Sumedh

Complete Math Formulae

(All 5 Units)

Unit - I

Sumedh

- Formation of PDE

$z = f(x, y)$ / $q = \phi(x, y, z)$

Type - 1 :-

Elimination of Arbitary Constant

Case - 1 :-

No of Independant variables = Arbitary Constants [1^{st} Order PDE]

Case - 2 : " < " [2^{nd} Order PDE]

We can solve sums by partially diffrentiating the equation w.r.t to x / y and eliminating constants accordingly

$$\frac{\partial z}{\partial x} = p \quad , \quad \frac{\partial z}{\partial y} = q$$ — for 1^{st} order

$$\frac{\partial z^2}{\partial x^2} = r \quad , \quad \frac{\partial z^2}{\partial x \, \partial y} = s \quad , \quad \frac{\partial z^2}{\partial y^2} = t$$ — In case of 2^{nd} order

Type - 2 :-

Elimination of arbitary functions

Case - 1 : $z = f(x, y, z)$

Here, partially diffrentiate w.r.t x & y then divide p/q to eliminate functions

Case - 2 :-

$f(u, v) = 0$

$u = f(x, y, z)$ & $v = f(x, y, z)$

eg $\phi(x^2 + y^2 + z^2 , x + y + z)$

here, find $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$

then
$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \quad \text{to find PDE}$$

Case - 3 :-

$$z = f(x,y) + g(x,y)$$

Find p, q, r, s, t & solve

- Lagrange's linear equation

1^{st} order linear PDE is of the form :-

$$Pp + Qq = R$$

P, Q, R are functions of x, y, z

eg $px^2 + qy^2 = z^2$

where $P = x^2$, $Q = y^2$, $R = z^2$

Step - 1 :-

Auxillary form : $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

Step - 2 :-

Solve the subsidary eq

$$u(x, y, z) = c_1$$
$$v(x, y, z) = c_2$$

Type 1 : Method of Grouping

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{P} = \frac{dy}{Q} \quad \& \quad \frac{dy}{Q} = \frac{dz}{R}$$

Integrate ⋮ c_1 Integrate ⋮ c_2

Type - 2 : Method of multipliers

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

(l, m, n) shld be such a ratio that $lP + mQ + nR = 0$

$\therefore$ $l\,dx + m\,dy + n\,dz = 0$

$\therefore$ do with 2 multipliers i.e 2 diff ratios
eg (0, 0, 0) (2, 2, 2)

Integrate to find c_1 & c_2

Step - 3 :-

$\therefore$ $c_1 = u$, $c_2 = v$

$f(u, v) = 0$

└ final ans

- Homogeneous PDE

- Complementary function

Type - 1 : $m_1 \neq m_2 \neq \dots m_n$

$$CF = f_1(y + m_1 x) + f_2(y + m_2 x) + \dots + f_n(y + m_n x)$$

Type - 2 : $m_1 = m_2 = \dots = m_n = m$

$$CF = f_1(y + mx) + x f_2(y + mx) + x^2 f_3(y + mx) + \dots x^{n-1} f_n(y + mx)$$

Type - 3 : $m_1 = m_2 \neq m_3$

$$CF = f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x)$$

- Particular Integral

$$PI = \frac{F(x, y)}{\phi(D, D')}$$

where,
coeff of $x = D$
coeff of $y = D'$

Type 1 :-

$$F(x, y) = e^{ax + by}$$

$$PI = \frac{e^{ax + by}}{\phi(D, D')}$$

$D = a$
$D' = b$

if $\phi(a, b) = 0$

then,

$$PI = \frac{x\left[e^{ax + by}\right]}{\phi'(D, D')}$$ wrt D

⋮

& so on

Type - 2

$$F(x,y) = \sin(ax+by) \;/\; \cos(ax+by)$$

$$\therefore \quad PI = \frac{\sin(ax+by) \;/\; c}{\phi(D, D')}$$

here,
$$D^2 = -a^2$$
$$D'^2 = -b^2$$
$$DD' = -ab$$

alternate method,

we know,

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$RP = \cos\theta$$
$$IP = \sin\theta$$

when $F(x,y) = \sin(ax+by)$

$$PI = IP\left\{ \frac{e^{i(ax+by)}}{\phi(D, D')} \right\}$$

Type - 1 $\quad D = ai$, $D' = bi$

when $PI = \cos(ax+by)$

$$PI = RP\left\{ \frac{e^{i(ax+by)}}{\phi(D, D')} \right\}$$

Type - 3

$$F(x,y) = x^m y^n$$

$$PI = \frac{x^m y^n}{\phi(D.D')}$$

$$= \frac{1}{D}\left[\frac{x^m y^n}{(1+f(DD'))}\right]$$

$$= \frac{x^m y^n}{D}\left(1+f(DD')\right)^{-1}$$

$$\frac{1}{D^2}(x)$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

where

$$D = \partial/\partial x \qquad D' = \partial/\partial y$$

$$\boxed{\therefore \text{ Gen Sol}^n = CF + PI}$$

- Solutions of PDE

· Types

1. $f(p,q) = 0$
2. $z = px + qy + f(p,q)$
3. $z(x,y,p,q) = 0$
4. $f(x,p,q) = 0$
5. $f(y,p,q) = 0$
6. $f(z,p,q) = 0$

Example 1

Solve $\sqrt{p}+\sqrt{q}=1$.

