Sumedh

- Formation of PDE

$z = f(x, y)$ / $q = \phi(x, y, z)$

Type - 1 :-

Elimination of Arbitary Constant

Case - 1 :-

No of Independant variables = Arbitary Constants [1st Order PDE]

Case - 2 · " < " [2nd Order PDE]

We can solve sums by partially diffrentiating the equation w.r.t to x / y and eliminating constants accordingly

$$\frac{\partial z}{\partial x} = p \quad , \quad \frac{\partial z}{\partial y} = q$$ — for 1st order

$$\frac{\partial z^2}{\partial x^2} = r \quad , \quad \frac{\partial z^2}{\partial x \, \partial y} = s \quad , \quad \frac{\partial z^2}{\partial y^2} = t$$ — In case of 2nd order

Type - 2 :-

Elimination of arbitary functions

Case - 1 : $z = f(x, y, z)$

Here, partially diffrentiate w.r.t x & y then divide p/q to eliminate functions

Case - 2 :-

$f(u, v) = 0$

$u = f(x, y, z)$ & $v = f(x, y, z)$

eg $\phi(x^2 + y^2 + z^2 \, , \, x + y + z)$

here, find $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$

then
$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \quad \text{to find PDE}$$

Case - 3 :-

$z = f(x,y) + g(x,y)$

Find p, q, r, s, t & solve

- Lagrange's linear equation

1^{st} order linear PDE is of the form :-

$$Pp + Qq = R$$

P, Q, R are functions of x, y, z

eg $px^2 + qy^2 = z^2$

where $P = x^2$, $Q = y^2$, $R = z^2$

Step - 1 :-

Auxillary form : $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

Step - 2 :-

Solve the subsidary eq

$u(x, y, z) = c_1$
$v(x, y, z) = c_2$

Type 1 : Method of Grouping

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{P} = \frac{dy}{Q} \quad \& \quad \frac{dy}{Q} = \frac{dz}{R}$$

Integrate   Integrate

c_1 c_2

Type - 2 : Method of multipliers

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

(l, m, n) shld be such a ratio that $lP + mQ + nR = 0$

$\therefore$ $l\,dx + m\,dy + n\,dz = 0$

$\therefore$ do with 2 multipliers i.e 2 diff ratios
eg (0, 0, 0) (2, 2, 2)

Integrate to find c_1 & c_2

Step - 3 :-

$\therefore$ $c_1 = u$, $c_2 = v$

$f(u, v) = 0$

final ans

- Homogeneous PDE

- Complementary function

Type - 1 : $m_1 \neq m_2 \neq \dots m_n$

$$CF = f_1(y + m_1 x) + f_2(y + m_2 x) + \dots + f_n(y + m_n x)$$

Type - 2 : $m_1 = m_2 = \dots = m_n = m$

$$CF = f_1(y + mx) + x f_2(y + mx) + x^2 f_3(y + mx) + \dots x^{n-1} f_n(y + mx)$$

Type - 3 : $m_1 = m_2 \neq m_3$

$$CF = f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x)$$

- Particular Integral

$$PI = \frac{F(x, y)}{\phi(D, D')}$$

where,
coeff of $x = D$
coeff of $y = D'$

Type 1 :-

$$F(x, y) = e^{ax + by}$$

$$PI = \frac{e^{ax+by}}{\phi(D, D')}$$

$D = a$
$D' = b$

if $\phi(a, b) = 0$

then,

$$PI = \frac{x\left[e^{ax+by}\right]}{\phi'(D, D')}$$ wrt D

⋮

& so on

Type - 2

$$F(x,y) = \sin(ax+by) \;/\; \cos(ax+by)$$

$$\therefore \quad PI = \frac{\sin(ax+by) \;/\; c}{\phi(D, D')}$$

here,
$$D^2 = -a^2$$
$$D'^2 = -b^2$$
$$DD' = -ab$$

alternate method.

we know,

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$RP = \cos\theta$$
$$IP = \sin\theta$$

when $F(x,y) = \sin(ax+by)$

$$PI = IP \left\{ \frac{e^{i(ax+by)}}{\phi(D, D')} \right\}$$

Type - 1 $\quad D = ai$

$D' = bi$

when $PI = \cos(ax+by)$

$$PI = RP \left\{ \frac{e^{i(ax+by)}}{\phi(D, D')} \right\}$$

Type - 3

$$F(x,y) = x^m y^n$$

$$PI = \frac{x^m y^n}{\phi(D.D')}$$

$$= \frac{1}{D}\left[\frac{x^m y^n}{(1+f(DD'))}\right]$$

$$= \frac{x^m y^n}{D}\left(1+f(DD')\right)^{-1}$$

$$\frac{1}{D^2}(x)$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

where

$D = \partial/\partial x \qquad D' = \partial/\partial y$

$$\boxed{\therefore \text{ Gen Sol}^n = CF + PI}$$

- **Solutions of PDE**

· Types

1. $f(p,q) = 0$
2. $z = px + qy + f(p,q)$
3. $z(x,y,p,q) = 0$
4. $f(x,p,q) = 0$
5. $f(y,p,q) = 0$
6. $f(z,p,q) = 0$

Solutions of first order partial differential equations : TYPE - I $\quad f(p,q)=0$

Example 1

Solve $\sqrt{p}+\sqrt{q}=1$.

