Sumedh

Curvature of curve : The rate of bending of a curve in any interval is called the curvature of the curve in that interval

Note : The curvature of a circle at any point on it is the same and is equal to the reciprocal of it's radius

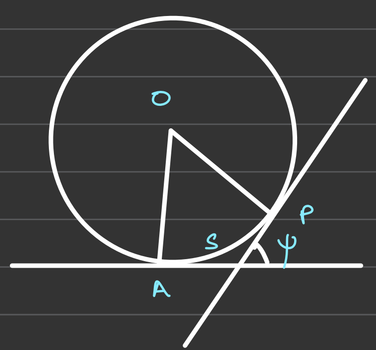
The curvature of a st. line is 0

Definition of curvature

Consider a curve $y = f(x)$, suppose that curve has a definite tangent at each point, let A be a fixed point on curve, let P be an arbitary pt. on curve, s denote the arc length AP

let ψ be the angle by the tangent with the x-axis with Tangent, then $\frac{d\psi}{ds}$ is called

curvature of curve at p, thus the curvature is the rate of turning of the tangents w/ respect to the arc length



Rad. of curvature : The reciprocal of the curvature of a curve at any pt. is called the rad. of cur & is denoted by ρ

$$\boxed{\begin{array}{l}\text{For Cartesian Curve} \\ \rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} \\ y_1 = y' \\ y_2 = y''\end{array}}$$

For parametric eq.

$x = f(t)$

$y = \phi(t)$

$$\boxed{\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}}$$

Note: To calc. ρ, when $\frac{dy}{dx} \to \infty$, we use :-

$$\boxed{\rho = \frac{(1+(y_1)^2)^{3/2}}{y''}}$$

eg find ROC of $y = e^x$ at $(0,1)$

Sol^n $\quad \rho = \left.\dfrac{(1+y_1^2)^{3/2}}{y_2}\right|_{(0,1)}$

$$= \frac{(1+e^{2x})^{3/2}}{e^x}$$

$$= \frac{(1+1)^{3/2}}{1}$$

$$= 2\sqrt{2}$$

eg Find ROC at $\sqrt{x} + \sqrt{y} = 1$ $\left(\tfrac{1}{4}, \tfrac{1}{4}\right)$

Sol^n

y' : $\quad \dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}} \cdot y' = 0$

$$y' = -\frac{\cancel{2}\sqrt{y}}{\cancel{2}\sqrt{x}}$$

$$y' = -1$$

$$y'' = \frac{vu' - uv'}{v^2}$$

$$y'' = 4$$

$$\rho = \frac{(+1)^{3/2}}{4} = \frac{(1+1)^{3/2}}{4} = \frac{2^{3/2}}{4}$$

$$= \frac{\sqrt{2}}{2}$$

eg find ROC $x^3 + y^3 = 3axy$ $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Sol^n $3x^2 + 3y^2 \cdot y' = 3ay + 3axy'$

$$x^2 - ay = y'(ax - y^2)$$

$$y' = \frac{ax - y^2}{x^2 - ay}$$

$$= \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}}$$

$$= \frac{-3a^2}{3a^2} = -1$$

$$y'' = \frac{ax - y^2}{x^2 - ay}$$

$$y'' = \frac{(a - 2y \cdot y')(x^2 - ay) - (ax - y^2)(2x - ay')}{(x^2 - ay)^2} \quad \left(\left(\frac{3a}{2}\right)^2 - a \cdot \left(\frac{3a}{2}\right)\right)$$

$$= \frac{(a + 3a)\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(3a - a)}{\left(\frac{3a^2}{2}\right)^2}$$

$$= \frac{(\cancel{4}a)\left(\frac{3a^2}{\cancel{4}}\right) + \left(\frac{3a^2}{\cancel{4}_2}\right)(\cancel{2}^1 a)}{}$$

$$= \frac{6a^3 + \frac{3a^2}{2}}{} \Big/ \frac{9a^4}{16}$$

$$= \frac{9a^3}{2\,9a^4}\cdot 16 \Rightarrow \frac{16}{2}\times\frac{1}{a}$$

$$= 9a^3/2 \,/\, 9a^4/16 \Rightarrow 8/a$$

$\vdots$

$$= \frac{-32}{3a}$$

find ROC $\left(1, \frac{\log x}{x}\right)$

Show that at any pt P on rect Hyper

$$xy = c^2$$

$$e = \frac{r^3}{2c^2}$$

then r is the dist. of P from centre of curve

eg show ROC attaining pt. of the cycloid

$x = a(\theta + \sin\theta)$; $y = a(1-\cos\theta)$ is $4a\cos\theta/2$

Soln
$$\frac{dx}{d\theta} = a(1+\cos\theta)$$

$$\frac{dy}{d\theta} = a(\sin\theta)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{a\sin\theta}{1+\cos\theta}$$

$$= \frac{\cancel{2}\sin\theta/2\ \cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$= \tan\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \left(\sec^2\theta/2\right)\cdot\left(\frac{1}{a(1+\cos\theta)}\right)$$

