A Duality Transform for Constructing Small Grid Embeddings of 3d Polytopes

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Abstract

We study the problem of how to obtain an integer realization of a 3d polytope when an integer realization of its dual polytope is given. We focus on grid embeddings with small coordinates and develop novel techniques based on Colin de Verdière matrices and the Maxwell–Cremona lifting method.

We show that every truncated 3d polytope with n vertices can be realized on a grid of size $O(n^{44})$. Moreover, for every simplicial 3d polytope with maximal vertex degree Δ and vertices placed on an $L \times L \times L$ grid, a dual polytope can be realized on an integer grid of size $L^{O(\Delta)}$. This implies that for a class $\mathcal{C}$ of simplicial 3d polytopes with bounded vertex degree and polynomial size grid embedding, the dual polytopes of $\mathcal{C}$ can be realized on a polynomial size grid as well.

1 Introduction

By Steinitz's theorem the graphs of convex 3d polytopes[1] are exactly the planar 3-connected graphs [16]. Several methods are known for realizing a planar 3-connected graph G as a polytope with graph G on the grid [4, 7, 11, 12, 13, 15]. It is challenging to find algorithms that produce polytopes with small integer coordinates. Having a realization with small grid size is a desirable feature, since then the polytope can be stored and processed efficiently. Moreover, grid embeddings imply good vertex and edge resolution. Hence, they produce "readable" drawings.

In 2d, every planar 3-connected graph with n vertices can be drawn with straight-line edges on an $O(n) \times O(n)$ grid without crossings [5], and a drawing with convex faces can be realized on an $O(n^{3/2} \times n^{3/2})$ grid [2]. For the realization as a polytope the currently best algorithm guarantees an integer embedding with coordinates of size at most $O(147.7^n)$ [3, 11]. The current best lower bound is $\Omega(n^{3/2})$ [1]. Closing this gap is an intriguing open problem in lower dimensional polytope theory.

Recently, progress has been made for a special class of 3d polytopes, the so-called *stacked polytopes*. A *stacking operation* replaces a triangular face of a polytope with a tetrahedron, while maintaining the convexity of the embedding (see Figure 1). A polytope that can be constructed from a tetrahedron and a sequence of stacking operation is called a stacked 3d polytope, or for the scope of this paper simply a *stacked polytope*. The graphs of stacked polytopes are planar 3-trees. Stacked polytopes can be embedded on a grid that is polynomial in n [6]. This is, however, the only nontrivial polytope class for which such an algorithm is known.

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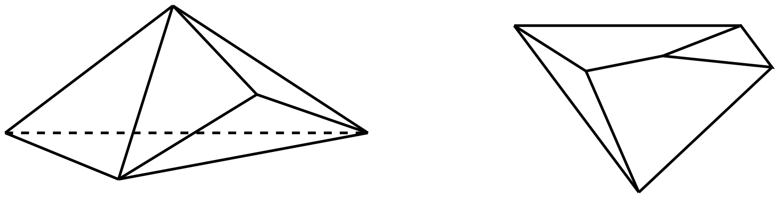
[1]In our terminology polytopes are always considered *convex*.



(a) A tetrahedron after one stacking operation.

(b) A tetrahedron after the corresponding truncation.

Figure 1

1.1 Our results

In this paper we introduce a duality transform that maintains a polynomial grid size. In other words, we provide a technique that takes a grid embedding of a polytope with graph G and generates a grid embedding of a polytope whose graph is G^*, the dual graph of G. We call a 3d polytope with graph G^* a *dual polytope.*

We prove the following result:

Theorem 8. *Let G be a triangulation and let $\mathcal{P} = (q_i)$ be a realization of G as a convex polytope with integer coordinates. Then there exists a realization $(\phi_f)_{f \in F(G)}$ of the dual graph G^* as a convex polytope with integer coordinates bounded by*[2]

$$|\phi_f| < \max |q_i|^{O(\Delta_G)}.$$

This in particular implies, that if we only consider simplicial polytopes with bounded vertex degrees and with integer coordinates bounded by a polynomial in n, then the dual polytope obtained with our techniques has also integer coordinates bounded by a (different) polynomial in n. Although our bound is not purely polynomial, it is in general an improvement over the standard approaches for constructing dual polytopes, see Sect. 1.2.

For the class of stacked polytopes (although their maximum vertex degree is not bounded) we can also apply our approach to show that all graphs dual to planar 3-trees can be embedded as polytopes on a polynomial size grid. These polytopes are known as truncated polytopes. Truncated polytopes are simple polytopes, which can be generated from a tetrahedron and a sequence of *vertex truncations.* A vertex truncation is the dual operation to stacking (Figure 1). This means that a degree-3 vertex of the polytope is cut off by adding a new bounding hyperplane that separates this vertex from the remaining vertices of the polytope. We prove the following theorem.

Theorem 4. *Any truncated 3d polytope with n vertices can be realized with integer coordinates of size $O(n^{44})$.*

Our approach for truncated polytopes is more direct than the approach that leads to Theorem 8, since stronger results for realizations of stacked polytopes on the grid are known [6].

As useful tools we also develop new techniques dealing with 2d equilibrium stresses, that are presented in the Sect. 5.1 of this paper.

1.2 Duality

There exist several natural approaches how to construct a dual polytope. To the best of our knowledge, all of them increase the coordinates of the original polytope in general by an exponential factor.

[2] Throughout the paper we use the notation $X^{O(\Delta)}$ to denote an upper bound of the form $X^{\kappa \cdot \Delta}$, where $\kappa > 0$ is a universal constant.

The most prominent construction is polarity. Let P be some polytope that contains the origin. Then $P^* = \{y \in \mathbb{R}^d \colon x^T y \leq 1 \text{ for all } x \in P\}$ is a polytope dual to P, called its *polar*. The vertices of P^* are intersection points of planes with integral normal vectors, and hence not necessarily integer points. In order to scale to integrality one has to multiply P^* with the product of all denominators of its vertex coordinates, which may cause an exponential increase of the grid size.