Solution : Given p.d.e is

$$\sqrt{p}+\sqrt{q}=1 \tag{1}$$

This is a first order p.d.e of the form $f(p,q)=0$
Therefore the trial solution is

$$\boxed{z=ax+by+c} \tag{2}$$

To Find : The Complete Integral (or) Complete Solution
We have to eliminate any one of the arbitrary constants in (2).
(Since, In a complete integral , Number of arbitrary constants must be equal to Number of independent variables.)
Diff. (2) p.w.r.t. x and y , We get

$\partial z \qquad \partial z$

That is,

$$p = a \quad \& \quad q = b$$

Use $p = a, q = b$ in (1), We get

$$\sqrt{a} + \sqrt{b} = 1 \quad \Rightarrow \quad \sqrt{b} = 1 - \sqrt{a}$$

$$b = \left(1 - \sqrt{a}\right)^2$$

Hence the **complete solution** is

$$\boxed{z = ax + \left(1 - \sqrt{a}\right)^2 y + c} \tag{3}$$

To Find : The Singular Integral (or) Singular Solution: we have $\frac{\partial z}{\partial a} = 0$ $\frac{\partial z}{\partial c} = 0$

2 conditions

Diff. (3) p.w.r.t. 'c' we get $\frac{\partial z}{\partial c} = \boxed{1 \longrightarrow 0 = 1,}$ which is not true.

Hence there is **No singular solution**.

for singular solution, partially differentiate the complete solution by constants

To Find : The General Integral (or) General Solution

Put $c = \phi(a)$ in the Complete Solution (5), We get

$$\boxed{z = ax + \left(1 - \sqrt{a}\right)^2 y + \phi(a)} \tag{4}$$

Diff. (4) p.w.r.t. 'a' and eliminating 'a', We get the general solution of the given p.d.e.

TYPE - II Clairaut's Form : $z = px + qy + f(p, q)$

Example 1

Solve $z = px + qy + pq$.

Solution : Given p.d.e is

$$z = px + qy + pq \tag{1}$$

This is in the Clairaut's form

$$z = px + qy + f(p, q)$$

The Complete Integral (or) Complete Solution of (1) which is in Clairaut's form is obtained by replacing p by a and q by b

$$\boxed{z = ax + by + ab} \tag{2}$$

To Find : The Singular Integral (or) Singular Solution

Diff. (2) p.w.r.t. 'a' and 'b' , We get

$$\frac{\partial z}{\partial a} = x + b \quad \& \quad \frac{\partial z}{\partial b} = y + a$$

$$0 = x + b \quad \& \quad 0 = y + a$$

$$b = -x \quad \& \quad a = -y \tag{3}$$

Use (3) in the complete solution (2) , We get

$$z = -xy - xy + xy$$

$$\boxed{z + xy = 0}$$

This is the required singular integral.

To Find : The General Integral (or) General Solution
Put $b = \phi(a)$ in the Complete Solution (2), We get

$$z = ax + \phi(a)y + a\phi(a) \tag{4}$$

Diff. (4) p.w.r.t. 'a' and eliminating 'a', We get the general solution of the given p.d.e.

Type ~~IV~~ III (Separable Equations): Unit-I

$f(x, y, p, q) = 0$

The first order PDE is said to be separable equation it it can put in the form $f(x, p) = \phi(y, p)$.

For such PDE the solutions can be obtained as:

- **Step I:** Put $f(x, p) = \phi(y, p) = a$
- **Step II:** Write $p = f_1(x, a)$ and $q = \phi_1(y, a)$
- **Step III:** Putting in $dz = pdx + qdy$ and integrating, get the complete integral as

$$z = \int f_1(x, a)dx + \int \phi_1(y, a)dy + b.$$

Example 1 TYPE - ~~III~~ IV

Solve $p = 2qx$ (a) $f(x,p,q)=0$ (b) $f(y,p,q)=0$ (c) $f(z,p,q)=0$

Solution : Given p.d.e is

$$p = 2qx \tag{1}$$

for $f(x,p,q)=0$
take $q=a$
$f(y,p,q)=0$
take $p=a$

This is a Type - III (a) p.d.e of the form

$$f(x,p,q)=0$$

Let z be a function of x and y

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$dz = p\,dx + q\,dy \tag{2}$$

Put $q=a$ in (1) and solve for p, We get

$$p = 2ax$$

Use p and q values in (2) , We get

$$dz = 2ax\ dx + a\ dy$$

Integrating on both the sides, We get

$$\int dz = \int 2ax\ dx + \int a\ dy$$

$$\boxed{z = ax^2 + ay + c} \tag{3}$$

This is the required complete solution.

