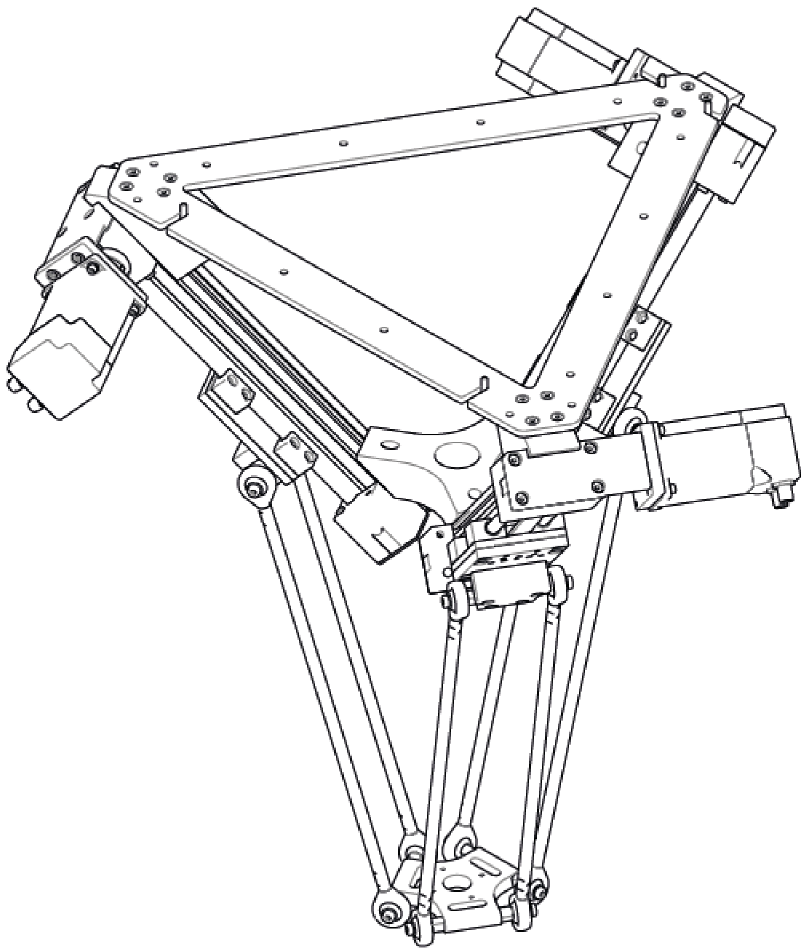


Parallel Manipulators: Kinematic Analyses of Delta 360 by IGUS

By

Prof. Dr. Mehmet İsmet Can Dede
Robotics
Management Center at Innsbruck

OVERVIEW

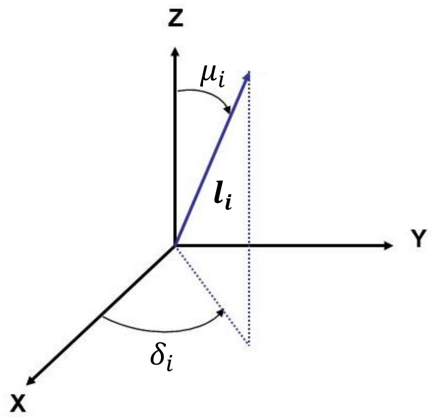
This series of lecture notes is prepared to show step by step kinematic analyses of an sTPM called Delta 360 by IGUS company. In the previous lecture notes, we have simplified the kinematic model of this manipulator as shown in Figure 6. In this part of the lecture notes, first of all we will identify the legs with their selected number L_i for $i = 1,2,3$. Then, we will define the local frames for each leg, which will simplify our solution steps.

As a result of this simplification, without the need of developing loop closure equations, we will be able to solve our forward kinematic problem where we determine the task space pose of the end-effector (moving platform) for a specified set of primary variables at the joint space. Also we will be able to solve our inverse kinematic problem where we determine the primary variables at the joint space for a specified set of task space variables.

For Delta 360 manipulator, we have selected our primary variables as the input variables received from the linear actuators (s_1, s_2, s_3). These variables are defined to be at zero position when the actuators are at their fully retracted positions as can be depicted from Figure 5.

The position of the moving platform is defined as the position of the origin O_m of the coordinate frame that is attached to the virtual moving platform. The measurement of its position is carried out with respect the origin O_b of the coordinate frame that is attached to the virtual fixed platform.

We will not need the secondary variables to be determined in our kinematic analyses hence, they were not stated in Figure 6. In fact, during the operation of Delta 360, you will not need to know their values. However, for the sake of completeness, we could resemble them by using the spherical coordinates defined appropriately for our selected frames. If we consider leg L_i for the figure on the right, the secondary variables associated with the L_i are δ_i and μ_i. Accordingly, we can resemble the components of $\boldsymbol{l_i}$ vector as follows:



$l_{ix} = l_i \sin\mu_i \cos\delta_i$; $l_{iy} = l_i \sin\mu_i \sin\delta_i$; $l_{iz} = l_i \cos\mu_i$

Consequently, we can list the primary and secondary variables for the simplified kinematic model of Delta 360 as $\boldsymbol{a^T} = [s_1 \quad s_2 \quad s_3]$ and $\boldsymbol{b^T} = [\mu_1 \quad \mu_2 \quad \mu_3 \quad \delta_1 \quad \delta_2 \quad \delta_3]$. Hence, all joint variables are listed as $\boldsymbol{q^T} = [\boldsymbol{a^T} \quad \boldsymbol{b^T}]$.

