Curvature

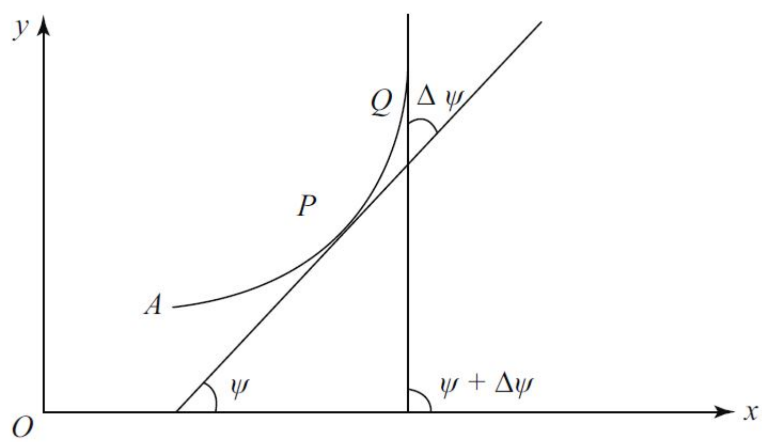
Let $y = f(x)$ be a curve that does not intersect itself and having tangents at each point. Let A be a fixed point on the curve from which arc length is measured. Let P be any point on a given curve and Q a neighbouring points sothat $AP = s$ and $AQ = s + \Delta s$.

Therefore Length arc $PQ = \Delta s$

Let the tangents at P and Q make an angles Ψ and $\Psi + \Delta\Psi$ respectively with positive direction of x-axis, so that the angle between the tangents at P and $Q = \Delta\Psi$. Thus for a change of Δs in the arcual length of the curve, the direction of the tangent to the curve changes by $\Delta\Psi$.

Hence $\dfrac{\Delta\Psi}{\Delta s}$ is the average rate of bending of the curve (or average rate of change of direction of the tangent to the curve in the arcual interval PQ) or average curvature of the arc PQ.

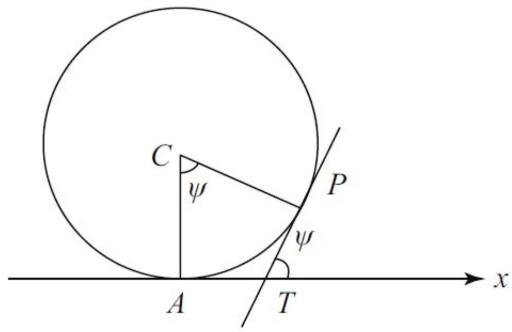
Therefore $\lim\limits_{\Delta s \to 0} \left(\dfrac{\Delta\Psi}{\Delta s}\right) = \dfrac{d\Psi}{ds}$ is the rate of bending of the curve with respect to arcual distance at P or the curvature of the curve at the point P. The curvature is denoted by κ.



Therefore average Curvature of arc $PQ = \dfrac{d\Psi}{ds}$

Find the curvature of a circle of radius at any point on it

Let the arcual distances of points on the circle be measured from A, the lowest point of the circle and let the tangent at A be chosen as the x-axis. Let $AP = s$ and let the tangent at P make an angle Ψ with the x-axis.



Then $s = a.A\hat{C}P = a\Psi$ [Since the angle between CA and CP equals the angle between the respective perpendiculars AT and PT.]
or $\Psi = \frac{1}{a}s$
Therefore $\frac{d\Psi}{ds} = \frac{1}{a}$
Thus the curvature of a circle at any point on it equals the reciprocal of its radius. Equivalently, the radius of a circle equals the reciprocal of the curvature at any point on it.

Radius of Curvature

Radius of curvature of a curve at any point on it is defined as the reciprocal of the curvature of the curve at that point and denoted by ρ. Thus $\rho = \frac{1}{\kappa} = \frac{ds}{d\Psi}$.
Note: To find ρ of a curve at any point on it, we should know the relation between s and Ψ for that curve, which is not easily derivable in most cases. Generally curves will be defined by means of their Cartesian, parametric or polar equations. Hence formulas for ρ in terms of cartesian, parametric or polar co-ordinates are given below.

(1) For cartesian curve $y = f(x)$:

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$.

(2) For parametric equations $x = f(t), y = g(t)$:

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} \text{ or } \rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

where $x' = \frac{dx}{dt}, y' = \frac{dy}{dt}, x'' = \frac{d^2x}{dt^2}, y'' = \frac{d^2y}{dt^2}$

(3) For polar coordinates $r = f(\theta)$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

where $r_1 = \dfrac{dr}{d\theta}$, $r_2 = \dfrac{d^2r}{d\theta^2}$.

Note:

- Curvature of a straight line is zero.
- Curvature of a circle is the reciprocal of its radius.
- To calculate ρ when $\dfrac{dy}{dx}$ becomes infinite, we can use the formula

$$\rho = \frac{\left[1 + \left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$$

Example 1: Find the radius of curvature at the point $\left(\dfrac{1}{4}, \dfrac{1}{4}\right)$ on the curve $\sqrt{x} + \sqrt{y} = 1$.