Diff. (3) p.w.r.t c , We get $0 = 1$ $[Absurd]$ Hence There is no singular Integral.

Put $c = \phi(a)$ in the Complete Solution (3), We get

$$\boxed{z = ax^2 + ay + \phi(a)} \tag{4}$$

Diff. (4) p.w.r.t. 'a' and eliminating 'a', We get the general solution.

Examples under Type - III (b) $f(y, p, q) = 0$

Solve the following partial differential equations :

(1). $pq = y$

(2). $p = 2qy$

(3). $q = py + p^2$

(4). $\sqrt{p} + \sqrt{q} = \sqrt{y}$

Example 1

Solve $pq = y$.

Solution : Given p.d.e is

$$pq = y \tag{1}$$

This is a Type - III (b) p.d.e of the form

$$f(y, p, q) = 0$$

Let z be a function of x and y

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

$$dz = p\ dx + q\ dy \tag{2}$$

Put $p = a$ in (1) and solve for q, We get

$$aq = y \quad \Rightarrow \quad q = \frac{y}{a}$$

Use p and q values in (2) , We get

$$dz = a\ dx + \frac{y}{a}\ dy$$

Integrating on both the sides, We get

$$\int dz = \int a\ dx + \int \frac{y}{a}\ dy$$

$$\boxed{z = ax + \frac{y^2}{2a} + c} \tag{3}$$

This is the required complete solution.

Diff. (4) p.w.r.t c , We get

$$0 = 1 \quad [Absurd]$$

Hence There is no singular Integral.

Put $c = \phi(a)$ in the Complete Solution (3), We get

$$\boxed{z = ax + \frac{y^2}{2a} + \phi(a)} \tag{4}$$

Examples under Type - III (c) $f(z,p,q)=0$

Solve the following partial differential equations :

(1). $p(1+q)=qz$

(2). $p\left(1+q^2\right)=q(z-a)$

(3). $z^2=1+p^2+q^2$

(4). $9\left(p^2z+q^2\right)=4$

(5). $z=p^2+q^2$

(6). $p\left(1-q^2\right)=q(1-z)$

$u = x+ay$

$p = \frac{dz}{du}$ $q = \frac{a\,dz}{du}$

Example 1

Solve $p(1+q) = qz$.

Solution : Given p.d.e is

$$p(1+q) = qz \tag{1}$$

This is a Type - III (c) p.d.e of the form $f(z,p,q) = 0$
Let z be a function of u where $u = x + ay$

$$\frac{\partial u}{\partial x} = 1 \quad \& \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} \quad \& \quad q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}$$

$$p = \frac{dz}{du} \quad \& \quad q = a\frac{dz}{du} \tag{2}$$

Use (2) in (1) , We get

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z$$

$$1 + a\frac{dz}{du} = az \rightarrow \frac{dz}{du} = \frac{az-1}{a}$$

$$du = \frac{a}{az-1}dz$$

Integrating on both the sides, We get

$$\int du = \int \frac{a}{az-1}dz$$

$$u = \log(az-1) + c$$

$x+ay$

Hence the complete solution is

$$\boxed{x + ay = \log(az-1) + c} \qquad (1)$$

This is the required complete solution. Diff. (3) p.w.r.t c , We get

$0 = 1 \quad [Absurd]$. Hence There is no singular Integral.

Put $c = \phi(a)$ in the Complete Solution (3), We get

$$\boxed{x + ay = \log(az - 1) + \phi(a)} \tag{4}$$

Diff. (4) p.w.r.t. 'a' and eliminating 'a', We get the general solution.

- Gen formulae

1. $\int \frac{1}{\sqrt{a^2-x^2}} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a)$

2. $\int \sqrt{a^2-x^2}\,.dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2}$

3. $\int \sqrt{x^2+a^2}\,.dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x^2+\sqrt{a^2+x^2}\right|$

4. $\int \sqrt{x^2-a^2}\,.dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x^2+\sqrt{x^2-a^2}\right|$