A second approach uses the classic Maxwell–Cremona correspondence technique (also known as lifting approach) [10], which is applied in many embedding algorithms for 3d polytope realization [13, Chapter 13.1]. The idea here is to first draw the graph of the polytope as a convex 2d embedding with an additional equilibrium condition. The equilibrium condition guarantees that the 2d drawing is a projection of a convex 3d polytope. Furthermore the polytope can be reconstructed from its projection in a canonical way (called lifting) in linear time.

There is a classical transformation that constructs for a 2d drawing of a graph in equilibrium a 2d drawing of its dual graph, also in equilibrium. This drawing is called the *reciprocal diagram*. The induced lifting realizes a dual polytope, but it does not provide small integer coordinates for two reasons. First, the weights that define the equilibrium of the reciprocal diagram are the reciprocals of the weights in the original graph. Since the weights influence the coordinates of the lifting in a direct way, it is hard to handle them without scaling by a large factor. Second, the lifting realizes the dual polytope in the projective space with one point "over the horizon". The second property can be "fixed" with a projective transformation. This, however, makes a large scaling factor for an integer embedding unavoidable in the general case.

1.3 Structure, Notation and Conventions

As a novelty we work with a by-product of Colin de Verdière matrices that we call CDV matrices to construct *small* grid embeddings. In order to make these techniques (as introduced by Lovász) applicable we extend this framework slightly, see Sect. 2. In Sect. 3 we present the main idea, combining the classical lifting approach with the methods of Sect. 2, which finds applications in the following sections, where the results on truncated polytopes (Sect. 4) and triangulations (Sect. 5) are presented. We also frequently use the notion of equilibrium stress, for which we develop novel techniques in Sect. 5.1.

We denote by G the graph of the original polytope, and by G^* its dual graph. For any graph H we write $V(H)$ for its vertex set, $E(H)$ for its edge set and $N(H, v)$ for the set of neighbors of a vertex v in H. Since we consider 3-connected planar graphs, the facial structure of the graph is predetermined up to a global reflection [17, Theorem 11]. The set of faces is therefore predetermined, and we name it $F(H)$. A face spanned by vertices v_i, v_j, and v_k is denoted as $(v_i v_j v_k)$. For an embedding $(p_i)_{1 \leq i \leq n}$ of a graph $G = (V, E)$ with the vertex set $V = (v_i)_{1 \leq i \leq n}$ we always assume that the point $p_k \in \mathbb{R}^d$ corresponds to the vertex v_k with the same index. A graph obtained from H by stacking a vertex v_1 on a face $(v_2 v_3 v_4)$, is denoted as $\mathrm{Stack}(H; v_1; v_2 v_3 v_4)$. We denote the maximum vertex degree of a graph G as Δ_{G}. Finally, we write $G[X]$ for the induced subgraph of a vertex set $X \subseteq V(G)$.

For convenience we use $|p|$ for the Euclidean norm of the vector p. Throughout the paper the graphs we consider are planar and 3-connected, every embedding into $\mathbb{R}^d$ is always understood as a straight-line embedding.

2 3d Representations with CDV Matrices

In this section we review some of the methods Lovász introduced in his paper on Steinitz representations [9].

Definition 1. *We call an embedding $(u_i)_{i \leq 1 \leq n} \in (\mathbb{R}^3)^n$ of a planar 3-connected graph G in $\mathbb{R}^3$ a* cone-convex embedding*, iff its projection onto the sphere $\left(\frac{u_i}{|u_i|}\right)_{1 \leq i \leq n}$ with edges drawn* *as geodesic arcs is a convex embedding of G into the sphere.*

We remark that the embedding of a graph G into the sphere is convex iff the faces of the embedding are the intersections of convex disjoint polyhedral cones with the sphere. So, in other words, an embedding is cone-convex iff the cones over its faces are convex and disjoint. We remark that the vertices of a cone-convex embedding are not supposed to form a convex polytope.

Definition 2. *Let $(u_i)_{1\leq i\leq n}$ be an embedding of a graph G into $\mathbb{R}^d$. We call a symmetric matrix $M = [M_{ij}]_{1\leq i,j\leq n}$ a* CDV matrix *of the embedding if*

1. $M_{ij} = 0$ *for* $i \neq j, (v_i v_j) \notin E(G)$, *and*
2. $\sum_{1\leq j\leq n} M_{ij} u_j = 0$ *for* $1 \leq i \leq n$.

We call a CDV matrix positive *if $M_{ij} > 0$ for all $(v_i v_j) \in E(G)$.*

We refer to the second condition in the above definition as the *CDV equilibrium condition.* The CDV equilibrium condition can also be expressed in a slightly different, more geometric form as

$$\sum_{v_j\in N(G,v_i)} M_{ij}u_j = -M_{ii}u_i \quad \text{for } 1 \leq i \leq n. \tag{1}$$

Hence, a positive CDV matrix witnesses that every vertex of the embedding can be represented as a convex combination of its neighbors using symmetric weights. The name CDV matrix was chosen in correspondence with Colin de Verdière matrices as defined in [9]. However, these two objects are of a different nature and only share the CDV equilibrium condition in their methodology. Since the CDV matrix is a natural 3d counterpart to the 2d notion of equilibrium stress (which we introduce in Sect. 3 and we show the connection between these two notions in Lemma 3), we refer to its entries as stresses.

The following lemma appears in [9], we include the proof since it illustrates how to construct a realization out of a CDV matrix.

Lemma 1 (Lemma 4, [9]). *Let $(u_i)_{1\leq i\leq n}$ be a cone-convex embedding of a planar 3-connected graph G with a positive CDV matrix M. Then every face f in G can be assigned with a vector ϕ_f, s.t. for each adjacent face g and separating edge $(v_i v_j)$*

$$\phi_f - \phi_g = M_{ij}(u_i \times u_j), \tag{2}$$

where f lies to the left and g lies to the right from $\overrightarrow{u_i u_j}$. The set of vectors (ϕ_f) is uniquely defined up to translations.

