Linear Differential Equations with Constant Coefficients

An equation of the form

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_n y=F(x),$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants, is called a Linear differential equation of degree n with constant coefficients.

Let $\frac{d}{dx}=D, \frac{d^2}{dx^2}=D^2$, etc. Then the above equation can be written as

$$(a_0D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a_n)y=F(x)$$

$$\text{i.e. } \phi(D)y=F(x) \tag{1}$$

The general or complete solution of (1) consists of two parts namely (i) Complementary Function (C.F.) and the (ii) Particular Integral (P.I).

i.e. $y=$C.F.+P.I.

To find the complementary function

Putting $D=m$ and $F(x)=0$ in (1).

Therefore the auxiliary equation of (1) is $\phi(m)=0$

i.e. $a_0m^n+a_1m^{n-1}+\ldots+a_n=0$.

By solving this equation, we get n roots say $m_1, m_2, m_3, \ldots, m_n$.

Case (i): If all the roots are real and unequal, i.e. if $m_1 \neq m_2 \neq m_3 \neq \ldots \neq m_n$, then

C.F.$=c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+\ldots+c_ne^{m_nx}$

Case (ii): If two roots are equal (i.e. $m_1=m_2=m$) and the remaining be real and unequal, then

C.F.$=(c_1+c_2x)e^{mx}+c_3e^{m_3x}+\ldots+c_ne^{m_nx}$

Case (iii): If three roots are equal (i.e. $m_1=m_2=m_3=m$) and the remaining be real and unequal, then

C.F.$=(c_1+c_2x+c_3x^2)e^{mx}+c_4e^{m_4x}+\ldots+c_ne^{m_nx}$

Case (iv): If all the roots are equal (i.e. $m_1=m_2=m_3=\ldots=m_n=m$) then

C.F.$=(c_1+c_2x+c_3x^2+\ldots+c_nx^{n-1})e^{mx}$

Case (v): If roots are imaginary i.e. if $m=\alpha \pm i\beta$, then

C.F.$=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$

To find the Particular integral (P.I.)

If the RHS is zero, i.e. $F(x) = 0$, then there is no particular integral. In this case the complementary function is the general solution of the given differential equation. On the other hand if $F(x) \neq 0$, then we have P.I. also.

The P.I. is given by P.I.$=\dfrac{1}{\phi(D)}F(x)$

where $F(x)$ is any one of the following form

(1) $F(x) = e^{ax}$

(2) $F(x) = \sin ax$ or $\cos ax$

(3) $F(x) = x^n$, where n is a constant (+ve integer)

(4) $F(x) = e^{ax}f(x)$, where $f(x) = x^n$ or $\sin ax$ or $\cos ax$

Type 1: If $F(x) = e^{ax}$

$$\begin{aligned} \text{P.I.} &= \frac{1}{\phi(D)}e^{ax} \\ &= \frac{1}{\phi(a)}e^{ax} \text{ provided } \phi(a) \neq 0 \end{aligned}$$

If $\phi(a) = 0$ then P.I.$=x.\dfrac{1}{\phi'(D)}e^{ax} = x.\dfrac{1}{\phi'(a)}e^{ax}$, provided $\phi'(a) \neq 0$

If $\phi'(a) = 0$ then P.I.$=x^2.\dfrac{1}{\phi''(D)}e^{ax} = x^2.\dfrac{1}{\phi''(a)}e^{ax}$, provided $\phi''(a) \neq 0$.

This process may be repeated till the denominator becoming non zero when replacing D by a.

Example 1: Solve $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$.

Solution: $(D^2 - 7D + 12)y = 0$

The auxiliary equation is $m^2 - 7m + 12 = 0$

$\Rightarrow (m-3)(m-4) = 0 \Rightarrow m = 3, 4$

The general solution is $y = Ae^{3x} + Be^{4x}$.

Example 2: Solve $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 0$.

Solution: $(D^2 - 6D + 13)y = 0$

The auxiliary equation is $m^2 - 6m + 13 = 0$

$\Rightarrow m = \dfrac{6 \pm \sqrt{36-52}}{2} = \dfrac{6 \pm \sqrt{-16}}{2}$

$\Rightarrow m = 3 \pm 2i$

The general solution is $y = e^{3x}(A\cos 2x + B\sin 2x)$.

Example 3: Solve $(D^2 - 4D + 4)y = 0$

Solution: The auxiliary equation is $m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

The general solution is $y = (A + Bx)e^{2x}$.

Example 4: Solve $(D^2 + 3D + 2)y = e^{5x}$

Solution:

The auxiliary equation is $m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$

The complementary function (C.F.) is $Ae^{-x} + Be^{-2x}$

To find Particular integral (P.I.):

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 + 3D + 2} e^{5x} \\ &= \frac{1}{5^2 + 3(5) + 2} e^{5x} \\ &= \frac{e^{5x}}{42}\end{aligned}$$

The general solution is

$$y = \text{C.F.} + \text{P.I.} = Ae^{-x} + Be^{-2x} + \frac{e^{5x}}{42}.$$

Example 5: Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = e^{-2x}$.

Solution: $(D^2 + 4D + 4)y = 0$

The auxiliary equation is $m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$

The complementary function (C.F.) is $(A + Bx)e^{-2x}$

To find Particular integral (P.I.):

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 + 4D + 4} e^{-2x} \\ &= \frac{1}{(-2)^2 + 4(-2) + 4} e^{-2x} \\ &= \frac{e^{-2x}}{0} = x.\frac{1}{2D + 4} e^{-2x} \\ &= \frac{1}{2(-2) + 4} e^{-2x} \\ &= \frac{e^{-2x}}{0} = \frac{x^2}{2} e^{-2x}\end{aligned}$$

The general solution is

$$y = \text{C.F.} + \text{P.I.} = (A + Bx)e^{-2x} + \frac{x^2 e^{-2x}}{2}.$$

Example 6: Solve $(D^2 + 2D + 1)y = e^{-x} + 3$

Solution:

The auxiliary equation is $m^2 + 2m + 1 = 0$

$\Rightarrow (m + 1)^2 = 0 \Rightarrow m = -1, -1$

The complementary function (C.F.) is $(A + Bx)e^{-x}$

To find Particular integral (P.I.):

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 + 2D + 1}(e^{-x} + 3e^{0x}) \\
&= \frac{1}{D^2 + 2D + 1}e^{-x} + 3.\frac{1}{D^2 + 2D + 1}e^{0x} \\
&= \frac{1}{(-1)^2 + 2(-1) + 1}e^{-x} + 3.\frac{1}{0 + 0 + 1}e^{0x} \\
&= \frac{e^{-x}}{0} + 3 = x.\frac{1}{2D + 2}e^{-x} + 3 \\
&= \frac{e^{-x}}{0} + 3 = \frac{x^2}{2}e^{-x} + 3
\end{aligned}$$

The general solution is

$$y = \text{C.F.} + \text{P.I.} = (A + Bx)e^{-x} + \frac{x^2}{2}e^{-x} + 3.$$

Example 7: Solve $(D^2 + 9)y = e^{-2x}$

Solution:

The auxiliary equation is $m^2 + 9 = 0$

$\Rightarrow m = - \pm 3i$

The complementary function (C.F.) is $A \cos 3x + B \sin 3x$

To find Particular integral (P.I.):

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 + 9}(e^{-2x} \\
&= \frac{1}{(-2)^2 + 9}e^{-2x} \\
&= \frac{1}{13}e^{-2x}
\end{aligned}$$

The general solution is

$$y = \text{C.F.} + \text{P.I.} = A\cos 3x + B\sin 3x + \frac{1}{13}e^{-2x}.$$

Type 2: If $F(x) = \sin ax$ (or) $\sin ax$

$$\begin{aligned}\text{P.I.} &= \frac{1}{\phi(D)}\sin ax \text{ (or) } \cos ax\\ &= \frac{1}{\phi(-a^2)}\sin ax \text{ (or) } \cos ax \text{ provided } \phi(-a^2) \neq 0\end{aligned}$$

If $\phi(D) = 0$ when $D^2 = -a^2$ then P.I.$=x.\dfrac{1}{\phi'(D)}\sin ax$ (or) $\cos ax$

$\Rightarrow$ P.I. $= x.\dfrac{1}{\phi'(-a^2)}\sin ax$ (or) $\cos ax$ provided $\phi'(-a^2) \neq 0$

This process may be repeated till the denominator becoming non zero when replacing D^2 by $-a^2$.