5. $\int \frac{1}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right.$

Department of Mathematics

21MAB201T-Transforms and Boundary Value Problems

Tutorial Sheet-Unit-1

Q. No.	Questions	Answers
	PART-A (5 x 8 marks)	
1.	(i) Form the partial differenial equation by eliminating the arbitrary constants a & b from $z=(x^2+a^2)(y^2+b^2)$, (ii) Find a complete integral of $x(1+y)p=y(1+x)q$.	4xyz = pq $z=a(logxy+x+y)+b$
2.	Find the partial differenial equation by eliminating the arbitrary functions f and ϕ from $z=f(y)+\phi(x+y+z)$.	$r(1+q)=s(1+p)$
3.	Solve $z=px+qy+p^2+q^2$.	$z=ax+by+a^2+b^2$ (CS) $4z+(x^2+y^2)=0$ (SS)
4.	Solve $(x^2-yz)p+(y^2-zx)q=(z^2-xy)$. wrong answer	$f(x+y+z, xyz)=0$
5.	Solve $(D+D')^2 z=e^{x-y}$. Correct . $z=f_1(y-x)+xf_2(y-x)+\frac{x^2}{2}e^{x-y}$	~~$z=f_1(y+2x)+xf_2(y-x)+\frac{x^2}{2}e^{x-y}$~~
	PART-B (3 x 15 marks)	
1.	(i) Form the partial differenial equation by eliminating the arbitrary function ϕ from $\phi(x^2+y^2+z^2, z^2-2xy)=0$, and (ii) Solve $pq=k$, where k is a constant.	$(p-q)z=y-x$ $z=ax+\frac{k}{a}y+c$
2.	Solve $(D^3-7DD'^2-6D'^3)z=sin(x+2y)+x^3$.	$z=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)+-\frac{1}{75}cox(x+2y)+\frac{x^6}{120}$
3.	(i) Solve $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$. (ii) Solve $z^2=1+p^2+q^2$	$f(x^2+y^2+z^2, xyz)=0$ $\cosh^{-1}z=\frac{1}{\sqrt{1+a^2}}(x+ay)+c$

Sumedh

- Tutor Sheet Unit - 1

1. (i)

Soln $z = (x^2+a^2)(y^2+b^2)$

$p = (y^2+b^2)(2x)$

$q = (x^2+a^2)(2y)$

$(y^2+b^2) = p/2x$

$(x^2+a^2) = q/2y$

$z = \frac{q}{2y}\frac{p}{2x}$

$pq = 4zxy$

1. (ii)

Soln $x(1+y)p = y(1+x)q$

Type - III $f(x, y, p, q) = 0$

$\frac{x}{(1+x)} \cdot p = \frac{y}{(1+y)} \cdot q = a$

$\frac{xp}{1+x} = a \qquad \frac{yq}{1+y} = a$

$p = \frac{a(1+x)}{x} \qquad q = \frac{a(1+y)}{y}$

$z = \int p\,dx + \int q\,dy$

$z = \int \frac{a(1+x)}{x} \cdot dx + \int \frac{a(1+y)}{y} \cdot dy$

$z = a\int \frac{1}{x} + 1 \; .dx + a\int \frac{1}{y} + 1 \; .dy$

$z = a[\log(x) + x + \log(y) + y + c]$

$z = a[\log(xy) + x + y + b]$

2.

Sol^n $z = f(y) + \phi(x+y+z)$ 1

$p = 0 + \phi'(x+y+z)(1+p)$ 2

$q = f'(y) + \phi'(x+y+z)(1+q)$ 3

$r = \phi''(x+y+z)(1+p)^2 + \phi'(x+y+z)(r)$ 4

$s = \phi''(x+y+z)(1+p)(1+q) + \phi'(x+y+z)(s)$ 5

$t = f''(y) + \phi''(x+y+z)(1+q)^2 + \phi'(x+y+z)(t)$ 6

$p = \phi'(x+y+z)(1+p)$

$\phi'(x+y+z) = \dfrac{p}{1+p}$

$r = \phi''(x+y+z)(1+p)^2 + \phi'(\ )(r)$

$r(1-\phi'(\ )) = \phi''(\ )(1+p)^2$ $\div$

$s(1-\phi'(\ )) = \phi''(\ )(1+p)(1+q)$

$\dfrac{r}{s} = \dfrac{1+p}{1+q}$

$e^{(\ )} f(x,y)$

$D = D+a$

$D' = D+$

3.

Solⁿ $z = px + qy + p^2 + q^2$

Type -2 : $z = px + qy + f(p, q)$

For Complete Solution ; $p = a$ & $q = b$

$$z = ax + by + a^2 + b^2$$

For Singular Solution

$$\frac{\partial z}{\partial a} = \frac{\partial z}{\partial b} = 0$$

$$\frac{\partial z}{\partial a} = x + 2a = 0$$
$$a = -x/2$$

$$\frac{\partial z}{\partial b} = y + 2b = 0$$
$$b = -y/2$$

$$z = ax + by + a^2 + b^2$$
$$z = \left(\frac{-x}{2}\right)x + \left(\frac{-y}{2}\right)y + \left(\frac{-x}{2}\right)^2 + \left(\frac{-y}{2}\right)^2$$

$$z = \frac{-x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

$$z = \frac{x^2 - 2x^2}{4} + \frac{y^2 - 2y^2}{4}$$

$$4z = -(x^2 + y^2)$$

$$4z + x^2 + y^2 = 0$$

For General Solution

$$b = \phi(a)$$

$$z = ax + by + a^2 + b^2$$
$$z = ax + \phi(a)y + a^2 + \phi^2(a)$$

$$\frac{\partial z}{\partial a} = a + \phi'(a)y + 2a + \phi'(a)2\phi(a)$$

4]