Proof. To construct the family of vectors (ϕ_f), we start by assigning an arbitrary value to ϕ_{f_0} (for an arbitrary face f_0); then we proceed iteratively. To prove the consistency of the construction, we show that the differences $(\phi_f - \phi_g)$ sum to zero over every cycle in G^*. Since G as well as G^* is planar and 3-connected, it suffices to check this condition for all elementary cycles of G^*, which are the faces of G^*. Let $\tau(i)$ denote the set of counterclockwise oriented edges of the face in G^* dual to $v_i \in V(G)$. Then, combining (1) and (2) yields

$$\sum_{(f,g)\in\tau(i)} (\phi_f - \phi_g) = \sum_{v_j\in N(G,v_i)} M_{ij}(u_i \times u_j) = u_i \times \left(\sum_{v_j\in N(G,v_i)} M_{ij}u_j \right) = u_i \times (-M_{ii}u_i) = 0.$$

The vectors (ϕ_f) are unique up to the initial choice for ϕ_{f_0}. $\square$

Note that there is a canonical way to derive a CDV matrix from a 3d polytope [9]. Every 3d embedding of a graph G as a polytope (u_i) possesses a positive CDV matrix defined by the vertices (ϕ_i) of its polar and equation (2). We refer to this matrix as the *canonical CDV matrix.*

The vectors constructed in Lemma 1 satisfy the following crucial property:

Lemma 2 (based on Lemma 5, [9]). *Let $(u_i)_{1\le i\le n}$ be a cone-convex embedding of a planar 3-connected graph G with a positive CDV matrix M. Then for any set of vectors $(\phi_f)_{f\in F(G)}$ fulfilling (2), the convex hull* $\mathrm{Conv}((\phi_f)_{f\in F(G)})$ *is a convex polytope with graph G^*; and the isomorphism between G^* and the skeleton of* $\mathrm{Conv}((\phi_f)_{f\in F(G)})$ *is given by $f \to \phi_f$.*

Proof. Lovász formulates this result (Lemma 5, [9]) with an additional constraint $|u_i| = 1$ for all v_is, that is for an embedding of a graph onto the sphere. Although he never uses this constraint in the proof we will provide an accurate reduction of Lemma 2 to Lovász's original formulation.

Let $(u_i)_{1\le i\le n}$ be a cone-convex embedding of G and $[M_{ij}]$ any of its positive CDV matrices. We set

$$\widetilde{u}_i := \frac{u_i}{|u_i|}, \qquad 1 \le i \le n,$$
$$\widetilde{M}_{ij} := |u_i||u_j|M_{ij}, \qquad 1 \le i, j \le n.$$

By definition, $(\widetilde{u}_i)$ is a convex embedding of G onto the sphere and clearly $[\widetilde{M}_{ij}]$ is one of its positive CDV matrices, since

$$\forall i \quad \sum_{1\le j\le n} \widetilde{M}_{ij}\widetilde{u}_j = \sum_{1\le j\le n} |u_i||u_j|M_{ij}\frac{u_j}{|u_j|} = |u_i| \sum_{1\le j\le n} M_{ij}u_j = 0.$$

Thus, $(\widetilde{u}_i)$ with $[\widetilde{M}_{ij}]$ fulfills the conditions of the lemma and, moreover, for every i we have $|\widetilde{u}_i| = 1$. By applying Lovász's Lemma [9, Lemma 5], we observe that the convex hull $\mathrm{Conv}((\phi_f))$ of any set of vectors $(\phi_f)_{f\in F(G)}$ fulfilling

$$\phi_f - \phi_g = \widetilde{M}_{ij}(\widetilde{u}_i \times \widetilde{u}_j), \quad \forall (f, g) \text{ dual to } (v_iv_j) - \text{edges of } G^* \text{ and } G$$

is a convex polytope with skeleton G^*. Since

$$\widetilde{M}_{ij}(\widetilde{u}_i \times \widetilde{u}_j) = |u_i||u_j|M_{ij}\left(\frac{u_i}{|u_i|} \times \frac{u_j}{|u_j|}\right) = M_{ij}(u_i \times u_j)$$

and, in particular,

$$\phi_f - \phi_g = M_{ij}(u_i \times u_j) \Leftrightarrow \phi_f - \phi_g = \widetilde{M}_{ij}(\widetilde{u}_i \times \widetilde{u}_j),$$

every system (ϕ_f) constructed from (u_i) and $[M_{ij}]$ coincides with a system constructed from $(\widetilde{u}_i)$ and $[\widetilde{M}_{ij}]$ and therefore spans the desired polytope. □

Lemmas 1 and 2 imply the following theorem which is the main tool in later constructions:

Theorem 1. *Let G be a planar 3-connected graph, let $(u_i)_{1\le i\le n}$ be its cone-convex embedding with integer coordinates and M - a positive CDV matrix with integer entries. Then there exists a realization $(\phi_f)_{f\in F(G)}$ of the dual graph G^* as a convex polytope with integer coordinates bounded by*

$$|\phi_f| < 2n \cdot \max_{(v_i,v_j)\in E(G)} |M_{ij}(q_i \times q_j)|.$$

Proof. We use Lemma 1 to construct $(\phi_f)_{f\in F(G)}$ that satisfy (2) and such that $\phi_{f_0} = (0, 0, 0)^T$ for a distinguished face $f_0 \in F(G)$. Lemma 2 guaranties that $(\phi_f)_{f\in F(G)}$ form a convex polytope with graph G^*. Since (ϕ_f) satisfy (2), $\phi_{f_0} = (0, 0, 0)^T$ and all M_{ij} as well as all u_i are integral, all ϕ_f have integer coordinates as well.