Solution : Given p.d.e is

$$\sqrt{p}+\sqrt{q}=1 \tag{1}$$

This is a first order p.d.e of the form $f(p,q)=0$
Therefore the trial solution is

$$\boxed{z=ax+by+c} \tag{2}$$

To Find : The Complete Integral (or) Complete Solution
We have to eliminate any one of the arbitrary constants in (2).
(Since, In a complete integral , Number of arbitrary constants must be equal to Number of independent variables.)
Diff. (2) p.w.r.t. x and y , We get

$$\partial z \qquad \partial z$$

That is,

$$p = a \quad \& \quad q = b$$

Use $p = a, q = b$ in (1), We get

$$\sqrt{a} + \sqrt{b} = 1 \quad \Rightarrow \quad \sqrt{b} = 1 - \sqrt{a}$$

$$b = \left(1 - \sqrt{a}\right)^2$$

Hence the **complete solution** is

$$\boxed{z = ax + \left(1 - \sqrt{a}\right)^2 y + c} \tag{3}$$

To Find : The Singular Integral (or) Singular Solution: we have $\frac{\partial z}{\partial a} = 0$ $\frac{\partial z}{\partial c} = 0$

2 conditions

Diff. (3) p.w.r.t. 'c' we get $\frac{\partial z}{\partial c} = \boxed{1 \longrightarrow 0 = 1,}$ which is not true.

Hence there is **No singular solution**.

for singular solution, partially differentiate the complete solution by constants

To Find : The General Integral (or) General Solution

Put $c = \phi(a)$ in the Complete Solution (5), We get

$$\boxed{z = ax + \left(1 - \sqrt{a}\right)^2 y + \phi(a)} \tag{4}$$

Diff. (4) p.w.r.t. 'a' and eliminating 'a', We get the general solution of the given p.d.e.

TYPE - II Clairaut's Form : $z = px + qy + f(p, q)$

Example 1

Solve $z = px + qy + pq$.

Solution : Given p.d.e is

$$z = px + qy + pq \tag{1}$$

This is in the Clairaut's form

$$z = px + qy + f(p, q)$$

The Complete Integral (or) Complete Solution of (1) which is in Clairaut's form is obtained by replacing p by a and q by b

$$\boxed{z = ax + by + ab} \tag{2}$$

To Find : The Singular Integral (or) Singular Solution
Diff. (2) p.w.r.t. 'a' and 'b' , We get

$$\frac{\partial z}{\partial a} = x + b \quad \& \quad \frac{\partial z}{\partial b} = y + a$$

$$0 = x + b \quad \& \quad 0 = y + a$$

$$b = -x \quad \& \quad a = -y \tag{3}$$

Use (3) in the complete solution (2) , We get

$$z = -xy - xy + xy$$

$$\boxed{z + xy = 0}$$

This is the required singular integral.

To Find : The General Integral (or) General Solution
Put $b = \phi(a)$ in the Complete Solution (2), We get

$$z = ax + \phi(a)y + a\phi(a) \tag{4}$$

Diff. (4) p.w.r.t. 'a' and eliminating 'a', We get the general solution of the given p.d.e.

Type ~~IV~~ III (Separable Equations): Unit-I

$f(x, y, p, q) = 0$

The first order PDE is said to be separable equation it it can put in the form $f(x, p) = \phi(y, p)$.

For such PDE the solutions can be obtained as:

- **Step I:** Put $f(x, p) = \phi(y, p) = a$
- **Step II:** Write $p = f_1(x, a)$ and $q = \phi_1(y, a)$
- **Step III:** Putting in $dz = pdx + qdy$ and integrating, get the complete integral as

$$z = \int f_1(x, a)dx + \int \phi_1(y, a)dy + b.$$

Example 1

Solve $p = 2qx$

TYPE - ~~III~~ IV

(a) $f(x,p,q) = 0$ (b) $f(y,p,q) = 0$ (c) $f(z,p,q) = 0$

Solution : Given p.d.e is

$$p = 2qx \tag{1}$$

for $f(x,p,q) = 0$
take $q = a$
$f(y,p,q) = 0$
take $p = a$

This is a Type - III (a) p.d.e of the form

$$f(x,p,q) = 0$$

Let z be a function of x and y

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$dz = p\,dx + q\,dy \tag{2}$$

Put $q = a$ in (1) and solve for p, We get

$$p = 2ax$$

Use p and q values in (2) , We get

$$dz = 2ax\ dx + a\ dy$$

Integrating on both the sides, We get

$$\int dz = \int 2ax\ dx + \int a\ dy$$

$$\boxed{z = ax^2 + ay + c} \quad (3)$$

This is the required complete solution.

Diff. (3) p.w.r.t c , We get $0 = 1$ $[Absurd]$ Hence There is no singular Integral.

Put $c = \phi(a)$ in the Complete Solution (3), We get

$$\boxed{z = ax^2 + ay + \phi(a)} \quad (4)$$

Diff. (4) p.w.r.t. 'a' and eliminating 'a', We get the general solution.

Examples under Type - III (b) $f(y, p, q) = 0$

Solve the following partial differential equations :

(1). $pq = y$

(2). $p = 2qy$

(3). $q = py + p^2$

(4). $\sqrt{p} + \sqrt{q} = \sqrt{y}$

Example 1

Solve $pq = y$.