Example 1: Solve $(D^2 + 3D + 2)y = \sin 3x$

Solution:

The auxiliary equation is $m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$

The complementary function (C.F.) is $Ae^{-x} + Be^{-2x}$

To find Particular integral (P.I.):

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 + 3D + 2}\sin 3x\\ &= \frac{1}{-(3)^2 + 3D + 2}\sin 3x\\ &= \frac{1}{3D - 7}\sin 3x\\ &= \frac{1}{(3D-7)}.\frac{(3D+7)}{(3D+7)}\sin 3x\\ &= \frac{(3D+7)}{9D^2 - 49}\sin 3x\\ &= \frac{[3D(\sin 3x) + 7\sin 3x]}{9(-(3^2)) - 49}\\ &= -\frac{1}{130}\left[9\cos 3x + 7\sin 3x\right]\end{aligned}$$

The general solution is

$$y = \text{C.F.} + \text{P.I.} = Ae^{-x} + Be^{-2x} - \frac{1}{130}\left[9\cos 3x + 7\sin 3x\right].$$

Example 2: Solve $(D^2+4)y = \cos 2x$

Solution:

The auxiliary equation is $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

The complementary function (C.F.) is $A\cos 2x + B\sin 2x$

To find Particular integral (P.I.):

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+4}\cos 2x \\
&= \frac{1}{-2^2+4}\cos 2x \\
&= x.\frac{1}{2D}\cos 2x = \frac{x}{4}\sin 2x
\end{aligned}$$

The general solution is

$$y = \text{C.F.} + \text{P.I.} = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x.$$

Example 3: Solve $(D^2+6D+8)y = e^{-2x} + \cos^2 x$

Solution:

The auxiliary equation is $m^2 + 6m + 8 = 0$

$\Rightarrow (m+2)(m+4) = 0 \Rightarrow m = -2, -4$

The complementary function (C.F.) is $Ae^{-2x} + Be^{-4x}$

To find Particular integral (P.I.):

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+6D+8}e^{-2x} + \cos^2 x \\
&= \frac{1}{D^2+6D+8}e^{-2x} + \frac{1}{D^2+6D+8}\left(\frac{1+\cos 2x}{2}\right) \\
&= \frac{1}{D^2+6D+8}e^{-2x} + \frac{1}{2}.\frac{1}{D^2+6D+8}.e^{0x} + \frac{1}{2}.\frac{1}{D^2+6D+8}\cos 2x \\
&= \text{P.I.}_1 + \text{P.I.}_2 + \text{P.I.}_3
\end{aligned}$$

$$\begin{aligned}
\text{P.I.}_1 &= \frac{1}{D^2+6D+8}e^{-2x} \\
&= \frac{1}{4-12+8}e^{-2x} \\
&= x.\frac{1}{2D+6}e^{-2x} \\
&= x.\frac{1}{-4+6}e^{-2x} = \frac{x}{2}e^{-2x}
\end{aligned}$$

$$\begin{aligned}
\text{P.I.}_2 &= \frac{1}{2}.\frac{1}{D^2+6D+8}e^{0x} \\
&= \frac{1}{2}.\frac{1}{8}e^{0x} = \frac{1}{16} \\
\text{P.I.}_3 &= \frac{1}{2}.\frac{1}{D^2+6D+8}\cos 2x \\
&= \frac{1}{2}.\frac{1}{-4+6D+8}\cos 2x \\
&= \frac{1}{2}.\frac{1}{6D+4}\cos 2x = \frac{1}{4}.\frac{1}{3D+2}\cos 2x \\
&= \frac{1}{4}.\frac{(3D-2)}{9D^2-4}\cos 2x \\
&= \frac{1}{4}.\frac{[3D(\cos 2x) - 2\cos 2x]}{-36-4} = \frac{1}{80}\left[3\sin 2x + \cos 2x\right]
\end{aligned}$$

The general solution is

$$y = Ae^{-2x} + Be^{-4x} + \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{1}{80}\left[3\sin 2x + \cos 2x\right].$$

Example 4: Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$

Solution:

The auxiliary equation is $m^2 - 4m + 3 = 0$

$\Rightarrow (m-1)(m-3) = 0 \Rightarrow m = 1, 3$

The complementary function (C.F.) is $Ae^x + Be^{3x}$

To find Particular integral (P.I.):

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2-4D+3}\sin 3x \cos 2x \\
&= \frac{1}{D^2-4D+3}.\frac{1}{2}\left[\sin 5x + \sin x\right] \\
&= \frac{1}{2}.\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{2}.\frac{1}{D^2-4D+3}\sin x \\
&= \text{P.I.}_1 + \text{P.I.}_2 \\
\text{P.I.}_1 &= \frac{1}{2}.\frac{1}{D^2-4D+3}\sin 5x \\
&= \frac{1}{2}.\frac{1}{-25-4D+3}\sin 5x \\
&= -\frac{1}{4}.\frac{1}{2D+11}\sin 5x \\
&= -\frac{1}{4}.\frac{(2D+11)}{4D^2-121}\sin 5x
\end{aligned}$$

$$
\begin{aligned}
\text{P.I.}_1 &= -\frac{1}{4}.\frac{[2D(\sin 5x)+11\sin 5x]}{-100-121}\\
&= \frac{1}{884}\,[10\cos 5x+11\sin 5x]\\
\text{P.I.}_2 &= \frac{1}{2}.\frac{1}{D^2-4D+3}\sin x\\
&= \frac{1}{2}.\frac{1}{-1-4D+3}\sin x\\
&= -\frac{1}{4}.\frac{1}{2D-1}\sin x\\
&= -\frac{1}{4}.\frac{(2D+1)}{4D^2-1}\sin x\\
&= -\frac{1}{4}.\frac{[2D(\sin x)+\sin x]}{-4-1}\\
&= \frac{1}{20}\,[2\cos x+\sin x]
\end{aligned}
$$

The general solution is

$$
y = Ae^{x}+Be^{3x}+\frac{1}{884}\,[10\cos 5x+11\sin 5x]+\frac{1}{20}\,[2\cos x+\sin x]\,.
$$

Formula:

(1) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

(2) $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

(3) $2\cos A\cos B = \cos(A+B)+\cos(A-B)$

(4) $2\sin A\sin B = \cos(A-B)-\cos(A+B)$

(5) $2\cos A\sin B = \sin(A+B)-\sin(A-B)$

(6) $2\sin A\cos B = \sin(A+B)+\sin(A-B)$

Type 3: $F(x)=x^n$ where n is +ve integer

P.I.$=\dfrac{1}{\phi(D)}x^n = \dfrac{1}{[1\pm f(D)]}x^n = [1\pm f(D)]^{-1}x^n$

Expand $[1\pm f(D)]^{-1}$ as a Binomial series.

Formula:.

(1) $(1+x)^{-1} = 1-x+x^2-x^3+\dots$

(2) $(1-x)^{-1} = 1+x+x^2+x^3+\dots$

(3) $(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots$

(4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

Example 1: Solve $(D^2+3D+2)y = x^2$.

Solution:

The auxiliary equation if $m^2+3m+2=0$

$\Rightarrow (m+1)(m+2)=0 \Rightarrow m=-1,-2$

C.F.$=Ae^{-x}+Be^{-2x}$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+3D+2}x^2 \\
&= \frac{1}{2}.\frac{1}{\left[1+\left(\dfrac{D^2+3D}{2}\right)\right]}x^2 \\
&= \frac{1}{2}.\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1}x^2 \\
&= \frac{1}{2}.\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2-\dots\right]x^2 \\
&= \frac{1}{2}.\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{9D^2}{4}\right]x^2 \\
&= \frac{1}{2}.\left[x^2-\frac{D^2(x^2)}{2}-\frac{3D(x^2)}{2}+\frac{9D^2(x^2)}{4}\right] \\
&= \frac{1}{2}.\left[x^2-\frac{1}{2}.2-\frac{3}{2}.2x+\frac{9}{4}.2\right] \\
&= \frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]
\end{aligned}$$

The general solution is

$$y = Ae^{-x}+Be^{-2x}+\frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]$$

Example 2: Solve $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y = x^2+3x-1$.

