Sumedh

$z = f(x)$

$\hookrightarrow$ independ. variable

$\rightarrow$ depend. variable

If $z = f(x,y)$, a function of 2 variables x & y, if we keep y as const and x vary alone, then z is a function of x only. The derivates of z w.r.t x treating y as const. which is called the partial derivation of z w.r.t x and is denoted by $\partial z / \partial x$, similary $\partial z / \partial y$

$z = f(x, y)$

here y = const.

$$\therefore \frac{\partial z}{\partial x}$$

Total diffrential co-efficient: If z is a func. of x & y and x & y are functions of t only then

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

$\hookrightarrow$ TDC of z

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

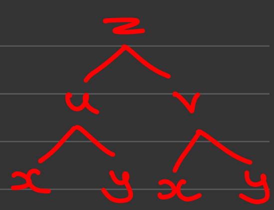
also,

Diffrentiation of implicit function : When x & y are connected by means of relation of the func. $f(x,y)=0$, x and y are said to be implicitly related or y is said to be the implicit func. of x

$$\frac{dy}{dx} = \frac{\partial f/\partial x}{\partial f/\partial y}$$

Note: if z is directly a func. of 2 var. u & v which r in turn func. of 2 other var. x & y then

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \end{aligned}}$$



it is called the partial derivative of 2 func.

eg :- $z = xy^2 + x^2y$, $x = at^2$ $y = 2at$

$$\frac{dz}{dt} = ?$$

Sol^n

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$z = xy^2 + x^2y$$

$$\frac{\partial z}{\partial x} = y^2 + 2xy \quad -(i)$$

$$\frac{\partial z}{\partial y} = 2xy + x^2 \quad -(ii)$$

$$x = at^2$$
$$\frac{dx}{dt} = 2at \quad \text{— (iii)}$$

$$y = 2at$$
$$\frac{dy}{dt} = 2a \quad \text{—— (iv)}$$

$$\therefore \quad \frac{dz}{dt} = (y^2 + 2xy) \cdot 2at + (2xy + y^2)\, 2a$$
$$= \left[(2at)^2 + 2(at^2)(2at)\right] 2at + \left[2(at^2)(2at) + (a\,t^2)\right] 2a$$
$$= \left[4a^2t^2 + 4a^2t^3\right] 2at + \left[4a^2t^3 + a^2t^4\right] 2a$$
$$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 8a^3t^4$$
$$= 16a^3t^2 + 10a^3t^4$$
$$= 2a^3t^3(8 + 5t)$$

eg $u = \sin(x/y)$

$x = e^t$, $y = t^2$ $\quad du/dt = ?$

solⁿ $\quad \frac{\partial u}{\partial x} = \frac{\cos(x/y)}{y}$

$$\frac{\partial u}{\partial y} = -\frac{\cos(x/y) \cdot x}{y^2}$$

$$\frac{dx}{dt} = e^t \qquad \frac{dy}{dt} = 2t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{du}{dt} = \frac{\cos(x/y)}{y} \cdot e^t - \frac{\cos(x/y)\, x}{y^2} \cdot 2t$$

$$= \cos(e^t/t^2) \frac{\cdot e^t}{t^2} - \frac{\cos(e^t/t^2)}{\cancel{t^4}\, t^3} \cdot 2\cancel{t} \cdot e^t$$

$$= \cos(e^t/t^2) \cdot \frac{e^t}{t^2} \left[1 - \frac{2}{t}\right]$$

eg If $u = f\left(x/y,\ y/z,\ z/x\right)$ then P.T

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = 0$$

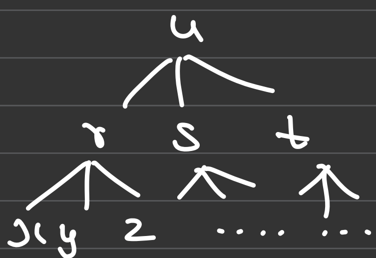
Soln if $u = f(r, s, t)$ then r, s, t r func. of x, y, z