The task space position of the moving platform is identified as the position of O_m measured relative to O_b and therefore, let's identify the moving platform's position as $\boldsymbol{\xi_E}^{\boldsymbol{T}} = [x \quad y \quad z]$. Since, Delta 360 manipulator's moving platform performs only translational motion, whenever it is necessary you can define a tool as your end-effector with a tip point away from the O_m by $[x_t \quad y_t \quad z_t]$. Hence, if you would like to track the tool's tip point, then you can modify $\boldsymbol{\xi_E}^{\boldsymbol{T}}$ to $\boldsymbol{\xi_t}^{\boldsymbol{T}} = [x + x_t \quad y + y_t \quad z + z_t]$.

The mobility analysis of the simplified kinematics (3PUU) is carried out in the lecture notes on Parallel Manipulators: Definitions and Characteristics. It is determined to be a 3-DoF manipulator.

The number of independent loops for the 3PUU is $n_{ikl} = 3 - 1 = 2$.

The possible independent kinematic loops can be formed by selecting the two pairs from the following pair of legs: $\{L_1, L_2\}$, $\{L_1, L_3\}$, $\{L_2, L_3\}$.

3PUU is a spatial manipulator therefore, we will receive 3 independent equations from each loop in x, y and z directions. Since $n_{ikl} = 2$, the total number of loop-closure equations (LCEs) is 6, which is equal to the number of secondary variables, n. This means, if we develop LCEs, we will be able to determine the secondary variables for a specified set of primary variables.

FORWARD KINEMATIC ANALYSIS OF THE DELTA 360 MANIPULATOR

Our aim in forward kinematic analysis of the Delta 360 manipulator is to write equations for calculating the task space positions of the tip point $\boldsymbol{\xi_E}$, in this case O_m, (x, y, z) in terms of primary variables (s_1, s_2, s_3). In other words, we want to relate the motion of the linear actuators to the motion of the moving platform.

In contrast to the procedure defined in Parallel Manipulators: Position Level Kinematics, we will skip Step 1 where we develop LCEs and determine the secondary variables in terms of primary variables.

Our first step is to specify a leg number for each leg and its local coordinate frame. The origin of all local coordinate frames is selected as O_b. Also, all local frames share the same z-axis definition with the Z-axis of VFP. The leg numbers and their local frames are identified in Figure 7 according to the following:

- Leg L_1 and its frame are denoted in black color. In fact, the local frame $\mathcal{F}_1\{O_b; x^{(1)}, y^{(1)}, Z\}$ of L_1 is aligned with the coordinate frame of the VFP, $\mathcal{F}_b\{O_b; X, Y, Z\}$.
- Leg L_2 and its frame are denoted in red color. In fact, the local frame $\mathcal{F}_2\{O_b; x^{(2)}, y^{(2)}, Z\}$ of L_2 is rotated from $\mathcal{F}_b$ about the Z-axis by and angle of $-120°$.
- Leg L_3 and its frame are denoted in green color. In fact, the local frame $\mathcal{F}_3\{O_b; x^{(3)}, y^{(3)}, Z\}$ of L_3 is rotated from $\mathcal{F}_b$ about the Z-axis by and angle of $+120°$.

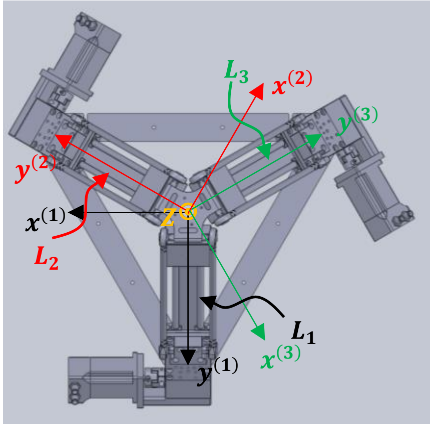


Figure 7. Leg numbers and their respective frames assigned for Delta 360 manipulator

The strategy we will follow in forward kinematic analysis is as follows:

Step A:

- Write forward kinematics equations for each leg starting from O_b and terminating at O_m. That is, write 3 equations for each leg in $\mathcal{F}_b\{O_b; X, Y, Z\}$.
- In these equations, instead of using $(l_i \sin\mu_i \cos\delta_i; l_i \sin\mu_i \sin\delta_i\ ; l_i \cos\mu_i)$ for the components of $\boldsymbol{l_i}$, we will use (l_{ix}, l_{iy}, l_{iz}). Therefore, we by-pass the information on the secondary variables.
- Leave the (l_{ix}, l_{iy}, l_{iz}) alone on one side of the equations.