Solution: Given

$$\sqrt{x} + \sqrt{y} = 1 \tag{1}$$

Differentiate (1) w.r.to x, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}.\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \tag{2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)} = -1$$

Again Differentiate (2) w.r.to x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}.\dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} - \sqrt{y}.\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2}\right]$$

$$= -\frac{1}{2x}\left[\frac{\sqrt{x}}{\sqrt{y}}.\left(-\frac{\sqrt{y}}{\sqrt{x}}\right) - \frac{\sqrt{y}}{\sqrt{x}}\right]$$

$$= \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)} = 4$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{[1+1]^{3/2}}{4} = \frac{1}{\sqrt{2}}$$

Example 2: Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the curve $x^3+y^3=3axy$.

Solution: Given

$$x^3+y^3=3axy \tag{3}$$

Differentiate (3) w.r.to x, we get

$$3x^2+3y^2\frac{dy}{dx} = 3a\left[y+x\frac{dy}{dx}\right]$$

$$\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax} \tag{4}$$

Again Differentiate (4) w.r.to x, we get

$$\frac{d^2y}{dx^2} = \frac{(y^2-ax)\left[a\frac{dy}{dx}-2x\right]-(ay-x^2)\left[2y\frac{dy}{dx}-a\right]}{(y^2-ax)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{\left[\frac{9a^2}{4}-\frac{3a^2}{2}\right](-a-3a)-\left[\frac{3a^2}{2}-\frac{9a^2}{4}\right](-3a-a)}{\left(\frac{9a^2}{4}-\frac{4a^2}{2}\right)^2}$$

$$= -\frac{32}{3a}$$

$$\rho = \frac{[1+(-1)^2]^{3/2}}{-32/3a} = -\frac{3a}{8\sqrt{2}}$$

Hence $|\rho| = \frac{3a}{8\sqrt{2}}$.

Example 3: Find the radius of curvature at $x=1$ on $y=\frac{\log x}{x}$.

Solution: Given $y=\frac{\log x}{x}$.

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{x.\frac{1}{x}-\log x}{x^2} = \frac{1-\log x}{x^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 1 \text{ since } \log 1 = 0$$

$$\frac{d^2y}{dx^2} = \frac{x^2.\left(-\frac{1}{x}\right) - (1-\log x).2x}{x^4} = \frac{-x-2x(1-\log x)}{x^4}$$
$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=1} = -3$$
$$\rho = \frac{(1+1)^{3/2}}{-3} = -\frac{2\sqrt{2}}{3} \Rightarrow |\rho| = \frac{2\sqrt{2}}{3}.$$

Example 4: Find the radius of curvature for the curve $y = c\cosh\frac{x}{c}$ at the point where the curve cross the $y-$ axis.

Solution: Put $x = 0$ in $y = c\cosh\frac{x}{c}$ Since the curve cuts $y-$ axis. Therefore the point is $(0, c)$. Since $\cosh 0 = 1$.

Diff. w.r.to x, we get
$$\frac{dy}{dx} = c\sinh\frac{x}{c}.\frac{1}{c} = \sinh\frac{x}{c}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,c)} = 0$$

Again diff. w.r.to x, we get $\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\frac{x}{c}$
$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{(0,c)} = \frac{1}{c}$$
$$\rho = \frac{(1+0)^{3/2}}{\frac{1}{c}} = c.$$

Example 5: Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ is $4a\cos\left(\frac{\theta}{2}\right)$.

Solution: Given $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$

DIff. w.r.to θ, we get $\frac{dx}{d\theta} = a(1+\cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$

Therefore $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right).\frac{d\theta}{dx}$$
$$= \frac{1}{2}\sec^2\frac{\theta}{2}.\frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2}.\frac{1}{a(1+\cos\theta)}$$
$$= \frac{1}{2a}\sec^2\frac{\theta}{2}.\frac{1}{2\cos^2\frac{\theta}{2}} = \frac{1}{4a}\sec^4\frac{\theta}{2}$$
$$\rho = \frac{\left[1+\tan^2\frac{\theta}{2}\right]^{3/2}}{\frac{1}{4a}\sec^4\frac{\theta}{2}} = \frac{(\sec^2\frac{\theta}{2})^{3/2}}{\sec^4\frac{\theta}{2}}$$
$$= 4a\cos\frac{\theta}{2}$$

Example 6: Find the radius of curvature at any point on the curve $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$.

Solution: Given $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$
$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t$, $\frac{dy}{dt} = at\sin t$

Therefore $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(\tan t).\frac{dt}{dx}$$
$$= \sec^2 t.\frac{dt}{dx} = \sec^2 t.\frac{1}{at\cos t} = \frac{1}{at}\sec^3 t$$
$$\rho = \frac{[1+\tan^2 t]^{3/2}}{\frac{1}{at}\sec^3 t} = \frac{\sec^3 t}{\sec^3 t}.at = at$$

Example 7: Show that the radius of curvature t the point θ on the $x = 3a\cos\theta - a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$ is $3a\sin\theta$.

Solution: Given $x = 3a\cos\theta - a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$ is $3a\sin\theta$.