let $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$

$$\frac{\partial r}{\partial x} = 1/y \qquad \frac{\partial s}{\partial x} = 0 \qquad \frac{\partial t}{\partial x} = -z/x^2$$

$$\frac{\partial r}{\partial y} = -x/y^2 \qquad \frac{\partial s}{\partial y} = 1/z \qquad \frac{\partial t}{\partial y} = 0$$

$$\frac{\partial r}{\partial z} = 0 \qquad \frac{\partial s}{\partial z} = -y/z^2 \qquad \frac{\partial t}{\partial z} = 1/x$$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial x} \cdot \frac{1}{y} + \frac{\partial u}{\partial s} \cdot 0 + \frac{\partial u}{\partial t} \cdot \left(\frac{-z}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{y} \frac{\partial u}{\partial \cancel{z} r} - \frac{z}{x^2} \frac{\partial u}{\partial t}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} \quad \text{— (i)}$$

Now $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + 0$$

$$y\cdot\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} \quad \text{— (ii)}$$

Now $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$\frac{\partial u}{\partial z} = 0 + \frac{\partial u}{\partial s}\cdot\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t} \quad \text{— (iii)}$$

add (i) + (ii) + (iii)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial r} \cdots$$

$$= 0$$

eg find $\frac{dy}{dx}$, if $xe^{-y} - 2ye^{x} = 1$

Given $f(x,y) = xe^{-y} - 2ye^{x} - 1 = 0$

sol^n $\frac{\partial f}{\partial x} = e^{-y} - 2ye^{x}$

$$\frac{\partial f}{\partial y} = -xe^{-y} - 2e^{x}$$

$$\boxed{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}}$$

$$= \frac{-e^{-y} + 2ye^{x}}{-xe^{-y} - 2e^{x}}$$

$$= \frac{e^{-y} - 2ye^{x}}{xe^{-y} + 2e^{x}}$$

eg $\frac{dy}{dx} = ?$ if $(\cos x)^{y} = (\sin y)^{x}$

sol^n Take log

$f(x,y) = y\log(\cos x) - x\log(\sin y) = 0$

$$\frac{\partial f}{\partial x} = -\frac{y\sin x}{\cos x} - \log(\sin y)$$

$$\frac{\partial f}{\partial y} = \log(\cos x) - \frac{x.\cos y}{\sin y}$$

$$\frac{dy}{dx} = -\left(\frac{-y\tan x - \log(\sin y)}{\log(\cos x) - x.\cot y}\right)$$

eg if $u = x.\log(xy)$ where $x^3 + y^3 + 3xy - 1 = 0$

then find du/dx

soln

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{\partial u}{\partial x} = x\left(\frac{1}{xy}\right) \cdot y + \log(xy) \quad \text{--- (i)}$$

$$= 1 + \log(xy)$$

$$\frac{\partial u}{\partial y} = x \frac{1}{\cancel{x}y} \cdot \cancel{x} = x/y \quad \text{--- (ii)}$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$\partial f/\partial x = 3x^2 + 3y$$

$$\partial f/\partial y = 3y^2 + 3x$$

$$\frac{dy}{dx} = -\frac{(x^2+y)}{(y^2+x)} \quad \text{--- (iii)}$$

Now $\frac{du}{dx} = 1 + \log(xy) - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$

eg if $x^y + y^x = c$ find dy/dx

eg if $z = f(y-z, z-x, x-y)$

P.T $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial z} = 0$

eg if $f(x,y) = x^2+y^2$ where $x = r\cos\theta$
$y = r\sin\theta$

find $\frac{df}{dr} = ?$ $\frac{\partial f}{\partial \theta} = ?$

Sol $\frac{df}{dr} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dr} + \frac{\partial f}{\partial y} \frac{dy}{dr}$

$\frac{\partial f}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = 2y$

$\frac{dx}{dr} = \cos x$ $\frac{dy}{dx} = \sin\theta$

$\frac{df}{dr} = 2x \cdot \cos\theta + 2y\sin\theta$

$= 2r\cos^2\theta + 2r\sin^2\theta$

$= 2r(1)$

$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \theta} \cdot \frac{d\theta}{\partial x} + \frac{\partial f}{\partial \theta}$

eg] $u = f(r)$ $\quad x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \left(\frac{1}{r}\right) f'(r)$$