Step B:

- Take the squares of the 3 equations for each leg separately and add them up. This will result in ${l_i}^2 = {l_{ix}}^2 + {l_{iy}}^2 + {l_{iz}}^2$ on one side of a single equation obtained for each leg.
- Finally, we have one equation for each leg which adds up to 3 equations and the unknown is the position of $\mathcal{F}_m\{O_m; x, y, z\}$. Hence, with 3 unknowns and 3 equations, we will be able to solve our forward kinematic problem.

In order to understand the forward kinematic equations for each leg, the parameters used are denoted once again in Figure 8. The screen shot of the Delta manipulator in Figure 8 is taken when all the linear actuators are at their mid-ranges ($s_1 = s_2 = s_3 = 72$ mm). Hence, looking from the top, O_m is aligned with O_b.

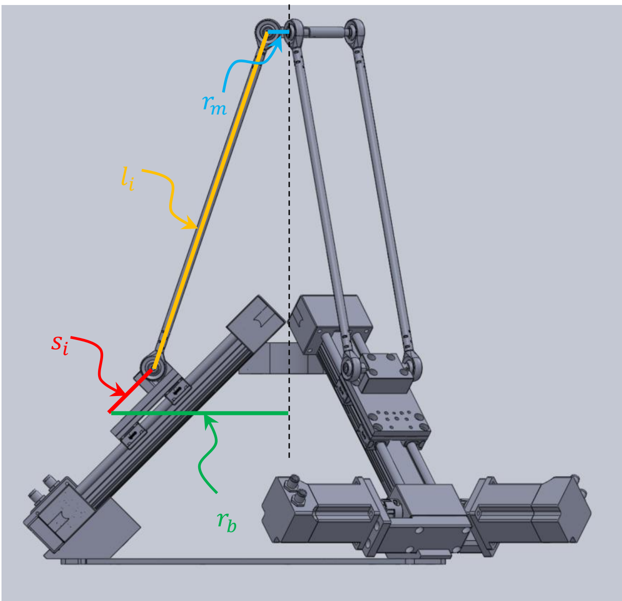


Figure 8. Kinematics parameters displayed on Delta 360 manipulator

Step A:

The forward kinematic equations for L_1:

It is easier to develop forward kinematic equations using the local frames. It happens that the local frame $\mathcal{F}_1$ chosen for L_1 has its basis vectors in the same direction of the basis vectors of the base frame $\mathcal{F}_b$. Hence, there are no transformations necessary from local frame to the base frame ➔ $x = x^{(1)}$ and $y = y^{(1)}$.

y-axis equation:

$$y = r_b - s_1 \cos(45°) + l_{1y} - r_m = y_1 + l_{1y} \tag{1}$$

where $y_1 = r_b - s_1 \cos(45°) - r_m$. Then, we can leave the y component of l_1 alone on the right side of the equation to receive the forward equation in y-axis for leg L_1.

$$(y - y_1) = l_{1y} \tag{2}$$

x-axis equation:

For the forward kinematic equation in x-axis for leg L_1, the only component from the links is the x-axis component of the link with length l_1.

$$x = l_{1x} \tag{3}$$

z-axis equation:

$$z = s_1 \sin(45°) + l_{1z} = z_1 + l_{1z} \tag{4}$$

where $z_1 = s_1 \sin(45°)$. Then, we can leave the z component of l_1 alone on the right side of the equation to receive the loop-closure equation in z-axis for leg L_1.

$$(z - z_1) = l_{1z} \tag{5}$$

The forward kinematic equations for L_2:

The local frame selected for leg L_2 is the frame that is rotated from the base frame about the Z-axis by an angle of -120°. The forward kinematic equations are first written in the selected local frame and then, they are transformed to the base frame resolution. The superscript represents the frame of resolution in the equations.

$$y^{(2)} = r_b - s_2 \cos(45°) + l_{2y^{(2)}} - r_m = y_2' + l_{2y^{(2)}} \tag{6}$$

where $y_2' = r_b - s_2 \cos(45°) - r_m$.

$$x^{(2)} = l_{2x^{(2)}} \tag{7}$$

y-axis equation:

The Equations (6) & (7) defined in the 2nd local frame are used to derive the equations in the base frame as follows:

$$y = -y^{(2)} \sin(30°) - x^{(2)} \cos(30°) = -y_2' \sin(30°) - l_{2y^{(2)}} \sin(30°) - l_{2x^{(2)}} \cos(30°) \tag{8}$$

where; $l_{2y} = -l_{2y^{(2)}} \sin(30°) - l_{2x^{(2)}} \cos(30°)$. Please refer to Figure 7 for this calculation.

Let's define $y_2 = -y_2' \sin(30°)$. Then, we can leave the y component of l_2 alone on the right side of the equation to receive the loop-closure equation in y-axis for leg L_2.

$$(y - y_2) = l_{2y} \tag{9}$$

x-axis equation:

The Equations (6) & (7) defined in the 2nd local frame are also used to derive the forward kinematic equation in x-axis of the base frame as follows:

$$x = y^{(2)} \cos(30°) - x^{(2)} \sin(30°) = y_2' \cos(30°) + l_{2y^{(2)}} \cos(30°) - l_{2x^{(2)}} \sin(30°) \tag{10}$$

where; $l_{2x} = l_{2y^{(2)}} \cos(30°) - l_{2x^{(2)}} \sin(30°)$. Please refer to Figure 7 for this calculation.