Solution : Given p.d.e is

$$pq = y \tag{1}$$

This is a Type - III (b) p.d.e of the form

$$f(y, p, q) = 0$$

Let z be a function of x and y

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

$$dz = p\ dx + q\ dy \tag{2}$$

Put $p = a$ in (1) and solve for q, We get

$$aq = y \quad \Rightarrow \quad q = \frac{y}{a}$$

Use p and q values in (2) , We get

$$dz = a\ dx + \frac{y}{a}\ dy$$

Integrating on both the sides, We get

$$\int dz = \int a\ dx + \int \frac{y}{a}\ dy$$

$$\boxed{z = ax + \frac{y^2}{2a} + c} \tag{3}$$

This is the required complete solution.

Diff. (4) p.w.r.t c , We get

$$0 = 1 \quad [Absurd]$$

Hence There is no singular Integral.

Put $c = \phi(a)$ in the Complete Solution (3), We get

$$\boxed{z = ax + \frac{y^2}{2a} + \phi(a)} \tag{4}$$

Examples under Type - III (c) $f(z, p, q) = 0$

Solve the following partial differential equations :

(1). $p(1+q) = qz$

(2). $p\left(1+q^2\right) = q(z-a)$

(3). $z^2 = 1 + p^2 + q^2$

(4). $9\left(p^2 z + q^2\right) = 4$

(5). $z = p^2 + q^2$

(6). $p\left(1-q^2\right) = q(1-z)$

$u = x + ay$

$p = \frac{dz}{du}$ $q = a\frac{dz}{du}$

Example 1

Solve $p\,(1+q) = qz$.

Solution : Given p.d.e is

$$p\,(1+q) = qz \tag{1}$$

This is a Type - III (c) p.d.e of the form $f\,(z,p,q) = 0$
Let z be a function of u where $u = x + ay$

$$\frac{\partial u}{\partial x} = 1 \quad \& \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} \quad \& \quad q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}$$

$$p = \frac{dz}{du} \quad \& \quad q = a\frac{dz}{du} \tag{2}$$

Use (2) in (1) , We get

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z$$

$$1 + a\frac{dz}{du} = az \rightarrow \frac{dz}{du} = \frac{az-1}{a}$$

$$du = \frac{a}{az-1}dz$$

Integrating on both the sides, We get

$$\int du = \int \frac{a}{az-1}dz$$

$$\underbrace{u}_{x+ay} = \log(az-1) + c$$

Hence the complete solution is

$$\boxed{x + ay = \log(az-1) + c} \quad (1)$$

This is the required complete solution. Diff. (3) p.w.r.t c , We get

$0 = 1 \quad [Absurd]$. Hence There is no singular Integral.

Put $c = \phi(a)$ in the Complete Solution (3), We get

$$\boxed{x + ay = \log\left(az - 1\right) + \phi(a)} \tag{4}$$

Diff. (4) p.w.r.t. 'a' and eliminating 'a', We get the general solution.

Unit - 2

Sumedh

- **Basic Formulae**

1. $$\int u.v\,dx = u\int v\,dx - \int\left(u'\int v\,dx\right).dx$$

OR

$$\int u.v\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 \cdots \left[\text{till } u^n = 0\right]$$

2. Even function

$$f(x) = f(-x)$$

here, $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

3. Odd function

$$f(-x) = -f(x)$$

here, $$\int_{-a}^{a} f(x)\,dx = 0$$

4. $$\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

5. $\sin n\pi = 0$

$\sin 2n\pi = 0$

$\cos n\pi = (-1)^n$ [n : even = 1, odd = -1]

$\cos 2n\pi = 1$

- Fourier Series

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where, $$a_0 = \frac{1}{2\pi} \int_{c}^{c+2\pi} f(x)\, dx$$

[for c: if $(0, 2\pi)$ then $c = 0$]

$$a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) . \cos nx\, dx$$

[for $[0, a]$ here $2l = a$, $l = a/2$, l instead of π]

$$b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \sin nx\, dx$$

eg if given $(0, 2\pi)$ at $f(x) = x$

here $$a_0 = \frac{1}{2\pi} \int_{0}^{2\pi} x . dx$$

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} x . \cos nx \quad ; \quad b_n = \frac{1}{\pi} \int_{0}^{2\pi} x . \sin nx\, dx$$

eg if given $$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

here $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} -\pi\, dx + \frac{1}{2\pi} \int_{0}^{\pi} x\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{0} -\pi . \cos nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} x . \cos nx\, dx$$

- **Half Range Series**

1. **Cosine series**

$$a_0 = \frac{1}{l} \int_0^l f(x)\, dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = 0$$

here, $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}\, dx$

2. **Sine series**

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} \cdot dx$$

here, $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot dx$

- **Parseval's Identity**

$$\frac{1}{2l} \int_c^{c+2l} [f(x)]^2\, dx = a_0^2 + \frac{1}{n} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$a_0 = \frac{1}{2l} \int_c^{c+2l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

here,

half range cosine $\frac{1}{\pi}\int_0^{\pi} (f(x))^2 dx = a_0 + \frac{1}{2}[a_1^2 + a_2^2 + \dots]$

sine $\frac{1}{\pi}\int_0^{\pi} (f(x))^2 dx = \frac{1}{2}[b_1^2 + b_2^2 + \dots]$