Solution:

Given $(D^2-5D+6)y = x^2+3x-1$

The auxiliary equation if $m^2+3m+2=0$

$\Rightarrow (m-2)(m-3)=0 \Rightarrow m=2,3$

C.F.$=Ae^{2x}+Be^{3x}$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2-5D+6}(x^2+3x-1)\\
&= \frac{1}{6}.\frac{1}{\left[1+\left(\dfrac{D^2-5D}{6}\right)\right]}(x^2+3x-1)\\
&= \frac{1}{6}.\left[1+\left(\frac{D^2-5D}{6}\right)\right]^{-1}(x^2+3x-1)\\
&= \frac{1}{6}.\left[1-\left(\frac{D^2-5D}{6}\right)+\left(\frac{D^2-5D}{6}\right)^2-\ldots\right](x^2+3x-1)\\
&= \frac{1}{6}.\left[1-\frac{D^2}{6}+\frac{5D}{6}+\frac{25D^2}{36}\right](x^2+3x-1)\\
&= \frac{1}{6}.\left[(x^2+3x-1)-\frac{D^2(x^2+3x-1)}{6}+\frac{5D(x^2+3x-1)}{6}+\frac{25D^2(x^2+3x-1)}{36}\right]\\
&= \frac{1}{6}.\left[x^2+3x-1-\frac{1}{6}.2+\frac{5}{6}.(2x+3)+\frac{25}{36}.2\right]\\
&= \frac{1}{6}\left[x^2+\frac{14x}{3}+\frac{23}{9}\right]
\end{aligned}$$

The general solution is
$y = Ae^{2x}+Be^{3x}+\dfrac{1}{6}\left[x^2+\dfrac{14x}{3}+\dfrac{23}{9}\right]$

Example 3: Solve $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+3y=5x^2$.

Solution:

Given $(D^2+2D+3)y=5x^2$

The auxiliary equation if $m^2+2m+3=0$
$\Rightarrow m = \dfrac{-2\pm\sqrt{4-12}}{2} = -1\pm i\sqrt{2}$
C.F.$=e^{-x}\left[A\cos\sqrt{2}x+B\sin\sqrt{2}x\right]$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+2D+3}.5x^2\\
&= \frac{5}{3}.\frac{1}{\left[1+\left(\dfrac{D^2+2D}{3}\right)\right]}x^2\\
&= \frac{5}{3}.\left[1+\left(\frac{D^2+2D}{3}\right)\right]^{-1}x^2\\
&= \frac{5}{3}.\left[1-\left(\frac{D^2+2D}{3}\right)+\left(\frac{D^2+2D}{3}\right)^2-\ldots\right]x^2
\end{aligned}$$

$$\text{P.I.} = \frac{5}{3}.\left[1 - \frac{D^2}{3} - \frac{2D}{3} + \frac{4D^2}{9}\right]x^2$$
$$= \frac{5}{3}.\left[x^2 - \frac{1}{3}.2 - \frac{2}{3}.2x + \frac{4}{9}.2\right]$$
$$= \frac{5}{3}\left[x^2 - \frac{4x}{3} + \frac{2}{9}\right]$$

The general solution is

$y = e^{-x}\left[A\cos\sqrt{2}x + B\sin\sqrt{2}x\right] + \frac{5}{3}\left[x^2 - \frac{4x}{3} + \frac{2}{9}\right]$

Type 4: If $F(x) = e^{ax}f(x)$ where $f(x) = x^n$ or $\sin ax$ or $\cos ax$ then

P.I.$=\frac{1}{\phi(D)}e^{ax}f(x) = e^{ax}\frac{1}{\phi(D+a)}f(x)$

Now $\frac{1}{\phi(D+a)}f(x)$ will be in any one of the previous forms.

Example 1: Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-x}\sin 2x$.

Solution: Given $(D^2 + 4D + 4)y = e^{-x}\sin 2x$

The auxiliary equation if $m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$

C.F.$=(A + Bx)e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2 + 4D + 4}e^{-x}\sin 2x$$
$$= e^{-x}\frac{1}{(D-1)^2 + 4(D-1) + 4}\sin 2x$$
$$= e^{-x}\frac{1}{D^2 + 2D + 1}\sin 2x$$
$$= e^{-x}\frac{1}{-4 + 2D + 1}\sin 2x$$
$$= e^{-x}\frac{(2D+3)}{4D^2 - 9}\sin 2x$$
$$= e^{-x}.\frac{[2D(\sin 2x) + 3\sin 2x]}{4(-4) - 9}$$
$$= -\frac{e^{-x}}{25}\left[4\cos 2x + 3\sin 2x\right]$$

The general solution is

$y = (A + Bx)e^{-2x} - \frac{e^{-x}}{25}\left[4\cos 2x + 3\sin 2x\right]$

Example 2: Solve: $(D^2 + D + 1)y = x^2e^{-x}$

Solution: The auxiliary equation if $m^2 + m + 1 = 0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

C.F.=$e^{-x/2}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right]$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+D+1}x^2e^{-x} \\
&= e^{-x}\frac{1}{(D-1)^2+D-1+1}x^2 \\
&= e^{-x}\frac{1}{D^2-D+1}x^2 \\
&= e^{-x}\left[1+(D^2-D)\right]^{-1}x^2 \\
&= e^{-x}\left[1-(D^2-D)+(D^2-D)^2-\dots\right]x^2 \\
&= e^{-x}\left[1-D^2+D+D^2\right]x^2 = e^{-x}(1+D)x^2 \\
&= (x^2+2x)e^{-x}
\end{aligned}$$

The general solution is

$y = e^{-x/2}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right] + (x^2+2x)e^{-x}$

Example 3: Solve: $(D^2+9)y = (x^2+1)e^{3x}$

The auxiliary equation is $m^2+9=0$

$\Rightarrow m = \pm i3$

C.F.=$A\cos 3x + B\sin 3x$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+9}(x^2+1)e^{3x} \\
&= e^{3x}\frac{1}{(D+3)^2+9}(x^2+1) \\
&= \frac{e^{3x}}{18}\frac{1}{1+\left(\frac{D^2+6D}{18}\right)}(x^2+1) \\
&= \frac{e^{3x}}{18}\left[1+\left(\frac{D^2+6D}{18}\right)\right]^{-1}(x^2+1) \\
&= \frac{e^{3x}}{18}\left[1-\left(\frac{D^2+6D}{18}\right)+\left(\frac{D^2+6D}{18}\right)^2+\dots\right](x^2+1) \\
&= \frac{e^{3x}}{18}\left[1-\frac{D^2}{18}-\frac{6D}{18}+\frac{36D^2}{324}\right](x^2+1)
\end{aligned}$$

$$\text{P.I.} = \frac{e^{3x}}{18}\left[x^2+1-\frac{D^2(x^2+1)}{18}-\frac{6D(x^2+1)}{18}+\frac{36D^2(x^2+1)}{324}\right]$$
$$= \frac{e^{3x}}{18}\left[x^2+1-\frac{1}{18}.2-\frac{6}{18}.2x+\frac{36}{3234}.2\right]$$
$$= \frac{e^{3x}}{18}\left(x^2-\frac{2x}{3}+\frac{10}{9}\right)$$

The complete solution is

$y = A\cos 3x + B\sin 3x + \dfrac{e^{3x}}{18}\left(x^2-\dfrac{2x}{3}+\dfrac{10}{9}\right)$

Example 4: Solve $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y = e^{-2x}+e^{3x}\sin x$