Let's define $x_2 = y_2' \cos(30°)$. Then, we can leave the x component of l_2 alone on the right side of the equation to receive the forward kinematic equation in x-axis for leg L_2.

$$(x - x_2) = l_{2x} \tag{11}$$

z-axis equation:

$$z = s_2 \sin(45°) + l_{2z} = z_2 + l_{2z} \tag{12}$$

where $z_2 = s_2 \sin(45°)$. Then, we can leave the z component of l_2 alone on the right side of the equation to receive the forward kinematic equation in z-axis for leg L_2.

$$(z - z_2) = l_{2z} \tag{13}$$

The forward kinematic equations for L_3:

The local frame selected for leg L_3 is the frame that is rotated from the base frame about the Z-axis by an angle of +120°. The forward kinematic equations are first written in the selected local frame and then, they are transformed to the base frame resolution. The superscript represents the frame of resolution in the equations.

$$y^{(3)} = r_b - s_3 \cos(45°) + l_{3y^{(2)}} - r_m = y_3' + l_{3y^{(3)}} \tag{14}$$

where $y_3' = r_b - s_3 \cos(45°) - r_m$.

$$x^{(3)} = l_{3x^{(3)}} \tag{15}$$

y-axis equation:

The Equations (14) & (15) defined in the 3rd local frame are used to derive the equations in the base frame as follows:

$$y = -y^{(3)} \sin(30°) + x^{(3)} \cos(30°) = -y_3' \sin(30°) - l_{3y^{(3)}} \sin(30°) + l_{3x^{(3)}} \cos(30°) \tag{16}$$

where; $l_{3y} = -l_{3y^{(3)}} \sin(30°) + l_{3x^{(3)}} \cos(30°)$. Please refer to Figure 7 for this calculation.

Let's define $y_3 = -y_3' \sin(30°)$. Then, we can leave the y component of l_3 alone on the right side of the equation to receive the forward kinematic equation in y-axis for leg L_3.

$$(y - y_3) = l_{3y} \tag{17}$$

x-axis equation:

The Equations (14) & (15) defined in the 3rd local frame are also used to derive the forward kinematic equation in x-axis of the base frame as follows:

$$x = -y^{(3)}\cos(30°) - x^{(3)}\sin(30°) = -y_3'\cos(30°) - l_{3y^{(3)}}\cos(30°) - l_{3x^{(3)}}\sin(30°) \tag{18}$$

where; $l_{3x} = -l_{3y^{(3)}}\cos(30°) - l_{3x^{(3)}}\sin(30°)$. Please refer to Figure 7 for this calculation.

Let's define $x_3 = -y_3'\cos(30°)$. Then, we can leave the x component of l_3 alone on the right side of the equation to receive the forward kinematic equation in x-axis for leg L_3.

$$(x - x_3) = l_{3x} \tag{19}$$

<u>z-axis equation:</u>

$$z = s_3\sin(45°) + l_{3z} = z_3 + l_{3z} \tag{20}$$

where $z_3 = s_3\sin(45°)$. Then, we can leave the z component of l_3 alone on the right side of the equation to receive the forward kinematic equation in z-axis for leg L_3.

$$(z - z_3) = l_{3z} \tag{21}$$

Step B:

In Step A, we received the forward kinematic equations by avoiding the use of secondary variables (μ_i, δ_i for $i = 1, 2, 3$). As it is discussed in the previous sections, we can use the fact that $l_i^2 = l_{ix}^2 + l_{iy}^2 + l_{iz}^2$ for $i = 1, 2, 3$ in the next set of equations which are based on Equations (2), (3) & (5) for leg L_1, Equations (9), (11) & (13) for leg L_2, and Equations (17), (19) & (21) for leg L_3.

$$x^2 + (y - y_1)^2 + (z - z_1)^2 = l_{1x}^2 + l_{1y}^2 + l_{1z}^2 = l_1^2 \tag{22}$$

$$(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 = l_{2x}^2 + l_{2y}^2 + l_{2z}^2 = l_2^2 \tag{23}$$

$$(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2 = l_{3x}^2 + l_{3y}^2 + l_{3z}^2 = l_3^2 \tag{24}$$

To have TPM motion, for this Delta 360 manipulator, link lengths are to be equal to each other. Otherwise, the moving and base platform would not stay parallel to each other. Therefore, as per the manufacturing specifications as well, $l_1 = l_2 = l_3 = l$. The above equations can be modified with respect to this information. Also, to carry out some mathematical manipulations, Equations (22), (23) & (24) are written in open form and reorganized as follows:

$$x^2 + y^2 + z^2 - 2yy_1 - 2zz_1 = l^2 - y_1^2 - z_1^2$$

$$x^2 + y^2 + z^2 - 2xx_2 - 2yy_2 - 2zz_2 = l^2 - x_2^2 - y_2^2 - z_2^2$$

$$x^2 + y^2 + z^2 - 2xx_3 - 2yy_3 - 2zz_3 = l^2 - x_3^2 - y_3^2 - z_3^2$$

Let $w_i = x_i^2 + y_i^2 + z_i^2$ for $i = 1, 2, 3$ in the above equation where $x_1^2 = 0$. We will use the next set of equations that are the modified versions of the above equations in the following steps.