$$= \frac{1}{2a(2\cos^2\theta/2)}\cdot\sec^2\theta/2$$

$$= \frac{1}{4a}\cos^4\theta/2$$

$$e = \left(\frac{1+(y')^2}{y''}\right)^{3/2}$$

$$= \frac{\left(1+(\tan\theta/2)^2\right)^{3/2}}{\cos^4\theta/2}\cdot 4a$$

$$= \quad \vdots$$

$$4a\cos\theta/2$$

eg $x = a(\cos t + t\sin t)$ $y = a(\sin t - t\cos t)$

Sol^n $\frac{dx}{dt} = (-a\cancel{\sin t} + \cancel{a\sin t} + at\cos t)$

$\frac{dy}{dt} = (a\cancel{\cos t} + at\sin t - \cancel{a\cos t})$

$\frac{dy}{dx} = \frac{\cancel{a}t\sin t}{\cancel{at}\cos}$

$\frac{dy}{dx} = -\tan t$

$\frac{d^2y}{dx^2} = -\sec^2\theta \cdot \frac{1}{at\cos\theta}$

$= \frac{-\sec^3\theta \cdot \cancel{at}}{at}$

$e = \frac{(1+\tan^2 t)^{3/2}}{-\sec^3\theta} \cdot at$

$= -\frac{\sec^3\theta}{\sec^3\theta} \cdot at$

$= -at$

eg : $x^{2/3} + y^{2/3} = a^{2/3}$ $\left(a\cos^3\theta , a\sin^3\theta \right)$

Solⁿ

$$\frac{\cancel{2}}{\cancel{3}} \frac{1}{x^{1/3}} + \frac{\cancel{2}}{\cancel{3}} \frac{y'}{y^{1/3}} = 0$$

$$y' = -\left(\frac{y}{x}\right)^{1/3}$$

$$y' = -\tan\theta$$

$$y'' = \cancel{-}\sec^2\theta \cdot \frac{1}{3a\cos^2\theta \cdot \sin\theta}$$

$$y'' = \frac{\sec^4\theta}{\sin\theta} \cdot \frac{1}{3a}$$

$$e = \frac{\cancel{\sec^3\theta}}{\sec^4\theta} \cdot \sin\theta \cdot 3a$$

$$= \frac{\sin\theta}{\sec\theta} \cdot 3a$$

- Formula for e in polar coordinates

$r = f(\theta)$

$$\boxed{\therefore \quad e = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - rr_2}}$$

$$r_1 = \frac{dr}{d\theta} , \quad r_2 = \frac{d^2r}{d\theta^2}$$

eg Find ROC (r,θ) $r = a\cos\theta$

Sol^n $r_1 = -a\sin\theta$

$r_2 = -a\cos\theta$

$$e = \frac{(a^2\cos^2\theta + a^2\sin^2\alpha)^{3/2}}{a^2\cos^2\theta + 2a^2\sin^2\theta + a^2\cos^2\theta}$$

$$= \frac{a^{3/2}}{2\cancel{a^2}}$$

$$= \frac{a}{2}$$

eg Find ROC (r,θ) on equiangular spiral

$r = ae^{\theta\cot\alpha}$

So^n $\log r = \log(ae^{\theta\cot\alpha})$

$\log r = \log a + \theta\cot\alpha$

r' : $\frac{1}{r}\cdot r' = \cot\alpha$

$r' = r\cot\alpha$

$r'' = \cot\alpha \cdot r'$

$= r\cot^2\alpha$

$$e = \frac{r^2 + r^2\cot^2\alpha}{r^2 + 2(r^2\cot\alpha)^2 - r^2\cot^2\alpha}$$

$$= \frac{r^3 \cdot \operatorname{cosec}^3\alpha}{r^2 (1+\cot^2\alpha)}$$

$$= \frac{r^3 \cdot \operatorname{cosec}^3\alpha}{r^2 + r^2\cot^2\alpha}$$

$$= r \operatorname{cosec}\alpha$$

eg $r^n = a^n \cos n\theta$ is $\dfrac{a^n r^{-n+1}}{n+1}$

prove ROC $r^2 = a^2 \cos 2\theta$ is $\dfrac{a^2}{3r}$

Sol^n $r^n = a^n \cos n\theta$

$$n \log r = n \log a + \log(\cos n\theta)$$

$$\frac{n}{r} \cdot r' = 0 - n \frac{1}{\cos n\theta} \sin n\theta$$

$$r' = -\frac{r}{\cancel{n}} \tan n \times \cancel{n}$$

$$r' = -r \tan n\theta$$

$$r'' = + \tan^2 n\theta \cdot r$$

$$e = \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r^2 \tan^2\theta}$$

$$= \frac{r^3 \sec^3\theta}{r^2 \sec^2\theta}$$

$$= r \sec\theta.$$

$$= \frac{\left(r^2(1+\tan^2 n\theta)\right)^{3/2}}{r^2 + r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{\left(r^2(\sec^2 n\theta)\right)^{3/2}}{r^2(1+\tan^2 n\theta) + nr^2\sec^2 n\theta}$$

$$= \frac{\left(\left((r\sec n\theta)^2\right)^{1/2}\right)^3}{r^2(\sec^2 n\theta) + nr^2\sec^2 n\theta}$$