Sol^n $\quad x = r\sin\theta$
$\quad y = r\cos\theta$

$\quad x^2 + y^2 = r$

Given $\quad u = f(r)$

$$\frac{du}{dr} = f'(r)$$

y' w.r.t x & y

$$2x = 2r\frac{\partial r}{\partial x} \qquad \frac{\partial r}{\partial x} = x/r$$

$$2y = 2r\frac{\partial r}{\partial y} \qquad \frac{\partial r}{\partial y} = y/r$$

also, $$\frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = f'(r) \,.\, x/r$$

$$= f'(r) \times x \times \frac{1}{r}$$

$$\begin{array}{l} \quad u \qquad v \qquad w \\ \frac{\partial^2 u}{\partial x^2} \Big[\; u'vw + uv'w + uvw' \end{array}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\frac{\partial r}{\partial x} \cdot x \cdot \frac{1}{r} + f'(r).(1)\frac{1}{r} + f'(r).x\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x}$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x)$$

- **Taylor's Thm for func. of 2 var.**

Taylor's Series expansion $f(x,y)$ in powers of $(x-a)$ & $(y-b)$ is given by :-

$$f(x,y) = f(a,b) + \frac{1}{\underline{1}}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{\underline{2}}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] +$$

$$\frac{1}{\underline{3}}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \ldots\ldots$$

(i)

Corollary :-

① Put $a = b = 0$ in (i), we get

$$f(x,y) = f(0,0) + \frac{1}{\underline{1}}\left[x f_x(0,0) + y f_y(0,0)\right] +$$

$$\frac{1}{\underline{2}}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] +$$

$$\frac{1}{\underline{3}}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^2 f_{yyy}(0,0)\right] + \ldots$$

Maclauren Series

It is used to expand $f(x,y)$ in the neighbourhood of a,b

It is used to expand $f(x,y)$ in powers of x & y i.e near origin $(0,0)$

eg Expand $x^2y + 3y - 2$ in powers of $(x-1)$ & $(y-2)$ using taylor's thm upto terms of 3rd degree

Sol^n Taylor series (x,y) in powers $(x-1)$ & $(y-2)$ is

$$f(x,y) = f(a,b) + \frac{1}{\angle 1}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{\angle 2}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] +$$

$$\frac{1}{\angle 3}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \ldots\ldots$$

$$f(x,y) = x^2y + 3y - 2$$

$$(x-1) = (x-a)$$
$$a = 1$$

$$(y+2) = (y-b)$$
$$b = -2$$

$$f(1,-2) = -10$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = 2xy \;, \quad f(1,-2) = -4$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = x^2+3 \;, \quad f(1,-2) = 4$$

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = 2y \;, \quad f(1,-2) = -4$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = 0 \;, \quad = 0$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial (x^2+3)}{\partial x} = 2x$$

$$f(1,-2) = 2$$

$$f_{xxx}(x,y) = \frac{\partial^3 f}{\partial x^3} = 0 \qquad f(1,-2) = 0$$

$$yyy \quad = \quad = 0 \qquad = 0$$

$$f_{xxy}(x,y) = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}(2x) = 2$$

$$f(1,-2) = 2$$

$$f_{xyy}(x,y) = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial x}(0) = 0$$

$$f(1,-2) = 0$$

Substituting

$$x^2y + 3y - 2 = -10 + 1\left[(x-1)(-4) + (y+2)(4)\right]$$

$$+ \frac{1}{2}\left[(x-1)^2(-4) + 2(x-1)(y+2)(2) + 0\right]$$

$$+ \frac{1}{\cancel{6}}\left[0 + \cancel{3}(x-1)^2(y+2)\cancel{(2)} + 0 + 0\right]$$

$$= -10 + (-4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2)$$

$$+ (x-1)^2(y+2) + \dots$$

eg Using Taylor's series $e^x \log(1+y)$ upto 3^{rd} degree in the neighbour of origin.