Diff. w.r.to θ, we get
$\frac{dx}{d\theta} = -3a\sin\theta + a\sin 3\theta$ and $\frac{dy}{d\theta} = 3a\cos\theta - 3a\cos 3\theta$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta - \cos 3\theta}{\sin 3\theta - \sin\theta}$

We know that $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$ and $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2\sin 2\theta \sin\theta}{2\cos 2\theta \sin\theta} = \tan 2\theta$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{d\theta}{dx} = \frac{d}{d\theta}(\tan 2\theta)\frac{d\theta}{dx} \\
&= \sec^2 2\theta .2.\frac{1}{3a(\sin 3\theta - \sin\theta)} \\
&= \frac{2sec^2 2\theta}{3a.2\cos 2\theta \sin\theta} = \frac{1}{3a}\sec^3 2\theta \csc\theta \\
\rho &= \frac{\left[1+\tan^2 2\theta\right]^{3/2}}{\dfrac{1}{3a}\sec^3 2\theta \csc\theta} = 3a\sin\theta
\end{aligned}$$

Example 8: Find the radius of curvature at $(a\cos^3\theta, a\sin^3\theta)$ on $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution: Given $x = a\cos^3\theta$, $y = a\sin^3\theta$

Diff. w.r.to θ, we get

$\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$ and $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{d\theta}{dx} = \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx} \\
&= -\sec^2 .\frac{-1}{3a\cos^2\theta\sin\theta} = \frac{1}{3a\sin\theta\cos^4\theta} \\
\rho &= \frac{\left[1+\tan^2 2\theta\right]^{3/2}}{\dfrac{1}{3a\sin\theta\cos^4\theta}} = 3a\sin\theta\cos\theta
\end{aligned}$$

Example 9: Show that at any point P on the rectangular hyperbola $xy = c^2, \rho = \dfrac{r^3}{2c^2}$ where r is the distance of P from the centre of the curve.

Solution: Given $xy = c^2$

Diff. w.r.to x, we have

$x\dfrac{dy}{dx} + y = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$

Again Diff. w.r.to x, we have

$$\frac{d^2y}{dx^2} = -\left[\frac{x.\dfrac{dy}{dx} - y}{x^2}\right] = \frac{2y}{x^2}$$

$$\rho = \frac{\left[1 + \dfrac{y^2}{x^2}\right]^{3/2}}{\dfrac{2y}{x^2}} = \frac{(x^2+y^2)^{3/2}}{2xy} = \frac{r^3}{2c^2}.$$

Example 10: Show that the radius of curvature at an end of the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to the semi-latus rectum.

Solution: Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Diff. w.r.to x, we have

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{xb^2}{ya^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(a,0)} = \infty$$

Consider $\frac{dx}{dy} = -\frac{ya^2}{xb^2}$, Therefore $\left(\frac{dy}{dx}\right)_{(a,0)} = 0$

Diff. w.r.to y, we have

$$\frac{d^2x}{dy^2} = -\frac{a^2}{b^2}\left[\frac{x.1 - y.\dfrac{dx}{dy}}{x^2}\right]$$

$$\Rightarrow \left(\frac{d^2x}{dy^2}\right)_{(a,0)} = -\frac{a}{b^2}$$

$$\rho = \frac{\left[1 + \dfrac{dx}{dy}\right]^{3/2}}{\dfrac{d^2x}{dy^2}} = -\frac{b^2}{a}$$

$\Rightarrow |\rho| = \frac{b^2}{a}$ =semi latus rectum.

Example 11: Find ρ at any point $P(at^2, 2at)$ on the parabola $y^2 = 4ax$, prove that if S is its focus, then ρ^2 varies as $(SP)^3$.

Solution: Given $x = at^2$,$y = 2at$

$$\Rightarrow \frac{dx}{dt} = 2at \text{ and } \frac{dy}{dt} = 2a$$

Therefore $\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{2a}{2at} = \dfrac{1}{t}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{t}\right).\frac{dt}{dx}$$
$$= \frac{-1}{t^2}\frac{dt}{dx} = \frac{-1}{t^2}.\frac{1}{2at} = -\frac{1}{2at^3}$$
$$\rho = \frac{\left[1+\dfrac{1}{t^2}\right]^{3/2}}{\left(-\dfrac{1}{2at^3}\right)} = -2a(t^2+1)^{3/2}$$

$$\rho^2 = 4a^2(t^2+1)^3 \tag{5}$$

Since the focus is $S(a, 0)$. Distance $SP = \sqrt{(at^2-a)^2+(2at-0)^2} = a(t^2+1)$

$$(SP)^3 = a^3(t^2+1)^3 \tag{6}$$

Dividing (5) by (6), we get $\dfrac{\rho^2}{(SP)^3} = \dfrac{4}{a}$ =constant. (or) ρ^2 varies as $(SP)^3$.

Example 12: Find the radius of curvature at any point (r, θ) for the curve $r = a\cos\theta$.

Solution: Given $r = a\cos\theta$

Differentiating w.r.to θ we get

$r_1 = \dfrac{dr}{d\theta} = -a\sin\theta$ and $r_2 = \dfrac{d^2r}{d\theta^2} = -a\cos\theta$

$r = \dfrac{(a^2\cos^2\theta + a^2\sin^2\theta)^{3/2}}{a^2\cos^2\theta + 2a^2\sin^2\theta + a^2\cos^2\theta} = \dfrac{a}{2}.$

Example 13: Find the radius of curvature of the curve $r = a(1+\cos\theta)$ at the point $\theta = \dfrac{\pi}{2}$.