Solution: Given $(D^2+4D+4)y = e^{-2x}+e^{3x}\sin x$

The auxiliary equation is $m^2+4m+4=0$

$\Rightarrow (m+2)^2 = 0 \Leftarrow m = -2, -2$

C.F.=$(A+Bx)e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2+4D+4}e^{-2x}+e^{3x}\sin x$$
$$= \frac{1}{D^2+4D+4}e^{-2x}+\frac{1}{D^2+4D+4}e^{3x}\sin x$$
$$= \frac{1}{4-8+4}e^{-2x}+e^{3x}\frac{1}{(D+3)^2+4(D+3)+4}\sin x$$
$$= x.\frac{1}{2D+4}e^{-2x}+e^{3x}\frac{1}{D^2+10D+25}\sin x$$
$$= x.\frac{1}{-4+4}e^{-2x}+e^{3x}\frac{1}{-1+10D+25}\sin x$$
$$= x^2.\frac{1}{2}e^{-2x}+\frac{e^{3x}}{2}.\frac{(5D-12)}{25D^2-144}\sin x$$
$$= \frac{x^2}{2}e^{-2x}-\frac{e^{3x}}{338}(5\cos x-12\sin x).$$

The complete solution is

$y = (A+Bx)e^{-2x}+\dfrac{x^2}{2}e^{-2x}-\dfrac{e^{3x}}{338}(5\cos x-12\sin x).$

Formula:

(1) $\sinh x = \dfrac{e^x-e^{-x}}{2}$

(2) $\cosh x = \dfrac{e^x+e^{-x}}{2}$

Example 1: Solve: $(D^2+4)y = \sinh 2x.$

Solution: The auxiliary equation is $m^2+4=0$

$\Rightarrow m = \pm 2i$

C.F.$=A\cos 2x + B\sin 2x$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+4}\sinh 2x \\
&= \frac{1}{D^2+4}\left(\frac{e^{2x}-e^{-2x}}{2}\right) \\
&= \frac{1}{2}\left[\frac{1}{D^2+4}e^{2x} - \frac{1}{D^2+4}e^{-2x}\right] \\
&= \frac{1}{2}\left[\frac{1}{8}e^{2x} - \frac{1}{8}e^{-2x}\right] \\
&= \frac{1}{8}\left(\frac{e^{2x}-e^{-2x}}{2}\right) = \frac{1}{8}\sinh 2x
\end{aligned}$$

The general solution is

$y = A\cos 2x + B\sin 2x + \dfrac{1}{8}\sinh 2x$

Example 2: Solve: $(D^2+1)y = x\sin hx$

Solution: The auxiliary equation is $m^2+1=0$

$\Rightarrow m = \pm i$

C.F.$=A\cos x + B\sin x$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+1}x\sin hx = \frac{1}{D^2+1}x\left(\frac{e^{x}-e^{-x}}{2}\right) \\
&= \frac{1}{2}\left[\frac{1}{D^2+1}xe^{x} - \frac{1}{D^2+1}xe^{-x}\right] \\
&= \frac{1}{2}\left[e^{x}\frac{1}{(D+1)^2+1}x - e^{-x}\frac{1}{(D-1)^2+1}x\right] \\
&= \frac{1}{2}\left[e^{x}\frac{1}{D^2+2D+2}x - e^{-x}\frac{1}{D^2-2D+2}x\right] \\
&= \frac{e^{x}}{4}.\frac{1}{1+\left(\frac{D^2+2D}{2}\right)}x - \frac{e^{-x}}{4}.\frac{1}{1+\left(\frac{D^2-2D}{2}\right)}x \\
&= \frac{e^{x}}{4}.\left[1+\left(\frac{D^2+2D}{2}\right)\right]^{-1}x - \frac{e^{-x}}{4}.\left[1+\left(\frac{D^2-2D}{2}\right)\right]^{-1}x \\
&= \frac{e^{x}}{4}\left(1-\frac{2D}{2}\right)x - \frac{e^{-x}}{4}\left(1+\frac{2D}{2}\right)x \\
&= \frac{e^{x}}{4}(x-1) - \frac{e^{-x}}{4}(x+1) \\
&= \frac{x}{2}\left[\frac{e^{x}-e^{-x}}{2}\right] - \frac{1}{2}\left[\frac{e^{x}+e^{-x}}{2}\right] = \frac{1}{2}\left[x\sinh x - \cos hx\right]
\end{aligned}$$

The general solution is

$y = A\cos x + B\sin x + \dfrac{1}{2}\left[x\sinh x - \cos hx\right]$

Type 5: $F(x) = x^n \sin ax$ **or** $x^n \cos ax$

P.I.$= \dfrac{1}{\phi(D)} x^n \sin ax$ or $x^n \cos ax$

$=$Imaginary part of $e^{iax}.\dfrac{1}{\phi(D+ia)}x^n$

or Real part of $e^{iax}.\dfrac{1}{\phi(D+ia)}x^n$

Example 1: Solve: $\dfrac{dy^2}{dx^2} - 2\dfrac{dy}{dx} + y = x\sin x.$

Solution: Given $(D^2 - 2D + 1)y = x\sin x$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

C.F.$=(A + Bx)e^x$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - 2D + 1} x \sin x \\
&= \text{Imp. part of } \frac{1}{D^2 - 2D + 1} x e^{ix} \\
&= \text{Imp. part of } e^{ix}\ \frac{1}{(D+i)^2 - 2(D+i) + 1} x \\
&= \text{Imp. part of } e^{ix}\ \frac{1}{D^2 + 2D(i-1) - 2i} x \\
&= \text{Imp. part of } \frac{-e^{ix}}{2i}\ \frac{1}{\left[1 - \frac{1}{2i}\left(D^2 + 2D(i-1)\right)\right]} x \\
&= \text{Imp. part of } \frac{-e^{ix}}{2i}\ \left[1 - \frac{1}{2i}\left(D^2 + 2D(i-1)\right)\right]^{-1} x \\
&= \text{Imp. part of } \frac{-e^{ix}}{2i}\ \left[1 + \frac{D(i-1)}{i}\right] x \\
&= \text{Imp. part of } \frac{-e^{ix}}{2i}\ \left(x + \frac{(i-1)}{i}\right) \\
&= \text{Imp. part of } \frac{e^{ix}}{2}(xi + i - 1) \\
&= \text{Imp. part of } \frac{1}{2}(\cos x + i\sin x)\left[i(x+1) - 1\right] \\
&= \frac{1}{2}(x+1)\cos x - \frac{1}{2}\sin x
\end{aligned}$$

The general solution is

$y = (A + Bx)e^x + \frac{1}{2}(x+1)\cos x - \frac{1}{2}\sin x$

Example 2: Solve: $(D^2 - 1)y = x^2 \cos x$

Solution: The auxiliary equation is $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F.$=Ae^{-x} + Be^x$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2-1}x^2\cos x \\
&= \text{Real part of } \frac{1}{D^2-1}x^2 e^{ix} \\
&= \text{Real part of } e^{ix}\ \frac{1}{(D+i)^2-1}x \\
&= \text{Real part of } e^{ix}\ \frac{1}{D^2+2Di-2}x \\
&= \text{Real part of } \frac{-e^{ix}}{2}\ \frac{1}{\left[1-\left(\frac{D^2+2Di}{2}\right)\right]}x^2 \\
&= \text{Real part of } \frac{-e^{ix}}{2}\ \left[1-\left(\frac{D^2+2Di}{2}\right)\right]^{-1}x^2 \\
&= \text{Real part of } \frac{-e^{ix}}{2}\ \left[1+\left(\frac{D^2+2Di}{2}\right)+\left(\frac{D^2+2Di}{2}\right)^2+\ldots\right]x^2 \\
&= \text{Real part of } \frac{-e^{ix}}{2}\ \left[1+\frac{D^2}{2}+\frac{2Di}{2}-\frac{4D^2}{4}\right]x^2 \\
&= \text{Real part of } \frac{-e^{ix}}{2}\left[x^2+\frac{D^2(x^2)}{2}+\frac{2iD(x^2)}{2}-\frac{4D^2(x^2)}{4}\right] \\
&= \text{Real part of } \frac{-e^{ix}}{2}\left[x^2+2xi-1\right] \\
&= \text{Real part of } \frac{-1}{2}(\cos x + i\sin x)\left[(x^2-1)+2xi\right] \\
&= -\frac{1}{2}(x^2-1)\cos x + x\sin x
\end{aligned}$$

