Advanced Control Engineering I

2. Exercise incl. solution

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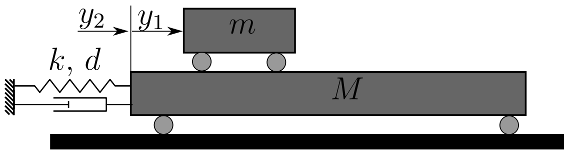
Task 1

- **Objective**: Minimal realizations (Canonical forms)
- **System Specification**: Calculate the known canonical form of the following system:



Model equation:

$$m\,(\ddot{y}_1 + \ddot{y}_2) = u$$
$$M\,\ddot{y}_2 + d\,\dot{y}_2 + k\,y_2 = -u.$$

State:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ \dot{y}_1 + \dot{y}_2 \\ y_2 \\ \dot{y}_2 \end{pmatrix}$$

State space representation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{m}u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\tfrac{1}{M}u - \tfrac{k}{M}x_3 - \tfrac{d}{M}x_4. \end{aligned}$$

$$y = y_1 = x_1 - x_3$$

- **Tasks for each system**:
  - Calculate the controllable canonical form
  - Calculate the observable canonical form
  - What is the relative degree of the system
  - Calculate the input-output canonical form
- **Note**: Calculation by hand. Also try if Matlab can help.

Solution to Task 1

System in state space representation

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k}{M} & -\frac{d}{M} \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \\ -\frac{1}{M} \end{pmatrix}, \quad \boldsymbol{c}^T = \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix}$$

Transfer function

$$G(s) = \boldsymbol{c}^T (s\mathrm{I} - \boldsymbol{A})^{-1} \boldsymbol{b} + d$$

$$(s\mathrm{I} - \boldsymbol{A})^{-1} = \begin{pmatrix} s & -1 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & -1 \\ 0 & 0 & \frac{k}{M} & s + \frac{d}{M} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s^2}\begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} & \mathbf{0} \\ \mathbf{0} & \frac{M}{M\,s^2 + d\,s + k}\begin{pmatrix} s + \frac{d}{M} & 1 \\ -\frac{k}{M} & s \end{pmatrix} \end{pmatrix}$$

$$\begin{aligned} G(s) &= \begin{pmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{-M\,s - d}{M\,s^2 + d\,s + k} & \frac{-M}{M\,s^2 + d\,s + k} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \\ -\frac{1}{M} \end{pmatrix} \\ &= \frac{1}{m\,s^2} + \frac{1}{M\,s^2 + d\,s + k} = \frac{(M+m)\,s^2 + d\,s + k}{M\,m\,s^4 + d\,m\,s^3 + k\,m\,s^2} \\ &= \frac{1}{m\,M} \frac{(M+m)\,s^2 + d\,s + k}{s^4 + \frac{d}{M}\,s^3 + \frac{k}{M}\,s^2} \end{aligned}$$

Controllable canonical form

$$\boldsymbol{A}_{\text{ctrb}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k}{M} & -\frac{d}{M} \end{pmatrix}, \quad \boldsymbol{b}_{\text{ctrb}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{c}_{\text{ctrb}}^T = \begin{pmatrix} \frac{k}{m\,M} & \frac{d}{m\,M} & \frac{m+M}{m\,M} & 0 \end{pmatrix}$$

Observable canonical form

$$\boldsymbol{A}_{\text{obsv}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{k}{M} \\ 0 & 0 & 1 & -\frac{d}{M} \end{pmatrix}, \quad \boldsymbol{b}_{\text{obsv}} = \begin{pmatrix} \frac{k}{m\,M} \\ \frac{d}{m\,M} \\ \frac{m+M}{m\,M} \\ 0 \end{pmatrix}, \quad \boldsymbol{c}_{\text{obsv}}^T = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

Input-output canonical form

From original state representation:

$$\begin{aligned} \xi_1 &= y = x_1 - x_3 \\ \xi_2 &= \dot{y} = x_2 - x_4 \\ \dot{\xi}_2 &= \ddot{y} = (\frac{1}{m} + \frac{1}{M})\,u + \frac{k}{M}\,x_3 + \frac{d}{M}\,x_4 \end{aligned}$$

$$\eta_1 = q^T \boldsymbol{x}$$

With q is the last row of the inverse of the controllability matrix C:

$$C = \begin{pmatrix} \boldsymbol{b} & A\,\boldsymbol{b} & A^2\,\boldsymbol{b} & A^3\,\boldsymbol{b} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} & 0 & 0 \\ \frac{1}{m} & 0 & 0 & 0 \\ 0 & -\frac{1}{M} & \frac{d}{M^2} & \frac{k}{M^2} - \frac{d^2}{M^3} \\ -\frac{1}{M} & \frac{d}{M^2} & \frac{k}{M^2} - \frac{d^2}{M^3} & -2\,\frac{k\,d}{M^3} + \frac{d^3}{M^4} \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} 0 & m & 0 & 0 \\ m & 0 & 0 & 0 \\ \frac{d\,m}{k} & \frac{m\,(M\,k - d^2)}{k^2} & \frac{2\,M\,k\,d - d^3}{k^2} & \frac{M\,(M\,k - d^2)}{k^2} \\ \frac{M\,m}{k} & -\frac{m\,M\,d}{k^2} & \frac{M\,(M\,k - d^2)}{k^2} & -\frac{M^2\,d}{k^2} \end{pmatrix}$$

$$q^T = \begin{pmatrix} \frac{M\,m}{k} & -\frac{m\,M\,d}{k^2} & \frac{M\,(M\,k - d^2)}{k^2} & -\frac{M^2\,d}{k^2} \end{pmatrix}$$

$$\eta_1 = q^T \boldsymbol{x} = \frac{M}{k} \left(m\,x_1 - \frac{m\,d}{k}\,x_2 + (M - \frac{d^2}{k})\,x_3 - \frac{M\,d}{k}\,x_4 \right)$$

$$\eta_2 = \frac{M}{k}\,(m\,x_2 + d\,x_3 + M\,x_4)$$

$$\dot{\eta}_2 = -M\,x_3$$

From controllable canonical form with state $\bar{\boldsymbol{x}}$:

$$\xi_1 = y = \frac{k}{m\,M}\,\bar{x}_1 + \frac{d}{m\,M}\,\bar{x}_2 + \frac{m+M}{m\,M}\,\bar{x}_3$$

$$\xi_2 = \dot{y} = \frac{k}{m\,M}\,\bar{x}_2 + \frac{d}{m\,M}\,\bar{x}_3 + \frac{m+M}{m\,M}\,\bar{x}_4$$

$$\dot{\xi}_2 = \ddot{y} = \frac{k}{m\,M}\,\bar{x}_3 + \frac{d}{m\,M}\,\bar{x}_4 + \frac{m+M}{m\,M} \left(\frac{-k}{M}\,\bar{x}_3 + \frac{-d}{M}\,\bar{x}_4 + u \right) = -\frac{k}{M^2}\,\bar{x}_3 - \frac{d}{M^2}\,\bar{x}_4 + \frac{m+M}{m\,M}\,u$$

$$\eta_1 = \bar{x}_1$$

$$\eta_2 = \dot{\eta}_1 = \bar{x}_2$$

$$\dot{\eta}_2 = \bar{x}_3$$

Rearrange to express all equations in the new state $\boldsymbol{x}_{\text{io}} = \begin{pmatrix} \xi_1 & \xi_2 & \eta_1 & \eta_2 \end{pmatrix}^T$:

$$\begin{aligned} \bar{x}_1 &= \eta_1 \\ \bar{x}_2 &= \eta_2 \\ \bar{x}_3 &= \frac{m\,M}{m+M}\,\xi_1 - \frac{k}{m+M}\,\eta_1 - \frac{d}{m+M}\,\eta_2 \\ \bar{x}_4 &= \frac{m\,M}{m+M}\,\xi_2 - \frac{k}{m+M}\,\eta_2 - \frac{d\,m\,M}{(m+M)^2}\,\xi_1 + \frac{k\,d}{(m+M)^2}\eta_1 + \frac{d^2}{(m+M)^2}\eta_2 \end{aligned}$$

The system in input-output canonical form is

$$A_{\mathrm{io}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{m\,k}{(m+M)\,M} + \frac{m\,d^2}{(m+M)^2\,M} & -\frac{m\,d}{(m+M)\,M} & \frac{k^2}{(m+M)\,M^2} - \frac{k\,d^2}{(m+M)^2\,M^2} & \frac{2\,k\,d}{(m+M)\,M^2} - \frac{d^3}{(m+M)^2\,M^2} \\ 0 & 0 & 0 & 1 \\ \frac{m\,M}{m+M} & 0 & -\frac{k}{m+M} & -\frac{d}{m+M} \end{pmatrix}$$

$$b_{\mathrm{io}} = \begin{pmatrix} 0 \\ \frac{m+M}{m\,M} \\ 0 \\ 0 \end{pmatrix}$$

$$c_{\mathrm{io}}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$