To finish the proof we estimate how large the vectors (ϕ_f) are. We evaluate ϕ_{f_k} for some face $f_k \in F(G)$. The following algebraic expression holds for all values ϕ_{f_i}:

$$\phi_{f_k} = \phi_{f_0} + (\phi_{f_1} - \phi_{f_0}) + \ldots + (\phi_{f_{k-1}} - \phi_{f_{k-2}}) + (\phi_{f_k} - \phi_{f_{k-1}}).$$

Let us now consider the shortest path $f_0, f_1, \ldots, f_k$ in G^* connecting the faces f_0 and f_k. Clearly, k is less than $2n$, and hence

$$|\phi_{f_k}| \leq 2n \cdot \max_{(f_a, f_b) \in E(G^*)} |\phi_{f_a} - \phi_{f_b}| = 2n \cdot \max_{(v_i, v_j) \in E(G)} |M_{ij}(q_i \times q_j)|.$$

□

3 From Plane to 3d Cone-Convex Embeddings

In this section we describe the connection between convex 2d embeddings with positive equilibrium stresses and cone-convex 3d embeddings with positive CDV matrices.

In the definition below we follow the presentation of [6]:

Definition 3. *An assignment $\omega\colon E(G) \to \mathbb{R}$ of scalars (denoted by $\omega(i,j) = \omega_{ij} = \omega_{ji}$) to the edges of a graph G is called a* stress. *A stress is an* equilibrium stress *for an embedding (u_i) of G into $\mathbb{R}^d$ if for every vertex $v_i \in V(G)$*

$$\sum_{v_j \in N(G, v_i)} \omega_{ij}(u_j - u_i) = 0.$$

We call an equilibrium stress of a 2d embedding with a distinguished boundary face f_0 positive *if it is positive on every edge that does not belong to f_0.*

The concept of equilibrium stress plays a central role in the classical Maxwell–Cremona lifting approach and it is also a crucial concept in our embedding algorithm. The following lemma establishes the connection between equilibrium stresses and CDV matrices.

Lemma 3. *Let $(q_i)_{1 \leq i \leq n}$ be an embedding of a graph G into $\mathbb{R}^3$. The following three statements hold:*

1. *Let G^+ be the graph G with one additional vertex v_0 connected with every vertex in G, and let $(q_0 = (0,0,0)^T, q_1, \ldots, q_n)$ be an embedding of G^+ into $\mathbb{R}^3$ equipped with an equilibrium stress (ω_{ij}). Then the assignment*

$$M_{ij} := \begin{cases} -\sum_{v_k \in N(G^+, v_i)} \omega_{ik}, & i = j, \\ \omega_{ij} & (v_i v_j) \in E(G), \\ 0 & \text{else} \end{cases}$$

   defines a CDV matrix $[M_{ij}]_{1 \leq i,j \leq n}$ for the embedding $(q_i)_{1 \leq i \leq n}$ of G.

2. *Let (ω_{ij}) be an equilibrium stress for the embedding $(q_i)_{1 \leq i \leq n}$ of G. Then the assignment*

$$M_{ij} := \begin{cases} -\sum_{v_k \in N(G, v_i)} \omega_{ik} & i = j, \\ \omega_{ij} & (v_i v_j) \in E(G), \\ 0 & \text{else} \end{cases}$$

   defines a CDV matrix $[M_{ij}]$ for $(q_i)_{1 \leq i \leq n}$.

3. *Let $(q_i)_{1 \leq i \leq n}$ be a flat embedding that lies in a plane not containing $(0,0,0)^T$. Let $[M_{ij}]$ be a CDV matrix for $(q_i)_{1 \leq i \leq n}$. Then $(M_{ij})_{(i,j) \in E(G)}$ defines an equilibrium stress for $(q_i)_{1 \leq i \leq n}$.*

Proof. 1. We check that the CDV equilibrium holds by noting

$$\forall i \quad \sum_{1\le j\le n} M_{ij}q_j = \sum_{1\le j\le n, j\ne i} M_{ij}q_j + M_{ii}q_i = \sum_{v_j\in N(G,v_i)} \omega_{ij}q_j + M_{ii}q_i$$
$$= \sum_{v_j\in N(G^+,v_i)} \omega_{ij}(q_j - q_i) - \omega_{i0}q_0 + \Big(\sum_{v_j\in N(G^+,v_i)} \omega_{ij} + M_{ii}\Big)q_i = 0.$$

The last transition holds since $\sum_{v_j\in N(G^+,v_i)} \omega_{ij}(q_j - q_i) = 0$ by the definition of (ω_{ij}), $q_0 = 0$, and $\sum_{v_j\in N(G^+,v_i)} \omega_{ij} + M_{ii} = 0$ due to the choice of M_{ii}.

2. We construct the embedding $(q_0 = (0,0,0)^T, q_1, \ldots, q_n)$ of the graph $G^+ := G + \{v_0\}$ and extend the equilibrium stress (ω_{ij}) of G to an equilibrium stress (ω^+_{ij}) of G^+ by assigning zeros to all the new edges $\omega^+_{i0} := 0$. Then we use the part (1) of the lemma to finish the proof.

3 . We denote by α the plane containing (q_i). We rewrite the CDV equilibrium condition:

$$0 = \sum_{1\le j\le n} M_{ij}q_j = \sum_{v_j\in N(G,v_i)} M_{ij}(q_j - q_i) + \left(M_{ii} + \sum_{v_j\in N(G,v_i)} M_{ij}\right) q_i.$$

and notice that, if nonzero, the first summand is parallel to the plane α while the second is not, so both must equal zero and

$$\sum_{v_j\in N(G,v_i)} M_{ij}(q_j - q_i) = 0$$

finishes the proof. □

The equilibrium stress on a realization of a complete graph arises as a "building block" in later constructions. The complete graph K_n, embedded in $\mathbb{R}^{n-2}$, has a unique equilibrium stress up to multiplication with a scalar. This stress has an easy expression in terms of volumes related to the embedding. We use the *square bracket notation*[3]

$$[q_iq_jq_kq_l] := \det\begin{pmatrix} x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \\ z_i & z_j & z_k & z_l \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \text{where } q = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

to obtain a formula for the equilibrium stress on the K_5 embedding.