Solution: Given $r = a(1+\cos\theta)$

Differentiating w.r.to θ we get

$r_1 = \dfrac{dr}{d\theta} = -a\sin\theta$ and $r_2 = \dfrac{d^2r}{d\theta^2} = -a\cos\theta$

At $\theta = \dfrac{\pi}{2}$, $r = a, r_1 = -a, r_2 = 0$

$r = \dfrac{(a^2+a^2)^{3/2}}{a^2+2a^2} = \dfrac{(2a^2)^{3/2}}{3a^2} = \dfrac{2\sqrt{2}a}{3}.$

Example 14: Show that the radius of curvature of the curve $r^n = a^n\cos n\theta$ is $\dfrac{a^n r^{-n+1}}{n+1}$. Hence prove that the radius of curvature of the curve $r^2 = a^2\cos 2\theta$ is $\dfrac{a^2}{3r}$.

Solution: Given

$$r^n = a^n\cos n\theta \tag{7}$$

Diff. w.r.to θ we have $nr^{n-1}\dfrac{dr}{d\theta} = -na^n \sin n\theta$

$\Rightarrow \dfrac{dr}{d\theta} = -\dfrac{a^n \sin n\theta}{r^{n-1}} = -\dfrac{a^n \sin n\theta}{r^n r^{-1}}$

$= -\dfrac{a^n \sin n\theta}{a^n \cos n\theta r^{-1}} = -r \tan n\theta$

Again diff. w.r.to θ, we get

$\Rightarrow \dfrac{d^2r}{d\theta^2} = -\dfrac{dr}{d\theta}.\tan n\theta - r\sec^2 n\theta.n = r\tan^2 n\theta - nr\sec^2 n\theta$

$$\begin{aligned}\text{Now } r^2 + 2r_1^2 - rr_2 &= r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta \\ &= r^2 + r^2\tan^2 n\theta + nr^2\sec^2 n\theta \\ &= r^1\sec^2 n\theta + nr^2\sec^2 n\theta = r^2(n+1)\sec^2 n\theta\end{aligned}$$

$$\text{and } r^2 + r_1^2 = r^2 + r^2\tan^2 n\theta = r^2\sec^2 n\theta$$

$$\begin{aligned}\rho &= \frac{\left(r^2\sec^2 n\theta\right)^{3/2}}{r^2(n+1)\sec^2 n\theta} = \frac{r\sec n\theta}{n+1} \\ &= \frac{r}{(n+1)\cos n\theta} = \frac{r}{(n+1)\dfrac{r^n}{a^n}} = \frac{a^n r^{1-n}}{n+1}\end{aligned}$$

Putting $n = 2$ in (7), we get $r^2 = a^2\cos 2\theta$. Hence $\rho = \dfrac{a^2 r^{-1}}{3} = \dfrac{a^2}{3r}$.

Example 15: Find the radius of curvature at any point (r, θ) on the equiangular spiral $r = ae^{\theta\cot\alpha}$.

Solution: Given $r = ae^{\theta\cot\alpha}$

Diff. w.r.to θ, we have

$\dfrac{dr}{d\theta} = ae^{\theta\cot\alpha}.\cot\alpha = r\cot\alpha$

Again Diff. w.r.to θ, we have

$\dfrac{d^2r}{d\theta^2} = ae^{\theta\cot\alpha}.\cot^2\alpha = r\cot^2\alpha$

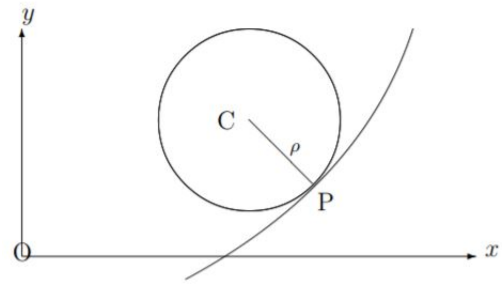
$$\text{Now } r^2 + 2r_1^2 - rr_2 = r^2 + 2r^2\cot^2\alpha - r^2\cot^2\alpha = r^2(1+\cot^2\alpha) = r^2\csc^2\alpha$$

$$\text{and } r^2 + r_1^2 = r^2 + r^2\cot^2\alpha = r^2\csc^2\alpha$$

$$\rho = \frac{(r^2\csc^2\alpha)^{3/2}}{r^2\csc^2\alpha} = r\csc\alpha$$

Centre of Curvature and Circle of Curvature

Let P be a point on the curve $y = f(x)$. Suppose we were to draw a circle which just touches the curve. The point C is called the **centre of curvature** at P for the curve. The circle whose centre is C and radius ρ is called the **circle of curvature** at P for the curve. [See Fig.]



Equation of the circle of curvature is $(x - \overline{x})^2 + (y - \overline{y})^2 = \rho^2$

Example 1: Find the coordinates of centre of curvature of the curve $y = x^3 - 6x^2 + 3x + 1$ at (1,-1).

