# Solid Object Visualisation Case Study

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**Abstract**

“Please pay particular attention to the preparation of your abstract; use the material in this reference as a guide. Every manuscript other than a discussion must be accompanied by an informative abstract of no more than one paragraph (200 to 300 words). The abstract should be self-contained. No references, figures, tables, or equations are allowed in an abstract. Do not use new terminology in an abstract unless it is defined or is well known from prior publications. SEG discourages the use of commercial names or parenthetical statements. The abstract must not simply list the topics covered in the paper but should (1) state the scope and principal objectives of the research, (2) describe the methods used, (3) summarize the results, and (4) state the principal conclusions. Do not refer to the paper itself in the abstract. For example, do not say, “In this paper, we will discuss”

The abstract must stand alone as a very short version of the paper rather than as a description of the contents. Remember that the abstract will be the most widely read portion of the paper. Various groups throughout the world publish abstracts of Geophysics papers. Readers and occasionally even reviewers may be influenced by the abstract to the point of final judgement before the body of the paper is read.”

Blah blah blah

# Chapter 1

# Introduction

> “The purpose of the introduction is to tell readers why they should want to read what follows the introduction. This section should provide sufficient background information to allow readers to understand the context and significance of the problem. This does not mean, however, that authors should use the introduction to rederive established results or to indulge in other needless repetition. The introduction should (1) present the nature and scope of the problem; (2) review the pertinent literature, within reason; (3) state the objectives; (4) describe the method of investigation; and (5) describe the principal results of the investigation.”

Blah blah blah

# Chapter 2

# Methods

Transformations are operations which can be carried out on geometric data. Usually in the form:

$$\mathbf{p'} = f(\mathbf{p})$$

Where $\mathbf{p}$ is a point in space, $f$ is a transformation function and $\mathbf{p'}$ is a point which results from that function.

## 2.1 Shift

Shift (or translation) is one of the most basic linear transformations. Let $\mathbf{p}$ be a point is space and $\mathbf{t}$ be a vector describing the offset from that point. The new point $\mathbf{p'}$ is then defined as:

$$\boxed{\mathbf{p'} = \mathbf{p} + \mathbf{t}}$$

Note that a point has a position and no direction, while a vector has a direction and no position. Because of that operations between points and vectors (such as addition) are not defined, however a point $\mathbf{p}$ can be expressed as a displacement $\mathbf{t_p}$ from the origin of the coordinate system $\mathbf{O}$. This property of points is often assumed and the displacement vector $\mathbf{t_p}$ is what is implied instead. The same convention is used in this paper.

### 2.1.1 Deriving a matrix

In the case where $\mathbf{p}, \mathbf{t} \in \mathbb{R}^3$ the above equation can also be expressed as a system of 3 equations (one for each axis).

$$\begin{aligned} p'_x &= p_x + t_x \\ p'_y &= p_y + t_y \\ p'_z &= p_z + t_z \end{aligned}$$

We can expand the equations slightly...

$$\begin{aligned}
p\prime_x &= p_x 1 + p_y 0 + p_z 0 + t_x \\
p\prime_y &= p_x 0 + p_y 1 + p_z 0 + t_y \\
p\prime_z &= p_x 0 + p_y 0 + p_z 1 + t_z
\end{aligned}$$

...to make it obvious how we can derive the homogeneous 3D shift matrix.

$$\begin{bmatrix} p\prime_x \\ p\prime_y \\ p\prime_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

## 2.2 Scale

Scale (or stretch) is another linear transformation which displaces a vector in certain proportion to its distance from the origin.

The simplest form of scale is:

$$\mathbf{p}\prime = \alpha \mathbf{p}$$

Where $\alpha$ is a scale factor. This function scales the original point $\mathbf{p}$ homogeneously in all directions.

If we wanted to scale is a specific direction, let it be defined as the unit vector $\hat{\mathbf{s}}$, we could do so by displacing the original point (in the direction $\hat{\mathbf{s}}$) by an amount equal to the projection of $\mathbf{p}$ onto $\hat{\mathbf{s}}$ scaled by a factor $\alpha - 1$ (see 2.1).