$$= \frac{\cancel{r^3}\sec^3 n\theta}{\cancel{r^2}\sec^2 n\theta\,(1+n)}$$

$$= \frac{r\sec n\theta}{1+n}$$

$$= \frac{r}{(1+n)\cos n\theta}$$

$$= \frac{r}{(1+n)\left(\frac{r^n}{a^n}\right)} \quad \left(r^n = a^n \cos n\theta\right)$$

$$= \frac{r^{1-n}}{\frac{1+n}{a^n}}$$

$$= \frac{a^n r^{1-n}}{1+n}$$

$$\therefore \rho = \frac{a^n r^{1-n}}{1+n}.$$

$r^n = a^n \cos n\theta$ is $\dfrac{a^n r^{1-n}}{1+n}$

$\Downarrow$ here $n=2$

$r^2 = a^2 \cos 2\theta$ is $\dfrac{a^2 r^{1-2}}{1+2} = \dfrac{a^2}{3r}$

Involutes & Evolutes :-

Let C be the centre of curvature corresponding to a point P of a given curve as P moves along the curve C will trace out a locus which is called evolute of the given curve.

If a curve β is a evolute of a curve α then α is said to be an envolute of β

eg Find the evolutes of the parabola $y^2 = 4ax$, the coords at any pt. on $y^2 = 4ax$ can be taken as $x = at^2$, $y = 2at$

Sol^n

$$x = at^2 \quad , \quad y = 2at$$

$$x' = 2at \quad , \quad y' = 2a$$

$$y_1 = \frac{dy}{dx} = \frac{2a}{2at} = 1/t$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}(1/t) \cdot \frac{1}{2at}$$

$$= -\frac{1}{2at^3}$$

$$\boxed{\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}} = at^2 - \frac{(1/t)(1+(1/t)^2)}{-1/2at^3}$$

$$= at^2 + 2at^2 + 2a$$

$$= 3at^2 + 2a$$

$$\boxed{\bar{y} = y + \frac{\left(1+(y_1)^2\right)}{y_2}}$$

$$= 2at + \left(\quad \right) / (\;)$$

$$= -2at^3$$

$$\bar{x} = 3at^2 + 2a \quad , \quad \bar{y} = -2at^3$$

$$\frac{\bar{x} - 2a}{3a} = t^2 \quad , \quad \frac{-\bar{y}}{2a} = t^3$$

$$\left(\frac{\bar{x} - 2a}{3a}\right)^3 = t^6 \quad , \quad \left(\frac{-\bar{y}}{2a}\right)^2 = t^6$$

$$\frac{(\bar{x} - 2a)^3}{27a^{\cancel{3}1}} = \frac{\bar{y}^2}{4\cancel{a^2}}$$

$$(\bar{x} - 2a)^3 = 27a\bar{y}^2$$

$$(\bar{x} - 2a)^3 - 27a\bar{y}^2 = 0$$

Now, locus of $(\bar{x}, \bar{y})$ is

$$4(x - 2a)^3 - 27a\, y^2 = 0$$

which is require evolute

eg find the eq. of the evolute of the parabola $x^2 = 4ay$

Sol^n $\quad x^2 = 4ay$

$$x = 2at$$
$$y = at^2$$

$$x' = 2a \qquad y' = 2at$$

$$y_1 = \frac{dy/dt}{dx/dt} = t$$

$$y_2 = 1 \times \frac{1}{2a} = 1/2a$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= \left[2at - t(1+t^2)\right] 2a$$

$$\bar{x} = 4a^2t - 2at - 2at^3 = -2at^3$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= at^2 + \left[(1+t^2)\right] 2a$$

$$= at^2 + 2a + 2at^2$$

$$\bar{y} = 3at^2 + 2a$$

$$\bar{x} = -2at^3$$

$$\bar{y} = 3at^2 + 2a$$

$$-\frac{\bar{x}}{2a} = t^3 \qquad \frac{\bar{y} - 2a}{3a} = t^2$$

$$\frac{\bar{x}^2}{4a^2} = t^6 \qquad \frac{(\bar{y} - 2a)^3}{27a^3} = t^6$$

$$27a^3\bar{x}^2 = 4a^2(\bar{y}-2a)^3$$

$$27a^3\bar{x}^2 - 4a^2(\bar{y}-2a)^3 = 0$$

$$27a\bar{x}^2 - 4(\bar{y}-2a)^3 = 0$$

$\therefore$ Now locus of $(\bar{x}, \bar{y})$ is

$27ax^2 - 4(y-2a)^3 = 0$ is req. evolute

eg find the eq. of the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol^n $\boxed{x = a\cos\theta \qquad y = b\sin\theta}$

$$x' = -a\sin\theta \qquad y' = b\cos\theta$$

$$y_1 = -\frac{b}{a}\cot\theta$$

$$y_2 = -\frac{b}{a}\operatorname{cosec}^2\theta \cdot \frac{1}{a\sin\theta} = \frac{-b}{a^2}\frac{1}{\sin^3\theta}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos\theta + \frac{b}{\cancel{a}}\cos\cancel{t}\theta \frac{(1 + b^2/a^2\cot^2\theta)}{-\cancel{b}} a^{\cancel{2}}\sin^{\cancel{3}}\theta$$

