Advanced Control Engineering I

1. Exercise incl. solution

Autor: Phillip Kronthaler

Last change: October 16, 2024

Task 1

- **Objective**: Minimal realizations (Canonical forms)
- **System Specification**: Create a minimal realization of the system given as transfer function:

$$G_1(s) = \frac{s}{s^3 + 5\,s + 2}$$

$$G_2(s) = \frac{s+1}{(s+1)\,(s+2)}$$

$$G_3(s) = \frac{s-2}{s^2 + 2\,s + 1}$$

$$G_4(s) = \frac{5\,s}{s^1 + 1}$$

$$G_5(s) = \frac{1}{s}$$

- **Tasks for each system**:
  - Calculate the controllable canonical form
  - Calculate the observable canonical form
  - What is the relative degree of the system
  - Calculate the input-output canonical form
- **Note**: Calculation by hand. Also try if Matlab can help.

Task 2

- **Objective**: Controllability / Observability
- **System Specification**: Linear drive with friction. Input is force on the cart, output is the velocity of the cart. Assume, that the position is also a state of the system!
- **Tasks**:
  - Calculate a state space model to describe the system
  - Is the system controllable?
  - Calculate the transformation to controllable canonical form, if possible.
  - Is the system observable?

- Calculate the transformation to observable canonical form, if possible.
- Calculate the transfer function. Is it always the same? Compare the transfer function with the different minimal realizations.

- **Note**: Try by hand. Also compute the transformations with Matlab.

Solution to Task 1

$$G_1(s) = \frac{s}{s^3 + 5\,s + 2}$$

The minimal realization is a order $P(s) = 3$ state space system:

$$A_{obsv} = A_{ctrb}^T = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & -5 \\ 0 & 1 & 0 \end{pmatrix}, \quad b_{obsv} = c_{ctrb} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad c_{obsv} = b_{ctrb} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Relative degree: $\rho = \deg P(s) - \deg Z(s) = 3 - 1 = 2$ Relative degree is not equal system order, thus an interpretation as input-output canonical form make sense: We use the system in controllable canonical form to get the following relations:

$$\begin{aligned} y &= \xi_1 = x_2 \\ \dot{y} = \dot{x}_2 &= \xi_2 = x_3 \\ \ddot{y} &= \dot{\xi}_2 = \dot{x}_3 = -2\,x_1 - 5\,x_2 + u \\ \eta_1 &= x_1 \\ \dot{\eta}_1 &= x_2 \end{aligned}$$

Transformation matrix by upper equations:

$$\begin{aligned} x_1 &= \eta_1 \\ x_2 &= \xi_1 \\ x_3 &= \xi_2 \end{aligned}$$

The system in input-output canonical form is given by

$$\begin{aligned} \dot{\mathbf{x}}_{\text{io}} &= \begin{pmatrix} 0 & 1 & 0 \\ -5 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{x}_{\text{io}} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \mathbf{x}_{\text{io}} \end{aligned}$$

$$G_2(s) = \frac{s+1}{(s+1)\,(s+2)} = \frac{1}{s+2}$$

The minimal realization is a order $\deg P(s) = 1$ state space system:

$$A_{obsv} = A_{ctrb}^T = -2\,, \quad b_{obsv} = c_{ctrb} = 1\,, \quad c_{obsv} = b_{ctrb} = 1\,.$$

Relative degree: $\rho = \deg P(s) - \deg Z(s) = 1 - 0 = 1$ Relative degree = system degree.

$$G_3(s) = \frac{s-2}{s^2 + 2\,s + 1}$$

The minimal realization is a order $\deg P(s) = 2$ state space system:

$$A_{obsv} = A_{ctrb}^T = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}, \quad b_{obsv} = c_{ctrb} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad c_{obsv} = b_{ctrb} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Relative degree: $\rho = \deg P(s) - \deg Z(s) = 2 - 1 = 1$ Relative degree is not equal system order, thus an interpretation as input-output canonical form make sense: We use the system in controllable canonical form to get the following relations:

$$\begin{aligned} y = \xi_1 &= -2\,x_1 + x_2 \\ \dot{y} = \dot{\xi}_1 &= -2\,\dot{x}_1 + \dot{x}_2 = -x_1 - 4\,x_2 + u \\ \eta_1 &= x_1 \\ \dot{\eta}_1 &= x_2 \end{aligned}$$