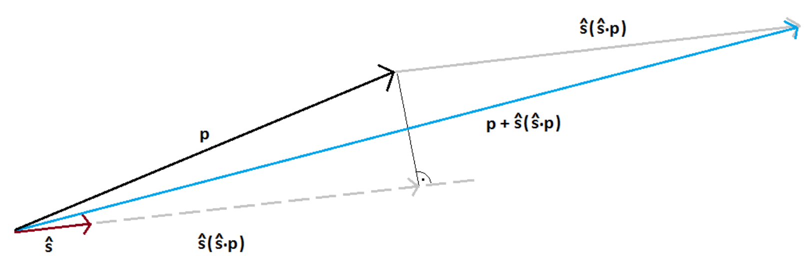


Figure 2.1: arbitrary axis scale

The resulting formula is this:

$$\boxed{\mathbf{p}\prime = \mathbf{p} + (\alpha - 1)\hat{\mathbf{s}}(\hat{\mathbf{s}} \cdot \mathbf{p})}$$

Let's consider the special case when $\mathbf{p} \perp \hat{\mathbf{s}}$. In that case $\hat{\mathbf{s}} \cdot \mathbf{p} = 0$, so we get the following equation:

$$\mathbf{p}\prime = \mathbf{p} + (\alpha - 1)\hat{\mathbf{s}}0 = \mathbf{p} + \mathbf{0} = \mathbf{p}$$

I.e. $\mathbf{p}$ is unaffected by the scaling operation, which is what we would expect.

Another interesting case is when $\mathbf{p} \parallel \hat{\mathbf{s}}$. In that case $\hat{\mathbf{s}} \cdot \mathbf{p} = |\mathbf{p}|$, which is the maximal value the expression can acquire, which means scaling does the most shift.

Let $\hat{\mathbf{s1}} \perp \hat{\mathbf{s2}}$ or $\hat{\mathbf{s1}} \parallel \hat{\mathbf{s2}}$. If we do two subsequent scale transformations on vector $\mathbf{p}$ along each of the two axis the resulting point $\mathbf{p\prime}$ the same regardless of the order of the operations. If none of the two conditions holds true then $\mathbf{p\prime}$ will be more affected by the direction vector to be used for the first scaling operation. Given this, let $\hat{\mathbf{x}} \perp \hat{\mathbf{y}} \perp \hat{\mathbf{z}}$; $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, \hat{\mathbf{p}} \in \mathbb{R}^3$ and $\alpha, \beta$ and $\gamma$ be scale actors along each vector respectively. We can combine 3 subsequent scaling operations in each direction into a single formula:

$$\mathbf{p\prime} = \alpha\hat{\mathbf{x}}(\mathbf{p} \cdot \hat{\mathbf{x}}) + \beta\hat{\mathbf{y}}(\mathbf{p} \cdot \hat{\mathbf{y}}) + \gamma\hat{\mathbf{z}}(\mathbf{p} \cdot \hat{\mathbf{z}})$$

### 2.2.1 Deriving a matrix

In the case where $\mathbf{p}, \hat{\mathbf{s}} \in \mathbb{R}^3$ we can decompose the arbitrary axis equation in the following system of 3 equations:

$$\begin{aligned}
p\prime_x &= p_x + (\alpha - 1)s_x(s_x p_x + s_y p_y + s_z p_z) \\
p\prime_y &= p_y + (\alpha - 1)s_y(s_x p_x + s_y p_y + s_z p_z) \\
p\prime_z &= p_z + (\alpha - 1)s_z(s_x p_x + s_y p_y + s_z p_z)
\end{aligned}$$

We can rearrange each so that we have only one occurrence of $p_x, p_y, p_z, p\prime_x, p\prime_y$ and $p\prime_z$. And we get the following:

$$\begin{aligned}
p\prime_x &= (1 + (\alpha - 1)s_x^2)p_x + (\alpha - 1)s_x s_y p_y + (\alpha - 1)s_x s_z p_z \\
p\prime_y &= (\alpha - 1)s_y s_x p_x + (1 + (\alpha - 1)s_y^2)p_y + (\alpha - 1)s_y s_z p_z \\
p\prime_z &= (\alpha - 1)s_z s_x p_x + (\alpha - 1)s_z s_y p_y + (1 + (\alpha - 1)s_z^2)p_z
\end{aligned}$$