- Harmonic Analysis

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

here 1st harmonic

$= \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta)$

where Amp $= \sqrt{a_1^2 + b_1^2}$

$$a_0 = \frac{2}{N}\sum Y$$

$$a_n = \frac{2}{N}\sum Y\cos\theta$$

$$b_n = \frac{2}{N}\sum Y\sin\theta$$

eg

x	0	1	2	3	4	5
y	9	18	24	28	26	20

here,

6 value

$\therefore \frac{360}{6} = 60$

$\therefore \theta = 60$

x	θ	Y	$Y\sin\theta$	$Y\cos\theta$
0	$60 \times 0 = 0$	9	⋮	⋮
1	$60 \times 1 = 60$	18		
2	120	24		
3	180	28		
4	240	26		
5	300	20		

Unit - 3

Sumedh

- Wave Equation

no particular formulae, explained thru diff eg. types

· Type - 1 (Released disp)

eg] $y(x,0) = k(lx - x^2)$ at $(0, l)$

Sol^n BC's

1. $y(0,t) = 0$
2. $y(l,t) = 0$
3. $\left(\dfrac{dy}{dt}\right)_{t=0} = 0$
4. $y(x,0) = k(lx - x^2)$

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \quad \text{— (i)}$$

BC 1 in (i)

$$y(0,t) = (A(1) + B(0))(\quad) = 0$$

$$A(\quad) = 0$$

$$\therefore A = 0$$

$$\therefore y(x,t) = B\sin\lambda x(\quad) \quad \text{— (ii)}$$

BC 2 in (ii)

$$y(l,t) = B\sin\lambda l(\quad) = 0$$

$$\sin\lambda l = 0$$

$$\lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l}$$

$$\therefore y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \quad \text{— (iii)}$$

BC 3 in (iii)

$$\left(\frac{dy}{dt}\right) = B\sin\left(\frac{n\pi x}{l}\right)\left(-C\sin\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right) + D\cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right)\right)$$

at $t = 0$

$$\frac{dy}{dt} = B\sin\left(\frac{n\pi x}{l}\right)\left(D\left(\frac{n\pi a}{l}\right)\right) = 0$$

$$\therefore \quad D = 0$$

$$y(x,t) = B\sin\left(\frac{n\pi x}{l}\right) \cdot C\cos\left(\frac{n\pi at}{l}\right)$$

Most gen. sol

$$y(x,t) = \sum B_n \sin(\ ) \cos(\ ) \quad -(iv)$$

BC 4 in (iv)

$$\sum B_n \sin\left(\frac{n\pi x}{l}\right) = K(lx - x^2)$$

$$\therefore \quad B_n = \frac{2}{l}\int_0^l F(x) \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{2K}{l}\int_0^l \underset{u}{(lx - x^2)} \cdot \underset{v}{\sin\left(\frac{n\pi x}{l}\right)}$$

$$u = (lx - x^2) \qquad v_1 = -\cos\left(\frac{n\pi x}{l}\right) \cdot \frac{l}{n\pi}$$

$$u' = l - 2x$$

$$u'' = -2 \qquad v_2 = -\sin\left(\frac{n\pi x}{l}\right) \cdot \frac{l^2}{n^2\pi^2}$$

$$v_3 = \cos\left(\frac{n\pi x}{l}\right) \cdot \frac{l^3}{(n\pi)^3}$$

$$= \frac{2K}{l}\left[(lx - x^2)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right) + (l - 2x)\left(+\sin\left(\frac{n\pi x}{l}\right) \cdot \frac{l^2}{(n\pi)^2}\right)\right.$$

$$\left. - \cancel{*}\,(+2)\left(\cos\left(\frac{n\pi x}{l}\right) \cdot \frac{l^3}{(n\pi)^3}\right)\right]_0^l$$

$$= \frac{2K}{l}\left[0 + 0 - \frac{2(-1)^n l^3}{(n\pi)^3} - \left(0 + 0 - \frac{2l^3}{(n\pi)^3}\right)\right.$$

$$B_n = \frac{2K}{l}\left[\frac{2l^3}{(n\pi)^3}\left(1 - (-1)^n\right)\right.$$

eg $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$ at $(0,l)$

Sol^n

$\vdots$

$D=0$

$\therefore \quad y(x,t) = B \sin\left(\frac{n\pi x}{l}\right) C \cos\left(\frac{n\pi a t}{l}\right)$

Most gen. sol^n

$y(x,t) = \sum B_n \sin(\ ) \cos(\ )$

By BC 4

$$\sum B_n \sin\left(\frac{n\pi x}{l}\right) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$\therefore \quad B_1 \sin\left(\frac{\pi x}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right) + B_3 \left(\sin \frac{3\pi x}{l}\right) + \dots$

$$= y_0 \left[\frac{3 \sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)}{4}\right]$$

here $B_1 = \frac{3y_0}{4}$, $B_2 = 0$, $B_3 = -\frac{y_0}{4}$

$\therefore \quad y(x,t) = B_1 \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi a t}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right) \cos\left(\frac{2\pi a t}{l}\right)$

$+ B_3 \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{3\pi a t}{l}\right)$

$= \frac{3y_0}{4} \sin(\ ) \cos(\ ) - \frac{y_0}{4} \sin(\ ) \cos(\ )$

- Type - 2 (Equillibrium disp)

eg $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 3x(l-x)$ at $(0, l)$

Sol^n BC 's

1. $y(0, t) = 0$
2. $y(l, t) = 0$
3. $y(x, 0) = 0$
4. $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 3x(l-x)$

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \quad -(i)$$

BC 1 in (i)

$$y(0,t) = (A(1) + B(0))(\quad) = 0$$

$\therefore$ $A = 0$

$$y(x,t) = (B\sin\lambda x)(\quad) \quad -(ii)$$

BC 2 in (ii)

$$y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left(C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right) \quad -(iii)$$