$$x^2 + y^2 + z^2 - 2yy_1 - 2zz_1 = l^2 - w_1 \tag{25}$$

$$x^2 + y^2 + z^2 - 2xx_2 - 2yy_2 - 2zz_2 = l^2 - w_2 \tag{26}$$

$$x^2 + y^2 + z^2 - 2xx_3 - 2yy_3 - 2zz_3 = l^2 - w_3 \tag{27}$$

Since Equations (25), (26) and (27) share common parameters (x, y, z, l), we can make use of this in the next set of mathematical modifications to isolate x and z coordinates of the origin of the frame O_m attached to the moving platform (tip point).

First, let's subtract Equation (26) from Equation (25) to receive:

$$xx_2 + y(y_2 - y_1) + z(z_2 - z_1) = \frac{w_2 - w_1}{2} \tag{28}$$

Then, let's subtract Equation (27) from Equation (25) to receive:

$$xx_3 + y(y_3 - y_1) + z(z_3 - z_1) = \frac{w_3 - w_1}{2} \tag{29}$$

To get rid of x variable in the equations and receive y as a function of z, let's multiply Equation (28) by x_3 and Equation (29) by x_2, and subtract them from each other.

$$x_3y(y_2 - y_1) + x_3z(z_2 - z_1) - x_2y(y_3 - y_1) - x_2z(z_3 - z_1) = \frac{x_3(w_2 - w_1)}{2} - \frac{x_2(w_3 - w_1)}{2}$$

Now we can isolate y on the left side of the equation and present it as a function of z:

$$y = \frac{\frac{x_3(w_2 - w_1) - x_2(w_3 - w_1)}{2} + [x_2(z_3 - z_1) - x_3(z_2 - z_1)]z}{x_3(y_2 - y_1) - x_2(y_3 - y_1)}$$

In short, we can re-write the above equations as follows:

$$y = \frac{\alpha_1 + \beta_1 z}{d} \quad (30)$$

where;

$\alpha_1 = \frac{x_3(w_2 - w_1) - x_2(w_3 - w_1)}{2}$

$\beta_1 = [x_2(z_3 - z_1) - x_3(z_2 - z_1)]$

$d = x_3(y_2 - y_1) - x_2(y_3 - y_1)$

This time, to get rid of y variable in the Equations (28) & (29) and receive x as a function of z, let's multiply Equation (28) by $(y_3 - y_1)$ and Equation (29) by $(y_2 - y_1)$, and subtract them from each other.

$$(y_3 - y_1)x_2x + (y_3 - y_1)(z_2 - z_1)z - (y_2 - y_1)x_3x - (y_2 - y_1)(z_3 - z_1)z = \frac{(y_3 - y_1)(w_2 - w_1)}{2} - \frac{(y_2 - y_1)(w_3 - w_1)}{2}$$

Now we can isolate x on the left side of the equation and present it as a function of z:

$$x = \frac{\frac{(y_2 - y_1)(w_3 - w_1) - (y_3 - y_1)(w_2 - w_1)}{2} + [(y_3 - y_1)(z_2 - z_1) - (y_2 - y_1)(z_3 - z_1)]z}{x_3(y_2 - y_1) - x_2(y_3 - y_1)}$$

In short, we can re-write the above equations as follows:

$$x = \frac{\alpha_2 + \beta_2 z}{d} \quad (31)$$

where;

$\alpha_2 = \frac{(y_2 - y_1)(w_3 - w_1) - (y_3 - y_1)(w_2 - w_1)}{2}$

$\beta_2 = [(y_3 - y_1)(z_2 - z_1) - (y_2 - y_1)(z_3 - z_1)]$

At this point, we have both x and y as functions of z. Now, we can substitute Equation (30) and (31) in Equation (25) and receive a quadratic equation for the z variable.

(25) ➔ $\frac{{\alpha_2}^2 + {\beta_2}^2 z^2 + 2\alpha_2\beta_2 z}{d^2} + \frac{{\alpha_1}^2 + {\beta_1}^2 z^2 + 2\alpha_1\beta_1 z}{d^2} + z^2 - 2zz_1 - 2y_1\left(\frac{\alpha_1 + \beta_1 z}{d}\right) = l^2 - w_1$

When we reconfigure the above equation to the quadratic form for z variable, we receive the following quadratic equation:

$$({\beta_2}^2 + d^2 + {\beta_1}^2)z^2 + 2(\alpha_2\beta_2 + \alpha_1\beta_1 - d^2 z_1 - y_1 d\beta_1)z + ({\alpha_2}^2 + {\alpha_1}^2 - 2y_1 d\alpha_1 - d^2 l^2 + d^2 w_1) = 0 \quad (32)$$

If we define the quadratic form as $az^2 + bz + c = 0$ then,

$a = {\beta_2}^2 + d^2 + {\beta_1}^2$

$b = 2(\alpha_2\beta_2 + \alpha_1\beta_1 - d^2 z_1 - y_1 d\beta_1)$

$c = {\alpha_2}^2 + {\alpha_1}^2 - 2y_1 d\alpha_1 - d^2 l^2 + d^2 w_1$

We receive a real solution if $\Delta = b^2 - 4ac \geq 0$. Therefore, **if $\Delta < 0$ then, this is a position level singularity of the Delta 360 encountered in the forward kinematic analysis ➔ position singularity of forward kinematics – PSFK**.