$$= a\cos\theta - a\cos\theta\cdot\sin^2\theta + \frac{b^2}{a^{\cancel{2}}}\cdot\cancel{a}\cot^2\theta\cdot\sin^2\theta\cdot\cos\theta$$

$$\bar{x} = a\cos\theta - a\cos\theta\cdot\sin^2\theta - \frac{b^2\cos^3\theta}{a}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= b\sin\theta - \frac{(1 + b^2/a^2 \cot^2\theta)}{b^2} \cdot a^2 \sin^3\theta$$

$$\bar{y} = b\sin\theta - \frac{a^2}{b}\sin^3\theta \;\bar{-}\; \cos^2\theta \cdot \sin\theta$$

$$\bar{x} = a\cos\theta - a\cos\theta \cdot \sin^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$\bar{x} = a\cos\theta\,(\cos^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= \left(\frac{a^2-b^2}{a}\right) \cdot \cos^3\theta$$

$$\bar{y} = b\sin\theta\,(\sin^2\theta) - \frac{a^2}{b}\sin^3\theta$$

$$\bar{y} = \left(\frac{b^2-a^2}{b^3}\right)\sin^3\theta$$

$$\frac{\bar{x} \cdot a}{(a^2-b^2)} = \cos^3\theta$$

$$\frac{\bar{y}\, b}{(b^2-a^2)} = \sin^3\theta$$

$$\left(\frac{\bar{x} \cdot a}{(a^2-b^2)}\right)^{1/3} = \cos\theta \qquad \text{||ly } \sin\theta$$

$\therefore$ ^{2} on both sides

$$\left(\frac{\bar{x}\cdot a}{a^2-b^2}\right)^{2/3} + \left(\frac{\bar{y}\cdot a}{b^2-a^2}\right)^{2/3} = 1$$

- **Envelope**

The terms corresponding to eq. $f(x,y,\alpha)=0$ for diff val. of α constitute a family of curves & alpha is called the parameter of the family

The envelope of a family of curves is the curve which touches each member of the family

If $f(x,y,\alpha)=0$ & $f(x,y,\alpha+\delta\alpha)=0$ be two consecutive members of a family of curves then the locus of their ultimate points of intersection is called the envelope of the family

Rule to find envelope of fam of curves $f(x,y,\alpha)=0$

① Eliminate α frm $f(x,y,\alpha)=0$ & $\frac{\partial f(x,y,\alpha)}{\partial\alpha}=0$

② let eq. of fam of curve be $A\alpha^2+B\alpha+C=0$ —(i) which is quad in α, then $B^2-4AC=0$ is the eq of the envelope of the fam of eq. —(i)

③ Evolute of a envelope of the normal of the curve

- **By Rule 1**

eg find the envelope of the fam of st. lines $y = mx \pm \sqrt{a^2m^2+b}$

Sol^n $y-mx = \pm\sqrt{a^2m^2+b^2}$

$(y-mx)^2 = a^2m^2+b^2$

$y^2 + m^2x^2 - 2amx - a^2m^2 - b^2$

$(x^2-a^2)m^2 - (2yx)m + (y^2-b^2)=0$

$b^2 - 4ac = 0$

$4y^2x^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$

$x^2b^2 + y^2a^2 - a^2b^2 = 0$

$-a^2 \cancel{b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) = 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

eg find envelope of st. lines rep by eq. $x\cos\alpha + y\sin\alpha = a\sec\alpha$