BC 3 in (iii)

$$y(x,0) = B\sin\left(\frac{n\pi x}{l}\right)\left(C(1) + D(0)\right)$$

$\therefore$ $C = 0$

$$\therefore y(x,t) = B\sin\left(\frac{n\pi x}{l}\right) D\sin\left(\frac{n\pi at}{l}\right)$$

Most gen sol^n

$$y(x,t) = B_n \sin(\quad)\sin(\quad) \quad -(iv)$$

BC 4 in (iv)

$$\frac{dy}{dt} = B_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right) \cdot \left(\frac{n\pi a}{l}\right)$$

at $t=0$

$$\left(\frac{dy}{dt}\right)_{t=0} = B_n \sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right) = 3\left(lx - x^2\right)$$

$$\therefore \quad B_n\left(\frac{n\pi a}{l}\right) = \frac{2}{l}\int_0^l 3\left(lx - x^2\right) \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$u = (lx - x^2)$ $\qquad v_1 = -\cos\left(\frac{n\pi x}{l}\right) \cdot \frac{l}{n\pi}$

$u' = (l - 2x)$

$u'' = -2$ $\qquad v_2 = -\sin\left(\frac{n\pi x}{l}\right) \cdot \frac{l^2}{(n\pi)^2}$

$v_3 = \cos\left(\frac{n\pi x}{l}\right) \cdot \frac{l^3}{(n\pi)^3}$

$$B_n\left(\frac{n\pi a}{l}\right) = \frac{6}{l}\left[-\left(lx - x^2\right)\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi} + \sin\left(\frac{n\pi x}{l}\right)\frac{l^2}{(n\pi)^2} - 2\cos\left(\frac{n\pi x}{l}\right)\frac{l^3}{(n\pi)^3}\right]_0^l$$

$$= \frac{6}{l}\left[0 + 0 - \frac{2(-1)^n l^3}{n\pi} - \left(-0 + 0 - \frac{2l^3}{(n\pi)^3}\right)\right]$$

$$= \frac{6}{l}\left[\frac{2l^3}{(n\pi)^3}\left(1 - (-1)^n\right)\right]$$

$$B_n\left(\frac{n\pi a}{l}\right) = \frac{12 l^2}{(n\pi)^3}\left[1 - (-1)^n\right]$$

$$B_n = \frac{12 a l^3}{(n\pi)^4 \cdot a}\left[1 - (-1)^n\right]$$

eg $\left(\frac{dy}{dt}\right)_{t=0} = V_0 \sin^3\left(\frac{\pi x}{l}\right)$ at $(0, l)$

Sol^n

$$y(x,t) = \sum B_n\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right) = V_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$$B_1\left(\frac{\pi a}{l}\right)\sin\left(\frac{\pi x}{l}\right) + B_2\left(\frac{2\pi a}{l}\right)\sin\left(\frac{2\pi x}{l}\right)$$

$$+ B_3\left(\frac{3\pi a}{l}\right)\sin\left(\frac{3\pi x}{l}\right) = V_0\left(\frac{3\sin(\pi x/l) - \sin\left(\frac{3\pi x}{l}\right)}{4}\right)$$

$$B_1 = \frac{3V_0 l}{4\pi a}, \quad B_2 = 0, \quad B_3 = \frac{-V_0 l}{12\pi a}$$

- Heat Equation

eg $l = 30\text{cm}$, T ends $= 20$ & $T = 80°$, both ends T reduced too

Sol^n $u(0,t) = 20$

$u(30,t) = 80$

$u_x = Ax + B$

$20 = A(0) + B$

$B = 20$

$u_x = Ax + B$

$80 = A(30) + 20$

$A = 2$

$\therefore u_x = 2x + 20 \rightarrow$ Steady state

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

BC's

1. $u(0,t) = 0$
2. $u(30,t) = 0$
3. $u(x,0) = 2x + 20$

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, C e^{-\alpha^2\lambda^2 t} \quad -(i)$$

BC 1 in (i)

$$u(0,t) = (A(1) + B(0))\, C e^{-\alpha^2\lambda^2 t} = 0$$

$$\therefore \quad A = 0$$

$$u(x,t) = (B\sin\lambda x)\, C e^{(\;)} \quad -(ii)$$

BC 2 in (ii)

$$u(30,t) = (B\sin\lambda 30)\, C e^{-\alpha^2 t\lambda^2} = 0$$

$$\sin\lambda 30 = 0$$
$$30\lambda = n\pi$$
$$\lambda = \frac{n\pi}{30}$$

$$\therefore \quad u(x,t) = (B\sin 30\lambda)\, C e^{-\alpha^2 t \frac{n^2\pi^2}{900}} \quad -(iii)$$

$$= B_n(\sin 30\lambda)(e^{(\;)})$$

BC 3

$$u(x,0) = B_n \sin 30\lambda = 2x + 20$$

$$\therefore \quad B_n = \frac{2}{l}\int_0^{30} (2x+20)\sin\left(\frac{n\pi x}{30}\right) .dx$$

$$u = (2x+20) \qquad V_1 = -\cos\left(\frac{n\pi x}{30}\right)\frac{30}{n\pi}$$
$$u' = 2$$