The solution of the quadratic equation suggests a double solution:

$$z = \frac{-b+\sigma_f\sqrt{\Delta}}{2a} \text{ where } \sigma_f = \pm 1$$

This double solution means that **there are two posture modes of the Delta 360 discovered in the forward kinematics analysis ➔ PMFKs**. Hence, for the same joint variables, the Delta 360 has two distinct assembly modes. For $\sigma_f = +1$ posture mode, the moving platform is at + displacement in the z-axis relative to the base platform. For $\sigma_f = -1$ posture mode, the moving platform is at - displacement in the y-axis relative to the base platform. $\sigma_f = +1$ is the desirable posture mode of our Delta 360 manipulator.

The only way you can switch between the two posture modes is to go into the condition where $\Delta = 0$. This pose was called posture mode changing pose – **PMCP** in the lecture on Parallel Manipulators: Position Level Kinematics. At this condition with a small external push +z or –z direction, the posture mode will be switched to $\sigma_f = +1$ or $\sigma_f = -1$, respectively. $\sigma_f = -1$ assembly mode is displayed in Figure 9 with red lines.

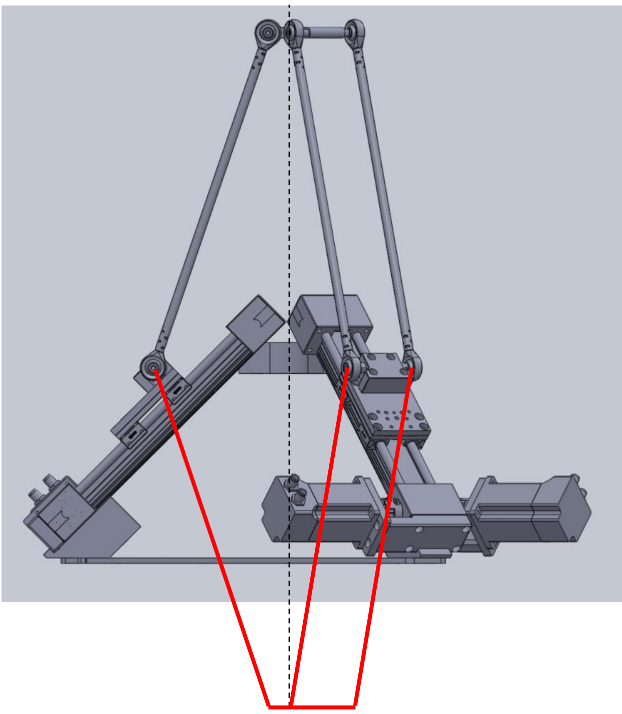


Figure 9. $\sigma_f = -1$ assembly mode of the Delta 360 manipulator

In summary, the forward kinematic equations with suitable posture mode for Delta 360 are tabulated below:

x-axis position of the tip point	$x = \frac{\alpha_2 + \beta_2 z}{d}$
y-axis position of the tip point	$y = \frac{\alpha_1 + \beta_1 z}{d}$
z-axis position of the tip point	$z = \frac{-b + \sqrt{\Delta}}{2a}$

INVERSE KINEMATIC ANALYSIS OF THE DELTA 360 MANIPULATOR

Our aim in inverse kinematic analysis of the Delta 360 manipulator is to write equations for calculating the primary variables (s_1, s_2, s_3) in terms of the task space positions of the tip point $\boldsymbol{\xi_E}$, in this case O_m, (x, y, z). In other words, we want to relate the motion of the linear actuators to the motion of the moving platform. Nevertheless, you can assign a tool to be placed on the moving platform which has a tip point that has an offset of $[x_t \quad y_t \quad z_t]$ from O_m. Hence, if you would like to track the tool's tip point, then you can modify the $\boldsymbol{\xi_E}^{\boldsymbol{T}}$ to $\boldsymbol{\xi_t}^{\boldsymbol{T}} = [x + x_t \quad y + y_t \quad z + z_t]$.

As defined in Parallel Manipulators: Position Level Kinematics, we will tackle the inverse kinematic problem for each leg separately in their own local frames $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$. Recall that $\mathcal{F}_1 \parallel \mathcal{F}_b \parallel \mathcal{F}_m$.