Lemma 4 (Rote, Santos, and Streinu [14])**.** *Let $(u_0, u_1, \ldots, u_4)$ be an integer embedding of the complete graph K_5 onto $\mathbb{R}^3$. Then the assignment:*

$$\omega_{ij} := [u_{i-2}u_{i-1}u_{i+1}u_{i+2}][u_{j-2}u_{j-1}u_{j+1}u_{j+2}]$$

(indices in cyclic notation) defines an integer equilibrium stress on this embedding.

Theorem 2. *Let $G_\uparrow$ be a planar 3-connected graph, $(p_i)_{2\le i\le n}$ — its convex planar embedding with integer coordinates, equipped with a positive integer equilibrium stress (ω_{ij}), and a designated triangular face $f_0 = (v_2v_3v_4)$ embedded as the boundary face. Then there exists a cone-convex embedding $(q_i)_{1\le i\le n}$ with integer coordinates of the graph $G = \mathrm{Stack}(G_\uparrow; v_1; v_2v_3v_4)$ equipped with an integer positive CDV matrix $[M_{ij}]_{1\le i,j\le n}$, such that*

$$M_{ij} = \omega_{ij} \quad \textit{for each internal edge } (v_iv_j) \textit{ of the original embedding of } G_\uparrow$$

and each entry of M is bounded by $O(n \max_{ij} |\omega_{ij}| \cdot \max_i |p_i|^6)$.

[3]For 2d vectors $[p_ip_jp_k]$ is defined similarly.

Proof. We can assume that $(0,0)^T$ lies inside the embedding of f_0. Let $(q_i)_{1\le i\le n}$ be the embedding of G, defined as follows: The embedding of $G_\uparrow$ is realized in the plane $\{z=1\}$ and the stacked vertex is placed at $(0,0,-1)^T$. The embedding is cone-convex since it describes a tetrahedron containing the origin with one face that is refined with a plane convex subdivision.

Following the structure of $G = \mathrm{Stack}(G_\uparrow; v_1; v_2v_3v_4)$, we decompose G into two subgraphs: $G_\uparrow = G[\{v_2,\ldots,v_n\}]$ and $G_\downarrow := G[\{v_1,v_2,v_3,v_4\}]$.

We first compute a CDV matrix $[M'_{ij}]_{2\le i,j\le n}$ for the embedding $(q_i)_{2\le i\le n}$ of $G_\uparrow$. The plane embedding $(p_i)_{2\le i\le n}$ of $G_\uparrow$ has an integer equilibrium stress $(\omega_{ij})_{2\le i,j\le n}$. Since $(q_i)_{2\le i\le n}$ is just a translation of $(p_i)_{2\le i\le n}$, clearly, $(\omega_{ij})_{2\le i,j\le n}$ is also an equilibrium stress for $(q_i)_{2\le i\le n}$ and we use the part (2) of Lemma 3 to transform it into the integer CDV matrix $[M'_{ij}]$.

As a second step we compute a CDV matrix $[M''_{ij}]_{1\le i,j\le 4}$ for the embedding of the tetrahedron $G_\downarrow$. We apply Lemma 4 for the embedding of the K_5 formed by $\{q_0 = (0,0,0)^T, q_1, q_2, q_3, q_4\}$ and receive an integer equilibrium stress $(\omega''_{ij})_{0\le i,j\le 4}$. We transform it to an integer CDV matrix $[M''_{ij}]_{1\le i,j\le 4}$ for the tetrahedron $\{q_1,q_2,q_3,q_4\}$ using the part (1) of Lemma 3. One can easily check that as soon as the origin lies inside the tetrahedron $\{q_1,q_2,q_3,q_4\}$ all entries M''_{ij} have the same sign. We can assume that $[M''_{ij}]$ is positive, otherwise we reorder the vertices $\{v_2,v_3,v_4\}$.

In the final step we extend the two CDV matrices M' and M'' to G and combine them. Clearly, a CDV matrix padded with zeros remains a CDV matrix. Furthermore, any linear combination of CDV matrices is again a CDV matrix. Thus, we form an integer CDV matrix for the whole embedding $(q_i)_{1\le i\le n}$ of G by setting:

$$M := M' + \lambda M'',$$

where λ is a positive integer chosen so that M is a positive CDV matrix. This can be done as follows.

Recall that (ω_{ij}) is a positive stress and M'' is a positive CDV matrix. Hence, the only six entries in M corresponding to edges of G that may be negative are: M_{23}, M_{34} and M_{42} (and their symmetric entries), for which $M_{ij} := M'_{ij} + \lambda M''_{ij}$ with $M'_{ij} < 0$ and $M''_{ij} > 0$. Thus, we choose λ such that M is positive at these entries. To satisfy this condition we pick

$$\lambda = \left\lfloor \max_{(i,j)\in\{(2,3),(3,4),(4,2)\}} (|M'_{ij}|/|M''_{ij}|) \right\rfloor + 1.$$

To bound M_{ij} we notice that the entries of M''_{ij} are strictly positive integers, so $\lambda = O(\max|M'_{ij}|)$, while $|M'_{ij}| = O(n\cdot\max|\omega_{ij}|)$ and $|M''_{ij}| = O(\max|\omega''_{ij}|) = O(\max|p_i|^6)$. The bound $|M_{ij}| = O(n\cdot\max_{ij}|\omega_{ij}|\cdot\max_i|p_i|^6)$ follows. □

4 Realizations of Truncated Polytopes

In this section we sum up previous results in Theorem 3 and present an embedding algorithm for truncated 3d polytopes in Theorem 4. We will apply Theorem 3 also in the more general setup of Sect. 5.