Solution: Given $y = x^3 - 6x^2 + 3x + 1$

Diff. w.r.to x, we get

$$\frac{dy}{dx} = 3x^2 - 12x + 3$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,-1)} = -6$$

Again diff. w.r.to x, we get

$$\frac{d^2y}{dx^2} = 6x - 12$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{(1,-1)} = -6$$

$$\overline{x} = x - \frac{y_1(1 + y_1^2)}{y_2} = 1 - \frac{(-6).37}{(-6)} = -36$$

$$\overline{y} = y + \frac{(1 + y_1^2)}{y_2} = -1 + \frac{37}{(-6)} = \frac{-43}{6}$$

Therefore centre of curvature is $\left(-36, \dfrac{-43}{6}\right)$.

Example 2: Find the coordinates of centre of curvature at the point (1, 1) on the curve $x^3 + y^3 = 2$.

Solution: Given $x^3 + y^3 = 2$

Diff. w.r.to x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = -1$$
Again diff. w.r.to x, we get
$$6x + 6y\frac{dy}{dx}.\frac{dy}{dx} + 3y^2\frac{d^2y}{dx^2} = 0$$
$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{(1,-1)} = -4$$
$$\overline{x} = 1 - \frac{(-1).2}{(-4)} = \frac{1}{2}$$
$$\overline{y} = 1 + \frac{2}{(-4)} = \frac{1}{2}$$
Therefore centre of curvature is $\left(-36, \frac{-43}{6}\right)$.

Example 3: Find the equation of circle of curvature of the parabola $y^2 = 12x$ at the point (3,6).

Solution: Given $y^2 = 12x$

Diff. w.r.to x, we get
$$2y\frac{dy}{dx} = 12 \Rightarrow \frac{dy}{dx} = \frac{6}{y}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(3,6)} = 1$$
Again diff. w.r.to x, we get
$$\frac{d^2y}{dx^2} = \frac{-6}{y^2}\frac{dy}{dx} \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{(3,6)} = \frac{-1}{6}$$
$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{2^{3/2}}{-\frac{1}{6}} = -12\sqrt{2}$$
$$\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 3 - \frac{2}{\frac{-1}{6}} = 15$$
$$\overline{y} = y + \frac{(1+y_1^2)}{y_2} = 6 + \frac{2}{\frac{-1}{6}} = -6$$
Circle of curvature is $(x-\overline{x})^2 + (y-\overline{y})^2 = \rho^2$

$\Rightarrow (x-15)^2 + (y+6)^2 = 288$

$\Rightarrow x^2 + y^2 - 30x + 12y - 27 = 0.$

Example 4: Find the equation of the circle of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left(\frac{a}{4},\frac{a}{4}\right)$.

Solution: Given $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Differentiating w.r.to x, we get

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}.\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=-1$$

Again Differentiating w.r.to x, we get

$$\frac{d^2y}{dx^2}=-\left[\frac{\sqrt{x}.\frac{1}{2\sqrt{y}}\frac{dy}{dx}-\sqrt{y}.\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}\right]$$

$$=-\frac{1}{2x}\left[\frac{\sqrt{x}}{\sqrt{y}}.\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)-\frac{\sqrt{y}}{\sqrt{x}}\right]$$

$$=\frac{\sqrt{x}+\sqrt{y}}{2x\sqrt{x}}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{4}{a}$$

$$\rho=\frac{[1+1]^{3/2}}{\frac{4}{a}}=\frac{a\sqrt{2}}{2}$$

$$\overline{x}=\frac{a}{4}-\frac{(-1).2}{\frac{4}{a}}=\frac{3a}{4}$$

$$\overline{y}=\frac{a}{4}+\frac{2}{\frac{4}{a}}=\frac{3a}{4}$$

$$\Rightarrow \left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{4}$$

Example 5: Find the equation of the circle of curvature of the rectangular hyperbola $xy=12$ at the point (3,4).

Example 6: Find the equation of the circle of curvature of the curve $x^3+y^3=3axy$ at the point $\left(\frac{3a}{2},\frac{3a}{2}\right)$.

Evolute

Let Q be the centre of curvature of a given curve C at the point P on it. When P moves on the curve C and takes different positions, Q will also take different positions and move on another curve C'. This curve C' is called the evolute of the curve C. Thus evolute can be defined as the locus of the centre of curvature. When C' is the evolute of the curve C, C is called the involute of the curve C'.

The procedure to find the equation of the evolute of a given curve is given below:

Let the equation of the given curve be

$$y = f(x) \tag{8}$$

If $(\overline{x}, \overline{y})$ is the centre of curvature corresponding to the point (x, y) on (8), then

$$\overline{x} = x - \frac{y_1(1 + y_1^2)}{y_2} \tag{9}$$

$$\overline{y} = y + \frac{(1 + y_1^2)}{y_2} \tag{10}$$

By eliminating x and y from (8), (9), (10), we get a relation between $\overline{x}$ and $\overline{y}$, which is the equation of the evolute.

Note: If the parametric co-ordinates of any point on the given curve are assumed, then we have to eliminate the parameter from Equations (9) and (10), which will simplify the procedure.

Example 1: Find the evolute of the parabola $y^2 = 4ax$.

Solution: The parametric equations of the given curve are $x = at^2, y = 2at$.