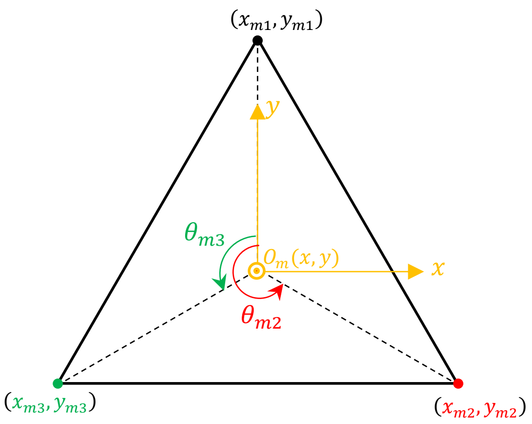
Similar to the forward kinematic equations, we will base our solution on the magnitude calculation of a 3-dimensional vector, $\boldsymbol{l_i}$ for $i = 1,2,3$.

For each leg, we will define position of the universal joint centers of the virtual link (that is drawn in red color in Figure 4) as:

- $\left\{x_{mi}^{(i)}, y_{mi}^{(i)}, z_{mi}^{(i)}\right\}$ for $i = 1,2,3$: universal joint centers on the moving platform as defined in frame $\mathcal{F}_i$
- $\left\{x_{si}^{(i)}, y_{si}^{(i)}, z_{si}^{(i)}\right\}$ for $i = 1,2,3$: universal joint centers on the slider of the actuator as defined in frame $\mathcal{F}_i$

Once these parameters are determined, the inverse kinematics solution for each leg is the same. For a given task space position of $O_m(x, y, z)$, the procedure for each leg is note in the equations below:

a) Calculate the universal joint center position on the moving platform in $\mathcal{F}_i$



At this point, we should reference the kinematic sketch of the moving platform on the left.

In this sketch, the locations of the universal joint centers on the moving platform (located at the corners of the equilateral triangle) are denoted with (x_{mi}, y_{mi}) for $i = 1,2,3$.

Each of these points are r_m away from the origin O_m of the moving platform's frame.

Here, $\theta_{m1} = 0°$, $\theta_{m2} = 240° = -120°$, $\theta_{m3} = 120°$

The reason why the z-axis information is not presented in the above sketch is that $z = z_{m1} = z_{m1} = z_{m3}$. This due to the fact that all these points are on the same moving platform, which does not rotate throughout its operation.

The common formula to calculate the universal joint center location calculation is carried out by using two-dimensional rotation matrix about z-axis:

$$\begin{bmatrix} x_{mi}^{(i)} \\ y_{mi}^{(i)} \end{bmatrix} = \begin{bmatrix} \cos\theta_{mi} & \sin\theta_{mi} \\ -\sin\theta_{mi} & \cos\theta_{mi} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ r_m \end{bmatrix} \tag{32}$$

Hence the location of each universal joint's center on the moving platform is determined in their respective local frames as follows;

$$x_{mi}^{(i)} = x\cos\theta_{mi} + y\sin\theta_{mi} \tag{33}$$

$$y_{mi}^{(i)} = y\cos\theta_{mi} - x\sin\theta_{mi} + r_m \tag{34}$$

$$z_{mi}^{(i)} = z \tag{35}$$

b) Calculate the universal joint's center position on the slider of the respective actuator for each leg

Since the previous step is also defined in base frame's origin O_b (recall that O_m is measured with respect to O_b), we will define the universal joint's center position on the slider of the respective actuator for each leg with respect to O_b as well. Please refer to Figure 8 to fully understand the calculations below.

$$x_{si}^{(i)} = 0 \tag{36}$$

$$y_{si}^{(i)} = r_b - s_i\cos 45° \tag{37}$$

$$z_{si}^{(i)} = s_i\sin 45° \tag{38}$$

c) Develop the equation to calculate the magnitude of the vector $\boldsymbol{l_i}$

We will use the Equations from (33) to (38) to write out the square of the magnitude of vector $\boldsymbol{l_i}$

$$\left(x_{mi}^{(i)} - x_{si}^{(i)}\right)^2 + \left(y_{mi}^{(i)} - y_{si}^{(i)}\right)^2 + \left(z_{mi}^{(i)} - z_{si}^{(i)}\right)^2 = l_i^{\,2} \tag{39}$$

d) Write the open form of $\left\{x_{si}^{(i)}, y_{si}^{(i)}, z_{si}^{(i)}\right\}$ in Equation (39) and solve for the primary variables s_i

Since we are trying to determine the primary variables s_i and it is not explicitly shown in Equation (39), let's write the open form of Equation (39) by using Equation (36), (37), (39).

$$x_{mi}^{(i)\,2} + y_{mi}^{(i)\,2} + r_b^{\,2} + (s_i\cos 45°)^2 - 2r_b(s_i\cos 45°) - 2y_{mi}^{(i)}(r_b - s_i\cos 45°) + z_{mi}^{(i)\,2} + (s_i\sin 45°)^2 - 2z_{mi}^{(i)}(s_i\sin 45°) = l_i^{\,2}$$

At this point we should recall that $l_i = l$ for $i = 1,2,3$.