Extracting a homogeneous 3D matrix from the above system of equations is easy.

$$\begin{bmatrix} p\prime_x \\ p\prime_y \\ p\prime_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + (\alpha - 1)s_x^2 & (\alpha - 1)s_x s_y & (\alpha - 1)s_x s_z & 0 \\ (\alpha - 1)s_x s_y & 1 + (\alpha - 1)s_y^2 & (\alpha - 1)s_y s_z & 0 \\ (\alpha - 1)s_x s_z & (\alpha - 1)s_y s_z & 1 + (\alpha - 1)s_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Now, let $\hat{\mathbf{x}} = (1, 0, 0); \hat{\mathbf{y}} = (0, 1, 0); \hat{\mathbf{z}} = (0, 0, 1)$ and $\alpha, \beta$ and $\gamma$ be scale factors in each direction respectively. If we replaced $\hat{\mathbf{s}}$ in the above matrix, with each and multiplied the 3 resulting matrices (after simplifying) we would get:

$$\mathbf{A} = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that the order of the multiplication does not matter as $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are mutually perpendicular.

## 2.3 Reflect

Reflection is a linear operation, where a point $\mathbf{p}$ is translated twice its distance to a reflector plane in a direction opposite to the normal vector of that plane. Consider the diagram 2.2:

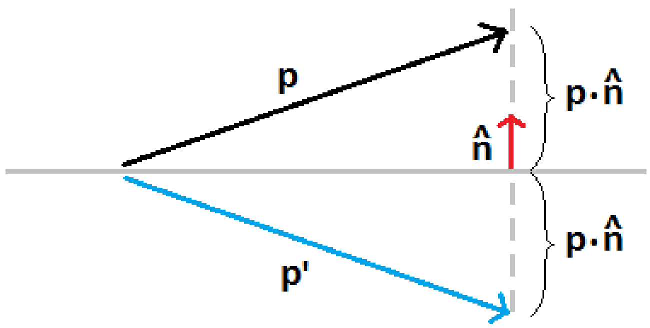


Figure 2.2: reflection

The formula then becomes obvious:

$$\boxed{\mathbf{p\prime} = \mathbf{p} - 2\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p})}$$

If $\mathbf{p}$ lies on the plane then $\hat{\mathbf{n}} \cdot \mathbf{p} = 0$, so we get that $\mathbf{p\prime} = \mathbf{p}$, which shouldn't be surprising.

Another thing we might want to consider is when $\mathbf{p}$ is behind the plane. In that case $\hat{\mathbf{n}} \cdot \mathbf{p} < 0$ which means we'll be going in the direction of the plane normal (rather than it's opposite), so $\mathbf{p\prime}$ will be in front of the plane. This also means that if we applied the same reflection transformation to a point twice (or any even number of times) we would get the same point.

An important thing to note is that we cannot define any plane simply by its normal direction, planes also have an offset from the origin as it could be expressed by the following plane equation:

$$Ap_x + Bp_y + Cp_z + D = \hat{\mathbf{n}} \cdot \mathbf{p} + D = 0$$

The above reflection equation assumes that $D = 0$. All we have to do to adapt it the formula to any plane is to translate the point $\mathbf{p}$ by $-D\hat{\mathbf{n}}$ (and effectively move the origin of the coordinate system to the plane) and then, after we reflect it, translate the resulting point $\mathbf{p\prime}$ back by $D\hat{\mathbf{n}}$.

