points, we have

$$1-\frac{3\lambda}{x^2}=0 \Rightarrow \lambda=\frac{x^2}{3} \tag{39}$$

Example 5: (Contd.)

$$1 - \frac{4\lambda}{y^2} = 0 \Rightarrow \lambda = \frac{y^2}{4} \tag{40}$$

$$1 - \frac{5\lambda}{z^2} = 0 \Rightarrow \lambda = \frac{z^2}{5} \tag{41}$$

From (39), (40) and (41), we get

$$\lambda = \frac{x^2}{3} = \frac{y^2}{4} = \frac{z^2}{5}$$

$$\Rightarrow x = \sqrt{3\lambda}, y = 2\sqrt{\lambda}, z = \sqrt{5\lambda} \tag{42}$$

Using (42) in (38), we get

$$\frac{3}{\sqrt{3\lambda}} + \frac{4}{\sqrt{\lambda}} + \frac{5}{\sqrt{5\lambda}} = 6$$

$$\Rightarrow \sqrt{\lambda} = \frac{\sqrt{3} + 2 + \sqrt{5}}{6}$$

Example 5: (Contd.)

Substituting $\sqrt{\lambda}$ in (42), we get

$$x = \frac{\sqrt{3}}{6}(\sqrt{3}+2+\sqrt{5})$$

$$y = \frac{1}{6}(\sqrt{3}+2+\sqrt{5})$$

$$z = \frac{\sqrt{5}}{6}(\sqrt{3}+2+\sqrt{5})$$

Example 6:

Find the maximum value of $x^m y^n z^p$ when $x+y+z=a$.

Solution: Given

$$g(x,y,z)=x+y+z-a=0 \tag{43}$$

The required function is $f(x,y,z)=x^m y^n z^p$
We define the function

$$F(x,y,z)=x^m y^n z^p+\lambda\,(x+y+z-a)$$

At the critical points, we have

$$mx^{m-1}y^n z^p+\lambda=0 \tag{44}$$

$$nx^m y^{n-1}z^p+\lambda=0 \tag{45}$$

$$px^m y^n z^{p-1}+\lambda=0 \tag{46}$$

Example 6: (Contd.)

From equations (44), (45) and (46), we get

$$-\lambda = mx^{m-1}y^nz^p = nx^my^{n-1}z^p = px^my^nz^{p-1}$$

$$\begin{aligned}\Rightarrow \frac{m}{x} = \frac{n}{y} &= \frac{p}{z} \\ &= \frac{m+n+p}{x+y+z} \\ &= \frac{m+n+p}{a} \quad \text{by} \quad (43)\end{aligned}$$

$$\therefore \quad x = \frac{am}{m+n+p}, \; y = \frac{an}{m+n+p}, \; z = \frac{ap}{m+n+p}$$

The maximum value of $f = \dfrac{a^{m+n+p}m^mn^np^p}{(m+n+p)^{m+n+p}}$.

Jacobians

If u and v are functions of the two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u, v with respect to x, y. It is denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$.

Jacobians: Note

The Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobians

Property 1:

If J_1 is the Jacobian of u, v with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v then $J_1 J_2 = 1$.
i.e. $\dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = 1$.

Proof:

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{vmatrix} \tag{47}$$

Property 1: (Contd.)

Let $u = u(x,y)$ and $v = v(x,y)$
Differentiating partially w.r.to u and v, we get

$$\left.\begin{aligned}
\frac{\partial u}{\partial u} &= 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}\\
\frac{\partial u}{\partial v} &= 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}\\
\frac{\partial v}{\partial u} &= 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}\\
\frac{\partial v}{\partial v} &= 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}
\end{aligned}\right\} \tag{48}$$

Substituting (48) in (47), we get

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Property 2:

If u, v are functions of r, s where r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

Proof: R.H.S

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} \end{vmatrix}$$

Property 1: (Contd.)

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial(u,v)}{\partial(x,y)} = L.H.S$$

Note:

1. $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,q)}{\partial(r,s,t)} \times \frac{\partial(r,s,t)}{\partial(x,y,z)}$
2. If u, v, w are functionally dependent functions of three independent variables x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$
3. If u, v, w are said to be functionally dependent, if each can be expressed interms of the others.

Example 1:

If $x = u^2 - v^2$ and $y = 2uv$, find the Jacobian of x and y with respect to u and v.

Solution:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= 4(u^2 + v^2)$$

Example 2:

If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

Proof:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Example 2: (Contd.)

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta\left\{\cos\theta\begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix}\right.$$

$$\left.+\sin\theta\begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix}\right\} \text{ by expanding the third row}$$

$$= r^2\sin\theta\left[\cos\theta\left(\cos\theta\cos^2\phi+\cos\theta\sin^2\phi\right)\right.$$

$$\left.+\sin\theta\left(\sin\theta\cos^2\phi+\sin\theta\sin^2\phi\right)\right] = r^2\sin\theta$$

Example 3:

If $u = \dfrac{yz}{x}, v = \dfrac{zx}{y}, w = \dfrac{xy}{z}$. Show that the Jacobian of u, v, w with respect to x, y, z is 4.

Proof:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

Example 3: (Contd.)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{x^2y^2z^2}\begin{vmatrix} -yz & zx & yx \\ zy & -zx & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{x^2y^2z^2}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 4$$

Example 4:

Are the functions $u = \sin^{-1} x + \sin^{-1} y$ and $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, functionally dependent? (Given $x^2 < 1, y^2 < 1$.)

Solution:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1}{\sqrt{1-x^2}} & \dfrac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \dfrac{xy}{\sqrt{1-x^2}} & \sqrt{1-x^2} - \dfrac{xy}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= 0$$

Therefore u and v are functionally dependent.