Diff. w.r.to t, we get $\dfrac{dx}{dt} = 2at$, $\dfrac{dy}{dt} = 2a$

Now $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1}{t}$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx}$$
$$= -\frac{1}{t^2}.\frac{1}{2at} = -\frac{1}{2at^3}$$

Let $(\overline{x}, \overline{y})$ be the corresponding centre of curvature. Then

$$\overline{x} = at^2 - \frac{\frac{1}{t}\left(1 + \frac{1}{t^2}\right)}{-\frac{1}{2at^3}} = at^2 + 2at\left(\frac{t^2+1}{t^2}\right) = 3at^2 + 2a \tag{11}$$

$$\overline{y} = 2at + \frac{\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}} = 2at - 2at^3\left(\frac{t^2+1}{t^2}\right) = -2at^3 \tag{12}$$

Hence the centre of curvature at any point t is $(3at^2 + 2a, -2at^3)$

Evolute of a curve is the locus of of its centres of curvature.

Now we have to eliminate t from (11) and (12).

From (11), we get $t^2 = \dfrac{\overline{x} - 2a}{3a}$

From (12), we get $t^3 = -\dfrac{\overline{y}}{2a}$

Now $t^6 = \dfrac{(\overline{x}-2a)^3}{27a^3}$ and $t^6 = \dfrac{\overline{y}^2}{4a^2}$

$\Rightarrow \dfrac{(\overline{x}-2a)^3}{27a^3} = \dfrac{\overline{y}^2}{4a^2} \Rightarrow 4(\overline{x}-2a)^3 = 27a\overline{y}^2$

Evolute is $4(x-2a)^3 = 27ay^2$.

Example 2: Find the equation of the evolute of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution: The parametric equations of the ellipse are $x = a\cos\theta$, $y = b\sin\theta$,

Diff. w.r.to θ, we get $\dfrac{dx}{d\theta} = -a\sin\theta$, $\dfrac{dy}{d\theta} = b\cos\theta$

Now $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\dfrac{b}{a}\cot\theta$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{d\theta}{dx} = \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\frac{d\theta}{dx}$$
$$= \frac{b}{a}cosec^2\theta.\frac{1}{-a\sin\theta} = -\frac{b}{a^2}cosec^3\theta$$

Let $(\overline{x}, \overline{y})$ be the corresponding centre of curvature. Then

$$\overline{x} = a\cos\theta - \frac{-\frac{b}{a}\cot\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{-\frac{b}{a^2}cosec^3\theta}$$
$$= a\cos\theta - a\cot\theta.\sin^3\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)$$
$$= a\cos\theta - a\sin^2\theta\cos\theta - \frac{b^2}{a}\cos^3\theta$$
$$= a\cos\theta(1-\sin^2\theta) - \frac{b^2}{a}\cos^3\theta$$
$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\overline{x} = \left(\frac{a^2 - b^2}{a}\right)\cos^3\theta \tag{13}$$

$$\overline{y} = b\sin\theta + \frac{1 + \dfrac{b^2}{a^2}\cot^2\theta}{-\dfrac{b}{a^2}cosec^3\theta}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta\cos^2\theta$$

$$= b\sin\theta(1 - \cos^2\theta) - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin^3\theta - \frac{a^2}{b}\sin^3\theta = \left(\frac{b^2 - a^2}{b}\right)\sin^3\theta$$

$$\overline{y} = -\left(\frac{a^2 - b^2}{a}\right)\sin^3\theta \tag{14}$$

To find the equation of the evolute we have to eliminate θ between (13) and (14).

From (13), we get $a\overline{x} = (a^2 - b^2)\cos^3\theta$

$\Rightarrow (a\overline{x})^{2/3} = (a^2 - b^2)^{2/3}\cos^2\theta$

$$\Rightarrow \cos^2\theta = \frac{(a\overline{x})^{2/3}}{(a^2 - b^2)^{2/3}} \tag{15}$$

From (14), we get $b\overline{y} = -(a^2 - b^2)\sin^3\theta$

$\Rightarrow (b\overline{y})^{2/3} = (a^2 - b^2)^{2/3}\sin^2\theta$

$$\Rightarrow \sin^2\theta = \frac{(b\overline{y})^{2/3}}{(a^2 - b^2)^{2/3}} \tag{16}$$

Adding (15) and (16), we get $(a\overline{x})^{2/3} + (b\overline{y})^{2/3}(a^2 - b^2)^{2/3}$.

Evolute is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

Example 3: Find the evolute of the parabola $x^2 = 4ay$.

Gamma and Beta Functions

Gamma Function: The definite integral $\int_0^\infty e^{-x} x^{n-1}\,dx$, for $n>0$ is said to be Gamma funtion and it is denoted by Γn.

ie, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$

Note: Gamma function is also called as Eulerian Integral of second kind.