Soln $(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy)$

Comparing with lagrange eq

$$Pp + Qq = R$$

$P = (x^2 - yz)$
$Q = (y^2 - zx)$
$R = (z^2 - xy)$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{(x^2 - yz)} = \frac{dy}{(y^2 - zx)} = \frac{dz}{(z^2 - xy)}$$

By Multipliers, $\dfrac{l dx + m dy + n dz}{lP + mQ + nR}$

i) (y, z, x)

$$\frac{y dx + z dy + x dz}{y(x^2 - yz) + z(y^2 - zx) + x(z^2 - xy)}$$

$$\cancel{yx^2} - \cancel{y^2 z} + \cancel{zy^2} - \cancel{z^2 x} + \cancel{xz^2} - \cancel{x^2 y}$$
$$= 0$$

$$\therefore \quad y dx + z dy + x dz = 0$$

$$\int y dx + \int z dy + \int x dz = 0$$

$$xy + zy + xz = C_1$$

ii) (z, x, y)

$$\frac{z dx + x dy + y dz}{z(x^2 - yz) + x(y^2 - zx) + y(z^2 - xy)}$$

$$\cancel{zx^2} - \cancel{yz^2} + \cancel{xy^2} - \cancel{zx^2} + \cancel{yz^2} - \cancel{xy^2} = 0$$

$$\therefore\ z\,dx + x\,dy + y\,dz = 0$$

$$\int z\,dx + \int x\,dy + \int y\,dz = 0$$

$$zx + xy + yz = C_2$$

$$\therefore\ f(xy + zy + xz,\ zx + xy + yz) = 0$$

5]

Sol^n $(D + D')^2 = e^{x-y}$

Aux eq

$$(m+1)^2 = 0$$
$$(m+1)(m+1) = 0$$

$$m = -1, -1$$

$$CF = f_1(y - x) + x f_2(y - x)$$

$$PI = \frac{e^{x-y}}{(D + D')^2}$$

e^{ax+by}

e^{x-b}

$a = 1$

$b = -1$

$\therefore\ D = 1$

$D' = -1$

$$PI = \frac{e^{x-y}}{(1-1)^2}$$

$$= \frac{e^{x-y}}{0}$$

$$= \frac{x e^{x-y}}{2(D + D')}$$

$D = 1$

$D' = -1$

$$= \frac{x e^{x-y}}{2(1-1)}$$

$$= \frac{x e^{x-y}}{0}$$

$$PI = \frac{x^2 e^{x-y}}{2}$$

$$\therefore \text{Gen Sol}^n = f_1(y-x) + x f_2(y-x) + \frac{x^2 e^{x-y}}{2}$$

Part - B

I i)

Sol^n $\phi(x^2+y^2+z^2, z^2-2xy) = 0$

$\phi(u, v)$

$u = x^2+y^2+z^2$ $\qquad$ $v = z^2-2xy$

$u_x = 2x + 2zp$ $\qquad$ $v_x = 2zp - 2y$

$u_y = 2y + 2zq$ $\qquad$ $v_y = 2zq - 2x$

$$\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = 0$$

$$\begin{vmatrix} 2x+2zp & 2y+2zq \\ 2zp-2y & 2zq-2x \end{vmatrix}$$

$$\begin{vmatrix} x+zp & y+zq \\ zp-y & zq-x \end{vmatrix} = 0$$

$(x+zp)(zq-x) - (y+zq)(zp-y)$

$xzq - x^2 + z^2pq - xzp - [yzp - y^2 + z^2pq - yzp]$

$xzq - x^2 + \cancel{z^2pq} - xzp - yzp + y^2 - \cancel{z^2pq} + yzq$

$y^2 - x^2 + xzq - xzp - yzp + yzq = 0$

$y^2 - x^2 + xz(q-p) + yz(q-p) = 0$

$y^2 - x^2 + (q-p)(z)(x+y) = 0$

$(x+y)(q-p)z = x^2 - y^2$

$(q-p)(z) = \frac{x^2-y^2}{x+y}$

$$(q-p)\,z = \cancel{\frac{x^2-y^2}{(x+y)}}\;\frac{(x-y)}{\cancel{(x-y)}}$$

$$(q-p)\,z = x-y$$

$$(p-q)\,z = y-x$$

1] ii

Sol^n

$$pq = k$$

$$f(p,q) = 0$$

For Complete Sol^n

$$z = ax + by + c$$

$$\frac{\partial z}{\partial x} = a = p \qquad \frac{\partial z}{\partial y} = b = q$$

$$pq = k$$
$$ab = k$$
$$b = k/a$$

$$\therefore\; z = ax + by + c$$
$$z = ax + (k/a)\,y + c$$

$$z = ax + \frac{ky}{a} + c$$

2]