$$V_2 = -\sin\left(\frac{n\pi x}{30}\right)\frac{900}{(n\pi)^2}$$

$$B_n = \frac{2}{30}\left[-(2x+20)\cos\left(\frac{n\pi x}{30}\right)\cdot\frac{30}{n\pi} + (2)\sin\left(\frac{n\pi x}{30}\right)\frac{900}{(n\pi)^2}\right]_0^{30}$$

$$= \frac{2}{30}\left[-80\frac{(-1)^n}{n\pi}\cdot 30 + 0 - \left(\frac{-20\cdot 30}{n\pi}\right)\right]$$

$$= \frac{2}{\cancel{30}}\left[-\frac{\overset{80}{\cancel{2400}}(-1)^n}{n\pi} + \frac{\overset{20}{\cancel{600}}}{n\pi}\right]$$

$$= -\frac{160(-1)^n}{n\pi} + \frac{40}{n\pi}$$

$$B_n = \frac{40}{n\pi}\left(1 - 4(-1)^n\right)$$

$$a_0 = \frac{2}{2}\int_0^3 x^2 \cdot dx$$

a_0

$$= \frac{4}{3}\left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{4}{3}\left[\right.$$

$$= 12/2 = 6$$

$$a_n = 4/3 \int_0^8 x^2 \cdot \cos\frac{2n\pi x}{3}$$

x^2

$2x$

2

Unit - 4

Sumedh

- **Fourier Transform**

of $f(x) = F(s)$

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} . dx$$

Inverse FT

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx} . dx$$

Parseval's Identity on FT

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

- **Fourier Cosine Transform (FCT)**

$$F_c[f(x)] = F_c(s)$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$$

Inverse FCT

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx\, ds$$

Parseval's Identity on FCT

$$\int_0^{\infty} |f(x)|^2 dx = \int_0^{\infty} |F_c(s)|^2 ds$$

$$\int_0^{\infty} f(x)\, g(x)\, dx = \int_0^{\infty} F_c(s)\, G_c(s)\, ds$$

- Fourier Sine Transform (FST)

$$F_s[f(x)] = F_s(s)$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$e^{-ax} \sin bx = \frac{b}{b^2+a^2}$$

$$e^{-ax} \cos bx = \frac{a}{a^2+n^2}$$

Inverse FST

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

PROPERTY.

(i) If $F[f(x)] = F(s)$, then

$$F[x \cdot f(x)] = (-i) \frac{d}{ds}[F(s)]$$

(ii) If $F_s[x \cdot f(x)] = -\frac{d}{ds}[F_c(s)]$

(iii) If $F_c[x \cdot f(x)] = \frac{d}{ds}[F_s(s)]$

Parseval's Identity on FST

$$\int_0^\infty |f(x)|^2 \, dx = \int_0^\infty |F_s(s)|^2 \, ds$$

$$\int_0^\infty f(x) g(x) \, dx = \int_0^\infty F_s(s) G_s(s) \, ds$$

Note :

$$e^{isx} = \cos sx + i \sin sx$$

$$e^{-isx} = \cos sx - i \sin sx$$

eg

Find the fourier transform of $f(x) = \begin{cases} 1 \, , & |x| \le a \\ 0 \, , & |x| > a \end{cases}$. Hence deduce that

(i) $\int_0^\infty \frac{\sin t}{t} \, dt = \frac{\pi}{2}$ (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Soln $f(x) = 1$ at $|x| \le a$ i.e $(-a, a)$

$$\therefore \; FT = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a 1 \, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a \overset{e}{\cos sx} + \cancel{i \overset{o}{\sin} sx} \overset{o}{} \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a \cos sx$$

$$F(s) = \frac{2}{\sqrt{2\pi}}\left[\frac{\sin as}{s}\right]$$

i) $\int_0^\infty \frac{\sin t}{t}\, dt = \pi/2$

inverse : $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s}\right) \cdot e^{-isx}\, ds$$

put $x = 0$, $a = 1$, $s = t$, $ds = dt$

$(-a, a)$

$$f(0) = \frac{1}{\cancel{\sqrt{2\pi}}} \times \sqrt{\frac{\cancel{2}}{\pi}} \int \frac{\sin t}{t} \cdot dt$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin t}{t} \cdot ds$$

0/0 = e

$$1 = \frac{2}{\pi} \int_0^\infty \frac{\sin t}{t}\, dt$$

ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

parsevals : $\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\, \frac{\sin as}{s}\right)^2 ds = \int_{-a}^{a} (1)^2\, dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds = \int_{-a}^{a} dx$$

put $a = 1$, $s = t$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \int_{-1}^{1} dx$$

$$\cancel{2}\, \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \cancel{2} \int_0^1 dx$$

- Self - reciprocal function

  $F[f(x)] = f(s)$

or

$F_s[f(x)] = f(s)$, $F_c[f(x)] = f(s)$

only eg : $F[e^{-x^2/2}] = e^{-s^2/2}$

eg

Find the Fourier transform of $e^{-a^2x^2}$ and hence deduce that $e^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier transform. (Or)
Show that $e^{-\frac{x^2}{2}}$ is self - reciprocal under Fourier transform.

