Conceptual problems 5
~~CSE~~ [MATH 208] → Section A/B

(1) It is impossible to find a 2x3 matrix A and a 3x2 matrix || 10/26/25
No such matrices exist $A : \mathbb{R}^3 \to \mathbb{R}^2$ & $B : \mathbb{R}^2 \to \mathbb{R}^3$. The three columns of A live in $\mathbb{R}^2$, so they are linearly dependent.

IMPOSSIBLE by Thm 3B
a linear map can't turn the dependent 3 columns in $\mathbb{R}^2$ into the independent columns of I_3

→ By theorem 3B, a linear map sends a dependent set to a dependent set), the columns of BA are dependent. But I_3 has independent columns, so $BA = I_3$ cannot occur.

→ IMPOSSIBLE

Moreover, if $BA = I_3$ then three columns of BA would be linearly independent, but by Theorem 3B no linear map can turn a dependent set (columns A in $\mathbb{R}^2$) into an independent one. So independence vs dependence directly contradicts.

(2) Row-reduce $\begin{bmatrix} 1 & 0 & 3 \\ -2 & x & 1 \\ 4 & -1 & 2 \end{bmatrix}$ $R_2 \leftarrow R_2 + 2R_1 \to (0, x, 7)$

$R_3 \leftarrow R_3 - 4R_1 \to (0, -1, -10)$ swap R_2, R_3; then

$R_3 \leftarrow R_3 + xR_2 \to (0, 0, 7-10x)$.

if $7-10x \neq 0$: three pivots → invertible. if $7-10x = 0$ like $x = \frac{7}{10}$: a free variable → not invertible.

NOT INVERTIBLE
→ forcing dependence by Theorem 1A
if $x = \frac{7}{10}$ row reduce → last pivot $7-10x$
use thm 1C + Thm 1A

→ By theorem 1C, row operations preserve solution sets AND theorem 1A, free variables → infinitely many homogeneous solutions. Therefore, it is not invertible only when $x = \frac{7}{10}$.

→ NOT INVERTIBLE

When $7-10x \neq 0$, the row-reduced form is I_3, so the matrix is row-equivalent to the identity; by theorem 1C, meaning the homogeneous system has only the trivial solution (no free variables), which is exactly the invertibility criterion - while at $x = \frac{7}{10}$ rank dr... to 2.

→ IMPOSSIBLE if T_2 isn't invertible (not onto or not one-to-one) then $T_2 \circ T_1$ can't be invertible by Def 3D (onto/1-1)

(3) This is impossible. If T_2 is not invertible, it is either not onto or not one-to-one. Not onto: range $(T_2 \circ T_1) \subseteq$ range $(T_2) \neq \mathbb{R}^3$, so the composition isn't onto based on Definition 3D (alternate).

Not one-to-one: pick $y_1 \neq y_2$ with $T_2(y_1) = T_2(y_2)$.

Since invertible T_1 is onto (Def 3D), choose x_i with $T_1(x_i) = y_i$.

Then $(T_2 \circ T_1)(x_1) = (T_2 \circ T_1)(x_2)$ with $x_1 \neq x_2$, so the composition isn't one-to-one.

→ IMPOSSIBLE

Either way $T_2 \circ T_1$ cannot be invertible.