Theorem 3. *Let $G = \mathrm{Stack}(G_\uparrow; v_1; v_2v_3v_4)$ and let $(p_i)_{2\le i\le n}$ be a planar embedding of $G_\uparrow$ with integer coordinates, boundary face $(v_2v_3v_4)$, and a positive integer equilibrium stress (ω_{ij}). Then there exists a realization $(\phi_f)_{f\in F(G^*)}$ of the graph G^*, dual to G, as a convex polytope with integer coordinates such that*

$$|\phi_f| = O(n^2\cdot\max|\omega_{ij}|\cdot\max|p_i|^8).$$

Proof. We first apply Theorem 2 to obtain a cone-convex embedding $(q_i)_{1\le i\le n}$ of G with integer coordinates and a positive integer CDV matrix $[M_{ij}]_{1\le i,j\le n}$. We then apply Theorem 1 and

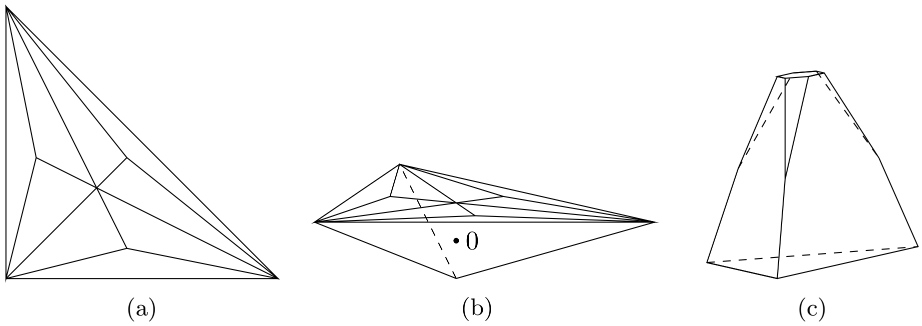


Figure 2: The 2d embedding of $G_\uparrow$ (a), the cone-convex embedding of G (b), and the resulting embedding of the dual (c).

obtain a family of vectors $(\phi_f)_{f\in F(G^*)}$ that form a desired realization of G^* as a convex polytope with integer coordinates.

To estimate how large the coordinates of the embedding (ϕ_f) are, we combine bounds for (ϕ_f) given by Theorem 1 with the bounds for the entries of M given by Theorem 2:

$$\begin{aligned}|\phi_f| &\le 2n \cdot \max_{(v_i,v_j)\in E(G)} |M_{ij}(q_i \times q_j)| \\ &= O(n \cdot (n \cdot \max|\omega_{ij}| \cdot \max|p_i|^6) \cdot \max|q_i|^2) = O(n^2 \cdot \max|\omega_{ij}| \cdot \max|p_i|^8).\end{aligned}$$

□

Next we apply Theorem 3 to construct an integer polynomial size grid embedding for truncated polytopes. To construct small integer 2d embeddings with a small integer equilibrium stress we use a result by Demaine and Schulz [6], which states that the graph of a stacked polytope with n vertices and any distinguished face f_0 can be embedded on a $10n^4 \times 10n^4$ grid with boundary face f_0 and with integral positive equilibrium stress (ω_{ij}) such that, for every edge $(v_i v_j)$, we have $|\omega_{ij}| = O(n^{10})$.

Theorem 4. *Any truncated 3d polytope with n vertices can be realized with integer coordinates of size $O(n^{44})$.*

Proof. Let G^* be the graph of the truncated polytope and $G := (G^*)^*$ its dual. Clearly, G is the graph of a stacked polytope with $(n+4)/2$ vertices. We denote the last stacking operation (for some sequence of stacking operations producing G) as the stacking of the vertex v_1 onto the face $(v_2v_3v_4)$ of the graph $G_\uparrow := G[V \setminus \{v_1\}]$. The graph $G_\uparrow$ is again a stacked graph, and hence, by the Lemma of Demaine and Schulz, there exists an embedding $(p_i)_{2\le i\le n}$ of $G_\uparrow$ into $\mathbb{Z}^2$ with an equilibrium stress (ω_{ij}) satisfying the properties of Theorem 3. We apply the theorem and obtain a polytope embedding (ϕ_f) of G^* with bound $|\phi_f| = O(n^2 \cdot \max|\omega_{ij}| \cdot \max|p_i|^8) = O(n^{44})$. □

Figure 2 shows an example of our algorithm. The computations for this example are presented in Sect. 6.

5 A Dual Transform for Simplicial Polytopes

As we have seen a small grid embedding of a 3d polytope can be computed when a small integer (though, not necessarily convex) embedding of its dual polytope with a small integral positive CDV matrix is known. However, if one wants to build a dual for an already embedded polytope, one usually does not possess such a matrix. The canonical CDV matrix associated with any embedding of a 3d polytope is not helpful, since its entries, when scaled to integers, might become exponentially large.

In this section of the paper we show how one can tackle this problem for simplicial polytopes. The section is divided into two parts. The first is devoted to new techniques regarding 2-dimensional equilibrium stresses. They immediately imply the result for a special subclass of simplicial polytopes, namely, for those containing a vertex of degree 3 and having some additional geometric properties. In the second part we use a more technical analysis to prove a more general theorem.

5.1 Equilibrium Stresses: Wheel Decomposition and the Reverse of the Maxwell-Cremona Lifting

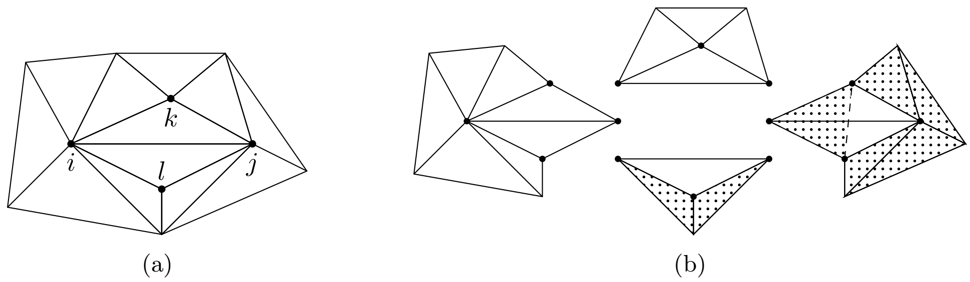


Figure 3: Part of a triangulation, participating in the wheel-decomposition of ω_{ij} (a); Wheels W_i, W_l, W_j, W_k ((b), c.c.w) with shadowed areas of the triangles participating in the definition of *large atomic stresses* for W_l and W_j.