The general solution is

$y = Ae^{-x} + Be^x - \frac{1}{2}(x^2-1)\cos x + x\sin x$

Example 3: Solve: $(D^2 - 2D + 1)y = xe^x \sin x$

Solution: The auxiliary equation is $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

C.F.$=(A+Bx)e^x$

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{D^2-2D+1} x e^x \sin x \\
&= e^x \frac{1}{(D+1)^2-2(D+1)+1} x \sin x \\
&= e^x \frac{1}{D^2} x \sin x \\
&= e^x \text{ Imp. part of } \frac{1}{D^2} x e^{ix} \\
&= e^x \text{ Imp. part of } e^{ix} \ \frac{1}{(D+i)^2} x \\
&= e^x \text{ Imp. part of } e^{ix} \ \frac{1}{D^2+2Di-1} x \\
&= e^x \text{ Imp. part of } \frac{e^{ix}}{-1} \ \frac{1}{[1-(D^2+2Di)]} x \\
&= e^x \text{ Imp. part of } \frac{e^{ix}}{-1} \ \left[1-(D^2+2Di)\right]^{-1} x \\
&= e^x \text{ Imp. part of } \frac{e^{ix}}{-1} (1+2Di) x \\
&= -e^x \text{ Imp. part of } (\cos x + i \sin x)(x+2i) \\
&= -2e^x \cos x - x e^x \sin x
\end{aligned}
$$

The general solution is

$y=(A+Bx)e^x-2e^x\cos x-xe^x\sin x$

Linear Differential Equations with Variable Coefficients

1. Cauchy's Homogeneous Linear Equations (Euler Type)

2. Homogeneous Equations of Legendre's Type

1. Cauchy's Homogeneous Linear Equations (Euler Type)

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = F(x) \tag{1}$$

where $a_1, a_2, \ldots a_n$ are constants and $F(x)$ is a function of x is called Cauchy's (Euler's) homogeneous linear differential equation.

Equation (1) can be transformed to a linear differential equation with constant coefficients by the transformation

$$x = e^z \text{ or } z = \log x \text{ and } \frac{dz}{dx} = \frac{1}{x}$$

Now $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$

Hence

$$xDy = D^{'}y \text{ where } D = \frac{d}{dx},\ D^{'} = \frac{d}{dz} \tag{2}$$

Also

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) \\
&= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{1}{x^2}\frac{dy}{dz} \\
&= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz} \\
&= \frac{1}{x}\frac{d^2y}{dz^2}\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz} \\
\Rightarrow x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{dz^2} - \frac{dy}{dz} \\
\Rightarrow x^2D^2y &= D^{'^2}y - D^{'}y
\end{aligned}$$

Hence

$$x^2D^2y = D^{'}(D^{'}-1)y \tag{3}$$

Similarly

$$x^3D^3y = D^{'}(D^{'}-1)(D^{'}-2)y \tag{4}$$

Substituting (2), (3), (4) and so on in (1), we get a linear differential equation with constant coefficients and can be solved by any one of the known method.

Example 1: Solve: $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 4\sin(\log x)$

Solution: Given

$$(x^2D^2 + xD + 1)y = 4\sin(\log x) \tag{5}$$

Let $x = e^z$ or $z = \log x$

sothat $xD = D^{'}$, $x^2D^2 = D^{'}(D^{'}-1)$ where $D = \dfrac{d}{dx}$, $D^{'} = \dfrac{d}{dz}$

Now equation (5) becomes

$$[D^{'}(D^{'}-1) + D^{'} + 1]y = 4\sin z$$

$$\Rightarrow (D^{'^2} + 1)y = 4\sin z$$

The auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

C.F.$=A\cos z + B\sin z$

$$\begin{aligned}\text{P.I.} &= \frac{1}{D'^2+1}4\sin z\\ &= 4\frac{1}{-1+1}\sin z\\ &= 4z\frac{1}{2D'}\sin z\\ &= 2z(-\cos z)\end{aligned}$$

The complete solution is

$y = A\cos z + B\sin z - 2z\cos z$

$\quad = A\cos\log x + B\sin\log x - 2\log x\cos\log x$

Example 2: Solve: $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 3y = x^2\log x$

Solution: Given

$$(x^2D^2 + xD - 3)y = x^2\log x \tag{6}$$

Let $x = e^z$ or $z = \log x$

sothat $xD = D^{'}$, $x^2D^2 = D^{'}(D^{'}-1)$ where $D = \dfrac{d}{dx}$, $D^{'} = \dfrac{d}{dz}$

Now equation (6) becomes

$$[D^{'}(D^{'}-1)+D^{'}-3]y = e^{2z}.z$$

$$\Rightarrow (D^{'^2}-3)y = ze^{2z}$$

The auxiliary equation is $m^2-3=0$

$\Rightarrow m = \pm\sqrt{3}$

C.F.$=Ae^{\sqrt{3}z}+Be^{-\sqrt{3}z}$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D'^2-3}ze^{2z}\\
&= e^{2z}\frac{1}{(D'+2)^2-3}z\\
&= e^{2z}\frac{1}{D'^2+4D'+1}z\\
&= e^{2z}\left[1+(D^{'^2}+4D^{'})\right]^{-1}z\\
&= e^{2z}\left[1-(D^{'^2}+4D^{'})+\ldots\right]z\\
&= e^{2z}(1-4D^{'})z = e^{2z}(z-4)
\end{aligned}$$

The complete solution is

$y = Ae^{\sqrt{3}z}+Be^{-\sqrt{3}z}+e^{2z}(z-4)$

$\quad = Ax^{\sqrt{3}}+Bx^{-\sqrt{3}}+(\log x-4)x^2$

Example 3: Solve: $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}+y=\dfrac{12\log x}{x^2}$

Solution:

Given $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}+y=\dfrac{12\log x}{x^2}$

Multiplying throughout by x^2, we get

$x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=12\log x$

$$\Rightarrow (x^2D^2+xD)y = 12\log x \tag{7}$$

Let $x=e^z$ or $z=\log x$

sothat $xD=D^{'}$, $x^2D^2=D^{'}(D^{'}-1)$ where $D=\dfrac{d}{dx}$, $D^{'}=\dfrac{d}{dz}$

Now equation (7) becomes

$$[D^{'}(D^{'}-1)+D^{'}]y = 12z$$

$$\Rightarrow D^{'^2}y = 12z$$

The auxiliary equation is $m^2 = 0$

$\Rightarrow m = 0, 0$

C.F.$=A + Bz$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D'^2} 12z \\ &= 12 \frac{1}{D'} \frac{z^2}{2} \\ &= 12 \frac{z^3}{6} = 2z^3 \end{aligned}$$

The complete solution is

$y = A + Bz + 2z^3$

$\quad = A + B \log x + 2(\log x)^3$

Example 4: Solve: $x^2 y'' - xy' + y = 0$

Solution: Given

$$(x^2 D^2 - xD + 1)y = 0 \tag{8}$$

Let $x = e^z$ or $z = \log x$

sothat $xD = D'$, $x^2 D^2 = D'(D' - 1)$ where $D = \frac{d}{dx}$, $D' = \frac{d}{dz}$

Now equation (8) becomes

$$[D'(D' - 1) - D' + 1]y = 0$$

$$\Rightarrow (D'^2 - 2D' + 1)y = 0$$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$\Rightarrow (m - 1)^2 = 0 \Rightarrow m = 1, 1$

C.F.$=(A + Bz)e^z$ The complete solution is

$y = (A + Bz)e^z$

$\quad = (A + B \log x)x$

2. Homogeneous Equations of Legendre's Type

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + p_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1} y} + \ldots + p_n y = F(x) \tag{9}$$

where $p_1, p_2, \ldots p_n$ are constants, is known as Legendre linear differential equation.