Sol^n $$(D^3 - 7DD'^2 - 6D'^3)\,z = \sin(x+2y) + x^3$$

Aux eq $\quad m^3 - 7m - 6 = 0$

$(-1)\quad (-1)^3 - 7(-1) - 6$

$(-1) + 7 - 6$

$= 0$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & 0 & -1 & 1 & +6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$$m^2 - m - 6 \qquad -6$$
$$m^2 - 3m + 2m - 6 = 0 \qquad -3 \quad +2$$
$$m(m-3) + 2(m-3)$$

$$\therefore\; m = 3, -2, -1$$

$$CF = f_1(y+3x) + f_2(y-x) + f_3(y-2x)$$

$$PI = \frac{\sin(x+2y) + x^3}{(\qquad\qquad)}$$

$$PI = PI_1 + PI_2$$

$$PI_1 = \frac{\sin(x+2y)}{(D^3 - 7DD'^2 - 6D'^3)}$$

$$= IP \left\{ \frac{e^{i(x+2y)}}{D^3 - 7DD'^2 - 6D'^3} \right\}$$

$$IP \left\{ \frac{e^{(xi + 2iy)}}{(\qquad\qquad)} \right\}$$

$D = i$
$D' = 2i$

$$= IP \left\{ \frac{e^{(ix + 2iy)}}{(i)^3 - 7(i)(2i)^2 - 6(2i)^3} \right\}$$

$$= IP \left\{ \frac{e^{(\qquad)}}{-i - 7(i)(4(-1)) - 6(8)(-i)} \right\}$$

$$= IP \left\{ \frac{e^{ix + 2iy}}{-i + 28i + 48i} \right\}$$

$$IP \left\{ \frac{e^{ix+2iy}}{75} \right\}$$

$$= IP \left\{ \frac{i\, e^{ix+2iy}}{75\, i \times i} \right\}$$

$$= IP \left\{ \frac{i(\qquad)}{-75} \right\}$$

$$= IP \left\{ \frac{-i}{75} \left[\cos(x+2y) + i \sin(x+2y) \right] \right.$$

$$\therefore PI_1 = -\frac{\cos(x+2y)}{75}$$

$$PI_2 = \frac{x^3}{D^3 - 7DD'^2 - 6D'^3}$$

$$= \frac{x^3}{D^3\left[1 - 7D'^2/D^2 - 6D'^3/D^3\right]}$$

$$= \frac{x^3}{D^3}\left[1 - \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right)\right]$$

$$= \frac{x^3}{D^3}\left[1 + \cancel{\frac{7D'^2}{D^2}} + \cancel{\frac{6D'^3}{D^3}} + \cdots\right]$$

$$= \frac{x^3}{D^3}$$

$$= \iiint x^3 \, . dx$$

$$= \iint \frac{x^4}{4} \, . dx$$

$$= \int \frac{x^5}{20} \, . dx$$

$$PI_2 = \frac{x^6}{120}$$

$$PI = \frac{-\cos(x+2y)}{75} + \frac{x^6}{120}$$

$$\therefore \text{Gen Sol}^n = f_1(y-x) + f_2(y-2x) + f_3(y-3x) + \frac{x^6}{120} - \frac{\cos(x+2y)}{75}$$

3] i)

Sol^n $\quad x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

Comparing with lagrange eq

$$Pp + Qq = R$$

$P = x(y^2 - z^2)$, $Q = y(z^2 - x^2)$, $R = z(x^2 - y^2)$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

By Multipliers

$$= \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

i) (x, y, z)

$$= \frac{x\,dx + y\,dy + z\,dz}{\cancel{x^2(y^2-z^2) + y^2(z^2-x^2) + z^2(x^2-y^2)}}$$

$$\therefore \quad dx + dy + dz = 0$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1$$

$$x^2 + y^2 + z^2 = C_1$$

ii) $(1/x, 1/y, 1/z)$

$$\frac{dx/x + dy/y + dz/z}{\frac{1}{\cancel{x}}\cancel{x}(\cancel{y^2}-\cancel{z^2}) + \frac{1}{\cancel{y}}\cancel{y}(\cancel{z^2}-\cancel{x^2}) + \frac{1}{\cancel{z}}z(\cancel{x^2}-\cancel{y^2})}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