### 2.3.1 Deriving a matrix

Deriving a 3D matrix is again quite easy. We first decompose the equation:

$$
\begin{aligned}
p\prime_x &= p_x - 2n_x(n_x p_x + n_y p_y + n_z p_z) \\
p\prime_y &= p_y - 2n_y(n_x p_x + n_y p_y + n_z p_z) \\
p\prime_z &= p_z - 2n_z(n_x p_x + n_y p_y + n_z p_z)
\end{aligned}
$$

Then rearrange so that we only have one occurrence of $p_x, p_y, p_z, p\prime_x, p\prime_y$ and $p\prime_z$and we arrive at:

$$
\begin{aligned}
p\prime_x &= (1 - 2n_x^2)p_x - 2n_x n_y p_y - 2n_x n_z p_z \\
p\prime_y &= -2n_x n_y p_x + (1 - 2n_y^2)p_y - 2n_y n_z p_z \\
p\prime_z &= -2n_x n_z p_x - 2n_y n_z p_y + (1 - 2n_z^2)p_z
\end{aligned}
$$

Which maps perfectly into a 3x3 matrix:

$$
\begin{bmatrix} p\prime_x \\ p\prime_y \\ p\prime_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 2n_x^2 & -2n_x n_y & -2n_x n_z & 0 \\ -2n_x n_y & 1 - 2n_y^2 & -2n_y n_z & 0 \\ -2n_x n_z & -2n_y n_z & 1 - 2n_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}
$$

## 2.4 Parallel Projection

Parallel projection is a linear transformation which maps a point in a 3D scene onto a projection plane, where the distance of an object from the projection plane does not affect its appearance.

We could achieve the effect of parallel projection similarly to the reflection transformation discussed earlier. The only difference would be instead of going twice the distance in the direction opposite to the surface normal, we go once. In other words using this formula:

$$\mathbf{p\prime} = \mathbf{p} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p})$$

This would effectively translate $\mathbf{p}$ onto the surface defined by the normal $\hat{\mathbf{n}}$, but that is a special case of parallel projection called orthographic projection, because the projection ray is perpendicular to the surface (and coincides with the surface normal). In the general case however this will not necessarily be true, take a look at 2.3. In this case $\mathbf{c}$ the viewing direction is not parallel to the plane normal $\hat{\mathbf{n}}$.

Because we know that $\mathbf{p\prime}$ lies in the plane, so it must satisfy the plane equation:

$$\mathbf{p\prime} \cdot \hat{\mathbf{n}} + D = 0$$

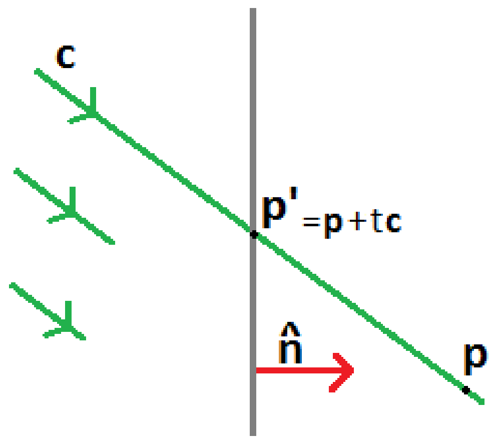


Figure 2.3: parallel projection

As with the reflection transform we will assume that the plane passes through the origin of the coordinate system (and will correct that later), which means we assume that $D = 0$.

We also know that the projected point $\mathbf{p\prime}$ will lie on a line which goes trough the original point $\mathbf{p}$ and has direction $\mathbf{c}$, so it can be expressed as the result of the line equation:

$$\mathbf{p\prime} = \mathbf{p} + t\mathbf{c}$$

Where $t$ is a specific value we want to find. Replacing $\mathbf{p\prime}$ back into the plane equation gives us:

$$(\mathbf{p} + t\mathbf{c}) \cdot \hat{\mathbf{n}} = (\mathbf{p} \cdot \hat{\mathbf{n}}) + t(\mathbf{c} \cdot \hat{\mathbf{n}}) = 0$$

And then we solve for $t$:

$$t = -\frac{\mathbf{p} \cdot \hat{\mathbf{n}}}{\mathbf{c} \cdot \hat{\mathbf{n}}}$$

Replacing $t$ into the line equation gives us the parallel projection equation:

$$\boxed{\mathbf{p\prime} = \mathbf{p} - \frac{\mathbf{p} \cdot \hat{\mathbf{n}}}{\mathbf{c} \cdot \hat{\mathbf{n}}}\mathbf{c}}$$

Now, lets consider orthographic projection again. When $\hat{\mathbf{n}} \parallel \mathbf{c}$ it is also true that $\hat{\mathbf{c}} = \frac{\mathbf{c}}{\mathbf{c} \cdot \hat{\mathbf{n}}} = \hat{\mathbf{n}}$. If we now specialized the equation we would get exactly our orthographic projection equation from before.

