Sumedh

- z - transform

$$z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$ (Unilateral)

$$z[f(t)] = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$z[f(n)] = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n}$$ (Bilateral)

$$\therefore\ z[f(n)] = F(z)$$

Series

$$(1-x)^{-1} = 1 + x + x^2 + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 \dots$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

eg $z[1]$

Sol^n $z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$z[1] = \sum \frac{1}{z}$$

$$\therefore\ 1 + \frac{1}{z} + \frac{1}{z^2} + \dots$$

$$z[1] = \left(1 - 1/z\right)^{-1}$$

$$= \frac{z}{z-1}$$

eg $z[a^n]$

Sol^n $z[f(n)] = \sum \frac{f(n)}{z^n}$

$z[a^n] = \sum \frac{a^n}{z}$

$= \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} \cdots$

$z[\ ] = (1 - a/z)^{-1}$

$z[a^n] = \frac{z}{z-a}$

Sim^lly $z[2^n] = \frac{z}{z-2}$

eg $z[n]$

Sol^n $z[f(n)] = \sum f(n) z^{-n}$

$z[n] = \sum \frac{n}{z^n}$

$= \frac{1}{z} + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + \cdots$

$= \frac{1}{z}\left(1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right)$

$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}$

$= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2}$

$= \frac{z^{\cancel{2}}}{\cancel{z}} \frac{1}{(z-1)^2}$

$z[n] = \frac{z}{(z-1)^2}$

eg $Z[1/n]$, $n>0$

Sol^n $$Z[f(n)] = \sum_{n=1}^{\infty} f(n) \cdot z^{-n}$$

$$Z[1/n] = \sum \frac{1}{n z^n}$$

$$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3$$

$$= \frac{(1/z)}{1} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \dots$$

$$= -\log(1 - 1/z)$$

$$= -\log\left(\frac{z-1}{z}\right)$$

$$= \log\left(\frac{z-1}{z}\right)^{-1}$$

$$Z[1/n] = \log\left(\frac{z}{z-1}\right)$$

- **Properties**

1. Change of Scale

$$Z[a^k f(k)] = F(z/a)$$

2. Shifting

$$Z[f(k+n)] = z^n F(z)$$

$$Z[f(k-n)] = z^{-n} F(z)$$

3. Multiplication

$$Z[k \cdot f(k)] = -z \frac{d}{dz} F(z)$$

$$Z[k^n f(k)] = \left(-z \frac{d}{dz}\right)^n F(z)$$

4. Division

$$Z\left[\frac{f(k)}{k}\right] = \int_z^{\infty} \frac{1}{z} F(z)\,dz$$

5. Convolution

$$f(n) * g(n) = \sum f(r) \cdot g(n-r)$$

eg $$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a} \times \frac{z}{z-b}\right]$$

$$= Z^{-1}\left[F(z) \cdot G(z)\right]$$

$$= Z^{-1}\left[F(z)\right] * Z^{-1}\left[G(z)\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] \times Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$\therefore a^n * b^n = \sum_{r=0}^{n} a^r b^{n-r}$$

$$= \sum a^r b^n b^{-r}$$

$$= b^n \sum_{r=0}^{n} \left(\frac{a}{b}\right)^r$$

$$= b^n \left(1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \cdots\right)$$

$$\left[1 + a + a^2 + \cdots = \frac{1 - a^{n+1}}{1 - a}\right]$$

$$= b^n \left[\frac{1 - (a/b)^{n+1}}{1 - a/b}\right]$$

$$= b^n \left[\frac{\dfrac{b^{n+1} - a^{n+1}}{b^{n+1}}}{\dfrac{b-a}{b}} \right]$$

$$= b^n \left[\frac{b}{b^{n+1}} \times \frac{b^{n+1} - a^{n+1}}{b-a} \right]$$

$$= \frac{b^{n+1} - a^{n+1}}{b-a}$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

gen formula

Imp Formulae

$$z[1] = \frac{z}{z-1}$$

$$z[a^n] = \frac{z}{z-a}$$

$$z[\sin \alpha n] = \frac{z \sin \alpha}{z^2 - 2z\cos\alpha + 1}$$

$$z[\cos \alpha n] = \frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$$

$$z[\sinh \alpha n] = \frac{z \cdot \sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$$

