Polytopes

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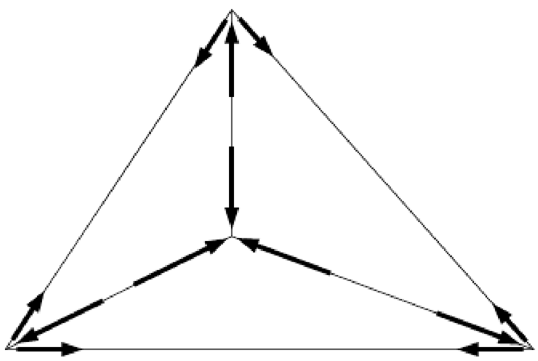
1 Introduction

2 Equilibrium Stresses

Theorem 1 (Maxwell, Whiteley). *Let G be a planar 3-connected graph with 2D drawing $\mathbf{p}$ and designated outer face f_0 . There exists a one-to-one correspondence between*

1. *equilibrium stresses w for G at $\mathbf{p}$; and*
2. *liftings in $\mathbb{R}$, where face f_0 remains in the $z = 0$ plane.*



3 The algorithms

The algorithm based on the article "Embedding Stacked Polytopes on a Polynomial-Size Grid" 2011 can be divided in:

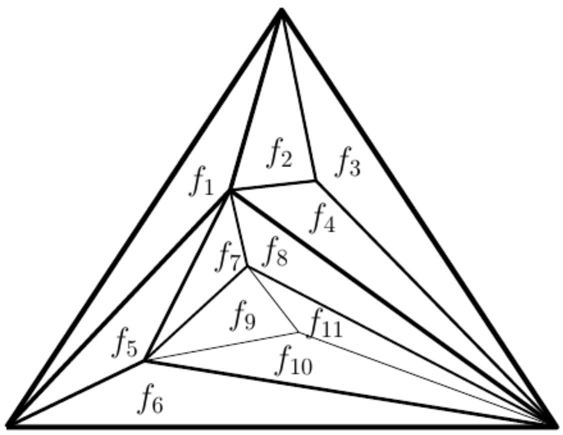
1. Receive the entry.
2. Heavy caterpillar decomposition.
3. Balance the tree.
4. Get the coefficients, or weights for the lifting.
5. Lift the graph to a polytope.

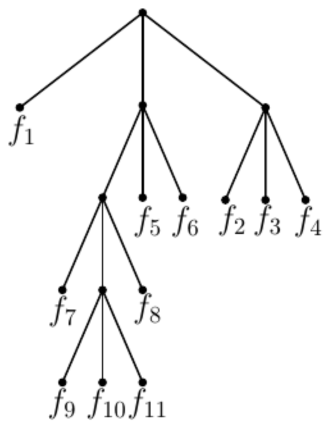
The algorithm presented in the definitives version was changed in the two last topics. It can be divided in:

1. Receive the entry.
2. Heavy caterpillar decomposition.
3. Balance the tree.
4. Lift the graph to a polytope.
5. Round the graph to grid points.

4 The Entry of the algorithm

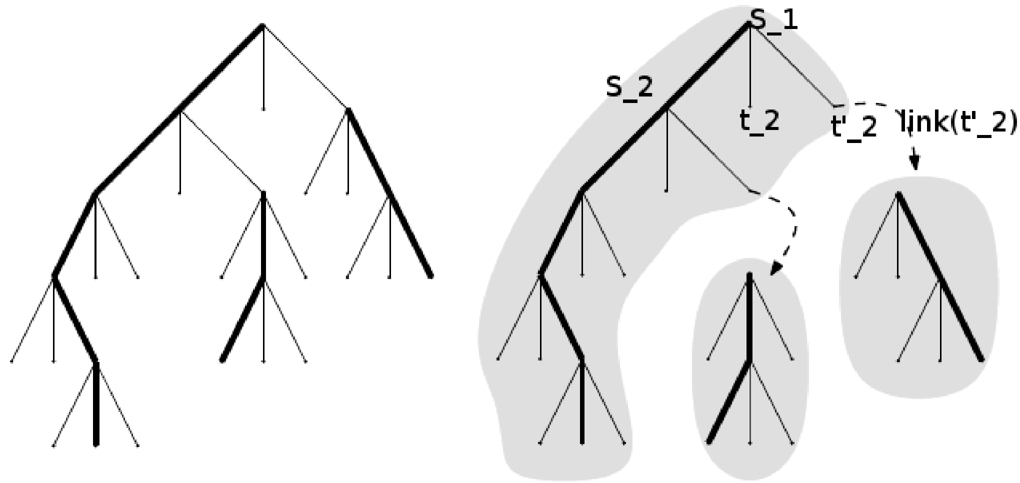
It is given to the algorithm a 3 connected planar graph and a tree representation of the graph, as showed below.