Sol^n $\quad e^x \log(1+y)$

$$f(x,y) = f(0,0) + \frac{1}{\angle 1}\left[x f_x(0,0) + y f_y(0,0)\right] +$$

$$\frac{1}{\angle 2}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] +$$

$$\frac{1}{\angle 3}\left[x^3 f_{xxx}(0,0) + 3x^2y\, f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \ldots$$

$f_x(x,y) = \frac{\partial f}{\partial x} = e^x \log(1+y)$	$f_x(0,0) = 0$
$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{e^x}{1+y}$	1
$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = e^x \log(1+y)$	0
$f_{xy}(x,y) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = e^x \log(1+y)$	1
$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \frac{-e^x}{(1+y)^2}$	-1
$f_{xxx}(x,y) = \frac{\partial^3 f}{\partial x^3} = e^x \log(1+y)$	0

$$f_{xxy} = \frac{\partial^3 f}{\partial x^2 \partial y}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{e^x}{1+y} \qquad 1$$

$$f_{xyy} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{-e^x}{(1+y)^2} \qquad -1$$

$$f_{yyy} = \frac{\partial^3 f}{\partial x^3} = \frac{2e^x}{(1+y)^2} \qquad 2$$

subs.

$$0 + 1[1] + \frac{1}{2}[2xy - y^2]$$

$$+\frac{1}{6}[3x^2y - 3xy^2 + 2y^3]$$

$$= y + xy - \frac{y^2}{2} + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3$$

eg :- expand $\tan^{-1}(y/x)$ using Taylor in $(1,1)$ upto quad terms

soln $f(1,1) = \pi/4$

$$f_x = \frac{y}{\cancel{x^2}} \times \frac{\cancel{x^2}}{x^2+y^2} = \frac{-y}{x^2+y^2} = \frac{-1}{2}$$

$$f_y = \frac{1}{\cancel{x}} \times \frac{\cancel{x^2}}{x^2+y^2} = \frac{x}{x^2+y^2} = \frac{1}{2}$$

$$f_{xx} = \frac{-y}{x^2+y^2}$$

$$= \frac{+y.2x.}{(x^2+y^2)^2}$$

$$\frac{2}{4} = \frac{1}{2}$$

$$f_{yy} = -\frac{x.2y}{(x^2+y^2)^2} = \frac{-1}{2}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} = 0$$

Sub

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] +$$

$$f(x,y) = \frac{\pi}{4} + \left[\frac{-(x-1)}{2} + \frac{(y-1)}{2}\right] + \frac{1}{2}\left[\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2}\right]$$

$$= \frac{\pi}{4}$$

1] $e^x \cos y$ at $(0,0)$

Sol^n at $(0,0)$ $= 1$

$f_x = e^x \cos y = 1$

$f_y = -e^x \sin y = 0$

$f_{xx} = e^x \cos y = 1$

$f_{yy} = -e^x \cos y = -1$

$f_{xy} = -e^x \sin y = 0$

$f_{xxx} = 1$

$f_{yyy} = e^x \sin y = 0$

$f_{xxy} = \quad = 1$

$f_{xyy} = \quad = -1$

$$A = 1 + 1[x] + \frac{1}{2}[x^2 - y^2] + \frac{1}{6}[x^3 + 3x^2y - 3y^2x]$$

2]
Sol^n e^{xy}

at $(1,1)$ $= e$

$f_x = e^{xy}$

$f_x = e^{xy}(y + xy')$

$e^{xy}(y + xy')$

3]

Sol^n

Maxima & Minima of functions of two variables

Procedure for finding the maximum or minimum values of $f(x,y)$

1) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ & equate to zero by solving the eqn $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$. We have the roots (x_1, y_1), (x_2, y_2), ... these points maybe maximum or minimum points. Now we need to verify this.

ep 2: Find the values of $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at these points.

ep 3: If $rt - s^2 > 0$ & $r < 0$ at that points then the function is maximum at that point.

ep 4: If $rt - s^2 > 0$ & $r > 0$ at that points then the function is minimum at that point.