### 2.4.1 Deriving a matrix

Following the standard procedure we decompose the equation into a system of three equation:

$$
\begin{aligned}
p\prime_x &= p_x - \frac{p_x n_x + p_y n_y + p_z n_z}{\hat{\mathbf{n}} \cdot \mathbf{c}} cx \\
p\prime_y &= p_y - \frac{p_x n_x + p_y n_y + p_z n_z}{\hat{\mathbf{n}} \cdot \mathbf{c}} cy \\
p\prime_z &= p_z - \frac{p_x n_x + p_y n_y + p_z n_z}{\hat{\mathbf{n}} \cdot \mathbf{c}} cz
\end{aligned}
$$

Then we rearrange and end up with three equations which look something like this:

$$
\begin{aligned}
p\prime_x &= (1 - \frac{n_x c_x}{\hat{\mathbf{n}} \cdot \mathbf{c}}) p_x - \frac{n_y c_x}{\hat{\mathbf{n}} \cdot \mathbf{c}} p_y - \frac{n_z c_x}{\hat{\mathbf{n}} \cdot \mathbf{c}} p_z \\
p\prime_y &= -\frac{n_x c_y}{\hat{\mathbf{n}} \cdot \mathbf{c}} p_x + (1 - \frac{n_y c_y}{\hat{\mathbf{n}} \cdot \mathbf{c}}) p_y - \frac{n_z c_y}{\hat{\mathbf{n}} \cdot \mathbf{c}} p_z \\
p\prime_z &= -\frac{n_x c_z}{\hat{\mathbf{n}} \cdot \mathbf{c}} p_x - \frac{n_y c_z}{\hat{\mathbf{n}} \cdot \mathbf{c}} p_y + (1 - \frac{n_z c_z}{\hat{\mathbf{n}} \cdot \mathbf{c}}) p_z
\end{aligned}
$$

And finally we construct our matrix:

$$
\begin{bmatrix} p\prime_x \\ p\prime_y \\ p\prime_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{n_x c_x}{\hat{\mathbf{n}} \cdot \mathbf{c}} & -\frac{n_y c_x}{\hat{\mathbf{n}} \cdot \mathbf{c}} & -\frac{n_z c_x}{\hat{\mathbf{n}} \cdot \mathbf{c}} & 0 \\ -\frac{n_x c_y}{\hat{\mathbf{n}} \cdot \mathbf{c}} & 1 - \frac{n_y c_y}{\hat{\mathbf{n}} \cdot \mathbf{c}} & -\frac{n_z c_y}{\hat{\mathbf{n}} \cdot \mathbf{c}} & 0 \\ -\frac{n_x c_z}{\hat{\mathbf{n}} \cdot \mathbf{c}} & -\frac{n_y c_z}{\hat{\mathbf{n}} \cdot \mathbf{c}} & 1 - \frac{n_z c_z}{\hat{\mathbf{n}} \cdot \mathbf{c}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}
$$

Notice that we divide by $\hat{\mathbf{n}} \cdot \mathbf{c}$ in every cell and that $-$ sign, we could factor these divisions into the $w$ parameter and have something which looks much nicer and requires less operations.

$$
\begin{bmatrix} p\prime_x \\ p\prime_y \\ p\prime_z \\ p\prime_w \end{bmatrix} = \begin{bmatrix} n_x c_x - \hat{\mathbf{n}} \cdot \mathbf{c} & n_y c_x & n_z c_x & 0 \\ n_x c_y & n_y c_y - \hat{\mathbf{n}} \cdot \mathbf{c} & n_z c_y & 0 \\ n_x c_z & n_y c_z & n_z c_z - \hat{\mathbf{n}} \cdot \mathbf{c} & 0 \\ 0 & 0 & 0 & -\hat{\mathbf{n}} \cdot \mathbf{c} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}
$$

