Sumedh

- Wave Equation

- Type - 1 (Released disp)

eg] $y(x,0) = k(lx - x^2)$ at $(0, l)$

Sol^n BC's

1. $y(0,t) = 0$
2. $y(l,t) = 0$
3. $\left(\frac{dy}{dt}\right)_{t=0} = 0$

4. $y(x,0) = k(lx - x^2)$

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \quad -(i)$$

BC 1 in (i)

$$y(0,t) = (A(1) + B(0))(\quad) = 0$$

$$A(\quad) = 0$$

$$\therefore A = 0$$

$$\therefore y(x,t) = B\sin\lambda x(\quad) \quad -(ii)$$

BC 2 in (ii)

$$y(l,t) = B\sin\lambda l(\quad) = 0$$

$$\sin\lambda l = 0$$

$$\lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l}$$

$$\therefore y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \quad -(iii)$$

BC 3 in (iii)

$$\left(\frac{dy}{dt}\right) = B\sin\left(\frac{n\pi x}{l}\right)\left(-C\sin\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right) + D\cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right)\right)$$

at $t = 0$

$$\frac{dy}{dt} = B\sin\left(\frac{n\pi x}{l}\right)\left(D\left(\frac{n\pi a}{l}\right)\right) = 0$$

$$\therefore \quad D = 0$$

$$y(x,t) = B\sin\left(\frac{n\pi x}{l}\right) \cdot C\cos\left(\frac{n\pi a t}{l}\right)$$

Most gen. sol

$$y(x,t) = \sum B_n \sin(\;)\cos(\;) \quad -(iv)$$

BC 4 in (iv)

$$\sum B_n \sin\left(\frac{n\pi x}{l}\right) = K(lx - x^2)$$

$$\therefore \quad B_n = \frac{2}{l}\int_0^l F(x) \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{2K}{l}\int_0^l \underset{u}{(lx - x^2)} \cdot \underset{v}{\sin\left(\frac{n\pi x}{l}\right)}$$

$$u = (lx - x^2) \qquad v_1 = -\cos\left(\frac{n\pi x}{l}\right) \cdot \frac{l}{n\pi}$$

$$u' = l - 2x$$

$$u'' = -2 \qquad v_2 = -\sin\left(\frac{n\pi x}{l}\right) \cdot \frac{l^2}{n^2\pi^2}$$

$$v_3 = \cos\left(\frac{n\pi x}{l}\right) \cdot \frac{l^3}{(n\pi)^3}$$

$$= \frac{2K}{l}\Bigg[(lx - x^2)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right) + (l-2x)\left(+\sin\left(\frac{n\pi x}{l}\right) \cdot \frac{l^2}{(n\pi)^2}\right)$$

$$- (+2)\left(\cos\left(\frac{n\pi x}{l}\right) \cdot \frac{l^3}{(n\pi)^3}\right)\Bigg]_0^l$$

$$= \frac{2K}{l}\left[0 + 0 - \frac{2(-1)^n l^3}{(n\pi)^3} - \left(0 + 0 - \frac{2l^3}{(n\pi)^3}\right)\right.$$

$$B_n = \frac{2K}{l}\left[\frac{2l^3}{(n\pi)^3}\left(1 - (-1)^n\right)\right.$$

eg $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$ at $(0, l)$

Sol^n

$\vdots$

$D = 0$

$\therefore \quad y(x,t) = B \sin\left(\frac{n\pi x}{l}\right) C \cos\left(\frac{n\pi a t}{l}\right)$

Most gen. sol^n

$y(x,t) = \sum B_n \sin(\;) \cos(\;)$

By BC 4

$$\sum B_n \sin\left(\frac{n\pi x}{l}\right) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$$\therefore \quad B_1 \sin\left(\frac{\pi x}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right) + B_3 \left(\sin \frac{3\pi x}{l}\right) + \cdots$$

$$= y_0 \left[\frac{3 \sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)}{4}\right]$$

here $B_1 = \frac{3 y_0}{4}$, $B_2 = 0$, $B_3 = -\frac{y_0}{4}$

$$\therefore \quad y(x,t) = B_1 \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi a t}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right) \cos\left(\frac{2\pi a t}{l}\right)$$

$$+ B_3 \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{3\pi a t}{l}\right)$$

$$= \frac{3y_0}{4} \sin(\;) \cos(\;) - \frac{y_0}{4} \sin(\;) \cos(\;)$$

- Type - 2 (Equillibrium disp)

eg $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 3x(l-x)$ at $(0, l)$

Sol^n BC 's

1. $y(0, t) = 0$
2. $y(l, t) = 0$
3. $y(x, 0) = 0$
4. $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 3x(l-x)$

