Lab III)

a) $y_0 = (5,4)$, $y_1 = (6,5)$

$m_{i\to j}(x_j) = \sum \tilde{\phi}_i(x_i)\ \psi_{ij}(x_i, x_j) \prod_{k \in N(j)} m_{k \to j}$.

where $\tilde{\phi}_i(x_i)$ is $\phi_i(x_i) . P_{Y_i|X_i}(y_i|x_i)$

$\tilde{\phi}_0 = \phi_0 . P_{Y_0|X_0}(y_0|x_0) = \frac{1}{12\times8} \times \frac{1}{5}$ if $x_0 = \left| \begin{matrix} (5,3)(5,5)\ (5,4) \\ (4,4)(6,4) \end{matrix} \right.$

$P_{X_1|X_0} = \{(5,3,\text{stay}),(5,4,\text{up}),(5,2,\text{down}),(4,3,\text{left}),(6,3,\text{right})\}$

$\{(5,6,\text{stay}),(5,7,\text{up}),(5,5,\text{down}),(4,5,\text{left}),(6,5,\text{right})\}$

$\{(5,4,\text{stay}),(5,5,\text{up}),(5,3,\text{down}),(4,4,\text{left}),(6,4,\text{right})\}$

$\{(4,4,\text{stay}),(4,5,\text{up}),(4,3,\text{down}),(3,4,\text{left}),(3,6,\text{right})\}$

$\{(6,4,\text{stay}),(6,5,\text{up}),(6,3,\text{down}),(5,3,\text{left}),(5,5,\text{right})\}$

→ for each of these x_1s the probability is equal

So for each of these x_1s, the message is

$1/12\times8 \times 1/5 \times 1/5 = \frac{1}{25} \times \frac{1}{12\times8}$.

$m_{1\to0}(x_0) = \sum \phi_1(x_1)\, P_{Y_1|X_1}(y_1|x_1) . P_{X_0|X_1}(x_0|x_1)$

→ $P_{Y_1|X_1}(y_1|x_1) = 0$ for all x_1 exept $x_1 = \left| \begin{matrix} (6,5),(6,6),(6,4) \\ (5,5),(7,5) \end{matrix} \right.$

→ $P_{X_0|X_1}(x_0|x_1) = 0$ except for $x_0 =$

$\{(6,5,\text{stay}),(6,4,\text{up}),(6,6,\text{down}),(5,5,\text{right})$

$(7,5,\text{left}),(6,6,\text{stay}),(6,7,\text{down}),(6,5,\text{up}),(5,6,\text{right})$

$(7,6,\text{left}),(6,4,\text{stay}),(6,5,\text{down}),(6,3,\text{up}),(5,4,\text{right})$

(7,4, left), (5,5, stay), (5,6, down), (5,4, up)
(4,5, right), (6,5, left), (7,5, stay), (6,5, right)
(8,7, left), (7,6, down), (7,4, up)}

- Since all of these are x_0s, the valid ones have equal probability cause they add up to one → P for each = 1/25.

→ Marginals: $P_{x_1 | Y_0, Y_1} = \frac{1}{2} \sum \phi_{x_0} m_{0 \to 1}(x_1)$

→ for | 1/3 : (6,5) 1/3 : (6,4) (7,5)
0 : (6,6) 1/3 : (5,5)

marginal $P_{x_0 | Y_0; Y_1} = \frac{1}{2} \sum \phi(x_1) M_{1 \to 0}(x_0)$

for | $0 \times \frac{1}{3}$ (6,5) $\frac{1}{6} \times \frac{1}{2}$ (5,5) $\frac{1}{6} \times \frac{1}{2}$ (6,4)
$0 \times \frac{1}{6}$ (6,6) $\frac{1}{6} \times 0$ (7,5) (5,4)

(5,4) : 1/3 (6,4) : 1/3 (5,5) : 1/3

b)

At time step 0:

- Most likely parts of the marginal at time 0:

((5,5, stay), 0.5), ((6, 4, stay), 0.5)

At time step 99:

- Most likely parts of the marginal at time step 99:

((11, 0, stay), 0.81), ((11, 0, right), 0.18),
((10, 1, down), 0.01))

c)

At time step 99:

- Most likely parts of the marginal at 99:

  ((3, 0, right), 0.9), ((2, 0, stay), 0.1)

- Most likely parts of the marginal at 0:

  ((2, 3, stay), 0.82), ((1, 3, stay), 0.18)

d)

$$P_{X_0, \dots, X_9 | Y_0, \dots Y_9}(x_0, \dots x_9 | y_0, \dots y_9)$$

$$= \sum_{i=1}^{99} \left[P_{X_1}(x_1) \prod_{i=2}^{99} P_{X_i | X_{i-1}}(x_i | x_{i-1}) \prod_{i=1}^{9} P_{Y_i | X_i}(y_i | x_i) \right]$$

$$= \sum_{i=1}^{9} \left[P_{X_1}(x_1) \prod_{i=2}^{9} P_{X_i | X_{i-1}}(x_i | x_{i-1}) \prod_{i=1}^{9} P_{Y_i | X_i}(y_i | x_i) \right]$$

$$\cdot \sum_{10}^{99} \left[P_{X_1}(x_1) \prod_{i=10}^{99} P_{X_i | X_{i-1}}(x_i | x_{i-1}) \prod_{i=10}^{99} P_{Y_i | X_i}(y_i | x_i) \right]$$

Since for this term, we'll be summing all possible values, they won't depend on $x_0 \dots x_9$ and will finally add upto 1. So we'll be able to ignor it. The first term only depends on on the distribution and observations.

e)

Last 10 hidden states in the MAP estimate:

(0, 5, up)

(0, 4, up)

(0, 3, up)

(0, 2, up)

(0, 1, up)

(0, 0, up)

(0, 0, stay

(1, 0, right)

(2, 0, right)

(3, 0, right)

f) optional