Example 5:

Verify whether the following functions are functionally dependent, and if so, find the relation between them.

$$u = \frac{x+y}{1-xy}, v = \tan^{-1} x + \tan^{-1} y$$

Solution:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= 0$$

Example 5: (Contd.)

Hence u, v are functionally dependent.
Now

$$\begin{aligned} v &= \tan^{-1} x + \tan^{-1} y \\ &= \tan^{-1} \left\{ \frac{x+y}{1-xy} \right\} \\ &= \tan^{-1} u \\ &\Rightarrow u = \tan v. \end{aligned}$$

Example 5:

$u = xy + yz + zx, v = x^2 + y^2 + z^2$ and $w = x + y + z$, determine whether there is a functional relationship between u, v, w and if so, find it.

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

Example 5: (Contd.)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \quad R_1 \to R_1 + R_2$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Hence the functional relationship exists between u, v and w.
Now

$$\begin{aligned} w^2 &= (x+y+z)^2 \\ &= x^2+y^2+z^2+2(xy+yz+zx) \\ &= v+2u \end{aligned}$$

Example 6:

If $u = y + z, v = x + 2z^2$ and $w = x - 4yz - 2y^2$, find the Jacobian of u, v, w with respect to x, y, z. Comment on the result.

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4y-4z & -4y \end{vmatrix}$$

Example 6: (Contd.)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -1(-4y-4z)+(-4y-4z)$$
$$= 0$$

Therefore u, v and w are functionally dependent.
Now

$$v - w = 2z^2 + 4yz + 2y^2$$
$$= 2(y+z)^2$$
$$= 2u^2.$$

Example 7:

If $u = xyz, v = xy + yz + zx$ and $w = x + y + z$, find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz & xz & xy \\ y+z & x+z & y+z \\ 1 & 1 & 1 \end{vmatrix}$$

Example 7: (Contd.)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} z(x-y) & x(y-z) & xy \\ x-y & y-z & y+x \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} z & x & xy \\ 1 & 1 & y+x \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)$$

Example 8:

Find the value of the Jacobian $\frac{\partial(u,v)}{\partial(r,\theta)}$, where $u = x^2 - y^2, v = 2xy$ and $x = r\cos\theta, y = r\sin\theta$.

Solution:

$$\begin{aligned}
\frac{\partial(u,v)}{\partial(r,\theta)} &= \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} \\
&= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= 4r^2.r = 4r^3
\end{aligned}$$

Example 9:

If $u = x + y + z, uv = y + z, uvw = z$, show that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.

Solution: Given $z = uvw$

$$\Rightarrow \frac{\partial z}{\partial u} = vw,\ \frac{\partial z}{\partial v} = vw,\ \frac{\partial z}{\partial v} = uv$$

and $y + z = uv \Rightarrow y + uvw = uv \Rightarrow y = uv - uvw$

$$\Rightarrow \frac{\partial y}{\partial u} = v - vw,\ \frac{\partial y}{\partial v} = u - uw,\ \frac{\partial y}{\partial v} = -uv$$

Also $u = x + y + z \Rightarrow x = u - y - z = u - uv - uvw \Rightarrow x = u - uv$

$$\Rightarrow \frac{\partial x}{\partial u} = 1 - v,\ \frac{\partial x}{\partial v} = -u,\ \frac{\partial x}{\partial v} = 0$$

Example 9: (Contd.)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1-vw & -uw & -uv \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} \quad R_1 \to R_1 + R_2; \quad R_2 \to R_2 + R_3$$

Example 9: (Contd.)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v \begin{vmatrix} 1-vw & -w & -1 \\ v & 1 & 0 \\ vw & w & 1 \end{vmatrix}$$

$$= u^2 v \begin{vmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ vw & w & 1 \end{vmatrix} \quad R_1 \to R_1 + R_3$$

$$= u^2 v$$

Example 10:

If $x = r\cos\theta, y = r\sin\theta$ verify that $\dfrac{\partial(x,y)}{\partial(r,\theta)} \times \dfrac{\partial(r,\theta)}{\partial(x,y)} = 1.$

Proof:

Now $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}\dfrac{y}{x}$

Differentiating partially w.r.t x, we get

$$2r\frac{\partial r}{\partial x} = 2x \text{ and } \frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}.\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ and } \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}$$

Similarly $\Rightarrow \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial \theta}{\partial y} = \dfrac{x}{r^2}$

Example 10: (Contd.)

$$\begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\ -\dfrac{y}{r^2} & \dfrac{x}{r^2} \end{vmatrix} = \frac{x^2+y^2}{r^3} = \frac{1}{r}$$

and

$$\begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix} = r \times \frac{1}{r} = 1$$

Example 11:

If $x = u(1-v), y = uv$ verify that $\dfrac{\partial(x,y)}{\partial(u,v)} \times \dfrac{\partial(u,v)}{\partial(x,y)} = 1$.

Proof:

Given $x = u(1-v) \Rightarrow u = x + uv \Rightarrow u = x + y$

$$\Rightarrow \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 1$$

and $y = uv \Rightarrow v = \dfrac{y}{u} \Rightarrow v = \dfrac{y}{x+y}$

$$\Rightarrow \frac{\partial v}{\partial y} = -\frac{y}{(x+y)^2}, \frac{\partial v}{\partial y} = \frac{x}{(x+y)^2}$$

$$\Rightarrow \frac{\partial v}{\partial y} = -\frac{y}{u^2}, \frac{\partial v}{\partial y} = \frac{x}{u^2}$$

Example 11: (Contd.)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{u^2} & \frac{x}{u^2} \end{vmatrix} = \frac{x+y}{u^2} = \frac{1}{u}$$

and

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-u & -u \\ v & u \end{vmatrix} = u$$

$$\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = u \times \frac{1}{u} = 1$$