5: If $rt - s^2 < 0$ for a certain points then the function is neither maximum nor minimum at that point. This point is known as Saddle point.

tep 6: If $rt - s^2 = 0$ at certain point then nothing can be said, further investigation is needed.

Note: Maximum or minimum value of a function is called its extreme value or stationary value.

eg :- Examine for extreme value $x^2 + y^2 + 6x + 12$.

Soln $f_x = 2x + 6$
$f_y = 2y$

$r = f_{xx} = 2$
$S = f_{xy} = 0$
$t = f_{yy} = 2$

Stationary pt i.e $f_x = 0$
$f_y = 0$

$2x + 6 = 0$ $\quad$ $2y = 0$
$x = -3$ $\quad$ $y = 0$

$\therefore (-3, 0)$

$r\big|_{(-3,0)} = 2$ $\quad$ $s\big|_{(-3,0)} = 0$ $\quad$ $t\big|_{(-3,0)} = 0$

$rt - s^2 = 2.2 - 0$
$= 4$
> 0

$\therefore$ minimum

$f(x,y) = x^2 + y^2 + 6x + 12$
$f(-3,0) = 3$

eg Examine for extreme values for $xy + \frac{a^3}{x} + \frac{a^3}{y}$

f_x $\quad y - \frac{a^3}{x^2}$

f_y $\quad x - \frac{a^3}{y^2}$

$r = \frac{2a^3}{x^3}$

xy = y' x ke terms me

$s = 1$

$t = \frac{2a^3}{y^3}$

Stat pts $f_x = 0$ $f_y = 0$

$y - \frac{a^3}{x^2} = 0$ $y = \frac{a^3}{x^2}$

$x - \frac{a^3}{y^2} = 0$

$x - \frac{a^3}{(a^3/x^2)^2}$

$x - \frac{x^4}{a^3}$

$x(a^3 - x^3) = 0$

$x = 0, a$

$x = 0$; & $x = a$

$y = R / \infty$, a

skip

$\therefore$ pts are $(0, \infty)$

(a, a)

