Advanced Control Engineering I

99. Exercise

Autor: Phillip Kronthaler

Last change: November 14, 2024

Task 1

Calculate the state-space model in controllable canonical form of the system represented by the transfer function

$$G(s) = \frac{2}{s^2 + 3\,s + 5}\,.$$

Task 2

Calculate the state-space model in controllable canonical form of the system represented by the transfer function

$$G(s) = \frac{2\,s^2}{s^2 + 3\,s + 5}\,.$$

Task 3

Calculate the state-space model in observable canonical form of the system represented by the transfer function

$$G(s) = \frac{2\,s}{s^2 + 3\,s + 5}\,.$$

Task 4

Calculate the state-space model in observable canonical form of the system represented by the transfer function

$$G(s) = \frac{s^2}{s^2 + 2\,s + 1}\,.$$

Task 5

Calculate the transfer function of the state-space model

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ y &= \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \boldsymbol{x}\,. \end{aligned}$$

Task 6

Name the order of the system and calculate the relative degree of

$$G(s) = \frac{2\,s}{s^2 + 3\,s + 5}\,.$$

Note: Explain why

Task 7

Name the order of the system and calculate the relative degree of

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ y &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \boldsymbol{x}\,.\end{aligned}$$

Note: Explain why

Task 8

Name the order of the system and calculate the relative degree of

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{pmatrix} -a\,\sqrt{x_1} \\ a\,\sqrt{x_1} - a\,\sqrt{x_2} \end{pmatrix} + \begin{pmatrix} b\,u \\ 0 \end{pmatrix} \\ y &= x_2\,,\end{aligned}$$

with a and b are constants.

Note: Explain why

Task 9

Create a prefilter to smooth a input step as reference trajectory for the system

$$G(s) = \frac{2}{s^2 + s}\,.$$

The eigenvalues of the filter should all be at $s_i = -2$. Select the lowest order which is needed.

Task 10

Create a feed-forward for the system

$$G(s) = \frac{2}{s^2 + s}\,.$$

Assume, that the auxiliary trajectory $\eta(t)$ is smooth enough. How can you compute the reference trajectory y_{ref} and the reference input trajectory u_{ref}? Write down the equations.

Task 11

The watertank system from the lab-exercise is given by the following mathematical system

$$\dot{\boldsymbol{x}} = \begin{pmatrix} -a\sqrt{x_1} \\ a\sqrt{x_1} - a\sqrt{x_2} \end{pmatrix} + \begin{pmatrix} b\,u \\ 0 \end{pmatrix}$$
$$y = x_2\,,$$

with a and b are constants. Explain, why x_2 is a flat output $y_f(t)$ of the system. Calculate the state and input variables by means of the flat output and its derivatives.

Task 12

A system is given by its transfer function

$$G(s) = \frac{2\,s}{s^2 + 3\,s + 5}\,.$$

Use the property of flatness to design an appropriate feed-forward. Use the auxiliary variable $y_f(t)$ to compute the reference trajectory $y_{ref}(t)$ and the reference input signal $u_{ref}(t)$. You do not need to calculate a polynomial transition, just compute the reference signals by means of the auxiliary variable and its derivatives.

Task 13

Is the following system controllable?

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x}\,,$$

If yes, use a state feedback control with the desired eigenvalues of the closed loop system at $s_1 = s_2 = -10$. Name the feedback gain $\boldsymbol{k}$.

Note: If the system is not in controllable canonical form, the Ackermann formula can also be used.

Task 14

Is the following system controllable?

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x}\,,$$

If yes, use a state feedback control with the desired eigenvalues of the closed loop system at $s_1 = s_2 = -10$. Name the feedback gain $\boldsymbol{k}$.

Note: If the system is not in controllable canonical form, the Ackermann formula can also be used.

Task 15

Is the following system observable?

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{pmatrix} 4 & 0 \\ -2 & 3 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x} \,, \end{aligned}$$

If yes, use a Luenberger observer with the desired eigenvalues at $s_1 = s_2 = -10$. Name the observer gain $\boldsymbol{\ell}$.

Note: If the system is not in observable canonical form, the Ackermann formula can also be used.

Task 16

Is the following system controllable?

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x} \,, \end{aligned}$$

If yes, use a Luenberger observer with the desired eigenvalues at $s_1 = s_2 = -10$. Name the observer gain $\boldsymbol{\ell}$.