## 2.5 Perspective Projection

Perspective projection is a transformation in which 3D points are mapped onto a 2D surface (or plane). It has the property to affect the size of the projected objects (typically the further an object is the smaller its projection is). This projection is based on the pinhole camera model where all rays cross a projection plane and converge into a single point. Consider 2.4:

It looks different to parallel projection and 2.3, but also bares a lot of resemblance. The most significant difference is than instead of a constant viewing direction $\mathbf{c}$ we now have a viewing direction that is different for every point. If we chose a focal point $\mathbf{v}$ we can compute that viewing direction as $\mathbf{p} - \mathbf{v}$. Then we can replace $\mathbf{c}$ with the new direction and we end up with:

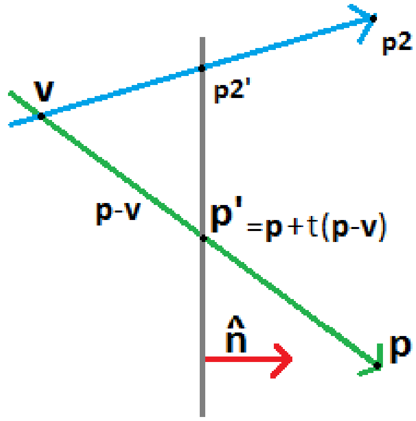


Figure 2.4: parallel projection

$$\boxed{\mathbf{p\prime} = \mathbf{p} - \frac{\mathbf{p} \cdot \hat{\mathbf{n}}}{(\mathbf{p} - \mathbf{v}) \cdot \hat{\mathbf{n}}}(\mathbf{p} - \mathbf{v})}$$

### 2.5.1 Deriving a matrix

Deriving a matrix is not so trivial this time, because the equation in not linear.

## 2.6 Rotate

In the 2D sense rotation can be expressed by the formula

$$\mathbf{p\prime} = \hat{\mathbf{x}}(\hat{\mathbf{x}} \cdot \mathbf{p}) \cos\theta + \hat{\mathbf{y}}(\hat{\mathbf{y}} \cdot \mathbf{p}) \sin\theta$$

Where $\hat{\mathbf{x}} \perp \hat{\mathbf{y}}$.

Suppose we wanted to rotate a point $\mathbf{p}$ at angle $\theta$ which lies on a plane defined by the normal $\hat{\mathbf{n}}$ (and for convenience passes through the origin). We need two perpendicular vectors on that plane to do the rotation - one is $\mathbf{p}$ the other one we could obtain by $\mathbf{p} \times \hat{\mathbf{n}}$, because we know that the product will it will be perpendicular $\mathbf{p}$ by definition and we know that it will lie in the plane (because it is also perpendicular to $\hat{\mathbf{n}}$). So we obtain the following formula:

$$\boxed{\mathbf{p\prime} = \mathbf{p} \cos\theta + \mathbf{n} \times \mathbf{p} \sin\theta}$$

What happens if $\mathbf{p}$ does not lie on a plane passing though the origin? In that case we simply shift the coordinate system so that the origin lies on the plane (i.e. shift the original point $\mathbf{p}$ in the opposite direction), and then after we do the operation we shift it back. A generic plane as already explained in previous sections is defined by a normal $\hat{\mathbf{n}}$ and an offset value $D$. Now, we want $D$ to be such a number that defines a the plane with a normal $\hat{\mathbf{n}}$ and also contains the point $\mathbf{p}$. In other words $D$ satisfies the plane equation:

$$\mathbf{p} \cdot \hat{\mathbf{n}} + D = 0$$

So we get:

$$D = -\mathbf{p} \cdot \hat{\mathbf{n}}$$

Now that we have found $D$ the translation vector we need to shift $\mathbf{p}$ by is $D\hat{\mathbf{n}}$ and the translation vector we need to shift the result of the rotation is $-D\hat{\mathbf{n}}$. We get two translations which look like this:

$$\begin{aligned}
\mathbf{p}\prime &= \mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \\
\mathbf{p}\prime\prime &= \mathbf{p}\prime \cos\theta + \mathbf{n} \times \mathbf{p}\prime \sin\theta \\
\mathbf{p}\prime\prime\prime &= \mathbf{p}\prime\prime + (\mathbf{p} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}
\end{aligned}$$