5 Heavy caterpillar decomposition

The tree representation is decomposed in *heavy paths*. Resulting sub-trees called *caterpillars*. When a node lies on a heavy path it is called a *spine node*, otherwise is a *tree node*. The spine nodes are labelled by s_1 (root), s_2, ..., s_i,...,$s_\perp$. The children from s_i are s_{i+1}, t_{i+1} and t'_{i+1}. In t are stored a pointer link(t) to a other caterpillar.



6 Balance the tree

The balacing of the Tree is given by the pseudo-code below:

function BALANCE(C)
    Input: A caterpillar C from the heavy caterpillar decomposition of $\mathcal{T}(G)$. All weights are equal 1.
    Output: Weights for the nodes of $\mathcal{T}(G)$.
    for all t_i, t_i' in C **do**
        BALANCE(link(t_i))
        BALANCE(link(t_i'))
        if $w(t_i) > w(t_i')$ **then**
            relabel $t_i \leftrightarrow t_i'$
        end if
        $w(t_i) = w(t_i')$
        add $(w(t_i) - w(t_i'))$ to the weight of $s_\perp$ in link (t_i)
        add $(w(t_i') - w(t_i))$ to the weight of $\text{link}^{-1}(C)$
    end for
end function

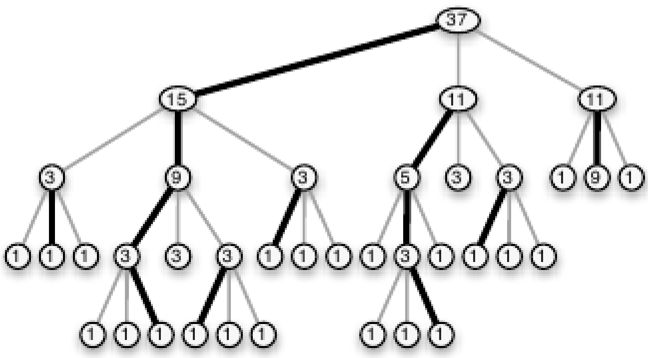
A balanced Tree will the have the properties:

$$w(s_{i-1}) = w(s_i) + w(t_i) + w(t_i')$$

$$w(s_i) \geq w(t_i), w(t_i')$$

$$w(t_i) = w(t_i')$$

A example of a balanced tree is show next:



7 First algorithm

7.1 Get the coefficients

If v_i is the vertex stacked on the face $v_j v_k v_l$ then:

$$\alpha_{ijkl} = \frac{1}{w(t_u)}$$

Where $w(t_u)$ is the weight of the subface $v_j v_k v_l$ obtained by the function BALANCE.

The weights are now updated with the coordinate values. First the coordinates from the vertices are rounded down. $r_i = (\lfloor x_i \rfloor, \lfloor y_i \rfloor)$.

The weights are:

$$\dot{w}_{ij} = \sum_{\{i,j,k,l\} \in S} \lfloor Y \alpha_{ijkl} \rfloor w_{ij}^{kl}$$

I have to find out how to make that sum

The coefficient α_{ijkl} is scaled by multiplying it with $Y = 4n^2$, n is the number of vertices.

$$w_{ij}^{kl} = [i, k, l][j, k, l]$$

$$[i, j, k] = \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ 1 & 1 & 1 \end{pmatrix}$$

7.2 The lifting

Each face i gains a parameter a_i in that way:

For the firs face is set:

$$a_0 = (0, 0, 0)^T$$

Having the parameter of the rigth face (the first is a_0) then we can find all the others with:

$$a_l = \dot{w}_{ij}(\mathbf{q}_i \times \mathbf{q}_j) + a_r$$

Where (i, j) is the common edge of f_i and f_l and $\mathbf{q}_i$ is defined as:

$$\mathbf{q}_i = (x_i, y_i, 1)^T$$

Now to calculate the heigth from a point $\mathbf{p}_i$ we have to make the inner product from $\mathbf{p}_i$ with a_k. Who is the parameter from face k whose one of the vertices is $\mathbf{p}_i$.

$$h_i = \langle \mathbf{p}_i, a_k \rangle$$

8 Second algorithm

8.1 Lift the graph

8.2 Round the graph to grid points