If we re-arrange the above equation as a quadratic form of s_i, we receive

$$s_i^{\,2} + s_i\left(-2z_{mi}^{(i)}\sin 45° - 2r_b\cos 45° + 2y_{mi}^{(i)}\cos 45°\right) + \left(x_{mi}^{(i)\,2} + y_{mi}^{(i)\,2} + z_{mi}^{(i)\,2} + r_b^{\,2} - 2y_{mi}^{(i)}r_b - l^2\right) = 0 \tag{40}$$

If we define the quadratic form as $a_i s_i^{\,2} + b_i s_i + c_i = 0$ then,

$a_i = 1$

$b_i = -2z_{mi}^{(i)}\sin 45° - 2r_b\cos 45° + 2y_{mi}^{(i)}\cos 45°$

$c_i = x_{mi}^{(i)\,2} + y_{mi}^{(i)\,2} + z_{mi}^{(i)\,2} + r_b^{\,2} - 2y_{mi}^{(i)}r_b - l^2$

We receive a real solution if $\Delta_i = b_i^{\,2} - 4a_i c_i \geq 0$. Therefore, **if $\boldsymbol{\Delta_i < 0}$ then, this is a position level singularity of the Delta 360 encountered in the inverse kinematic analysis for leg $\mathbf{L}_i$ ➔ position singularity of inverse kinematics – PSIK**. Recall from the lecture notes on Parallel Manipulators: Position Level Kinematics that each leg's singularity condition is independent from the other. A single leg may come to a singularity condition while the others are not in singularity conditions. Additionally, two or even all legs may come to a singular position simultaneously. The PSIK for Delta 360's legs is the condition when the link l_i is aligned with the linear actuator's axis of operation. These PSIKs are avoided by selecting suitable range of the linear actuators not to get into such position.

The solution of the quadratic equation suggests a double solution:

$$s_i = \frac{-b_i + \sigma_{ri}\sqrt{\Delta_i}}{2a_i} \text{ where } \sigma_{ri} = \pm 1 \text{ for } i = 1,2,3$$

This double solution means that **there are two posture modes associated with each leg of the Delta 360 discovered in the inverse kinematics analysis ➔ PMIKs**. Since there are 3 legs, this result in $2^3 = 8$ PMIK combinations. Hence, for the same task space position of the moving platform, each leg has 2 PMIK and Delta 360 manipulator has a total of 8 different assembly modes. For $\sigma_{ri} = +1$, s_i gets the large value, which means the slider of the actuator is placed at a further distance from the base platform. For $\sigma_{ri} = -1$, s_i gets the small value, which means the slider of the actuator is placed at a closer distance from the base platform. These two situations are sketched in Figure 10 for a single leg.

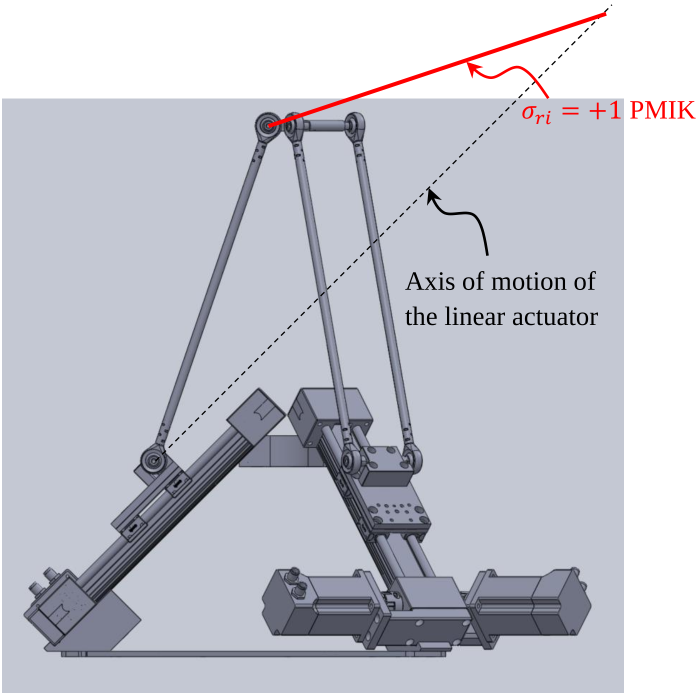


Figure 10. Two possible PMIKs of a single leg of Delta 360 manipulator

In Figure 10, PMIK for $\sigma_{ri} = -1$ is the actual CAD drawing and PMIK for $\sigma_{ri} = +1$ is drawn with red color. It is clear that this PMIK for any of the legs is undesirable. It is also impossible to get into the $\sigma_{ri} = +1$ assembly mode due to the range limitation of the linear actuators. Therefore, in our calculations, we will use the $\sigma_{ri} = -1$ assembly mode for all the legs.

Theoretically, the way you can switch between the two posture modes of each link is to go into the condition where $\Delta_i = 0$. This pose was called posture mode changing pose – **PMCP** in the lecture on Parallel Manipulators: Position Level Kinematics. In this pose, the axis between the centers of the universal joints at both ends of the links becomes perpendicular to the motion axis of the respective linear actuator. At this condition, only by driving the linear actuator, we can change the assembly mode.

The summary of inverse kinematic analysis is as follows:

$$s_i = \frac{-b_i - \sqrt{\Delta_i}}{2} \text{ for } i = 1,2,3 \text{ when } \theta_{m1} = 0°, \theta_{m2} = 240° = -120°, \theta_{m3} = 120° \quad (41)$$