$$z[\cosh \alpha n] = \frac{z^2 - z\cosh\alpha}{z^2 - 2z\cosh\alpha + 1}$$

- Inverse z- transform

· By partial function

eg find $z^{-1}\ \dfrac{z}{(z-1)(z-2)}$ if $|z| \geq 2$

Sol^n $F(z) = \dfrac{z}{(z-1)(z-2)}$

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)}$$

$$\frac{1}{(\quad)(\quad)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$\frac{1}{\cancel{(\quad)(\quad)}} = \frac{A(z-2) + B(z-1)}{\cancel{(\quad)(\quad)}}$$

let $z = 2$

$B = 1$

at $z = 1$

$A = -1$

$$\therefore \frac{F(z)}{z} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$F(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

$$z^{-1}[F(z)] = z^{-1}\left[\frac{z}{z-2}\right] - z^{-1}\left[\frac{z}{z-2}\right]$$

at $z \geq 2$

$$= 2^n - 1^n$$

$$= 2^n - 1$$

if does not satisfy then just write same answer with -ve sign

• Inversion Integral Method

eg find $f(n)$ if $F(z) = \dfrac{10z}{(z-1)(z-2)}$

Sol^n here pole $z = 1, 2$

consider $F(z).z^{n-1}$ $= \dfrac{10z.z^{n-1}}{(\quad)(\quad)}$

$= \dfrac{10z^n}{(\quad)(\quad)}$

find residue

$F(z)\,z^{n-1}$ at $z=1$

$= \left.\cancel{(z-1)}\dfrac{10z^n}{\cancel{(z-1)}(z-2)}\right|_{z=1}$

$= \dfrac{10(1)}{-1}$

$= -10$

at $z=2$

$\left.\cancel{(z-2)}\dfrac{10z^n}{(z-1)\cancel{(z-2)}}\right|_{z=2}$

10.2^n

$f(n) = -10 + 10.2^n$

$= 10(2^n - 1)$

Power Series

eg find $z^{-1}\left(\frac{1}{z-a}\right)$ when $|z| < |a|$

Soln

make sure < 1 eg if $|z| > |a|$

$1 > \frac{|a|}{|z|}$

$\therefore \quad \left|\frac{z}{a}\right| < 1$

$$F(z) = \frac{1}{z-a}$$

$$= \frac{1}{a\,(z/a - 1)}$$

To satisfy

$$F(z) = \frac{1}{-a\,(1 - z/a)}$$

$$= \frac{1}{-a}\left[1 + \frac{z}{a} + \frac{z^2}{a^2} + \dots\right]$$

$$= -\left[\frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \frac{z^3}{a^4}\right]$$

$$= -\left[\frac{z^n}{a^{n+1}}\right]$$

$$= -\left[z^n \cdot (a)^{-n-1}\right]$$

coeff of z^n $= -\,a^{-(n+1)}$

of z^{-n} $= -\,a^{-n+1}$

answer

· Long - division method

eg $x[z] = \dfrac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$, $|z|>1$ — causal

