Sumedh

- ## Fourier Transform

of $f(x) = F(s)$

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} . dx$$

Inverse FT

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} . dx$$

Parseval's Identity on FT

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

- ## Fourier Cosine Transform (FCT)

$$F_c[f(x)] = F_c(s)$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

Inverse FCT

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx \, ds$$

Parseval's Identity on FCT

$$\int_0^{\infty} |f(x)|^2 dx = \int_0^{\infty} |F_c(s)|^2 ds$$

$$\int_0^{\infty} f(x) g(x) dx = \int_0^{\infty} F_c(s) G_c(s) ds$$

- Fourier Sine Transform (FST)

$$F_s[f(x)] = F_s(s)$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$e^{-ax} \sin bx = \frac{b}{b^2+a^2}$$

$$e^{-ax} \cos bx = \frac{a}{a^2+n^2}$$

Inverse FST

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

PROPERTY.

(i) If $F[f(x)] = F(s)$, then

$F[x \cdot f(x)] = (-i) \frac{d}{ds}[F(s)]$

(ii) If $F_s[x \cdot f(x)] = -\frac{d}{ds}[F_c(s)]$

(iii) If $F_c[x \cdot f(x)] = \frac{d}{ds}[F_s(s)]$

Parseval's Identity on FST

$$\int_0^\infty |f(x)|^2 \, dx = \int_0^\infty |F_s(s)|^2 \, ds$$

$$\int_0^\infty f(x) \, g(x) \, dx = \int_0^\infty F_s(s) \, G_s(s) \, ds$$

Note : $e^{isx} = \cos sx + i \sin sx$

$e^{-isx} = \cos sx - i \sin sx$

eg

Find the fourier transform of $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$. Hence deduce that

(i) $\int_0^\infty \frac{\sin t}{t} \, dt = \frac{\pi}{2}$ (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Soln $f(x) = 1$ at $|x| \le a$ i.e $(-a, a)$

$$\therefore \text{FT} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \overset{e}{\cos sx} + \overset{o}{\cancel{i \sin sx}} \overset{o}{} \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a \cos sx$$

$$F(s) = \frac{2}{\sqrt{2\pi}}\left[\frac{\sin as}{s}\right]$$

i) $\displaystyle\int_0^\infty \frac{\sin t}{t}\, dt = \pi/2$

inverse : $\displaystyle f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s}\right) \cdot e^{-isx}\, ds$$

put $x = 0$, $a = 1$, $s = t$, $ds = dt$

$(-a, a)$

$$f(0) = \frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}} \times \sqrt{\frac{\cancel{2}}{\pi}} \int \frac{\sin t}{t} \cdot dt$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin t}{t} \cdot ds$$

o/o = e

$$1 = \frac{2}{\pi} \int_0^\infty \frac{\sin t}{t}\, dt$$

ii) $\displaystyle\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

parsevals : $\displaystyle\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\, \frac{\sin as}{s}\right)^2 ds = \int_{-a}^{a} (1)^2\, dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds = \int_{-a}^{a} dx$$

put $a = 1$, $s = t$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \int_{-1}^{1} dx$$

$$\cancel{2}\, \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \cancel{2} \int_0^1 dx$$

- Self - reciprocal function

$$F[f(x)] = f(s)$$

or

$$F_s[f(x)] = f(s) \quad , \quad F_c[f(x)] = f(s)$$

only eg : $F[e^{-x^2/2}] = e^{-s^2/2}$

eg

Find the Fourier transform of $e^{-a^2x^2}$ and hence deduce that $e^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier transform. *(Or)*
Show that $e^{-\frac{x^2}{2}}$ is self - reciprocal under Fourier transform.

Soln

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}$$

$$(A-B)^2 = A^2 + B^2 - 2AB$$

$$(A-B)^2 - B^2 = A^2 - 2AB$$

$$a^2x^2 - isx$$

$$A = ax$$

$$2AB = isx$$

$$2ax\,B = isx$$

$$B = \frac{is}{2a}$$

$$\therefore A^2 - 2AB = \left(ax - \frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$$

$$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}$$

$$\therefore = \int_{-\infty}^{\infty} e^{-\left\{\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right\}}$$

const

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} \cdot e^{-(s^2/4a^2)} \cdot dx$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