Sol $x\cos\alpha + y\sin\alpha = \frac{a}{\cos\alpha}$

$x + y\tan\alpha = a\sec^2\alpha$

$x + y\tan\alpha = a(1+\tan^2\alpha)$

$(a)\tan^2\alpha - (y)\tan\alpha + (a-x) = 0$

$y^2 - 4a(a-x) = 0$

$y^2 - 4a^2 + 4ax = 0$

$\frac{y^2}{a^2} - 4 + \frac{4x}{a} = 0$

$\frac{y^2}{a^2} + \frac{4x}{a} = 4$

eg find enve $\frac{x}{a} + \frac{y}{b} = 1$ wher a, b = para / $ab = c^2$

where c is known

Sol^n

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{a} + \frac{ya}{c^2} = 1$$

$$c^2x - ya^2 = ac^2$$

$$(y)a^2 + a(c^2) - c^2x$$

$$c^4 - 4(y)c^2x$$

$$c^4 - 4c^2yx = 0$$

$$c^2 = 4xy$$

eg find envelope of st. lines $x/a + y/b = 1$
a, b = par, $a+b=c$

Sol^n

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$a+b = c$$

$$b = c-a$$

$$\frac{x}{a} + \frac{y}{c-a} = 1$$

$$xc - ax + ay = ac - a^2$$

$$a^2 + (y - x - c)a + xc = 0$$

$$b^2 - 4ac = 0$$

$$(y - x - c)^2 - 4xc = 0$$

don't open

$\cancel{y^2+x^2}+c^2-\cancel{2yx}+\cancel{2xc}-2yc\cancel{-4xc}=0$

$(y-x-c)^2 = 4xc$

$y-x-c = 2\sqrt{x}\sqrt{c}$

$y = x+c+2\sqrt{x}\sqrt{c}$

$y = (\sqrt{x}\pm\sqrt{c})^2$

$\sqrt{y} = \sqrt{x}\pm\sqrt{c}$

- By rule ②

eg find envelope of st. line $x\cos\alpha + y\sin\alpha = a\sin\alpha\cos\alpha$

Sol^n $x\cos\alpha + y\sin\alpha = a\sin\alpha\cos\alpha$

Div by $\sin\alpha.\cos\alpha$

$x\operatorname{cosec}\alpha + y\sec\alpha = a$

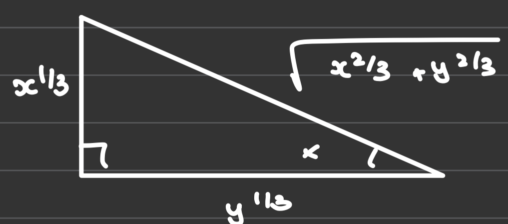
Partiall diff w.r.t α

$x[-\operatorname{cosec}\alpha.\cot\alpha] + y[\sec\alpha.\tan\alpha] = 0$

$x\left[-\frac{\cos\alpha}{\sin^2\alpha}\right] + y\left(\frac{\sin\alpha}{\cos^2\alpha}\right) = 0$

$\frac{x}{y} = \frac{\sin^3\alpha}{\cos^3\alpha}$

$\tan\alpha = (x/y)^{1/3}$



$\operatorname{cosec}\alpha = \frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}}$ $\quad\sec = \frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}}$

$$x\left(\frac{\sqrt{\quad}}{x^{1/3}}\right) + y\left(\frac{\sqrt{\quad}}{y^{1/3}}\right) = a$$

$$x^{2/3}\sqrt{\quad} + y^{2/3}\sqrt{\quad}$$

$$\sqrt{\quad}\,(x^{2/3} + y^{2/3})$$

$$(x^{2/3} + y^{2/3})^{3/2} = a$$

sq. & 1/3

$$x^{2/3} + y^{2/3} = a^{2/3}$$

eg find envelope of fam of line $x\cos^3\alpha + y\sin^3\alpha = a$

Ans : $x^2y^2 = a^2(x^2+y^2)$

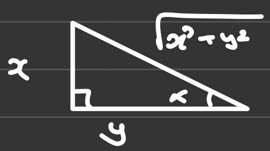
Sol^n $x\cos^3\alpha + y\sin^3\alpha = a$

diff par

$$-\cancel{3}x\cos^2\alpha.\sin\alpha + \cancel{3}y\sin^2\alpha.\cos\alpha = 0$$

$$\frac{x}{y} = \frac{\sin^{\cancel{2}}\alpha.\cancel{\cos\alpha}}{\cos^{\cancel{2}}\alpha.\cancel{\sin\alpha}}$$

$$\frac{x}{y} = \tan\alpha$$



$$\sin\alpha = \frac{x}{\sqrt{x^2+y^2}}$$

$\frac{1}{2}+3$

$$\cos\alpha = \frac{y}{\sqrt{x^2+y^2}}$$

$7/2$

$$x.\frac{x^3}{(\quad)^{7/2}} + \frac{y\;y^3}{(\quad)^{7/2}} = a$$

$$x^4 + y^4 = a(\quad)^{7/2}$$

- By rule ⑧

Envelope of two parameter fam of curves

eg Find the envelope of fam of ellipses $x^2/a^2 + y^2/b^2 = 1$ where the 2 parameter r connected by relation $a+b=c$

Sol^n

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \quad -(i)$$

$$a+b = c \quad -(ii)$$

diff (i) & (ii) w.r.t to parameter b regarding $a = f(b)$

$$-\cancel{2}\frac{x^2}{a^3} \cdot \frac{da}{db} - \cancel{2}\frac{y^2}{b^3} = 0$$

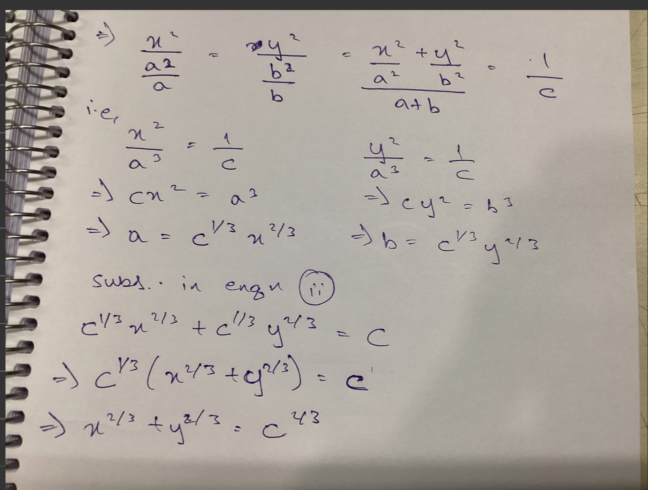
$$\frac{da}{db} + 1 = 0$$

$$\frac{da}{db} = -1$$

$$-\frac{x^2}{a^3} + \frac{y^2}{b^3} = 0$$

$$\frac{y^2}{b^3} = \frac{x^2}{a^3}$$

$$\frac{y^2}{\frac{b^2}{b}} = \frac{x^2}{\frac{a^2}{a}}$$



$$\Rightarrow \frac{x^2}{\frac{a^2}{a}} = \frac{y^2}{\frac{b^2}{b}} = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a+b} = \frac{1}{c}$$

i.e, $\frac{x^2}{a^3} = \frac{1}{c}$ $\qquad$ $\frac{y^2}{a^3} = \frac{1}{c}$

$\Rightarrow cx^2 = a^3$ $\qquad$ $\Rightarrow cy^2 = b^3$

$\Rightarrow a = c^{1/3} x^{2/3}$ $\qquad$ $\Rightarrow b = c^{1/3} y^{2/3}$

Subs. in eqn (ii)