$|z|<1$: Anticausal

Sol^n

$$1+4z^{-1}+7z^{-2}+10z^{-3}+13z^{-4}+\dots$$

$$1-2z^{-1}+z^{-2} \,\big)\, 1+2z^{-1}$$

$$\underline{-\;1 \mp 2z^{-1} + z^{-2}}$$

$$0 \quad \cancel{4z^{-1}} - z^{-2}$$

$$\cancel{4z^{-1}} - 8z^{-2} + 4z^{-3}$$

$$\underline{- \qquad + \qquad -}$$

$$\cancel{7z^{-2}} - 4z^{-3}$$

$$\cancel{7z^{-2}} - 14z^{-3} + 7z^{-4}$$

$$\underline{- \qquad + \qquad -}$$

$$\cancel{10z^{-3}} - 7z^{-4}$$

$$\cancel{10z^{-3}} - 20z^{-4} + 10z^{-5}$$

$$\underline{- \qquad + \qquad -}$$

$$13z^{-4} - 10z^{-5}$$

$$\vdots$$

$X(z) =$ answer

- Difference equation

$$Z[y(n)] = F(z)$$

$$Z[y(n+1)] = zF(z) - zy(0)$$

$$Z[y(n+2)] = z^2F(z) - z^2y(0) - zy(1)$$

$$Z[y(n+3)] = z^3F(z) - z^3y(0) - z^2y(1) - zy(2)$$

eg Solve $U_{n+2} - 5U_{n+1} + 6U_n = 4^n$, $U_0 = 0$, $U_1 = 1$

Sol^n $y(n+2) - 5y(n+1) + 6y(n) = 4^n$

$$z[y(n)] = F(z)$$

$$z[\qquad\qquad] = z[4^n]$$

$$z[y(n+2)] - 5z[y(n+1)] + 6z[y(n)] = z(4^n)$$

$$\left[z^2F(z) - z^2\cancel{y(0)}^{0} - zy(1)\right] - 5\left[zF(z) - z\cancel{y(0)}^{0}\right]$$

$$+ 6[F(z)] = \frac{z}{z-4}$$

$$\left[z^2F(z) - z - 5zF(z) + 6F(z)\right] = \frac{z}{z-4}$$

$$(z^2 - 5z + 6)F(z) - z = \frac{z}{z-4}$$

$$(z-2)(z-3)F(z) = z + \frac{z}{z-4}$$

$$(z-2)(z-3)F(z) = \frac{z^2 - 4z + z}{z-4}$$

$$(z-2)\cancel{(z-3)}F(z) = \frac{z\cancel{(z-3)}}{(z-4)}$$

$$F(z) = \frac{z}{(z-4)(z-2)}$$ → now Partial fract.

Part A		Marks: 8 X 5 = 40
SI. No	Question	Answers
1	Find Z-transform of $r^n \sin n\theta$.	$\dfrac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}$
2	Use convolution theorem to find the inverse Z transform of $\dfrac{8z^2}{(2z-1)(4z+1)}$	$\frac{2}{3}(1/2)^n + \frac{1}{3}(-1/4)^n$
3	Use Long Division Method to find the inverse Z transform of $\frac{4z}{(z-1)^3}$	$2(n-1)nU(n)$
4	Use Partial Fraction Method to find the inverse Z transform of $\dfrac{3^2 - 18z + 26}{(z-2)(z-3)(z-4)}$	$\dfrac{1}{2}2^n + \dfrac{1}{3}3^n + \dfrac{1}{4}4^n$
5	Use Method of Residues to find the inverse Z transform of $\dfrac{z}{z^2 + 2z + 2}$	$(\sqrt{2})^n sin\dfrac{3n\pi}{4}$
Part B		Marks: 15 X 3 = 45
1	Solve the difference equation using Z transform $y(n+2) - 7y(n+1) + 12y(n) = 2^n$, given that $y_0 = y_1 = 0$	$\mathrm{y(n)} = \frac{1}{2}2^n - 3^n + \frac{1}{2}4^n$
2	Solve the difference equation using Z transform $y(n+3) - 3y(n+1) + 2y(n) = 2^n$, given that $y_0 = 4,\ y_1 = 0, y_2 = 8$	$\mathrm{y(n)} = \frac{8}{3} + \frac{4}{3}(-2)^n$
3	Solve the difference equation using Z transform $f(n) + 3f(n-1) - 4f(n-2) = 0,\ \ n \geq 2$ given that $f(0) = 3, f(1) = -2$	$f(n) = (-4)^n + 2$

1

Soln $r^n \sin n\theta$

$$z[f(n)] = \sum_{x=0}^{\infty} f(n) z^{-n}$$

$$a = e^{i\theta}$$
$$a^n = e^{ni\theta}$$
$$a^n = e^{in\theta} = z[f(n)]$$

$$z[(e^{i\theta})^n] = \frac{z}{z - e^{i\theta}}$$

$$= \frac{z}{z - [\cos\theta + i\sin\theta]}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta} \cdot \frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$= \frac{z(\quad) + (\quad)}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{z(\ ) + (\ )}{z^2 - 2z\cos\theta + 1}$$

$$z[(e^{i\theta})^n] = \frac{z^2 - z\cos\theta + zi\sin\theta}{z^2 - 2z\cos + 1}$$

$$z[e^{in\theta}] =$$

$$z[\cos n\theta + i\sin n\theta] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos + 1} + i\frac{z\sin\theta}{z^2 - 2z\cos + 1}$$

$$\therefore z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\therefore \quad Z[r^n \sin n\theta] = \frac{(z/r)\sin\theta}{((z/r)^2 - 2(z/r)\cos\theta + 1)}$$

$$= \frac{(z/r)\sin\theta}{\frac{z^2}{r^2} - \frac{2z\cos\theta}{r} + 1}$$

$$= \frac{z\sin\theta}{\frac{z^2}{r} - 2z\cos\theta + r}$$

$$Z[r^n \sin n\theta] = \frac{z\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