$$\cdots\cdots \int_{-\infty}^{\infty} e^{-t^2} \cdot dt = \sqrt{\pi}$$

$$t = ax - \frac{is}{2a}$$

$$dt = a \cdot dx$$

$$= (\quad) \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$\frac{1}{a} (\quad) \left[\sqrt{\pi}\right]$$

$$= \frac{e^{-s^2/4a}}{a \sqrt{2\cancel{\pi}}} \times \sqrt{\cancel{\pi}}$$

$$F\left[e^{-a^2x^2}\right] \quad \frac{e^{-s^2/4a}}{a\sqrt{2}}$$

$$F\left[e^{-x^2/2}\right]$$

$$a^2 = 1/2$$

$$a = 1/\sqrt{2}$$

$$\therefore \quad F\left[e^{-x^2/2}\right] = \frac{e^{-s^2/4(1/2)}}{1/\cancel{\sqrt{2}} \times \cancel{\sqrt{2}}}$$

$$= e^{-s^2/2}$$

Tutorial Sheet- Unit - 4

	Part A	Marks: 8 X 5 = 40
Sl. No	**Question**	**Answers**
1	Find the Fourier transform of f(x) given by $$f(x) = \begin{cases} 1 - x^2, & for\ \lvert x\rvert < 1 \\ 0, & for\ \lvert x\rvert > 1 \end{cases}$$	$\frac{4}{\sqrt{2\pi}s^3}(\sin s - s\cos s)$
2	Find the Fourier cosine transform of f(x) defined as $$f(x) = \begin{cases} x\ , & for\ \ 0 < x < 1 \\ 2 - x, & for\ 1 < x < 2 \\ 0, & for\ x > 2 \end{cases}$$	$\frac{2\sqrt{2}\cos s\ (1-\cos s)}{\sqrt{\pi}s^2}$
3	Find the inverse Fourier transform of F(s) given by $F(s) = \begin{cases} \pi, if\ \lvert s\rvert < a \\ 0, if\ \lvert s\rvert > a \end{cases}$; and hence prove that $\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx = a\pi$.	$\sqrt{2\pi}\,\frac{\sin ax}{x}$ Then use Parseval's identity
4	Using Parseval's identity for Fourier cosine transform of e^{-ax}, evaluate $\int_0^{\infty} \frac{1}{(a^2+x^2)^2} dx$.	$\frac{\pi}{4a^3}$
5	Find the Fourier transform of $f(x) = \begin{cases} 1, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{cases}$ And hence find the value of $\int_0^{\infty} \frac{\sin x}{x} dx$	$F\{f(x)\} = \sqrt{\frac{2}{\pi}}\frac{\sin as}{s}$ $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$
	Part B	**Marks: 15 X 3 = 45**
1	Find the inverse Fourier transform of $\bar{f}(s)$ given by $$\bar{f}(s) = \begin{cases} a - \lvert s\rvert, & for \lvert s\rvert \le a \\ 0, & for\ \lvert s\rvert > a \end{cases}$$ Hence show that $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$	$\frac{a^2}{2\pi}\left(\frac{\sin\frac{ax}{2}}{\frac{ax}{2}}\right)^2$ Then use the definition of Fourier transform, and then let s→ $0, and$ put a= 2.
2	Find the Fourier transform of f(x), if $$f(x) = \begin{cases} 1 - \lvert x\rvert, & for\ \lvert x\rvert < 1 \\ 0, & for\ \ \ \lvert x\rvert > 1 \end{cases}$$ Hence prove that $\int_0^{\infty} \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$	$F\{f(x)\} = \frac{2\sqrt{2}\sin^2\frac{s}{2}}{\sqrt{\pi}s^2}$ Then use Parseval's identity
3	Find the Fourier transform of f(x) given by $$f(x) = \begin{cases} a^2 - x^2, & for\ \lvert x\rvert < a \\ 0, & for\ \lvert x\rvert > a \end{cases}$$ Hence, evaluate $\int_0^{\infty} (\frac{\sin x - x\cos x}{x^3})^2 dx$	$F\{f(x)\} = \frac{4}{\sqrt{2\pi}}[\frac{\sin as - as\cos as}{s^3}]$ $\int_0^{\infty} (\frac{\sin x - x\cos x}{x^3})^2 dx = \frac{\pi}{15}$