$\therefore \quad f(c_1, c_2) = 0$

$f(x^2+y^2+z^2, \quad xyz) = 0$

3] ii)
Sol^n

$z^2 = 1 + p^2 + q^2$

$f(z, p, q) = 0$

$z = f(u)$

$u = x + ay$

$p = \frac{dz}{du}$, $q = \frac{a\,dz}{du}$

$z^2 = 1 + p^2 + q^2$

$z^2 = 1 + \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$

$(z^2 - 1) = \left(\frac{dz}{du}\right)^2 [1 + a^2]$

$\left(\frac{dz}{du}\right)^2 = \frac{z^2-1}{1+a^2}$

$\frac{dz}{du} = \sqrt{\frac{z^2-1}{1+a^2}}$

$du = \sqrt{1+a^2}\left(\frac{1}{\sqrt{z^2-1}}\right).dz$

$\int du = \sqrt{1+a^2}\int \frac{1}{\sqrt{z^2-1}}.dz$

$\frac{1}{\sqrt{z-1}} = \ln|x + \sqrt{x^2-1}| + c$

$\therefore \int \frac{1}{\sqrt{z-1}} = \ln|z + \sqrt{z^2-1}| + c$

$u = \sqrt{1+a^2}\left[\ln|z + \sqrt{z^2-1}|\right] + c$

$u = x + ay$

$$x + ay = \sqrt{a^2+1}\left(\ln\left|z + \sqrt{z^2-1}\right|\right) + c$$

No singular solution

Part – A (2 x 8 = 16 Marks) Answer any TWO the Questions.					
Q. No	Question	Marks	BL	CO	PO
1	Form a PDE by eliminating the arbitrary constant from $\phi(xy,\ x+y+z)=0$	8	1	1	2
2	Solve $p^2+pq=z^2$	8	3	1	2
3	Solve using Lagrange's method $(3z-4y)p+(4x-2z)q=2y-3x$	8	3	1	2

Part – B (1 x 14 = 14 Marks) Answer ALL Questions.					
Q. No	Question	Marks	BL	CO	PO
4	Solve $(D^2-7DD'+6D'^2)z=e^{x+y}+xy$	14	2	1	2

1)

Sol^n $\phi(xy,\ x+y+z)$

$u = xy$

$v = x+y+z$

$u_x = y \qquad v_x = 1+p$

$u_y = x \qquad v_y = 1+q$

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$$

$$\begin{vmatrix} y & x \\ 1+p & 1+q \end{vmatrix} = 0$$

$y(1+q) - x(1+p) = 0$

$y + yq - x - xp = 0$

$y - x - yq - xp = 0$

2]

Sol^n $\quad p^2 + pq = z^2$

$u = x + ay$

$p = \frac{dz}{du}$, $q = \frac{a\,dz}{du}$

$\left(\frac{dz}{du}\right)^2 + \left(\frac{dz}{du}\right) a \frac{dz}{du} = z^2$

$\left(\frac{dz}{du}\right)^2 \left[1 + a\right] = z^2$

$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{1+a}$

$\left(\frac{du}{dz}\right)^2 = \frac{1+a}{z^2}$

$\frac{du}{dz} = \frac{\sqrt{1+a}}{z}$

$\int du = \int \frac{\sqrt{1+a}}{z} dz$

$u = \sqrt{1+a} \int \frac{1}{z} dz$

$u = \sqrt{1+a} \log(z) + c$

$\therefore \quad x + ay = \sqrt{1+a} \log(z) + c$

└ Complete solution

For Singular solution

$\frac{dz}{da} = \frac{dz}{dc} = 0$

$0 + 0 = 0 + 1 \qquad 1 \neq 0$

$\therefore$ No Singular solution

For General Solution

$c = \phi(a)$

$$x + ay = \sqrt{1+a}\,\log(z) + \phi(a)$$

eliminating a, $\therefore \frac{dz}{da}$

$$0 + y = \frac{-1}{2\sqrt{1+a}}\log(z) + \frac{\sqrt{1+a}}{z}.p + \phi'(a)$$

3]

Sol^n $(3z-4y)p + (4x-2z)q = 2y-3x$

Acc. to lagrange linear pdE

$$Pp + Qq = R$$

$$P = (3z-4y)$$
$$Q = (4x-2z)$$
$$R = (2y-3x)$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}$$

Acc to multipliers

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

i) (x, y, z)

$$= \frac{x\,dx + y\,dy + z\,dz}{x(3z-4y) + y(4x-2z) + z(2y-3x)}$$

$$= \frac{\quad}{0}$$

$\therefore \quad x\,dx + y\,dy + z\,dz = 0$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + c = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

ii) $(2, 3, 4)$

$$= \frac{2\,dx + 3\,dy + 4\,dz}{2(3z - 4y) + 3(4x - 2z) + 4(2y - 3x)}$$

$$= \frac{}{0}$$

$$\int 2\,dx + \int 3\,dy + \int 4\,dz = 0$$

$$2x + 3y + 4z = c_2$$

$$f(c_1, c_2) = 0$$

$$f\left(\frac{x^2+y^2+z^2}{2}, 2x+3y+4z\right) = 0$$

4] Sol^n $\quad (D^2 - 7DD' + 6D'^2)z = e^{x+y} + xy$

Aux eq :-

$$m^2 - 7m + 6 = 0$$
$$m^2 - 6m - m + 6 = 0$$
$$m(m-6) - 1(m-6) = 0$$
$$(m-1)(m-6) = 0$$