$c^{1/3}x^{2/3} + c^{1/3}y^{2/3} = c$

$\Rightarrow c^{1/3}(x^{2/3} + y^{2/3}) = c$

$\Rightarrow x^{2/3} + y^{2/3} = c^{2/3}$

eg enve of $x/a + y/b = 1$ $ab = c^2$

Soln $\frac{x}{a} + \frac{y}{b} = 1$ — (i)

$ab = c^2$ — (ii)

diff w.r.t b as $a = f(b)$

$$-\frac{x}{a^2} \cdot \frac{da}{db} - \frac{y}{b^2} = 1$$

$$a + \frac{da}{db} \cdot b = 0$$

$$\frac{da}{db} = \frac{-a}{b}$$

$$\cancel{-}\frac{x}{a^2} \cdot \cancel{-}\frac{a}{b} - \frac{y}{b^2} = 0$$

$$\frac{x}{a} = \frac{y}{b}$$

$$\frac{x}{a}\Big/1 = \frac{y}{b}\Big/1$$

by comp-divi

$$= \frac{\left(x/a + y/b\right)^{1}}{1+1} = \frac{1}{2}$$

$$\frac{\frac{x}{a}}{1} = \frac{\frac{y}{b}}{1} = \left(\quad\right)^{1} = 1/2$$

$$\frac{x}{a} = \frac{y}{b} = \frac{1}{2}$$

$a = 2x \quad , \quad b = 2y$

a, b sub in $ab = c^2$

$(2x)(2y) = c^2$

$4xy = c^2$

- Gamma & Beta func.

The definite integral $\int_0^\infty e^{-x} x^{n-1}.dx$ exist only when $n > 0$ & when it exists its a f(n) & called gamma func & denoted by gamm of n

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}.dx}$$

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}.dx$ exists only when $m>0$ & $n>0$ & when it exists it's a f(m,n) & called beta func. & denoted by $\beta(m,n)$

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}.dx}$$

eg $\Gamma(1)$

Sol^n $\int_0^\infty e^{-x}.dx$

$$\left.\frac{e^{-x}}{-1}\right|_0^\infty$$

$$= (-e^{-x})_0^\infty = 1$$

- Recurrance formula for gamma func.

$$\boxed{\Gamma(n) = (n-1)\,\Gamma(n-1)}$$

This formula is valid only when $n > 1$ as $\Gamma(n-1)$ exists only when $n > 1$

Corollary 1 imp

$$\boxed{\Gamma(n+1) = \angle n} \quad (\text{where } n = +ve)$$

proof :- $\Gamma(n+1) = n\,\Gamma(n)$

$= n(n-1)\,\Gamma(n-1)$

$= n(n-1)(n-2)\,\Gamma(n-2)$

$= n(n-1)(n-2)\cdots 3.2.1\,\cancel{\Gamma 1}^{\,1}$

$= \angle n$

② Symmetry of Beta func

$$\beta(m,n) = \beta(n,m)$$

③ Relation b/w gama & beta func

$$\beta(m,n) = \frac{\Gamma(m)\ \Gamma(n)}{\Gamma(m+n)}$$

④ Trigo form of Beta func.

$$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \cdot d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

eg evaluate using g & B func

i) $\int_0^1 x^7 (1-x)^8 \cdot dx$

Sol^n $\int_0^1 x^{m-1} (1-x)^{n-1}$

By comparing

$m-1 = 7 \qquad n-1 = 8$

$m = 8 \qquad n = 9$

$\therefore \beta(8,9)$

$$\therefore \beta(8,9) = \frac{\Gamma 8\ \Gamma 9}{\Gamma(8+9)}$$

$$= \frac{\Gamma 8\ \Gamma 9}{\Gamma 17}$$

$$= \frac{7! \cdot 8!}{16!}$$

ii) $\int_0^{\pi/2} \sin^7\theta \cdot \cos^5\theta \cdot d\theta$

Sol^n $\frac{1}{2}\beta\left(\frac{7+1}{2}, \frac{5+1}{2}\right)$

$\frac{1}{2}\beta(4,3)$

$= \frac{1}{2}\frac{\Gamma 4\,\Gamma 3}{\Gamma 7}$

$= \frac{1}{2}\times\frac{3!\times 2!}{6!}$

Proof $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\cdot dx$

Soln Put $x = \sin^2\theta$ $\quad (1-x) = 1-s^2\theta$
$= \cos^2\theta$

$\frac{dx}{d\theta} = 2\sin\theta\cdot\cos\theta$

$dx = 2\sin\theta\cdot\cos\theta\cdot d\theta$

$\int_0^1 x^{m-1}\cdot(1-x)^{n-1}\cdot dx$

$= \int_0^{\pi/2} \sin^{2m-2}\cdot\cos^{2n-2}\cdot 2\sin\theta\cdot\cos\theta\cdot d\theta$