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \quad -(i)$$

BC 1 in (i)

$$y(0,t) = (A(1) + B(0))(\quad) = 0$$

$$\therefore A = 0$$

$$y(x,t) = (B\sin\lambda x)(\quad) \quad -(ii)$$

BC 2 in (ii)

$$y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left(C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right) \quad -(iii)$$

BC 3 in (iii)

$$y(x,0) = B\sin\left(\frac{n\pi x}{l}\right)\left(C(1) + D(0)\right)$$

$$\therefore C = 0$$

$$\therefore y(x,t) = B\sin\left(\frac{n\pi x}{l}\right) D\sin\left(\frac{n\pi at}{l}\right)$$

Most gen sol^n

$$y(x,t) = B_n \sin(\quad)\sin(\quad) \quad -(iv)$$

BC 4 in (iv)

$$\frac{dy}{dt} = B_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right) \cdot \left(\frac{n\pi a}{l}\right)$$

at $t = 0$

$$\left(\frac{dy}{dt}\right)_{t=0} = B_n \sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right) = 3(lx - x^2)$$

$$\therefore \quad B_n\left(\frac{n\pi a}{l}\right) = \frac{2}{l}\int_0^l 3(lx - x^2)\cdot \sin\left(\frac{n\pi x}{l}\right)$$

$$\begin{aligned} u &= (lx - x^2) & v_1 &= -\cos\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi} \\ u' &= (l - 2x) & v_2 &= -\sin\left(\frac{n\pi x}{l}\right)\cdot\frac{l^2}{(n\pi)^2} \\ u'' &= -2 & v_3 &= \cos\left(\frac{n\pi x}{l}\right)\cdot\frac{l^3}{(n\pi)^3} \end{aligned}$$

$$B_n\left(\frac{n\pi a}{l}\right) = \frac{6}{l}\left[-(lx - x^2)\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi} + \sin\left(\frac{n\pi x}{l}\right)\frac{l^2}{(n\pi)^2} - 2\cos\left(\frac{n\pi x}{l}\right)\frac{l^3}{(n\pi)^3}\right]_0^l$$

$$= \frac{6}{l}\left[0 + 0 - \frac{2(-1)^n l^3}{n\pi} - \left(-0 + 0 - \frac{2l^3}{(n\pi)^3}\right)\right]$$

$$= \frac{6}{l}\left[\frac{2l^3}{(n\pi)^3}\left(1 - (-1)^n\right)\right]$$

$$B_n\left(\frac{n\pi a}{l}\right) = \frac{12 l^2}{(n\pi)^3}\left[1 - (-1)^n\right]$$

$$B_n = \frac{12 a l^3}{(n\pi)^4 \cdot a}\left[1 - (-1)^n\right]$$

eg $\left(\frac{dy}{dt}\right)_{t=0} = V_0 \sin^3\left(\frac{\pi x}{l}\right)$ at $(0, l)$

Sol^n

$$y(x,t) = \sum B_n\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right) = V_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$$B_1\left(\frac{\pi a}{l}\right)\sin\left(\frac{\pi x}{l}\right) + B_2\left(\frac{2\pi a}{l}\right)\sin\left(\frac{2\pi x}{l}\right)$$

$$+ B_3\left(\frac{3\pi a}{l}\right)\sin\left(\frac{3\pi x}{l}\right) = V_0\left(\frac{3\sin(\pi x/l) - \sin\left(\frac{3\pi x}{l}\right)}{4}\right)$$

$$B_1 = \frac{3V_0 l}{4\pi a}, \quad B_2 = 0, \quad B_3 = \frac{-V_0 l}{12\pi a}$$