Before proceeding, let us review how the canonical stress associated with an orthogonal projection of a 3d polytope in the $\{z = 0\}$ plane can be described. The assignment of heights to the interior vertices of a 2d embedding resulting in a polyhedral surface is called a (polyhedral) *lifting*. By the Maxwell-Cremona correspondence the equilibrium stresses of a 2d embedding of a planar 3-connected graph and its liftings (modulo projection preserving affine transformations of the space) are in 1-1 correspondence. Moreover, the bijection between liftings and stresses can be defined as follows. Let (p_i) be a 2d drawing of a triangulation and let (q_i) be the 3d embedding induced by some lifting. We map this lifting to the equilibrium stress (ω_{ij}) by assigning to every edge $(v_i v_j)$ separating the faces $(v_i v_j v_k)$ (on the left) and $(v_i v_j v_l)$ (on the right)

$$\omega_{ij} := \frac{[q_i q_j q_k q_l]}{[p_i p_j p_k][p_l p_j p_i]}. \tag{3}$$

This mapping gives the desired bijection. The expression (3) is a slight reformulation of the form presented in Hopcroft and Kahn [8, Equation 11].

We continue by studying the spaces of equilibrium stresses for triangulations. A graph formed by a vertex v_0, called *center*, connected to every vertex of a cycle $v_1, \ldots, v_n$ is called a *wheel* (Figure 3); we denote it as $\mathrm{W}(v_0; v_1 \ldots v_n)$. A wheel that is a subgraph of a triangulation G with $v_i \in V(G)$ as a center is denoted by W_i. Every triangulation can be "covered" with a set of wheels $(W_i)_{v_i \in V(G)}$, so that every edge is covered four times (Figure 3).

Lemma 5. *Let $(p_i)_{0 \le i \le n}$ be an embedding of a wheel $\mathrm{W}(v_0; v_1 \ldots v_n)$ in $\mathbb{R}^2$. Then the following expression defines an equilibrium stress:*

$$\omega_{ij} = \begin{cases} -1/[p_i p_{i+1} p_0] & j = i+1, 1 \le i \le n, \\ [p_{i-1} p_i p_{i+1}]/([p_{i-1} p_i p_0][p_i p_{i+1} p_0]) & j = 0, 1 \le i \le n. \end{cases}$$

The equilibrium stress for the embedding (p_i) is unique up to a renormalization.

Proof. This stress coincides with (3) from the lifting of W with $z_0 = 1$ and $z_i = 0$ for $1 \leq i \leq n$ and so it is an equilibrium stress. The space of the stresses is 1-dimensional, since the space of the polyhedral liftings is 1-dimensional. $\square$

Definition 4. *1. For a wheel W embedded in the plane we refer to the equilibrium stress defined in Lemma 5 as its* small atomic stress *and denote it as $(\omega^a_{ij}(W))$.*

2. We call the stress $(\Omega^a_{ij}(W))$ that is obtained by the multiplication of $(\omega^a_{ij}(W))$ by the factor $\prod_{1\leq j\leq n}[p_j p_{j+1} p_0]$, the large atomic stress *of W.*

We point out that the large atomic stresses are products of $\deg(v_0) - 1$ triangle areas multiplied by 2, and so, all stresses $\Omega^a_{ij}(W)$ are integers if W is realized with integer coordinates.

Theorem 5 (Wheel-decomposition)**.** *Let G be a triangulation. Every equilibrium stress $\omega = (\omega_{ij})$ of an embedding $(p_i)_{1\leq i\leq n}$ of G can be expressed as a linear combination of the small atomic stresses on the wheels $(W_i)_{1\leq i\leq n}$*

$$\omega = \sum_{1\leq i\leq n} \alpha_i \omega^a(W_i),$$

where the coefficients α_i are the heights (i.e., z-coordinates) of the corresponding vertices v_i in the Maxwell–Cremona lifting of $(p_i)_{1\leq i\leq n}$ induced by (ω_{ij}).

Proof. Let $(q_i)_{1\leq i\leq n}$ be the Maxwell-Cremona lifting of (p_i) by means of the stress (ω_{ij}). We rewrite this stress in terms of (q_i) and (p_i) as done in (3) using that for any four points $(a_i)_{1\leq i\leq 4}$ in $\mathbb{R}^3$ and their projections $(b_i)_{1\leq i\leq 4}$ to the (x, y)-plane

$$[a_1 a_2 a_3 a_4] = \sum_{1\leq i\leq 4} (-1)^{i+1} z_i [b_{i+1} b_{i+2} b_{i+3}]$$

(with cyclic notation for indices and for z_i being for the z-coordinate of a_i) and obtain

$$\begin{aligned}\omega_{ij} =& z_i \frac{[p_j p_k p_l]}{[p_i p_j p_k][p_l p_j p_i]} + z_j \frac{[p_l p_k p_i]}{[p_i p_j p_k][p_l p_j p_i]} - z_k \frac{1}{[p_i p_j p_k]} - z_l \frac{1}{[p_l p_j p_i]} \\ =& z_i \omega^a_{ij}(W_i) + z_j \omega^a_{ij}(W_j) + z_k \omega^a_{ij}(W_k) + z_l \omega^a_{ij}(W_l)\end{aligned}$$

for z_i being for the z-coordinate of q_i, which is exactly the decomposition of ω_{ij} into the small atomic stresses. $\square$

The following theorem is a part of the algorithm that we use to prove the more general Theorem 8, but it also is of independent interest since it provides an elegant construction that is inverse to the Maxwell-Cremona lifting procedure. The direct way to reverse the lifting procedure would be to project the 3d embedding of a graph to a plane and to calculate stresses using the equation (3). If the 3d embedding has polynomial size integer coordinates, the calculated stresses are, generally speaking, rational, and when scaled to integer may become exponentially large. The theorem provides a more careful method by allowing a small perturbation of the canonical stress as given by (3).