2]

Sol^n Inverse of $\frac{8z^2}{(2z-1)(4z+1)}$

$$F(z) = \frac{8z^2}{(2z-1)(4z+1)}$$

$$\frac{F(z)}{z} = \frac{8z}{(2z-1)(4z+1)}$$

$$\frac{8z}{(\quad)(\quad)} = \frac{A}{(2z-1)} + \frac{B}{(4z+1)}$$

$$\cancel{\frac{8z}{(\quad)(\quad)}} = \frac{A(4z+1) + B(2z-1)}{\cancel{(2z-1)(4z+1)}}$$

$z = 1/2$

$4 = 3A$

$A = \left(\frac{4}{3}\right)$

$z = -1/4$ $\quad -2 = B\left(\frac{-3}{2}\right)$

$B = 4/3$

$$\frac{F(z)}{z} = \frac{4}{3(2z-1)} + \frac{4}{3(4z+1)}$$

$$z^{-1}\left[F(z)\right] = \frac{4}{6} z^{-1}\left[\frac{z}{z-1/2}\right] + \frac{4}{12} z^{-1}\left[\frac{z}{z+1/4}\right]$$

$$= \frac{4}{6}\left(1/2\right)^n + \frac{4}{12}\left(-1/4\right)^n$$

$$= \frac{2}{3}\left(1/2\right)^n + \frac{1}{3}\left(-1/4\right)^n$$

By Convolution

$$F(z) = \frac{8z^2}{(2z-1)(4z+1)}$$

$$= 8 \quad z(f(n)) \times z(g(n))$$

$$= 8 \times z\left[\frac{z}{(2z-1)}\right] \times z\left[\frac{z}{(4z+1)}\right]$$

$$= \cancel{\frac{8}{8}}$$

3]

Soln $\frac{4z}{(z-1)^3}$

$$F(z) = \frac{4z}{(z-1)^3}$$

$$= \frac{4z}{z^3 - 3z^2(-1) + 3(z)(-1)^2 - (-1)^3}$$

$$= \frac{4z}{z^3 + 3z^2 + 3z + 1}$$

$$1+3z+3z^2+z^3 \overline{\Big)\, 4z} \quad 4z-12z^2$$

$$4z + 12z^2 + 12z^3 + 4z^4$$

$$- \quad - \quad - \quad -$$

$$0 - 12z^2 - 12z^3 - 4z^4$$

$$-12z^2 - 36z^3 - 36z^4 - 36z^5$$

$$+ \quad + \quad + \quad +$$

$$24z^3$$

4]
Soln

$$\frac{3^2 - 18z + 26}{(z-2)(z-3)(z-4)}$$

$$F(z) = \frac{z^2 - 18z + 26}{(z-2)(z-3)(z-4)}$$

$$\frac{F(z)}{z} = \frac{z - 18 + 26/z}{(\;)(\;)(\;)}$$

$$z - 18 + \frac{26}{z} = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$$

at $z = 2$

$$2 - 18 + 13 = A(-1)(-2)$$

$$-3/2 = A$$

at $z = 3$

$$3 - 18 + \frac{26}{3} = B(1)(-1)$$

$$\frac{-45+26}{3} = -B$$

$$+19/3 = +B$$

at $z = 4$

$$4 - 18 + \frac{26}{4} = C(2)(1)$$

$$\frac{-56+26}{4} = 2C$$

$$-15/4 = C$$

$$\frac{F(z)}{z} = \frac{-3}{2(z-2)} + \frac{19}{3(z-3)} - \frac{15}{4(z-4)}$$

$$z^{-1}[F(z)] = \frac{-3}{2} z^{-1}\left[\frac{z}{z-2}\right] + \frac{19}{3} z^{-1}\left[\frac{z}{z-3}\right] - \frac{15}{4} z^{-1}\left[\frac{z}{z-4}\right]$$

$$= -\frac{3}{2}(2)^n + \frac{19}{3}(3)^n - \frac{15}{4}(4)^n$$

$$= -3(2)^{n-1} + 19(3)^{n-1} - 15(4)^{n-1}$$

5]