### 2.6.1 Deriving a matrix

Decomposing the 2D formula gives us:

$$\begin{aligned}
p\prime_x &= p_x \cos\theta + (n_y p_z - n_z p_y) \sin\theta \\
p\prime_y &= p_y \cos\theta + (n_z p_x - n_x p_z) \sin\theta \\
p\prime_z &= p_z \cos\theta + (n_x p_y - n_y p_x) \sin\theta
\end{aligned}$$

Rearranging gives us:

$$\begin{aligned}
p\prime_x &= p_x \cos\theta - n_z p_y \sin\theta + n_y p_z \sin\theta \\
p\prime_y &= n_z p_x \sin\theta + p_y \cos\theta - n_x p_z \sin\theta \\
p\prime_z &= -n_y p_x \sin\theta + n_x p_y \sin\theta + p_z \cos\theta
\end{aligned}$$

So our matrix looks like this:

$$\begin{bmatrix} p\prime_x \\ p\prime_y \\ p\prime_z \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -n_z \sin\theta & n_y \sin\theta & 0 \\ n_z \sin\theta & \cos\theta & -n_x \sin\theta & 0 \\ -n_y \sin\theta & n_x \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

And to account for the offset we need to do a translation matrices for:

$$\mathbf{p}\prime = \mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$$

and

$$\mathbf{p}\prime = \mathbf{p} + (\mathbf{p} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$$

Decomposition of the first gives us:

$$\begin{aligned}
p\prime_x &= p_x - (n_x p_x + n_y p_y + n_z p_z) \\
p\prime_y &= p_y - (n_x p_x + n_y p_y + n_z p_z) \\
p\prime_z &= p_z - (n_x p_x + n_y p_y + n_z p_z)
\end{aligned}$$

Rearranging that gives us:

$$p\prime_x = (1 - n_x)p_x - n_y p_y - n_z p_z$$
$$p\prime_y = -n_x p_x + (1 - n_y)p_y + n_z p_z$$
$$p\prime_z = -n_x p_x - n_y p_y + (1 - n_z)p_z$$

So our matrix looks like this:

$$\begin{bmatrix} p\prime_x \\ p\prime_y \\ p\prime_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - n_x & -n_y & -n_z & 0 \\ -n_x & 1 - n_y & -n_z & 0 \\ -n_x & -n_y & 1 - n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

The second equation would produce a similar matrix, but instead of minus there would be a plus sign. We could also say that the second equation will compute the inverse matrix of the first one. If our rotation matrix is $\mathbf{R}$ and our first translation matrix is $\mathbf{T}$ we can compute the final matrix by multiplying in the following order:

$$\mathbf{A} = \mathbf{TRT^{-}1}$$

## 2.7 Shear

Shearing is a linear transformation which could be explained as shifting parallel planes in distance relative to the plane's distance from the origin (analogy with a deck of cards which is slides). See 2.5:

The further $\mathbf{p}$ is from the "ground" plane the more it is displaced in direction $\hat{\mathbf{v}}$. The distance of $\mathbf{p}$ from the ground plane can be obtained trough from $\mathbf{p} \cdot \hat{\mathbf{n}}$. The shift is in the direction of $\hat{\mathbf{v}}$ and is proportionate to the height. Note that vector $\mathbf{v}$ must be such that is from the plane, so $\mathbf{v} \perp \hat{\mathbf{n}}$. Our formula is:

$$\boxed{\mathbf{p\prime} = \mathbf{p} + (\mathbf{p} \cdot \hat{\mathbf{n}})\mathbf{v}}$$

The magnitude of $\mathbf{v}$ expresses the magnitude of the shear per height unit. A zero vector $\mathbf{v}$ would quite obviously mean that the resulting $\mathbf{p\prime}$ is the same as $\mathbf{p}$ and the higher $|\mathbf{v}|$ is the further $\mathbf{p\prime}$ would be from $\mathbf{p}$.