Equation (9) can be reduced to the linear differential equation with constant coefficients by putting $ax + b = e^z$ or $z = \log(ax + b)$ so that $\frac{dz}{dx} = \frac{a}{ax + b}$

Now $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{dy}{dz}.\frac{a}{ax+b}$

$\Rightarrow (ax+b)\frac{dy}{dx} = a\frac{dy}{dz}$

$\Rightarrow (ax+b)Dy = aD^{'}y$ where $D = \frac{d}{dx}, D^{'} = \frac{d}{dz}$

Similarly $(ax+b)^2D^2y = a^2D^{'}(D^{'}-1)y, (ax+b)^3D^3y = a^3D^{'}(D^{'}-1)(D^{'}-2)y$ and so on.

Substituting these in (9), we get a linear differential equation with constant coefficients which can be solved by one of the known methods.

Example 1: Solve: $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\left[\log(1+x)\right]$

Solution: Given

$$\left[(1+x)^2D^2 + (1+x)D + 1\right]y = 4\cos\left[\log(1+x)\right] \tag{10}$$

Let $1+x = e^z$ or $z = \log(1+x)$

sothat $(1+x)D = D^{'}, (1+x)^2D^2 = D^{'}(D^{'}-1)$ where $D = \frac{d}{dx}, D^{'} = \frac{d}{dz}$

Now equation (10) becomes

$$\left[D^{'}(D^{'}-1) + D^{'} + 1\right]y = 4\cos z$$
$$\Rightarrow (D^{'2}+1)y = 4\cos z$$

The auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

C.F.$=A\cos z + B\sin z$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^{'2}+1}4\cos z \\ &= 4\frac{1}{-1+1}\cos z \\ &= 4z\frac{1}{2D^{'}}\cos z \\ &= 2z\sin z \end{aligned}$$

The complete solution is

$y = A\cos z + B\sin z + 2z\sin z$

$=A\cos\log(1+x) + B\sin\log(1+x) + 2\log(1+x)\sin\log(1+x)$

Example 2: Solve: $(2x+5)^2\dfrac{d^2y}{dx^2}-6(2x+5)\dfrac{dy}{dx}+8y=0$

Solution: Given

$$\left[(2x+5)^2D^2-6(2x+5)D+8\right]y=0 \tag{11}$$

Let $2x+5=e^z$ or $z=\log(2x+5)$

sothat $(2x+5)D=2D^{'}, (2x+5)^2D^2=2^2D^{'}(D^{'}-1)$ where $D=\dfrac{d}{dx}, D^{'}=\dfrac{d}{dz}$

Now equation (11) becomes

$$\left[4D^{'}(D^{'}-1)-12D^{'}+8\right]y=0$$
$$\Rightarrow (4D^{'2}-16D^{'}+8)y=0$$

The auxiliary equation is $4m^2-16m+8=0$

$\Rightarrow m^2-4m+2=0$

$\Rightarrow m=\dfrac{4\pm\sqrt{16-8}}{2}=2\pm\sqrt{2}$

C.F.$=Ae^{(2+\sqrt{2})z}+Be^{(2-\sqrt{2})z}$

The complete solution is

$y=A(2x+5)^{(2+\sqrt{2})}+B(2x+5)^{(2-\sqrt{2})}$

Example 3: Solve: $(3x+2)^2\dfrac{d^2y}{dx^2}+3(3x+2)\dfrac{dy}{dx}-36y=3x^2+4x+1$

Solution: Given

$$\left[(3x+2)^2D^2+3(3x+2)D-36\right]y=3x^2+4x+1 \tag{12}$$

Let $3x+2=e^z$ or $z=\log(3x+2)$

sothat $(3x+2)D=3D^{'}, (3x+2)^2D^2=3^2D^{'}(D^{'}-1)$ where $D=\dfrac{d}{dx}, D^{'}=\dfrac{d}{dz}$

Now equation (12) becomes

$$\left[9D^{'}(D^{'}-1)+9D^{'}-36\right]y=3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1$$
$$\Rightarrow (D^{'2}-4)y=\frac{1}{27}(e^{2z}-1)$$

The auxiliary equation is $m^2-4=0$

$\Rightarrow m=\pm 2$

C.F.=$Ae^{2z} + Be^{-2z}$

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{D'^2 - 4} . \frac{1}{27}(e^{2z} - 1) \\
&= \frac{1}{27}\left[\frac{1}{D'^2 - 4}e^{2z} - \frac{1}{D'^2 - 4}e^{0z}\right] \\
&= \frac{1}{27}\left[\frac{1}{4 - 4}e^{2z} + \frac{1}{4}\right] \\
&= \frac{1}{27}\left[z\frac{1}{2D'}e^{2z} + \frac{1}{4}\right] \\
&= \frac{1}{27}\left[\frac{ze^{2z}}{4} + \frac{1}{4}\right] = \frac{1}{108}(ze^{2z} + 1)
\end{aligned}
$$

The complete solution is

$y = Ae^{2z} + Be^{-2z} + \dfrac{1}{108}(ze^{2z} + 1)$

$\quad = A(3x + 2)^2 + B(3x + 2)^{-2} + \dfrac{1}{108}\left[\log(3x + 2)(3x + 2)^2 + 1\right]$

Method of variation of parameters

This method is very useful for finding the particular integral of a second order linear differential equation whose complementary function is known.

Consider the equation

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = F(x) \tag{1}$$

where a_1, a_2 are constants, $F(x)$ is a function of x. Let the complementary function of (1) is

$$\text{C.F.} = c_1 f_1 + c_2 f_2$$

where c_1, c_2 are constants and f_1, f_2 are functions of x. Then

$$\text{P.I.} = P f_1 + Q f_2 \tag{2}$$

where

$$P = -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} F(x)\, dx$$

and

$$Q = \int \frac{f_1}{f_1 f_2' - f_2 f_1'} F(x)\, dx$$

Substituting P and Q in (2), we get th P.I.

Hence the complete solution is $y = C.F. + P.I.$

Example 1: Solve: $\dfrac{d^2y}{dx^2} + y = \sec x$ by the method of variation of parameters.

Solution: Given $(D^2 + 1)y = \sec x$

The auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

C.F.$= c_1 \cos x + c_2 \sin x = c_1 f_1 + c_2 f_2$

Here $f_1 = \cos x, f_2 = \sin x$ sothat $f_1' = -\sin x, f_2' = \cos x$

$\Rightarrow f_1 f_2' - f_2 f_1' = \cos^2 x + \sin^2 x = 1$

Let $PI = P f_1 + Q f_2 = P \cos x + Q \sin x$ where

$$\begin{aligned} P &= -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} F(x)\, dx \\ &= -\int \frac{\sin x}{1} \sec x\, dx \end{aligned}$$

$$
\begin{aligned}
P &= -\int \tan x \, dx \\
&= -\int \frac{\sin x}{\cos x} \, dx \\
&= \log(\cos x)
\end{aligned}
$$

and

$$
\begin{aligned}
Q &= \int \frac{f_1}{f_1 f_2' - f_2 f_1'} F(x) \, dx \\
&= \int \frac{\cos x}{1} \sec x \, dx \\
&= \int dx \\
&= x
\end{aligned}
$$

$\Rightarrow$ P.I. $= \cos x \log(\cos x) + x \sin x$

Hence the complete solution is $y = c_1 \cos x + c_2 \sin x + \cos x \log(\cos x) + x \sin x$

Example 2: Solve: $\dfrac{d^2y}{dx^2} + 4y = 4 \tan 2x$ by the method of variation of parameters.