Transformation matrix by upper equations:

$$\begin{aligned} x_1 &= \eta_1 \\ x_2 &= \xi_1 + 2\,\eta_1 \end{aligned}$$

The system in input-output canonical form is given by

$$\begin{aligned} \dot{\mathbf{x}}_{\text{io}} &= \begin{pmatrix} -4 & -9 \\ 1 & 2 \end{pmatrix} \mathbf{x}_{\text{io}} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}_{\text{io}} \end{aligned}$$

$$G_4(s) = \frac{5\,s}{s+1}$$

The minimal realization is a order $\deg P(s) = 1$ state space system. We have a $b_n = b_1 = 5$, i.e. we have to compute the $\tilde{b}_i$ values: $\tilde{b}_0 = b_0 - a_0\,b_1 = 0 - 5 = -5$.

$$A_{obsv} = A_{ctrb}^T = -1\,, \quad b_{obsv} = c_{ctrb} = -5\,, \quad c_{obsv} = b_{ctrb} = 1\,, \quad d_{obsv} = d_{ctrb} = b_n = 5\,.$$

Relative degree: $\rho = \deg P(s) - \deg Z(s) = 1 - 1 = 0$ Relative degree is zeros, thus no input-output canonical form is existing!

$$G_5(s) = \frac{1}{s}$$

The minimal realization is a order $\deg P(s) = 1$ state space system:

$$A_{obsv} = A_{ctrb}^T = 0\,, \quad b_{obsv} = c_{ctrb} = 1\,, \quad c_{obsv} = b_{ctrb} = 1\,.$$

Relative degree: $\rho = \deg P(s) - \deg Z(s) = 1 - 0 = 1$ Relative degree = system degree.

Solution to Task 2

Mathematical system:

$$\begin{aligned} m\,\ddot{x} + d\,\dot{x} &= u \\ y &= \dot{x} \end{aligned}$$

The state space model with the state vector $\mathbf{x} = \begin{pmatrix} x & \dot{x} \end{pmatrix}^T$ is

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{d}{m} \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x}$$

Compute the controllability matrix:

$$\boldsymbol{C} = \begin{pmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & -\frac{d}{m^2} \end{pmatrix}$$

System is controllable and in a scaled controllable canonical form, since introducing $u = m\,u^*$ results in controllable canonical form with the input u^*. Computing the transformation matrix results in:

$$\boldsymbol{C}^{-1} = \begin{pmatrix} d & m \\ m & 0 \end{pmatrix} \quad \rightarrow \quad \boldsymbol{q}^T = \begin{pmatrix} m & 0 \end{pmatrix} \quad \rightarrow \quad \boldsymbol{T} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

The transformation matrix is the identity matrix multiplied by m. The transformed system is given by:

$$\boldsymbol{A}_{\text{ctrb}} = \boldsymbol{T}\,\boldsymbol{A}\,\boldsymbol{T}^{-1}\,, \quad \boldsymbol{b} = \boldsymbol{T}\,\boldsymbol{b}\,, \quad \boldsymbol{c}^T = \boldsymbol{c}^T\,\boldsymbol{T}^{-1}$$

Compute the observability matrix:

$$\boldsymbol{O} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Observability matrix is not regular (not invertible), thus the system is not observable.

The transfer function of both system interpretations are equal:

$$G = \frac{1}{m\,s + d}$$

The transfer function shows only the system order of $n = \deg P(s) = 1$, which clarifies the statement of order reduction, if a system is not controllable / not observable.