$6 \to -6, -1$

$$m = 1, 6$$

$$CF = f_1(y+x) + f_2(y+6x)$$

$$PI = \frac{e^{x+y} + xy}{D^2 - 7DD' + 6D'^2}$$

$$PI = \frac{e^{x+y}}{(\quad)} + \frac{xy}{(\ )}$$

$$PI = PI_1 + PI_2$$

$$PI_1 = \frac{e^{x+y}}{D^2 - 7DD' + 6D'^2} \qquad D = 1,\ D' = 1$$

$$= \frac{e^{x+y}}{1 - 7(1) + 6(1)}$$

$$= \frac{e^{x+y}}{0}$$

$$= \frac{x\ e^{x+y}}{2D - 7D'}$$

$$= \frac{xe^{x+y}}{2-7}$$

$$PI_1 = -\frac{xe^{x+y}}{5}$$

$$PI_2 = \frac{xy}{D^2 - 7DD' + 6D'^2}$$

$$= \frac{xy}{D^2\left(1 - \frac{7D'}{D} + \frac{6D'^2}{D^2}\right)}$$

$$= \frac{xy}{D^2}\left(1 - \frac{7D'}{D} + \frac{6D'^2}{D^2}\right)^{-1}$$

$$= \frac{xy}{D^2}\left(1 + \frac{7D'}{D} + \frac{6D'^2}{D^2} + \cdots\cdots\right)$$

$$= \frac{xy}{D^2} + \frac{7xD'(y)}{D^3} + \frac{6x\,D'^2(y)}{D^4}$$

$$= \frac{xy}{D^2} + \frac{7x}{D^3} + 0$$

$$= y \iint x \cdot dx + 7 \iiint x \cdot dx$$

$$PI_2 = \frac{yx^3}{6} + \frac{7x^4}{24}$$

$$\therefore \text{Gen Sol}^n = f_1(y+x) + f_2(y+6x) - \frac{xe^{x+y}}{5} + \frac{yx^3}{6} + \frac{7x^4}{20}$$

3]

soln $(y^2+z^2)p - xyq = -xz$

$$Pp + Qq = R$$

$P = y^2+z^2$
$Q = -xy$
$R = -xz$

Acc to multipliers

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

i) (x, y, z)

$$\frac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2)+y(-xy)+z(-xz)}$$

$$\frac{x\,dx + y\,dy + z\,dz}{\cancel{xy^2} + \cancel{xz^2} - \cancel{xy^2} - \cancel{xz^2}}$$

$$\therefore \int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

ii) $\left(1, \frac{y}{x}, \frac{z}{x}\right)$

$$= \frac{1\,dx + y/x\,dy + z/x\,dz}{\cancel{y^2}+\cancel{z^2} + \frac{y}{\cancel{x}}(-\cancel{x}y) + \frac{z}{\cancel{x}}(-\cancel{x}z)}$$

$$\int dx + \int \frac{y}{x}\,dy + \int \frac{z}{x}\,dz = 0$$

$$x + \frac{y^2}{2x} + \frac{z^2}{2x} = c_2$$

$$\therefore f\left(\frac{x^2+y^2+z^2}{2}, x + \frac{y^2}{2x} + \frac{z^2}{2x}\right) = 0$$

$$xy^2 + xz^2 - y^2x$$

eg $\quad py^2 - xyq = x(z-2y)$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{xz - 2yx}$$

$$\int \frac{dx}{y^2} = \int \frac{dy}{-xy}$$

$$\frac{x}{y^2} = -\frac{1}{x}\log(y) + c$$

$$\frac{x}{y^2} - \frac{\log(y)}{x} = c_1$$

$$\int \frac{dy}{-xy} = \int \frac{dz}{xz - 2yx}$$

$$-\frac{\log(y)}{x} = \int \frac{1}{xz}.dz - \int \frac{1}{2yx}.dz$$

$$c_2 = \frac{\log(z)}{x} - \frac{z}{2xy} + \frac{\log(y)}{x}$$

$$f(c_1, c_2) = 0$$

$$f\left(\frac{x}{y^2} - \frac{\log(y)}{x}, \frac{\log(z)}{x} - \frac{z}{2xy} + \frac{\log(y)}{x} \right) = 0$$

$$x^2 + y^2 + (z-a)^2 = b^2$$

$$-\cancel{p}x = \cancel{2}(z-\cancel{a})p = 0$$

$$-\cancel{q}y = \cancel{2}(z-\cancel{a})q = 0$$

$$-\frac{x}{y} = \frac{p}{q}$$

$$u = x + ay$$

$$\frac{d\cancel{z}}{du}\left(1 + \frac{a\,dz}{du}\right) = a\frac{d\cancel{z}}{d\cancel{u}} = z$$

$$a\frac{dz}{du} = az - 1$$

$$\frac{du}{dz} = \frac{a}{az-1}$$

$$du = a\int \frac{1}{az-1}\,.dz$$

$$u = \frac{\cancel{a}\log(az-1)}{\cancel{a}}$$

$$u = \log(az-1) + c$$

$$\therefore x + ay = \log(az-1) + c$$