$r = 2$ $t = 2$ $s = 1$

$\therefore$ $rt - s^2 = 3 > 0$

$\llcorner$ min

$f(a,a) = 3a^2$
min val

eg:- Examine $f(x,y) = x^3 + y^3 - 3axy$ for max & min

$f_x = 3x^2 - 3ay$

$f_y = 3y^2 - 3ax$

$r = 6x$

$s = -3a$

$t = 6y$

$x^2 - ay = 0$

$y^2 - ax = 0$

$y = x^2/a$

$x^4 - a^3 x = 0$

$x(x^3 - a) = 0$

$x = 0, a$

$y = 0, a$

$(x,y) = (0,0) \quad , \quad (a,a)$

$-9a^2 > 0 \qquad 27a^2 > 0$

saddle

also if r is +ve when a is +ve
r is -ve $\quad a$ is -ve

i.e the pt. (a,a) is a minima $a > 0$
& (a,a) is max^m if $a < 0$

eg identify the saddle pt & extreme pt. $x^4 - y^4 - 2x^2 + 2y^2$

$f_x = 4x^3 - 4x$

$f_y = -4y^3 + 4y$

$r = 12x^2 - 4$

$s = 0$

$t = -12y^2 + 4$

$x^3 - x = 0 \qquad x(x^2 - 1) = 0$

$x = 0, +1, -1$

$y(1 - y^2) = 0 \qquad y = 0, 1, -1$

np coz even powers

$\therefore (0,0) \quad (\pm 1, 0) \quad (0, \pm 1) \quad (\pm 1, \pm 1)$

$-16 < 0$ saddle $\quad 32 > 0$ min $\quad 32 > 0$ max $\quad -64 < 0$ saddle

coz $r < 0$

eg:- examine $x^3 + y^3 - 12x - 3y + 20$ for extreme val

eg:- find max min of $\sin x + \sin y + \sin(x+y)$

soln $\quad f_x = 3x^2 - 12 = 0 \qquad x = \pm 2$

$f_y = 3y^2 - 3 = 0 \qquad y = \pm 1$

$r = 6x$

$s = 0$

$t = 6y$

eg $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

Sol^n

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2}$$

$$= \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= 0$$

$\therefore$ u & v are functionally dependant

eg if $x = u^2 - v^2$, $y = 2uv$ find the jacobian w.r.t u & v

eg $x = r\cos\theta$, $y = r\sin\theta$ $\quad \dfrac{\partial(x,y)}{\partial(r,\theta)} \times \dfrac{\partial(r,\theta)}{\partial(x,y)}$

eg $x = r\cos\theta$, $y = r\sin\theta$ $\quad \dfrac{\partial(x, y)}{\partial(r, \theta)} \times \dfrac{\partial(r, \theta)}{\partial(x, y)}$

So:

- Lagrange's method of undetermined m

This method is to find the max^m / min^n val. of a func. of 3 or more variables given the constraints

let $f(x, y, z)$ be a func. of x, y, z which is to be tested for max^m / min^n val.

The variables x, y, z connected by some relation

let $F = f + \lambda\phi$, at the max^m & min^m point F satisfies,

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

Solve for x, y, z using above equation along with $\phi(x, y, z) = 0$ which will give us max^m min^n pt. where λ is called lagrange's multiplier

eg Rectange box open at the top is to have vol. of 32 cu. ft, find the dimensions that in order the total surface is min^m

Sol^n Give vol = 32

$\therefore$ $xyz = 32$

$\phi(x, y, z) = xyz - 32 = 0$

The req. func. is the TSA i.e

$$f(x, y, z) = 2(xy + yz + zx) - xy$$
$$= xy + 2(yz + zx)$$

At the Max^m / Min^n point $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

let $F = (xy + 2yz + 2xz) + \lambda(xyz - 32)$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = (y + 2z) + \lambda(zy)$$

$$= (x + 2z) + \lambda(xz)$$

$$= (z + 2x) + \lambda(xy)$$

0

$x = y \qquad y = 2z$

also $\quad xyz = 32$

$$\left(x \times x \times \frac{x}{2}\right) - 32 = 0$$

$x = 4 \quad y = 4 \quad z = 2$

eg find the dimension of rectangle box open at the top of maximum cap is 432 find min Vol

sol^n $\quad \phi(x, y, z) = xy + 2(yz + xz) - 432 = 0$

$f(xyz) = xyz$

$$F = f + \lambda\phi$$

At max / min pt. F

satisfies $\dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial y} = \dfrac{\partial F}{\partial z} = 0$

$$\therefore \frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\overline{y}$$

$$\overline{z}$$

$$yz + x(y + 2z) = 0$$

$$xz + \lambda(x + 2z) = 0$$

$$yx + \lambda(2y + 2x) = 0$$

$$\frac{y\cancel{z}}{y+2z} = \frac{xz}{x+2z} = \frac{yx}{2x+2y} = \lambda$$

$$\cancel{xy} + \cancel{2}yz = \cancel{xy} + \cancel{2}xz \qquad 2xz + \cancel{2y}z = xy + \cancel{2yz}$$

$$y\cancel{z} = x\cancel{z} \qquad 2\cancel{x}z = \cancel{x}y$$

$$y = 2z$$

$$x^2 + \frac{\cancel{2}x^2}{\cancel{2}} + x^2 = 432$$

$$x^2 = 144$$

$$x = 12$$

eg Find the vol. of the largest rectangular parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol^n

$$\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$f(x, y, z) = 8xyz$$

The req. func. is the vol. of the $\parallel^{\circ}$pide given by and dimension $2x, 2y, 2z$

$$F = f + \lambda\phi$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$\vdots$ $= 0$