Recurrence formula for Γn: $\Gamma(n+1) = n\,\Gamma n$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n}\,dx, \quad n>0$$

$u = x^n \qquad dv = e^{-x}\,dx$

$du = nx^{n-1} \qquad v = -e^{-x}$

$$\Gamma(n+1) = \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty e^{-x} x^{n-1}\,dx$$

$$= n\,\Gamma n$$

Corollary 1: $\Gamma n+1 = n!$ If n is a positive integer

Proof:

$$\Gamma n+1 = n\,\Gamma n$$

$$= n(n-1)\Gamma(n-1) = n(n-1)(n-2)\ldots\Gamma(1)$$

Now $\Gamma_1 = \int_0^\infty e^{-x}\,dx = 1$

$$\therefore \Gamma(n+1) = n(n-1)(n-2)\ldots 1 = n!$$

Corollary 2: Γn is undefined, If n is a negative integer.

Corollary 3: Γn is defined, If n is a negative fraction

For example:

$$\Gamma(-1/2) = \frac{\Gamma(-1/2+1)}{-1/2} \qquad \left[\because \Gamma(n+1) = n\Gamma(n)\right.$$

$$\frac{\Gamma(n+1)}{n} = \Gamma n$$

$$= -2\Gamma 1/2$$

Corollary 4: $\Gamma 1/2 = \sqrt{\pi}$

Examples:

1) Evaluate $\int_0^\infty e^{-x} x^7 dx$

W.K.T $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$$\therefore \int_0^\infty e^{-x} x^{8-1} dx = \Gamma(8) = 7! \qquad \because \Gamma(n+1) = n!$$

2) Evaluate $\int_0^\infty e^{-x} \sqrt{x}\, dx = \int_0^\infty e^{-x} x^{1/2} dx$

$$= \int_0^\infty e^{-x} x^{3/2 - 1} dx = \Gamma 3/2$$

$$= (3/2 - 1)\Gamma(3/2 - 1) \qquad \left[\because \Gamma n = (n-1)\Gamma(n-1)\right.$$

$$= \frac{1}{2}\Gamma 1/2 = \frac{1}{2}\sqrt{\pi}$$

3.) $\int_0^\infty x^6 e^{-3x} dx$

Put $3x = t \Rightarrow 3dx = dt$

$$= \int_0^\infty \frac{e^{-t} t^6 dt}{3^7} = \frac{1}{3^7}\Gamma 7 = \frac{6!}{3^7}$$

4.) $\int_0^\infty x e^{-x^2} dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$= \int_0^\infty e^{-t} \frac{dt}{2} = \frac{1}{2}$

5.) $\int_0^\infty e^{-x^3} dx$

Put $t = x^3 \Rightarrow dt = 3x^2 dx$ $\qquad n-1 = -2/3$

$= \int_0^\infty e^{-t} \frac{t^{-2/3}}{3} dt = \frac{1}{3}\Gamma\frac{1}{3}$ $\qquad n = 1/3$.

6.) $\int_0^\infty x^{16} e^{-4x} dx$ $\qquad$ Ans: $\frac{\Gamma(17)}{4^{17}} = \frac{16!}{4^{17}}$

7.) Find the value of $\Gamma(3\frac{1}{2})$

Sol: $\Gamma(3\frac{1}{2}) = \Gamma\frac{7}{2} = \frac{5}{2}\Gamma\frac{5}{2}$ $\qquad \because \Gamma(n+1) = n\Gamma n$

$= \frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2}$ $\qquad \Gamma n = (n-1)\Gamma n-1$

$\qquad \Gamma\frac{7}{2} = (\frac{7}{2}-1)\Gamma\frac{7}{2}-1$

$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{15}{8}\sqrt{\pi}$

8.) Evaluate $\int_0^1 x^2 [\log(\frac{1}{x})]^3 dx = \int_0^1 x^2 [\log 1 - \log x]^3 dx$

$= \int_0^1 x^2 (-\log x)^3 dx$

put $\log x = -t$

$x = e^{-t} \Rightarrow dx = -e^{-t}dt$ $\qquad \therefore \log 0 = -\infty$

when $x = 0 \Rightarrow t = \infty$ when $x = 1 \Rightarrow t = 0$

$= \int_\infty^0 e^{-2t} t^3 (-e^{-t}) dt = \int_0^\infty e^{-3t} t^3 dt$

when $x = 0$
$\Rightarrow 0 = e^{-t}$
$0 = \frac{1}{e^t}$
$e^t = \frac{1}{0} = \infty$
$= e^{\infty}$

Put $3t = u \Rightarrow dt = \frac{du}{3}$

$$= \int_0^\infty e^{-u} \left(\frac{u}{3}\right)^3 \frac{du}{3} = \frac{1}{3^4} \int_0^\infty e^{-u} u^3 du$$

$$= \frac{1}{3^4} \Gamma_4 = \frac{3!}{3^4}$$

<u>Beta function</u>: The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ for $m>0, n>0$ is said to be the Beta function and is denoted by $\beta(m,n)$.

$$\text{ie, } \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad m>n>0.$$

<u>Properties of Beta function</u>

1) The function $\beta(m,n)$ is symmetric in its parameter,

$$\text{ie, } \beta(m,n) = \beta(n,m)$$

2) Other form of Beta function is

(i) $$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy \quad \left[\text{put } x = \frac{y}{1+y}\right]$$

(ii) $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad [\text{put } x = \sin^2\theta]$$

<u>Note</u>: If $I_{m,n} = \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta$ then

$$I_{m,n} = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

<u>Proof</u>:

$2m-1 \to t$

$m = \frac{t+1}{2}$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = 2\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta$$