- Heat Equation

eg $l = 30$ cm, T ends $= 20$ & $T = 80°$, both ends T reduced to o

Sol^n $u(0,t) = 20$

$u(30,t) = 80$

$u_x = Ax + B$

$20 = A(0) + B$

$B = 20$

$u_x = Ax + B$

$80 = A(30) + 20$

$A = 2$

$\therefore u_x = 2x + 20 \longrightarrow$ Steady state

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

BC's

1. $u(0,t) = 0$
2. $u(30,t) = 0$
3. $u(x,0) = 2x + 20$

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, C e^{-\alpha^2\lambda^2 t} \quad -(i)$$

BC 1 in (i)

$$u(0,t) = (A(1) + B(0))\, C e^{-\alpha^2\lambda^2 t} = 0$$

$$\therefore \quad A = 0$$

$$u(x,t) = (B\sin\lambda x)\, C e^{(\ )} \quad -(ii)$$

BC 2 in (ii)

$$u(30,t) = (B\sin\lambda 30)\, C e^{-\alpha^2 t\lambda^2} = 0$$

$$\sin\lambda 30 = 0$$
$$30\lambda = n\pi$$
$$\lambda = \frac{n\pi}{30}$$

$$\therefore \quad u(x,t) = (B\sin 30\lambda)\, C e^{-\alpha^2 t \frac{n^2\pi^2}{900}} \quad -(iii)$$

$$= B_n(\sin 30\lambda)(e^{(\ )})$$

BC 3

$$u(x,0) = B_n \sin 30\lambda = 2x + 20$$

$$\therefore \quad B_n = \frac{2}{l}\int_0^{30} (2x+20)\sin\left(\frac{n\pi x}{30}\right) dx$$

$$u = (2x+20) \qquad V_1 = -\cos\left(\frac{n\pi x}{30}\right)\frac{30}{n\pi}$$
$$u' = 2$$

$$V_2 = -\sin\left(\frac{n\pi x}{30}\right)\frac{900}{(n\pi)^2}$$

$$B_n = \frac{2}{30}\left[-(2x+20)\cos\left(\frac{n\pi x}{30}\right)\cdot\frac{30}{n\pi} + (2)\sin\left(\frac{n\pi x}{30}\right)\frac{900}{(n\pi)^2}\right]_0^{30}$$

$$= \frac{2}{30}\left[-80\,\frac{(-1)^n}{n\pi}\cdot 30 + 0 - \left(\frac{-20\cdot 30}{n\pi}\right)\right]$$

$$= \frac{2}{\cancel{30}}\left[-\frac{\overset{80}{\cancel{2400}}(-1)^n}{n\pi} + \frac{\overset{20}{\cancel{600}}}{n\pi}\right]$$

$$= -\frac{160(-1)^n}{n\pi} + \frac{40}{n\pi}$$

$$B_n = \frac{40}{n\pi}\left(1 - 4(-1)^n\right)$$

$$a_0 = \frac{2}{l}\int_0^3 x^2 \cdot dx$$

a_0

$$= \frac{4}{3}\left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{4}{3}\Big[$$

$$= 12/2 = 6$$

$$a_n = 4/3 \int_0^{8} x^2 \cdot \cos\frac{2n\pi x}{3}$$

x^2

$2x$

2

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(f(x))\, e^{-isx}\, ds$$

$$= \frac{1}{2\pi} \int \sqrt{\frac{2}{\pi}} \frac{\sin(sa)}{s} e^{-isx} \cdot dx$$

$$f(0) = 1/\pi \int_{-\infty}^{\infty} \frac{\sin(sa)}{s}\, ds$$

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin x}{x/a} \cdot \frac{dx}{a}$$

$$\therefore \int_{-\infty}^{\infty} (f(x))^2 = \int_{-\infty}^{\infty} F(f(x))^2\, ds$$

$$\int_{-a}^{a} 1^2\, dx = \int_{-\infty}^{\infty} \left[\sqrt{2/\pi}\, \frac{\sin(sa)}{s}\right]^2 ds$$

$$x\Big]_{-a}^{a} = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(s a)}{s^2}$$

$$= \pi/2$$

Q.] Find the fourier tranform $f(x) : \begin{cases} a - |x| & |x| < a \\ 0 & |x| > a \end{cases}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a - |x|)(\cos + i\sin x)$$

$$= \frac{1}{\sqrt{2\pi}} \int \underbrace{(\ )(\cos x)}_{\text{even}} + \int \underbrace{(\ )(\sin x)}_{\text{odd}}$$

$$= \frac{1}{\sqrt{2\pi}} \int \qquad + \quad 0$$