$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\cdot d\theta$

take $p = 2m-1$
$q = 2n-1$

$m = p+1/2$
$n = q+1/2$

$$\frac{1}{2}\beta\left(p+1/2,\ q+1/2\right) = \int_0^{\pi/2} \sin^p\theta \,.\, \cos^q\theta \,.\, d\theta$$

put $m = n = 1/2$ in $\beta(m,n) = \dfrac{\Gamma m\ \Gamma n}{\Gamma m+n}$

$$\beta\left(1/2, 1/2\right) = \frac{\Gamma 1/2\ \Gamma 1/2}{\Gamma 1} = \left(\Gamma 1/2\right)^2 \quad \text{— (i)}$$

also

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \,.\, \cos^{2n-1}\theta \, d\theta$$

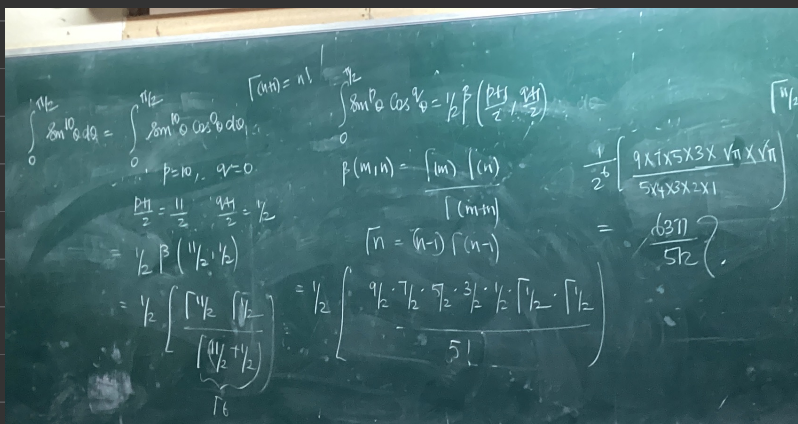
$=$

$\therefore$ from (i) + (ii)

$$\left(\Gamma 1/2\right)^2 = \pi$$

$$\Gamma 1/2 = \sqrt{\pi}$$

$$\int_0^{\pi/2} \sin^{10}\theta \,.\, d\theta =$$



eg) $\int_0^{\pi/2} \sqrt{\tan\theta}\,.d\theta$

Sol^n $\int_0^{\pi/2} \sin^{1/2}\theta . \cos^{-1/2}\theta \,.d\theta$

$p = 1/2 \quad , q = -1/2$

$\frac{p+1}{2} \quad , \quad \frac{q+1}{2}$

$3/4 \qquad\qquad 1/4$

$= \frac{1}{2}\beta\left(3/4 , 1/4\right)$

$= \frac{1}{2}\left[\frac{\Gamma 3/4 \, \Gamma 1/4}{\Gamma 1}\right]$

$= \frac{1}{2}\Gamma 3/4\,\Gamma 1/4 = \frac{1}{2}\,\Gamma 1/4\,\Gamma(1-1/4)$

$\frac{1}{2}\Gamma n\,\Gamma(1-n$

$\Gamma n\,\Gamma(1-n) = \beta(n, 1-n) = \int_0^{\infty}\frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$

eg $\int_0^{\infty} e^{-x}x^4 .dx$

So^n $n-1 = 4$

$n = 5$

$\Gamma(5) = 4!$

eg $\int_0^\infty e^{-x^2} . dx$

$x^2 = t$
$2x . dx = dt$
$dx = \frac{dt}{2x}$
$= \frac{dt}{2\sqrt{t}}$

$x = 0 \; , \; t = 0$
$x = \infty \; , \; t = \infty$

$= \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}}$

$= \frac{1}{2} \int_0^\infty e^{-t} . t^{-1/2} dt$

$n-1 = -1/2$

$n = 1/2$

$\Gamma 1/2 = \frac{1}{2}\left(\Gamma 1/2\right)$

$= \frac{1}{2} \sqrt{\pi} \quad = \sqrt{\pi}/2$

eg $\int_0^\infty e^{-3x} x^2 . dx$

Sol^n $3x = t$
$3 . dx = dt$
$dx = \frac{dt}{3}$

$\int e^{-t} . x^2 . \underline{dt}$

eg Envelope by rule 3

Find the envelope of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$a^2 + b^2 = c^2$

Sheet 1

1]
Sol:

$y = e^{x}$

$$e = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$= \left(1 + \frac{e^{2x}}{e^{x}}\right)^{3/2}$$

$$= \frac{(1+1)^{3/2}}{1} = 2^{3/2}$$

3]
Sol:

$x = at^2$
$x' = 2at$

$y' = 2a$

$$y_1 = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$y_2 = -\frac{1}{t^2} \frac{1}{2at} = -\frac{1}{2at^3}$$

$$e = -\left(1 + 1/t^2\right)^{3/2} \cdot 2at^3$$

$$= -\frac{(t^2+1)^{3/2}}{\cancel{(t^2)^{3/2}}} \cdot 2a\cancel{t^3}$$

$$= -(t^2+1)^{3/2} \cdot 2a$$

4)