Solution: Given $(D^2 + 4)y = 4 \tan 2x$

The auxiliary equation is $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

C.F.$= c_1 \cos 2x + c_2 \sin 2x = c_1 f_1 + c_2 f_2$

Here $f_1 = \cos 2x, f_2 = \sin 2x$ sothat $f_1' = -2 \sin 2x, f_2' = 2 \cos 2x$

$\Rightarrow f_1 f_2' - f_2 f_1' = 2 \cos^2 2x + 2 \sin^2 2x = 2$

Let $PI = P f_1 + Q f_2 = P \cos 2x + Q \sin 2x$ where

$$
\begin{aligned}
P &= -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} F(x) \, dx \\
&= -\int \frac{\sin 2x}{2} 4 \tan 2x \, dx \\
&= -2 \int \frac{\sin^2 2x}{\cos 2x} \, dx \\
&= -2 \int \frac{(1 - \cos^2 x)}{\cos 2x} \, dx \\
&= -2 \int \sec 2x \, dx + 2 \int \cos 2x \, dx \\
&= -2 . \frac{1}{2} \log(\sec 2x + \tan 2x) + 2 . \frac{\sin 2x}{2}
\end{aligned}
$$

and

$$\begin{aligned}
Q &= \int \frac{f_1}{f_1 f_2' - f_2 f_1'} F(x)\, dx \\
&= \int \frac{\cos 2x}{2} 4 \tan 2x\, dx \\
&= 2 \int \sin 2x\, dx = -\cos 2x
\end{aligned}$$

$\Rightarrow$ P.I. $= -\cos 2x \log(\sec 2x + \tan 2x) + \sin 2x \cos 2x - \sin 2x \cos 2x$

$= -\cos 2x \log(\sec 2x + \tan 2x)$

Hence the complete solution is $y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$

Example 3: Solve: $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = e^{-x} \tan x$ by the method of variation of parameters.

Solution: Given $(D^2 + 2D + 5)y = e^{-x} \tan x$

The auxiliary equation is $m^2 + 2m + 5 = 0$

$\Rightarrow m = \dfrac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$

C.F.$=e^{-x} [c_1 \cos 2x + c_2 \sin 2x] = c_1 f_1 + c_2 f_2$

Here $f_1 = e^{-x} \cos 2x, f_2 = e^{-x} \sin 2x$

sothat $f_1' = -2e^{-x} \sin 2x - e^{-x} \cos 2x, f_2' = 2e^{-x} \cos 2x - e^{-x} \sin 2x$

$\Rightarrow f_1 f_2' - f_2 f_1' = e^{-x} \cos 2x \left(2e^{-x} \cos 2x - e^{-x} \sin 2x\right) + e^{-x} \sin 2x \left(2e^{-x} \sin 2x + e^{-x} \cos 2x\right)$

$= 2e^{-x}$

Let $PI = Pf_1 + Qf_2 = Pe^{-x} \cos 2x + Qe^{-x} \sin 2x$ where

$$\begin{aligned}
P &= -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} F(x)\, dx \\
&= -\int \frac{e^{-x} \sin 2x}{2e^{-2x}} e^{-x} \tan x\, dx \\
&= -\frac{1}{2} \int \sin 2x \tan x\, dx \\
&= -\frac{1}{2} \int 2 \sin x \cos x \left(\frac{\sin x}{\cos x}\right) dx \\
&= -\int \sin^2 x\, dx \\
&= -\int \left(\frac{1 - \cos 2x}{2}\right) dx \\
&= -\frac{x}{2} + \frac{\sin 2x}{4}
\end{aligned}$$

and

$$\begin{aligned}
Q &= \int \frac{f_1}{f_1 f_2' - f_2 f_1'} F(x)\, dx \\
&= \int \frac{e^{-x}\cos 2x}{2e^{-2x}} e^{-x} \tan x \, dx \\
&= \frac{1}{2}\int (2\cos^2 x - 1)\left(\frac{\sin x}{\cos x}\right) dx \\
&= \frac{1}{2}\int (2\cos x \sin x - \tan x)\, dx \\
&= \frac{1}{2}\int (\sin 2x - \tan x)\, dx \\
&= -\frac{\cos 2x}{4} + \frac{1}{2}\log(\cos x)
\end{aligned}$$

$\Rightarrow$ P.I. $= \left(-\frac{x}{2} + \frac{\sin 2x}{4}\right) e^{-x}\cos 2x + \left(-\frac{\cos 2x}{4} + \frac{1}{2}\log(\cos x)\right) e^{-x} \sin 2x$

Hence the complete solution is

$y = e^{-x}\left[c_1 \cos 2x + c_2 \sin 2x\right] + \left(-\frac{x}{2} + \frac{\sin 2x}{4}\right) e^{-x}\cos 2x + \left(-\frac{\cos 2x}{4} + \frac{1}{2}\log(\cos x)\right) e^{-x}\sin 2x$

Example 4: Solve: $\frac{d^2y}{dx^2} + y = \csc x$ by the method of variation of parameters.

Solution: Given $(D^2+1)y = \csc x$

The auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

C.F.$= c_1 \cos x + c_2 \sin x = c_1 f_1 + c_2 f_2$

Here $f_1 = \cos x, f_2 = \sin x$

sothat $f_1' = -\sin x, f_2' = \cos x$

$\Rightarrow f_1 f_2' - f_2 f_1' = \cos^2 x + \sin^2 x = 1$

Let $PI = Pf_1 + Qf_2 = P\cos x + Q \sin x$ where

$$\begin{aligned}
P &= -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} F(x)\, dx \\
&= -\int \frac{\sin x}{1} \csc x \, dx \\
&= -\int dx = -x
\end{aligned}$$

and

$$\begin{aligned} Q &= \int \frac{f_1}{f_1 f_2' - f_2 f_1'} F(x)\,dx \\ &= \int \frac{\cos x}{1} \csc x\,dx \\ &= \int \frac{\cos x}{\sin x}\,dx \\ &= \int \cot x\,dx = \log(\sin x) \end{aligned}$$

$\Rightarrow$ P.I. $= -x\cos x + \sin x \log(\sin x)$

Hence the complete solution is $y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log(\sin x)$

Example 5: Solve: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \log x$ by the method of variation of parameters.

Solution: Given $(D^2 - 2D + 1)y = e^x \log x$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

C.F.$=(c_1 + c_2 x)e^x = c_1 f_1 + c_2 f_2$

Here $f_1 = e^x, f_2 = xe^x$

sothat $f_1' = e^x, f_2' = e^x + xe^x$

$\Rightarrow f_1 f_2' - f_2 f_1' = e^x(e^x + xe^x) - xe^x e^x = e^{2x}$

Let $PI = Pf_1 + Qf_2 = Pe^x + Qxe^x$ where

$$\begin{aligned} P &= -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} F(x)\,dx \\ &= -\int \frac{xe^x}{e^{2x}} e^x \log x\,dx \\ &= -\int x \log x\,dx \\ &= -\left[\log x . \frac{x^2}{2} - \int \frac{x^2}{2} . \frac{1}{x}\,dx\right] \text{ since } u = \log x, \;\; dv = x\,dx \\ &= -\frac{x^2}{2} \log x + \frac{x^2}{4} \end{aligned}$$

and

$$\begin{aligned} Q &= \int \frac{f_1}{f_1 f_2' - f_2 f_1'} F(x)\,dx \\ &= \int \frac{e^x}{e^{2x}} e^x \log x\,dx \\ &= x(\log x - 1) \text{ since } u = \log x, \;\; dv = \, dx \end{aligned}$$

$$\Rightarrow \text{P.I.} = \left(-\frac{x^2}{2}\log x + \frac{x^2}{4}\right)e^x + x(\log x - 1).xe^x = \frac{1}{2}x^2e^x\left(\log x - \frac{3}{2}\right)$$

Hence the complete solution is $y = (c_1 + c_2x)e^x + \frac{1}{2}x^2e^x\left(\log x - \frac{3}{2}\right)$

Self Practice

1. $\frac{d^2y}{dx^2} + y = x$
2. $(D^2 + 1)y = x\sin x$
3. $(D^2 + 3D + 2)y = x^2$
4. $y^{''} + y = \tan x$

Simultaneous Linear Differential Equations with Constant Co-efficients

Here we discuss differential eqns in which there is one independent variable and two or more dependent variables. such eqns are termed as simultaneous eqns.

1.) Solve: $\frac{dx}{dt}+2y=\sin 2t$, $\frac{dy}{dt}-2x=\cos 2t$.