y $= 0$

z $= 0$

$$8yz + \lambda\left(\frac{2x}{a^2}\right) = 0$$

$$8xz + \lambda\left(\frac{2y}{b^2}\right) = 0$$

$$8yx + \lambda\left(\frac{2z}{c^2}\right) = 0$$

$$\frac{8yz}{2x} \cdot a^2 = \frac{8xz}{2y} \cdot b^2 = \frac{8yx}{2z} \cdot c^2 = \lambda$$

$$ay = xb \qquad bz = cy$$

$$\frac{y}{b} = \frac{x}{a} \qquad \frac{z}{c} = \frac{y}{b}$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} \qquad \frac{z^2}{c^2} = \frac{y^2}{b^2}$$

$$3\frac{x^2}{a^2} = 1$$

$$\frac{x}{a} = 1/\sqrt{3}$$

$$x = a/\sqrt{3}$$

$$y = b/\sqrt{3} \qquad z = c/\sqrt{3}$$

$$V = xyz$$

$$= \frac{8}{3\sqrt{3}} abc$$

$\therefore$ max vol of $\|^{o}$piped is $8/\sqrt{3}^3 \, abc$

eg the temp at any pt x, y, z in space
$T = 400xyz^2 \rightarrow$ sphere $x^2 + y^2 + z^2 = 0$
$\rightarrow$ highest temp on surface

$$\phi = x^2 + y^2 + z^2 = 0$$

$$f = 400xyz^2$$

$$F = f + \lambda\phi$$

$$400yz^2 + \lambda(2x) = 0$$

$$400xz^2 + \lambda(2y) = 0$$

$$800xy + \lambda(2z) = 0$$

$$\frac{yz^2}{x} = \frac{xz^2}{y} = \frac{2xyz}{z}$$

$$x = y \qquad z^3 = 2y^2$$

$$z$$

$$x^2 + x^2 + 2x^2 - 1 = 0$$

$$x = 1/2$$

$$y = 1/2$$

eg find the min + max dist of the pt. (3, 4, 12)