Sol^n

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}$$

$(A-B)^2 = A^2 + B^2 - 2AB$

$(A-B)^2 - B^2 = A^2 - 2AB$

$a^2x^2 - isx$

$A = ax$

$2AB = isx$

$2axB = isx$

$B = \frac{is}{2a}$

$\therefore A^2 - 2AB = \left(ax - \frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$

$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}$

$$\therefore = \int_{-\infty}^{\infty} e^{-\left\{\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right\}}$$

const

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} \cdot e^{-(s^2/4a^2)} \cdot dx$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

$$\ldots\ldots \int_{-\infty}^{\infty} e^{-t^2} \cdot dt = \sqrt{\pi}$$

$$t = ax - \frac{is}{2a}$$

$$dt = a \cdot dx$$

$$= \left(\qquad \right) \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$\frac{1}{a} \left(\quad \right) \left[\sqrt{\pi} \right]$$

$$= \frac{e^{-s^2/4a}}{a \sqrt{2\cancel{\pi}}} \times \sqrt{\cancel{\pi}}$$

$$F\left[e^{-a^2x^2}\right] \quad \frac{e^{-s^2/4a}}{a\sqrt{2}}$$

$$F\left[e^{-x^2/2}\right]$$

$$a^2 = 1/2$$

$$a = 1/\sqrt{2}$$

$$\therefore \; F\left[e^{-x^2/2}\right] = \frac{e^{-s^2/4(1/2)}}{1/\cancel{\sqrt{2}} \times \cancel{\sqrt{2}}}$$

$$= e^{-s^2/2}$$

Unit - 5

Sumedh

- z - transform

$$z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n} \quad \text{(Unilateral)}$$

$$z[f(t)] = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$z[f(n)] = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n} \quad \text{(Bilateral)}$$

$$\therefore \; z[f(n)] = F(z)$$

Series

$$(1-x)^{-1} = 1 + x + x^2 + \ldots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 \ldots$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$$

eg $z[1]$

Sol^n $z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$z[1] = \sum \frac{1}{z}$$

1

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots$$

$$z[1] = \left(1 - 1/z\right)^{-1}$$

$$= \frac{z}{z-1}$$

eg $z[a^n]$

Sol^n $z[f(n)] = \sum \frac{f(n)}{z^n}$

$z[a^n] = \sum \frac{a^n}{z}$

$= \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} \dots$

$z[\ ] = (1 - a/z)^{-1}$

$z[a^n] = \frac{z}{z-a}$

Sim^{lly} $z[2^n] = \frac{z}{z-2}$

eg $z[n]$

Sol^n $z[f(n)] = \sum f(n) z^{-n}$

$z[n] = \sum \frac{n}{z^n}$

$= \frac{1}{z} + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + \dots$

$= \frac{1}{z}\left(1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \dots\right)$

$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}$

$\therefore \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2}$

$= \frac{z\cancel{^2}}{\cancel{z}} \frac{1}{(z-1)^2}$

$z[n] = \frac{z}{(z-1)^2}$

eg $z[1/n]$, $n>0$

Sol^n $$z[f(n)] = \sum_{n=1}^{\infty} f(n) \cdot z^{-n}$$

$$z[1/n] = \sum \frac{1}{n z^n}$$

$$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3$$

$$= \frac{(1/z)}{1} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \dots$$

$$= -\log(1 - 1/z)$$

$$= -\log\left(\frac{z-1}{z}\right)$$

$$= \log\left(\frac{z-1}{z}\right)^{-1}$$

$$z[1/n] = \log\left(\frac{z}{z-1}\right)$$

- Properties

1. Change of Scale

$$z[a^k f(k)] = F(z/a)$$

2. Shifting

$$z[f(k+n)] = z^n F(z)$$

$$z[f(k-n)] = z^{-n} F(z)$$

3. Multiplication

$$z[k \cdot f(k)] = -z \frac{d}{dz} F(z)$$

$$z[k^n f(k)] = \left(-z \frac{d}{dz}\right)^n F(z)$$

4. Divisio

$$Z\left[\frac{f(k)}{k}\right] = \int_{z}^{\infty} \frac{1}{z} F(z)\, dz$$

5. Convolution

$$f(n) * g(n) = \sum f(r) \cdot g(n-r)$$

eg $$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a} \times \frac{z}{z-b}\right]$$

$$= Z^{-1}\left[F(z) \cdot G(z)\right]$$

$$= Z^{-1}\left[F(z)\right] * Z^{-1}\left[G(z)\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] \times Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$\therefore\ a^n * b^n = \sum_{r=0}^{n} a^r b^{n-r}$$

$$= \sum a^r b^n b^{-r}$$

$$= b^n \sum_{r=0}^{n} \left(\frac{a}{b}\right)^r$$

$$= b^n \left(1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \dots\right)$$

$$\left[1 + a + a^2 + \dots = \frac{1 - a^{n+1}}{1-a}\right]$$

$$= b^n \left[\frac{1 - (a/b)^{n+1}}{1 - a/b}\right]$$

$$= b^n \left[\frac{\frac{b^{n+1} - a^{n+1}}{b^{n+1}}}{\frac{b-a}{b}} \right]$$

$$= b^n \left[\frac{b}{b^{n+1}} \times \frac{b^{n+1} - a^{n+1}}{b-a} \right]$$

$$= \frac{b^{n+1} - a^{n+1}}{b-a}$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

gen formula

Imp Formulae

$$z[1] = \frac{z}{z-1}$$

$$z[a^n] = \frac{z}{z-a}$$

$$z[\sin \alpha n] = \frac{z \sin \alpha}{z^2 - 2z\cos\alpha + 1}$$

$$z[\cos \alpha n] = \frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$$

$$z[\sinh \alpha n] = \frac{z \cdot \sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$$