Sol:

$Dx+2y=\sin 2t \longrightarrow (1)$

$Dy-2x=\cos 2t \longrightarrow (2)$ where $D=\frac{d}{dt}$

$(1)\times D \Rightarrow D^2x+2Dy=2\cos 2t$ $\quad \because D(\sin 2t)=2\cos 2t$

$(2)\times 2 \Rightarrow -4x+2Dy=2\cos 2t$

$\quad (+) \quad (-) \quad (-)$

$(D^2+4)x=0$

The auxiliary eqn is $m^2+4=0$

$\Rightarrow m^2=-4 \Rightarrow m=\pm 2i$

$C.F = c_1\cos 2t + c_2\sin 2t$

The complete solution is

$x=c_1\cos 2t+c_2\sin 2t \longrightarrow (3)$

Substituting (3) in (1), we get

$D[c_1\cos 2t+c_2\sin 2t]+2y=\sin 2t$

$-2c_1\sin 2t+2c_2\cos 2t+2y=\sin 2t$

$2y=\sin 2t+2c_1\sin 2t-2c_2\cos 2t$

$y=\frac{\sin 2t}{2}+c_1\sin 2t-c_2\cos 2t$

The solutions are

$$x = C_1 \cos 2t + C_2 \sin 2t$$

$$y = \left(\tfrac{1}{2}+C_1\right)\sin 2t - C_2 \cos 2t$$

2) Solve: $\frac{dx}{dt} + 2x - 3y = 5t$, $\frac{dy}{dt} - 3x + 2y = 2e^{2t}$

Sol:

$$Dx + 2x - 3y = 5t \quad , \quad Dy - 3x + 2 = 2e^{2t}$$

$$(D+2)x - 3y = 5t \longrightarrow (1)$$

$$-3x + (D+2)y = 2e^{2t} \longrightarrow (2)$$

$$(1) \times 3 \Rightarrow 3(D+\cancel{2})x - 9y = 15t$$

$$(2) \times (D+2) \Rightarrow -3(\cancel{D}+2)x + (D+2)^2 y = (D+2)2e^{2t}$$

$$-9y + (D+2)^2 y = 15t + 2(D+2)e^{2t}$$

$$\Rightarrow -9y + (D^2+4D+4)y = 15t + 2\left[2e^{2t} + 2e^{2t}\right]$$

$$\Rightarrow (D^2 + 4D - 5)y = 8e^{2t} + 15t$$

The auxiliary eqn is $m^2 + 4m - 5 = 0$

$$(m-1)(m+5) = 0 \Rightarrow m = 1, -5$$

$$C.F. = C_1 e^{t} + C_2 e^{-5t}$$

$$P.I = \frac{1}{D^2+4D-5}\left(8e^{2t} + 15t\right)$$

$$= 8 \cdot \frac{1}{D^2+4D-5} e^{2t} + 15 \cdot \frac{1}{D^2+4D-5} t$$

$$= 8 \cdot \frac{1}{4+8-5} e^{2t} + \frac{15}{-5} \cdot \frac{1}{\left[1 - \left(\frac{D^2+4D}{5}\right)\right]} t$$

$$= \frac{8}{7}e^{2t} - 3\left[1-\left(\frac{D^2+4D}{5}\right)\right]^{-1} t$$

$$= \frac{8}{7}e^{2t} - 3\left(1+\frac{4D}{5}\right)t$$

$$= \frac{8}{7}e^{2t} - 3\left(t+\frac{4}{5}\right) = \frac{8}{7}e^{2t} - 3t - \frac{12}{5}$$

The complete solution is $y = C.F. + P.I$

$$y = C_1e^{t} + C_2e^{-5t} + \frac{8}{7}e^{2t} - 3t - \frac{12}{5}$$

Now ① $\times (D+2) \Rightarrow (D+2)^2 x - \cancel{3(D+2)y} = (D+2).5t$

② $\times$ 3 $\Rightarrow$ $-9x + \cancel{3(D+2)y} = 6e^{2t}$

$$(D+2)^2 x - 9x = 5(1+2t) + 6e^{2t}$$

$$\Rightarrow (D^2+4D-5)x = 6e^{2t} + 10t + 5$$

The auxiliary eqn is $m^2+4m-5=0$

$(m-1)(m+5)=0 \Rightarrow m = 1, -5$

$C.F = C_1e^{t} + C_2e^{-5t}$

$$P.I = \frac{1}{D^2+4D-5}\left[6e^{2t}+10t+5\right]$$

$$= \frac{1}{D^2+4D-5}6.e^{2t} + \frac{1}{D^2+4D-5}10t + \frac{1}{D^2+4D-5}5e^{0t}$$

$$= P.I_1 + P.I_2 + PI_3$$

$$P.I_1 = \frac{1}{D^2+4D-5}6e^{2t} = 6.\frac{1}{4+8-5}e^{2t} = \frac{6}{7}e^{2t}$$

$$P.I_2 = \frac{1}{D^2+4D-5}10t = \frac{10}{-5}.\frac{1}{\left[1-\left(\frac{D^2+4D}{5}\right)\right]}t$$

$$= -2\left[1-\left(\frac{D^2+4D}{5}\right)\right]^{-1}t$$

(4)

$$P.I_2 = -2\left(1+\frac{4D}{5}\right)t$$
$$= -2\left(t+\frac{4}{5}\right) = -2t-\frac{8}{5}$$
$$P.I_3 = \frac{1}{D^2+4D-5}\,5e^{0t} = 5\cdot\frac{1}{-5} = -1$$

$P.I = P.I_1 + P.I_2 + P.I_3$

$$= \frac{6}{7}e^{2t} - 2t - \frac{8}{5} - 1 = \frac{6}{7}e^{2t} - 2t - \frac{13}{5}$$

The complete soln is $x = CF + P.I$

$$\Rightarrow x = c_1e^{t} + c_2e^{-5t} + \frac{6}{7}e^{2t} - 2t - \frac{13}{5}$$

The required solutions are

$$x = c_1e^{t} + c_2e^{-5t} + \frac{6}{7}e^{2t} - 2t - \frac{13}{5}$$
$$y = c_1e^{t} + c_2e^{-5t} + \frac{8}{7}e^{2t} - 3t + \frac{12}{5}$$

3) Solve: $\frac{dx}{dt} + 7x - y = 0$, $\frac{dy}{dt} + 2x + 5y = 0$

Sol:

$Dx + 7x - y = 0$, $Dy + 2x + 5y = 0$

$\Rightarrow (D+7)x - y = 0 \longrightarrow (1)$

$\Rightarrow 2x + (D+5)y = 0 \longrightarrow (2)$

$(1)\times 2 \Rightarrow 2(D+7)x - 2y = 0$

$(2)\times(D+7) \Rightarrow \underset{(-)}{2(D+7)x} \underset{(-)}{+} (D+7)(D+5)y = 0$

$-2y - (D+7)(D+5)y = 0$

$\Rightarrow (D^2+12D+37)y = 0$

The auxiliary eqn is $m^2+12m+37=0$

$$\Rightarrow m = \frac{-12\pm\sqrt{144-148}}{2} = -6\pm i$$

$C.F = e^{-6t}(C_1 \cos t + C_2 \sin t)$

The complete solution is $y = CF$

$\Rightarrow y = e^{-6t}(C_1 \cos t + C_2 \sin t)$

$① \times (D+5) \Rightarrow (D+7)(D+5)x - \cancel{(D+5)}y = 0$

$② \times 1 \Rightarrow \qquad 2x + (D+5)y = 0$

$(D+7)(D+5)x + 2x = 0$

$\Rightarrow (D^2 + 12D + 37)x = 0$

The auxiliary eqn is $m^2 + 12m + 37 = 0$

$\Rightarrow m = -6 \pm i$

$C.F = e^{-6t}[C_3 \cos t + C_4 \sin t]$

The complete soln is $x = C.F.$

$\Rightarrow x = e^{-6t}[C_3 \cos t + C_4 \sin t]$

The required soln is

$x = e^{-6t}(C_3 \cos t + C_4 \sin t)$

$y = e^{-6t}(C_1 \cos t + C_2 \sin t)$