Note: If the system is not in observable canonical form, the Ackermann formula can also be used.

Task 17

Define a mathematical model for a constant disturbance on the output $y(t)$. Extend the model below by the disturbance and observe the unknown disturbance by a Luenberger observer.

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x} \,, \end{aligned}$$

Note: You do not have to calculate the observer gain or anything, the task is just to extend the mathematical model and show the observer design to estimate the disturbance.

Theory

Task 18

Explain the idea behind zero-vibration input-shaping.

Task 19

Explain the system property "flatness".

Task 20

Explain the definition of the relative degree?

Task 21

Why is a double-s velocity profile a time-optimal solution for rigid systems?

Task 22

Name the minimum order of a polynomial to calculate a transition for a system of order n. Explain why.

Task 23

How to scale a prototyping function for a transition from $y_0 = 1\,\mathrm{m}$ to $y_E = 10\,\mathrm{m}$ within $T = 10\,\mathrm{s}$, if the prototyping function is given by

$$\varphi(\tau) = 10\,\tau^3 - 15\,\tau^4 + 6\,\tau^5\,.$$

Write down the reference trajectory for the flat output $\eta(t)$ and its first derivative $\dot{\eta}(t)$. Note: The flat output should be defined for all t.

Task 24

Define a differential equation to model a constant disturbance.

Task 1 –

Task 1:

$$G(s) = \frac{2}{s^2 + 3s + 5}$$

$b_0 = 2$
$a_1 = 3$
$a_0 = 5$

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 0 \end{bmatrix} \vec{x}$$

Task 2:

$$G(s) = \frac{2s^2}{s^2 + 3s + 5}$$

$b_n = 2 \rightarrow b_0 = -10$
$a_0 = 5 \qquad b_1 = -6$
$a_1 = 3$

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -10 & -6 \end{bmatrix} x + 2u$$

Task 3:

$$G(s) = \frac{2s}{s^2 + 3s + 5}$$

$b_1 = 2 \qquad a_1 = 3$
$b_0 = 0 \qquad a_0 = 5$

$$\dot{\vec{x}} = \begin{bmatrix} 0 & -5 \\ 1 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \vec{x}$$

Task 4:

$$G(s) = \frac{s^2}{s^2 + 2s + 1}$$

$b_n = 1 \qquad \tilde{b}_1 = -2$
$a_1 = 2 \qquad \tilde{b}_0 = -1$
$a_0 = 1$

Observable

$$\dot{\vec{x}} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \cdot \vec{x} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \vec{x} + u$$

Controllable

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & -2 \end{bmatrix} \vec{x} + u$$

Task 5:

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

$$G(s) = \frac{4s^2 + 3s + 2}{s^3 + s^2 + 3s + 5}$$

$b_0 = 2 \qquad a_0 = 5$
$b_1 = 3 \qquad a_1 = 3$
$b_2 = 4 \qquad a_2 = 1$

Task 6:

$$G(s) = \frac{2s}{s^2 + 3s + 5}$$

The order is two.

Relative degree is one.

$$s^2 y + 3sy + 5y = 2su$$

$$sy + 3y + 5\frac{y}{s} = 2u$$

$y' + 3y + 5 \cdot \int y = 2u \rightarrow$ max derivative is one
↳ relative degree is one

Task 7:

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

Order 3

Relative degree 2

Task 8:

$$\dot{\vec{x}} = \begin{bmatrix} -a\sqrt{x_1} \\ a\sqrt{x_1} - a\sqrt{x_2} \end{bmatrix} + \begin{bmatrix} b u \\ 0 \end{bmatrix}$$

$$y = x_2$$

↳ $\dot{y} = \dot{x}_2 = a\sqrt{x_1} - a\sqrt{x_2}$

$\ddot{y} = \ddot{x}_2 = a \cdot \frac{1}{2\sqrt{x_1}} \cdot \dot{x}_1 - a \frac{1}{2\sqrt{x_2}} \cdot \dot{x}_2$

↳ this depends on the input → $\ddot{y}$ is the second derivative so relative degree is two

→ order is two too because we have two states

Task 9:

$$G(s) = \frac{2}{s^2+s} \qquad \dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 0 \end{bmatrix} \vec{x}$$