$x^2 + y^2 + z^2 = 1$

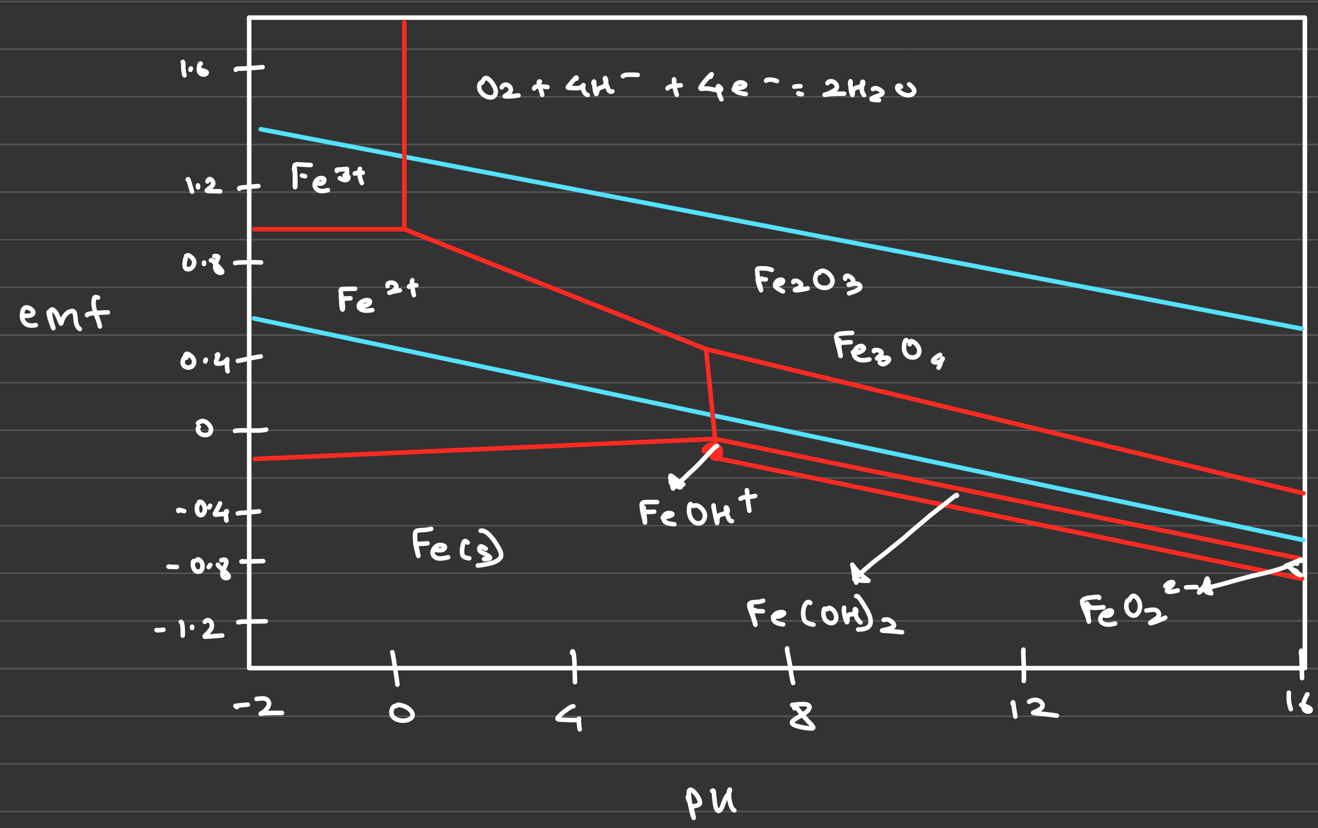
Soln $\phi = x^2 + y^2 + z^2 - 1 = 0$

Jacob

eg find the value of the Jacobian $u = x^2 - y^2$ $v = 2xy$
$x = r\cos\theta$ $y = r\sin\theta$ $x = r\cos\theta$

eg $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)$

- Pourbaix diagram



$Fe^{+2} + 2e^- \longrightarrow Fe(s)$ (redox, no pH dep.)

$Fe^{+3} + e^- \longrightarrow Fe^{2+}$ (redox, no pH dep)

$2Fe^{3+} + 3H_2O \longrightarrow Fe_2O_3 + 6H^+$ (acid-base)

$2Fe^{2+} + 3H_2O \rightarrow Fe_2O_3 + 6H^+ + 2e^-$ (both redox & ph dep)

$2Fe_3O_{4(s)} + H_2O \rightarrow 2Fe_2O_{3(s)} + 2H^+ + 2e^-$ (" ")

$y' = -59.2 \times (6/2) = -178\ mV/pH$

$y' = -59.2 \times (2/2) = -59.2\ mV/pH$

I

Zn (Anode) Au (catho)

Unit 1 → Periodic Properties
- Definition of each
- Factors
- Trends (exception)

→ HASB
- Examples (3 for each)
- Definition
- Principle
- Diff. betn
- Limitations

Unit 2 → Extensive, Intensive, Path fn, state fn, Entropy, Enthalpy, Int. Energy
- Define, Explain, Examples

→ Ist & IInd Law
- Statement & Explanation
- Numericals based on Ist Law
- Significance of Entropy

→ Derivation
- ΔG, Gibbs Helmhos, ΔA, Nernst (Titration Acid-Base)
→ Derive with application (Freq.)

→ Pourbaix Diag.
- Labelled Diag., 5 balanced eq., Features Significance, Limitation, Description of 5 curves & 5 zones

→ Numericals
- $\Delta S = \Delta S_p - \Delta S_r$
- ΔG
- $\Delta G = \Delta H - T\Delta S$
- $\Delta G = -nFE$
- ΔG

→ Dry & Wet Corrosion
- Short note, Examples, Difference (6M)

→ Construction of Electrochemical cell

- Neat - Labelled Diag. Oxdn.
- Eqn, Cell Notation
- EMF of the cell

→ Redox Potentiometric

- Cell set-up, Cell notation
- Chemical rxns
- Exp. for EMF
- 3 plots (graph)