• Tutorial

1] $f(x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & |x|>1 \end{cases}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left[(1-x^2) \cdot \cos sx + i \sin sx \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left[(1-x^2) \cos sx + \cancel{(1-x^2)\, i \sin sx}^{\,0} \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left(\cos sx - x^2 \cos x \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \Big[$$

$$\underset{u}{x^2} \underset{v}{\cos x}$$

$$uv_1 - u'v_2 + u''v_3 -$$

$$\frac{x^2 \sin sx}{s} + \left[\frac{2x \cos sx}{s^2} \right] \mp - \frac{2 \sin sx}{s^3}$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{\sin sx}{s} - \frac{x^2 \sin sx}{s} - \frac{2x \cos sx}{s^2} \mp \frac{2 \sin sx}{s^3} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left(\cancel{\frac{\sin s}{s}} - \cancel{\frac{\sin s}{s}} - \frac{2 \cos s}{s^3} + \frac{2 \sin s}{s^3} \right.$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}} \left(\frac{\cos s}{s^2} \right.$$

$$= \frac{4}{\sqrt{2\pi}\, s^3} \left(\sin s - s \cos s \right)$$

3]

Soln $f(x) = \begin{cases} \pi & |s| < a \\ 0 & |s| > a \end{cases}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \pi (\cos sx + i \sin sx) \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\pi \cos sx + \overset{0}{\cancel{i\pi \sin sx}}) \, ds$$

$$= \frac{2\pi}{\sqrt{2\pi}} \int_{0}^{a} \cos sx \, ds$$

$$= \sqrt{2\pi} \frac{\sin ax}{x}$$

Parseval

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-\infty}^{\infty} \left(\sqrt{2\pi} \frac{\sin ax}{x} \right)^2 dx = \int_{-a}^{a} \pi^2 \, ds$$

$$2\pi \int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx = \left[\pi^2 s \right]_{-a}^{a}$$

$$2\pi \left(\qquad \right) = \pi^2 a + \pi^2 a$$

$$\cancel{2\pi} \left(\quad \right) = \cancel{2} \pi^{\cancel{2}} a$$

$$= \pi a$$

4]

Sol^n FCT e^{-ax}

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+x^2} \right)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+x^2}$$

$$\int_0^\infty (e^{-ax})^2 \, dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}\right)^2 ds$$

$$\int e^{-2ax} \, dx = \frac{2a}{\pi} \int \frac{1}{(a^2+x^2)^2} ds$$

$$\left[\frac{e^{-2ax}}{-2a} \right]_0^\infty = \frac{2a^2}{\pi} \int_0^\infty \frac{ds}{(a^2+x^2)^2}$$

$$+\frac{1}{2a} \times \frac{\pi}{2a^2} = (\quad)$$

$$= \frac{\pi}{4a^3}$$

5]

Sol^n $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{+isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + \cancel{i \sin sx}^{\,0}) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{a} \cos sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \frac{\sin as}{s}$$

Inverse

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \frac{\sin as}{s} \bar{e}^{isx} \, ds$$

$$= \frac{1}{\pi} \int \frac{\sin as}{s} e^{-isx} \, dx$$

$$\int_{0}^{\infty} \frac{\sin x}{x} \, dx = \pi/2$$

put $x = 0$ $a = 1$ $s = x$

6]

Sol^n $F(s) = \begin{cases} a - |s| & |s| \le a \\ 0 & |s| > a \end{cases}$

$O \times O = E$

$O \times E = O$

Inverse :

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a - |s|) e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \left[\overset{E}{(a - |s|)} \overset{E}{\cos sx} - \cancel{\overset{E}{(a - |s|)} \overset{O}{\sin sx}}^{\,O} \right] ds$$

$$= \frac{1}{\sqrt{2\pi}}\, 2 \int_{0}^{a} (a - s) \cos sx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(a-s) \sin sx}{x} - \frac{\cos sx}{x^2} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{\cos ax}{x^2} - 0 + \frac{1}{x^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{x^2} - \frac{\cos ax}{x^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{x^2} \left[1 - \cos ax \right]$$

$$\sin^2 \frac{ax}{2}$$

7]

Sol^n $\quad f(x) = \begin{cases} 1-|x| & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left[(1-|x|) \cos sx + \cancel{(1-|x|) i \sin sx}^{\,0}\right] dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{1} (1-x) \cos sx \cdot dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(1-x) \sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{\cos s}{s^2} + \frac{1}{s^2}\right]$$

$=$

8]

Soln $f(x) = \begin{cases} a^2 - x^2 & |x| < a \\ 0 & |x| > a \end{cases}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}$$