Also note that we assume that the plane crosses trough the origin. This could easily be adapted to any scenario if we shifted the initial point $\mathbf{p}$ by $-\hat{\mathbf{n}}D$, and then the resulting point $\mathbf{p\prime}$ $\hat{\mathbf{n}}D$, where D is the forth plane parameter in the plane equation (as explained earlier).

### 2.7.1 Deriving a matrix

If we decomposed the formula in a system of 3 equations we would get:

$$p\prime_x = p_x + (n_x p_x + n_y p_y + n_z p_z)v_x$$
$$p\prime_y = p_y + (n_x p_x + n_y p_y + n_z p_z)v_y$$
$$p\prime_z = p_z + (n_x p_x + n_y p_y + n_z p_z)v_z$$

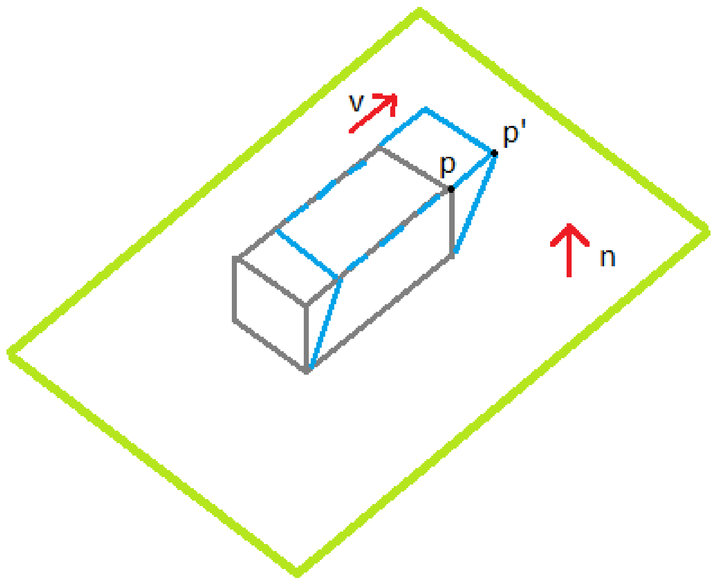


Figure 2.5: shear

Now if we rearranged it, we would get:

$$
\begin{aligned}
p\prime_x &= (1 + v_x n_x)p_x + v_x n_y p_y + v_x n_z p_z \\
p\prime_y &= v_y n_x p_x + (1 + v_y n_y)p_y + v_y n_z p_z \\
p\prime_z &= v_z n_x p_x + v_z n_y p_y + (1 + v_z n_z)p_z
\end{aligned}
$$

Creating a matrix is then straightforward:

$$
\begin{bmatrix} p\prime_x \\ p\prime_y \\ p\prime_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + v_x n_x & v_x n_y & v_x n_z & 0 \\ v_y n_x & 1 + v_y n_y & v_y n_z & 0 \\ v_z n_x & v_z n_y & 1 + v_z n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}
$$

# Chapter 3

# Results

> "The results section contains applications of the methodology described above. The results of experiments (either physical or computational) are data and can be presented as tables or figures and analyses. Whenever possible, include at least one example of recorded data to illustrate the technology or concept being proposed. Case-history results are usually geologic interpretations.
>
> Selective presentation of results is important. Redundancy should be avoided, and results of minor variations on the principal experiment should be summarized rather than included. Details appearing in figure captions and table heads should not be restated in the text. In a well-written paper, the results section is often the shortest."

## 3.1 Transformations

Blah blah blah

# Chapter 4

# Conclusion

> "The conclusion section should include (1) principles, relationships, and generalizations inferred from the results (but not a repetition of the results); (2) any exceptions to or problems with those principles, relationships, and generalizations, as indicated by the results; (3) agreements or disagreements with previously published work; (4) theoretical implications and possible practical applications of the work; and (5) conclusions drawn (especially regarding significance). In particular, with reference to item (1) above, a conclusion that only summarizes the results is not acceptable."

Blah blah blah

# Bibliography