Sol^n $x' = a(1 + \cos\theta)$

$y' = a - a\cos\theta$

$= a\sin\theta$

$y_1 = \dfrac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)}$

$y = \dfrac{\cancel{2}\sin\theta/2 \cdot \cancel{\cos\theta/2}}{\cancel{2}\cos^2\theta/2}$

$y_1 = \cancel{a}\tan\theta/2$

$y_2 = \dfrac{1}{2}\sec^2\theta/2 \cdot \dfrac{1}{a(1+\cos\theta)}$

$= \dfrac{1}{2}\sec^2\theta/2 \cdot \dfrac{1}{2\cos^2\theta/2 \times a}$

$y_2 = \dfrac{1}{4a} \cdot \dfrac{1}{\cos^4\theta/2}$

$e = (1 + \tan^2\theta/2)^{3/2} \cdot 4a\cos^4\theta/2$

$= (\sec^{\cancel{2}}\theta/2)^{3/\cancel{2}}$

$= \sec^3\theta/2 \cdot 4a\cos^4\theta/2$

$= 4a\cos\theta/2$

Part - B

1)

Sol^n $\sqrt{x} + \sqrt{y} = \sqrt{a}$ $\left(\dfrac{a}{4}, \dfrac{a}{4}\right)$

$-\dfrac{1}{2\sqrt{x}} \cdot \dfrac{1}{2\sqrt{y}} \cdot y' = 0$

$y' = -\dfrac{\sqrt{y}}{\sqrt{x}} = \dfrac{\sqrt{a/4}}{\sqrt{a/4}} = -\dfrac{\sqrt{a}/2}{\sqrt{a}/2} = -1$

$$y'' = \frac{+\frac{1}{2\sqrt{x}} \cdot \sqrt{y} + \frac{1}{2\sqrt{y}} \cdot \sqrt{x}}{x}$$

$$y'' = \frac{\frac{\sqrt{y}}{2\sqrt{x}} + \frac{\sqrt{x}}{2\sqrt{y}}}{x}$$

$$= \frac{\frac{1}{2} + \frac{1}{2}}{a/4} = \frac{1}{a/4} = \frac{4}{a}$$

$$e = \frac{(1+(-1)^2)^{3/2}}{} \cdot \frac{a}{4}$$

$$e = \frac{a}{4} \cdot 2^{3/2} \cdot 2^{-2} \qquad \frac{3}{2} - 2$$

$$\frac{3-4}{2} \quad -\frac{1}{2}$$

$$= \frac{a}{\sqrt{2}}$$

$$\bar{x} = x + \frac{y_1(1+(y_1)^2)}{y_2}$$

$$= \frac{a}{4} + \frac{-1(1+1)}{} \cdot \frac{a}{4}$$

$$= \frac{a}{4} - \frac{a}{2}$$

$$= -\frac{a}{4}$$

$$\bar{y} = \frac{a}{4} - \frac{(2)}{} \cdot \frac{a}{4}$$

$$= -\frac{a}{4}$$

$$e^2 = (x-\bar{x})^2 + (y-\bar{y})^2$$

$$\frac{a^2}{2} = \left(x-\frac{a}{4}\right)^2 + \left(y-\frac{a}{4}\right)^2$$

$$\frac{a^2}{2} = x^2 - \frac{xa}{2} + \frac{a^2}{16} + y^2 - \frac{ya}{2} + \frac{a^2}{16}$$

$$\frac{a^2}{2} = x^2+y^2 - \frac{a(x+y)}{2} + \frac{a^2}{8}$$

$$\frac{a^2}{2} - \frac{a^2}{8}$$

$$\frac{3a^2}{8} = x^2+y^2-\frac{a}{2}(x+y)$$

Sheet - 2

i) $y = mx + \frac{1}{m}$

$$my = m^2x + 1$$

$$xm^2 - my + 1$$

$$b^2 - 4ac = 0$$

$$y^2 - 4x = 0$$

$$y^2 = 4x$$

ii)

Sol: $x\cos\alpha + y\sin\alpha = a\sec\alpha$

$$x\cos^2\alpha + y\sin\alpha.\cos\alpha = a$$

$$-2x.\cos\alpha.\sin\alpha + y\cos^2\alpha - y\sin^2\alpha = \cancel{a}\ 0$$

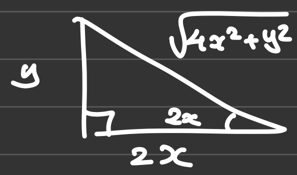
$$-2x.\cos\alpha.\sin\alpha + y(\cos^2\alpha - \sin^2\alpha) = 0$$

$$\frac{y}{x} = \frac{2\cos\alpha.\sin\alpha}{\cos^2\alpha - \sin^2\alpha}$$

$$= \frac{2\sin x \cdot \cos x}{\cos 2x}$$

$$\frac{y}{x} = \frac{\sin 2x}{\cos 2x}$$

$$\frac{y}{2x} = \tan 2x$$



$$\sin 2x = \frac{y}{\sqrt{4x^2+y^2}}$$

$$\cos 2x = \frac{2x}{\sqrt{4x^2+y^2}}$$