$$z[\cosh \alpha n] = \frac{z^2 - z\cosh\alpha}{z^2 - 2z\cosh\alpha + 1}$$

- Inverse z- transform

- By partial function

eg find z^{-1} $\dfrac{z}{(z-1)(z-2)}$ if $|z| \geq 2$

Sol^n $F(z) = \dfrac{z}{(z-1)(z-2)}$

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)}$$

$$\frac{1}{(\quad)(\quad)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$\frac{1}{\cancel{(\quad)(\quad)}} = \frac{A(z-2) + B(z-1)}{\cancel{(\quad)(\quad)}}$$

let $z = 2$

$B = 1$

at $z = 1$

$A = -1$

$$\therefore \frac{F(z)}{z} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$F(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

$$z^{-1}[F(z)] = z^{-1}\left[\frac{z}{z-2}\right] - z^{-1}\left[\frac{z}{z-2}\right]$$

at $z \geq 2$

$= 2^n - 1^n$

$= 2^n - 1$

if does not satisfy then just write same answer with -ve sign

- Inversion Integral Method

eg find $f(n)$ if $F(z) = \dfrac{10z}{(z-1)(z-2)}$

soln here pole $z = 1, 2$

consider $F(z) \cdot z^{n-1} = \dfrac{10z \cdot z^{n-1}}{(\quad)(\quad)}$

$= \dfrac{10z^n}{(\quad)(\quad)}$

find residue

$F(z) z^{n-1}$ at $z = 1$

$= \left. \cancel{(z-1)} \dfrac{10z^n}{\cancel{(z-1)}(z-2)} \right|_{z=1}$

$= \dfrac{10(1)}{-1}$

$= -10$

at $z = 2$

$\left. \cancel{(z-2)} \dfrac{10z^n}{(z-1)\cancel{(z-2)}} \right|_{z=2}$

10.2^n

$f(n) = -10 + 10 \cdot 2^n$

$= 10(2^n - 1)$

- Power Series

eg find $z^{-1}\left(\frac{1}{z-a}\right)$ when $|z| < |a|$

Soln make sure < 1 eg if $|z| > |a|$

$1 > \frac{|a|}{|z|}$

$\therefore \quad \left|\frac{z}{a}\right| < 1$

$$F(z) = \frac{1}{z-a}$$

$$= \frac{1}{a\,(z/a - 1)}$$

To satisfy

$$F(z) = \frac{1}{-a\,(1 - z/a)}$$

$$= \frac{1}{-a}\left[1 + \frac{z}{a} + \frac{z^2}{a^2} + \dots\right]$$

$$= -\left[\frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \frac{z^3}{a^4}\right.$$

$$= -\left[\frac{z^n}{a^{n+1}}\right]$$

$$= -\left[z^n . (a)^{-n-1}\right]$$

coeff of $z^n = -a^{-(n+1)}$

of $z^{-n} = -a^{-n+1}$

answer

· Long - division method

eg $x[z] = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$, $|z|>1$

causal

$|z|<1$: Anticausal

Sol^n

$$1 + 4z^{-1} + 7z^{-2} + 10z^{-3} + 13z^{-4} + \dots$$

$$1-2z^{-1}+z^{-2} \;\Big)\; 1+2z^{-1}$$

$$-\;1 - 2z^{-1} + z^{-2}$$

$$0 \quad \cancel{4z^{-1}} - z^{-2}$$

$$\cancel{4z^{-1}} - 8z^{-2} + 4z^{-3}$$

$$- \quad + \quad -$$

$$\cancel{7z^{-2}} - 4z^{-3}$$

$$\cancel{7z^{-2}} - 14z^{-3} + 7z^{-4}$$

$$- \quad + \quad -$$

$$\cancel{10z^{-3}} - 7z^{-4}$$

$$\cancel{10z^{-3}} - 20z^{-4} + 10z^{-5}$$

$$- \quad + \quad -$$

$$13z^{-4} - 10z^{-5}$$

$$\vdots$$

$X(z) =$ answer

- Difference equation

$$Z[y(n)] = F(z)$$

$$Z[y(n+1)] = zF(z) - zy(0)$$

$$Z[y(n+2)] = z^2F(z) - z^2y(0) - zy(1)$$

$$Z[y(n+3)] = z^3F(z) - z^3y(0) - z^2y(1) - zy(2)$$

eg Solve $U_{n+2} - 5U_{n+1} + 6U_n = 4^n$, $u_0 = 0$, $u_1 = 1$

Sol^n $y(n+2) - 5y(n+1) + 6y(n) = 4^n$

$$z[y(n)] = F(z)$$

$$z[\qquad\qquad] = z[4^n]$$

$$z[y(n+2)] - 5z[y(n+1)] + 6z[y(n)] = z(4^n)$$

$$\left[z^2F(z) - z^2\cancel{y(0)}^{0} - zy(1)\right] - 5\left[zF(z) - z\cancel{y(0)}\right]$$

$$+6[F(z)] = \frac{z}{z-4}$$

$$\left[z^2F(z) - z - 5zF(z) + 6F(z)\right] = \frac{z}{z-4}$$

$$(z^2 - 5z + 6)F(z) - z = \frac{z}{z-4}$$

$$(z-2)(z-3)F(z) = z + \frac{z}{z-4}$$

$$(z-2)(z-3)\,F(z) = \frac{z^2 - 4z + z}{z-4}$$

$$(z-2)\cancel{(z-3)}\,F(z) = \frac{z\cancel{(z-3)}}{(z-4)}$$

$$F(z) = \frac{z}{(z-4)(z-2)}$$ → now Partial fract.