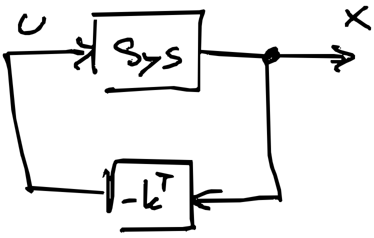
First we calculate the state feedback control to get desired poles.

char. polynom:

$$0 = s^2 + s = \underbrace{s \cdot (s+1)}$$

$$s_1 = 0$$

$$s_2 = -1$$



$$u = -k^T \cdot x$$

$$\dot{\vec{x}} = \underbrace{(A - b \cdot k^T)}_{\hat{A}} \vec{x} = \begin{bmatrix} 0 & 1 \\ -a_0 - k_1 & -a_1 - k_2 \end{bmatrix}$$

$$-\hat{a}_0 \qquad -\hat{a}_1$$

desired poles: $0 = (s+2)^2 = s^2 + 4s + 4$

$$\hat{a}_1 \qquad \hat{a}_0$$

$$\left.\begin{aligned} 4 &= \hat{a}_0 = 0 + k_1 \rightarrow k_1 = 4 \\ 4 &= \hat{a}_1 = +1 + k_2 \rightarrow k_2 = 3 \end{aligned}\right\} \quad k^T = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$\rightarrow$ continou in Simulink

Second way Ackermann

$$M_c = (b, Ab)$$

$$M_c = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$M_c^{-1} = \frac{1}{-1} \cdot \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} +1 & +1 \\ \boxed{+1 \quad 0} \end{bmatrix} \rightarrow q^T$$

$$k^T = q^T \cdot [\hat{a}_0 \cdot I + \hat{a}_1 \cdot A + A^n]$$

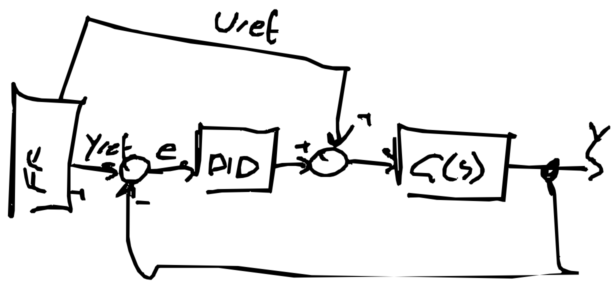
$$\hat{a}_0 = 4 \qquad \hat{a}_1 = 4$$

$$k^T = \begin{bmatrix} +1 & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \right)$$

$$k^T = \begin{bmatrix} +1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \end{bmatrix}$$

Task 10:

$$G(s) = \frac{2}{s^2+s}$$



$n = 2$

degree of polynom $= 2 \cdot n + 1 = 5$

$$Y_{desired} = a_5 \cdot t^5 + a_4 \cdot t^4 + a_3 \cdot t^3 + a_2 \cdot t^2 + a_1 \cdot t + a_0$$

$$\left.\begin{array}{ll} Y_{des}(0) = 0 & Y_{des}(T) = 1 \\ Y_{des}'(0) = 0 & Y_{des}'(T) = 0 \\ Y_{des}''(0) = 0 & Y_{des}''(T) = 0 \end{array}\right\} \text{assumptions} \quad T = 1s$$

$$Y_{des}' = 5a_5 t^4 + 4a_4 t^3 + 3a_3 t^2 + 2a_2 t + a_1$$

$$Y_{des}'' = 20a_5 t^3 + 12a_4 t^2 + 6a_3 t + 2a_2$$

$\underline{a_0, a_1, a_2 = 0}$ → results out of boundary conditions

$$1 = a_5 + a_4 + a_3 \quad \rightarrow \quad 1 = a_5 - \frac{5}{2}a_5 + a_3$$

$$\mapsto a_3 = 1 + \frac{3}{2}a_5$$

$$0 = 5a_5 + 4a_4 + 3a_3$$

$$0 = 20a_5 + 12a_4 + 6a_3$$

III − 2·II:

$$0 = 10a_5 + 4a_4 \rightarrow a_4 = -\frac{5}{2}a_5$$

II:

$$0 = 5a_5 - \frac{20}{2}a_5 + 3 + \frac{9}{2}a_5$$

$$= 5a_5 - 10a_5 + 3 + \frac{9}{2}a_5$$

$$-3 = a_5 \cdot \left(5 - 10 + \frac{9}{2}\right) = a_5 \cdot \left(-\frac{1}{2}\right)$$