Relation between Beta and Gamma Functions

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

Result: $\Gamma \frac{1}{2} = \sqrt{\pi}$

Proof: put $m = \frac{1}{2}$, $n = \frac{1}{2}$

$$\therefore \beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma \frac{1}{2} \, \Gamma \frac{1}{2}}{\Gamma 1} = \left(\Gamma \tfrac{1}{2}\right)^2 \longrightarrow ①$$

W.K.T $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$$\therefore \beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = 2\left[\theta\right]_0^{\pi/2} = \pi \longrightarrow ②$$

Equating ① and ②

$$\left[\Gamma \tfrac{1}{2}\right]^2 = \pi \Rightarrow \Gamma \tfrac{1}{2} = \sqrt{\pi}$$

Other Standard Results:

1) $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

Note:

(i) If $p=0$ then $$\int_0^{\pi/2} \cos^q\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q+2}{2}\right)} = \frac{\Gamma\left(\frac{q+1}{2}\right) \sqrt{\pi}}{2\,\Gamma\left(\frac{q+2}{2}\right)}$$

(ii) If $q=0$ then $$\int_0^{\pi/2} \sin^p\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\frac{1}{2}}{\Gamma\left(\frac{p+2}{2}\right)} = \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \sqrt{\pi}}{2\,\Gamma\left(\frac{p+2}{2}\right)}$$

2.) $\beta(m+1, n) + \beta(m, n+1) = \beta(m,n)$

3.) $\dfrac{\beta(m, n+1)}{n} = \dfrac{\beta(m+1, n)}{m} = \dfrac{\beta(m,n)}{m+n}$

4.) $\Gamma n \, \Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

1.) Evaluate $\int_0^1 x^6 (1-x)^9 \, dx$

~~Proof~~ Sol:

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{m-1} \, dx, \quad m, n > 0$$

Comparing $m-1 = 6 \Rightarrow m = 7, \quad n-1 = 9 \Rightarrow n = 10$

$$\therefore \int_0^1 x^6 (1-x)^9 \, dx = \beta(7, 10) = \frac{\Gamma 7 \, \Gamma 10}{\Gamma 17} \quad \left| \because \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \right.$$

$$= \frac{6! \, 9!}{16!}$$

2.) Evaluate $\int_0^{\pi/2} \sin^6 \theta \cos^{10} \theta \, d\theta$

~~Pro~~ Sol:

W.K.T $\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

$$\therefore \int_0^{\pi/2} \sin^6 \theta \cos^{10} \theta \, d\theta = \frac{1}{2} \beta\left(\tfrac{7}{2}, \tfrac{11}{2}\right)$$

$$= \frac{1}{2} \left[\frac{\Gamma(7/2) \, \Gamma(11/2)}{\Gamma 17/2} \right]$$

$$= \frac{1}{2} \frac{\left(\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma \frac{1}{2}\right)\left(\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma \frac{1}{2}\right)}{8!}$$

$$= \frac{225 \times 63 \, \pi}{512 \cdot 8!} = \frac{2835}{4128768} \pi$$

3.) Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

Sol:

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\frac{3}{4} \cdot \Gamma\frac{1}{4}}{\Gamma 1} \qquad \left[\because \Gamma n \Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

4.) Evaluate $\int_0^{\pi/2} \cos^8\theta \, d\theta$

Sol:

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$m=0,\ n=8$

$$= \frac{1}{2}\beta\left(\frac{1}{2}, \frac{9}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\frac{9}{2}\, \Gamma\frac{1}{2}}{\Gamma 5} = \frac{105\pi}{\cancel{4}68}$$

5.) Evaluate $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

Sol:

$$I = \int_0^{\pi/2} (\cos\theta)^{1/2} (\sin\theta)^{-1/2} \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} \frac{\Gamma\frac{3}{4}\, \Gamma\frac{1}{4}}{\Gamma 1} = \frac{1}{2} \cdot \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

6) $\int_0^{\pi/2} \sin^5\theta \, d\theta = \frac{1}{2}\beta\left(\frac{5+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta(3, \tfrac{1}{2})$

$$= \frac{1}{2} \frac{\Gamma 3 \, \Gamma 1/2}{\Gamma 7/2} = \frac{1}{2} \cdot \frac{2! \, \Gamma 1/2}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma 1/2} = \frac{8}{15}$$

7) Prove $\int_0^{\pi/2} \sqrt{\sin\theta} \, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

Proof:

$$\int_0^{\pi/2} \sqrt{\sin\theta} \, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \frac{1}{4}\beta\left(\frac{3}{4}, \frac{1}{2}\right)\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\Gamma 3/4 \, \Gamma 1/2}{\Gamma 5/4} \cdot \frac{\Gamma 1/4 \cdot \Gamma 1/2}{\Gamma 3/4}$$

$$= \frac{1}{4} \frac{\Gamma 1/2 \cdot \Gamma 1/2 \cdot \Gamma 1/4}{\Gamma 5/4} = \frac{1}{4} \frac{\Gamma 1/2 \cdot \Gamma 1/2 \cdot \Gamma 1/4}{1/4 \, \Gamma 1/4} = (\sqrt{\pi})^2 = \pi$$