Soln

$$\frac{z}{z^2+2z+2}$$

$$\frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

$$z = (-1-i) \qquad z = (-1+i)$$

$$F(z)\,z^{n-1} = \frac{z \times z^{n-1}}{[z-(-1-i)]\,[z-(-1+i)]}$$

Res at $z = (-1-i)$

$$\left.\cancel{[z-(-1-i)]} \times \frac{z^n}{\cancel{[\quad]}\,[z-(-1+i)]}\right|_{z=(-1-i)}$$

$$= \frac{(-1-i)^n}{\cancel{-1}-i\ \cancel{+1}-i}$$

$$= \frac{(\quad)^n}{-2i}$$

Res $z = -1+i$

$$\left.\frac{[z-\cancel{(-1+i)}] \times z^n}{[z-(-1-i)]\ \cancel{(\quad)}}\right|_{z=(-1+i)}$$

$$= \frac{(-1+i)^n}{\cancel{-1}+i+\cancel{1}+i}$$

$$= \frac{(-1+i)^n}{2i} \qquad \frac{(-1-i)^n}{2i}$$

$$= \frac{(-1+i)^n + (-1-i)^n}{2i}$$

$$= (-i^n$$

$$Z[y(n)] = F(z)$$

$$Z[y(n+1)] = zF(z) - zy(0)$$

$$Z[y(n+2)] = z^2F(z) - z^2y(0) - zy(1)$$

$$Z[y(n+3)] = z^3F(z) - z^3y(0) - z^2y(1) - zy(2)$$

6]

Soln

Solve the difference equation using Z transform

$y(n+2) - 7y(n+1) + 12y(n) = 2^n$, given that $y_0 = y_1 = 0$

$$Z[y(n+2) - 7y(n+1) + 12y(n)] = Z[2^n]$$

$$Z[y(n+2)] - 7Z[y(n+1)] + 12Z[y(n)] = \frac{z}{z-2}$$

$$\left[z^2F(z) - z^2\cancel{y(0)}^{0} - z\cancel{y(1)}^{0}\right] - 7\left[zF(z) - z\cancel{y(0)}^{0}\right] + 12[F(z)]$$

$$F(z)\left[z^2 - 7z + 12\right] = \frac{z}{z-2}$$

$$F(z)\left[z^2 - 4z - 3z + 12\right] = \frac{z}{(z-2)}$$

$$F(z)(z-4)(z-3) = \frac{z}{(z-2)}$$

$$\frac{F(z)}{z} = \frac{1}{(z-2)(z-3)(z-4)}$$

$$\frac{1}{(\ )(\ )(\ )} = \frac{A}{(z-2)} + \frac{B}{(z-3)} + \frac{C}{(z-4)}$$

$$\frac{1}{\cancel{(\ \ )}} = \frac{A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)}{\cancel{(\ \ )}}$$

$$1 = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$$

at $z = 2$

$1 = A(-1)(-2)$

$A = 1/2$

at $z = 3$

$1 = B(1)(-1)$

$B = -1$

at $z = 4$

$1 = C(2)(1)$

$C = 1/2$

$$\frac{F(z)}{z} = \frac{1}{2(z-2)} - \frac{1}{(z-3)} + \frac{1}{2(z-4)}$$

$$z^{-1}\left[F(z)\right] = \frac{1}{2} z^{-1}\left[\frac{z}{z-2}\right] - z^{-1}\left[\frac{z}{z-3}\right] + \frac{1}{2} z^{-1}\left[\frac{z}{z-4}\right]$$

$$= \frac{1}{2}(2)^n - (3)^n + \frac{1}{2}(4)^n$$

8] Solve the difference equation using Z transform $y(n+3) - 3y(n+1) + 2y(n) = 2^n$, given that $y_0 = 4,\ y_1 = 0, y_2 = 8$

Sol^n

$$z\left[y(n+3) - 3y(n+1) + 2y(n)\right] = Z[2^n$$