$$\rightarrow a_5 = 6$$

$$a_3 = 1 + \frac{3}{2} \cdot 6 = 10$$

$$a_4 = -\frac{5}{2} 6 = -15$$

$$Y_{des} = 6 \cdot t^5 - 15t^4 + 10t^3$$

$$\eta_{ref} = \frac{Y_{des}}{\underset{\hookrightarrow 2}{b_0}} = 3t^5 - 7.5 \cdot t^4 + 5t^3$$

$$\eta_{ref}' = 15t^4 - 30 \cdot t^3 + 15t^2$$

$$\eta_{ref}'' = 60 \cdot t^3 - 90t^2 + 30t$$

$$U_{ref} = \eta_{ref}'' + \overset{1}{a_1} \cdot \eta_{ref}' + \overset{0}{a_0} \cdot \eta_{ref}$$

$$U_{ref} = \overbrace{60 \cdot t^3 - 90t^2 + 45t}^{\eta_{ref}''} + \overbrace{15 \cdot t^4 - 30 \cdot t^3 + 15t^2}^{a_1 \cdot \eta_{ref}'}$$

$$= 15 \cdot t^4 + 30t^3 - 75t^2 + 30t$$

$$Y_{ref} = b_0 \cdot \eta_{ref} = 6 \cdot t^5 - 15 \cdot t^4 + 10 \cdot t^3$$

→ continue Simulink

Task 11:

$$\dot{\vec{x}} = \begin{bmatrix} -a\sqrt{x_1} \\ a\sqrt{x_1} - a\sqrt{x_2} \end{bmatrix} + \begin{bmatrix} bu \\ 0 \end{bmatrix}$$

$$y = x_2$$

$$\dot{x}_2 = a \cdot \sqrt{x_1} - a\sqrt{x_2}$$

$$\ddot{x}_2 = a \cdot \frac{1}{2\sqrt{x_1}}\left(-a\sqrt{x_1} + bu\right) - a\frac{1}{2\sqrt{x_2}} \cdot \left(a\sqrt{x_1} - a\sqrt{x_2}\right)$$

↳ relative degree 2

$$\rightarrow x_1 = \left(\frac{\dot{x}_2}{a} - \sqrt{x_2}\right)^2 = \left(\frac{\dot{x}_2}{a}\right)^2 - 2 \cdot \frac{\dot{x}_2}{a} \cdot \sqrt{x_2} + x_2$$

$$\dot{x}_1 = -a\sqrt{x_1} + bu \rightarrow u = \frac{\dot{x}_1}{b} + \frac{a}{b} \cdot \sqrt{x_1}$$

Too lazy to compute $u(x_2, \dot{x}_2, \ddot{x}_2)$...

With substitution u is only depending on x_2.

↳ Theirfor all states and inputs can be represented by only x_2. $\rightarrow$ x_2: flat output

Task 12

$$G(s) = \frac{2s}{s^2 + 3s + 5} = \frac{y_f}{u}$$ this is already my flat output

$y_f = y_{ref}$

$$u = \frac{1}{2} \int (\ddot{y}_f + 3\dot{y}_f + 5y_f)\, dt$$

Task 13

$$\dot{\vec{x}} = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \vec{x}$$

To check if thy system is controllable, it is necessary to compute the <u>controllability matrix</u> and check its rank.

$$M_c = (b, Ab)$$

$$Ab = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \det(M_c) = 4 - 4 = 0$$

$\Rightarrow$ <u>not controllable</u>

Some states can not be influenced by the input. Therefore I can not use state feedback control!

Task 14

$$\dot{\vec{x}} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = [1 \; 0] \, \vec{x}$$

$$M_c = [b, \; Ab]$$

$$M_c = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \det(M_c) = 1 \rightarrow \text{controllable}$$

$$M_c^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad q^T = [-2 \; 1]$$

$$k^T = q^T \cdot [\hat{a}_0 I + \hat{a}_1 \cdot A + A^2]$$

wish polynom: $(s+10)^2 = 0$

$$\underset{\hat{a}_1}{\ } s^2 + \underbrace{20}_{\hat{a}_1} s + \underbrace{100}_{\hat{a}_0} = 0$$

$k^T =$