Theorem 6 (Reverse of the Maxwell-Cremona Lifting)**.** *Let $(q_i)_{1\leq i\leq n}$ be an embedding of a triangulation G into $\mathbb{Z}^3$, whose projection $(p_i)_{1\leq i\leq n}$ to the plane $\{z = 0\}$ is noncrossing with boundary face $(v_1 v_2 v_3)$. Then one can construct a positive integer equilibrium stress (ω_{ij}) for $(p_i)_{1\leq i\leq n}$ such that*

$$|\omega_{ij}| < (\max_{i\leq n} |q_i|)^{2\,\Delta_G + 5}.$$

Proof. We assume that all the coordinates of the embedding are nonnegative. Let L be the largest coordinate of the embedding. We start with the equilibrium stress $(\widetilde{\omega}_{ij})$ as specified by (3) for the embedding (p_i). Since all the coordinates are integers, all stresses are bounded by

$$\frac{1}{\mathrm{L}^4} \leq \frac{1}{|[p_ip_jp_k]||[p_lp_jp_i]|} \leq |\widetilde{\omega}_{ij}| \leq |[q_iq_jq_kq_l]| \leq \mathrm{L}^3\,.$$

We are left with making these stresses integral while preserving a polynomial bound. The stress $(\widetilde{\omega}_{ij})$ can be written as a linear combination of the *large* atomic stresses of the wheels W_i by means of the Wheel-decomposition Theorem,

$$\widetilde{\omega}_{ij} = \alpha_i\Omega^a_{ij}(W_i) + \alpha_j\Omega^a_{ij}(W_j) + \alpha_k\Omega^a_{ij}(W_k) + \alpha_l\Omega^a_{ij}(W_l).$$

We remark that since we use *large* atomic stresses, coefficients α_k are not the z coordinates of q_k, but these z coordinates divided by the multiplicative factor from the definition of the large atomic stress. Since all the points p_i have integer coordinates, the large atomic stresses are integers as well. Moreover, each of them, as a product of $\deg(v_k)-1$ triangle areas, is bounded by $|\Omega^a_{ij}(W_k)| \leq \mathrm{L}^{2(\Delta_\mathrm{G}-1)}$.

To make the $\widetilde{\omega}_{ij}$s integral we round the coefficients α_i down. To guarantee that the rounding does not alter the signs of the stress, we scale the atomic stresses (before rounding) with the factor

$$C = 4\max_{i,j,k}|\Omega^a_{ij}(W_k)|/\min_{i,j}|\widetilde{\omega}_{ij}|$$

and define as the new stress:

$$\omega_{ij} := \lfloor C\alpha_i\rfloor\Omega^a_{ij}(W_i) + \lfloor C\alpha_j\rfloor\Omega^a_{ij}(W_j) + \lfloor C\alpha_k\rfloor\Omega^a_{ij}(W_k) + \lfloor C\alpha_l\rfloor\Omega^a_{ij}(W_l).$$

Clearly,

$$\begin{aligned}|\omega_{ij} - C\widetilde{\omega}_{ij}| &= \left|\sum_{\tau=i,j,k,l}(\lfloor C\alpha_\tau\rfloor\ - C\alpha_\tau)\Omega^a_{ij}(W_\tau)\right| \\ &< \sum_{\tau=i,j,k,l}|\Omega^a_{ij}(W_\tau)| \leq 4\max_{i,j,k}|\Omega^a_{ij}(W_k)| = C\min_{i,j}|\widetilde{\omega}_{ij}| \leq C|\widetilde{\omega}_{ij}|\end{aligned}$$

and so $\mathrm{sign}(\omega_{ij}) = \mathrm{sign}(C\widetilde{\omega}_{ij}) = \mathrm{sign}(\widetilde{\omega}_{ij})$. From the last equation it also follows that none of the stresses ω_{ij} are nonzero.

Therefore, the constructed equilibrium stress (ω_{ij}) is integral and positive. We conclude the proof with an upper bound on its size. Since $C < 4\,\mathrm{L}^{2(\Delta_\mathrm{G}-1)}\,\mathrm{L}^4$,

$$\begin{aligned}|\omega_{ij}| &\leq \left|\sum_{\tau=i,j,k,l}(C\alpha_\tau \pm 1)\Omega^a_{ij}(W_\tau)\right| \leq C|\widetilde{\omega}_{ij}| + \sum_{\tau=i,j,k,l}|\Omega^a_{ij}(W_\tau)| \\ &\leq C\max|\widetilde{\omega}_{ij}| + 4\max|\Omega^a_{ij}(W_k)| \leq 4\,\mathrm{L}^{2\,\Delta_\mathrm{G}+2}\cdot\mathrm{L}^3 + 4\,\mathrm{L}^{2\,\Delta_\mathrm{G}-2} = O(\mathrm{L}^{2\,\Delta_\mathrm{G}+5}).\end{aligned}$$

□

The following theorem, which is a special case of the more general Theorem 8 is a direct implication of Theorem 6 and Theorem 3:

Theorem 7. *Let $G_\uparrow$ be a triangulation and let $(q_i)_{2\leq i\leq n}$ be its realization as a convex polytope with integer coordinates, such that its orthogonal projection into the plane $\{z=0\}$ is a planar embedding $(p_i)_{2\leq i\leq n}$ of $G_\uparrow$ with boundary face $(v_2v_3v_4)$. Then the exists a realization $(\phi_f)_{f\in F(G)}$ of a graph dual to $G = \mathrm{Stack}(G_\uparrow; v_1; v_2v_3v_4)$ with integer coordinates bounded by*

$$|\phi_f| = O(n^2\max|q_i|^{2\,\Delta_\mathrm{G}+13}).$$

We remark that the algorithms following the lifting approach, e.g. [13], generate embeddings that fulfill the conditions of the above theorem.

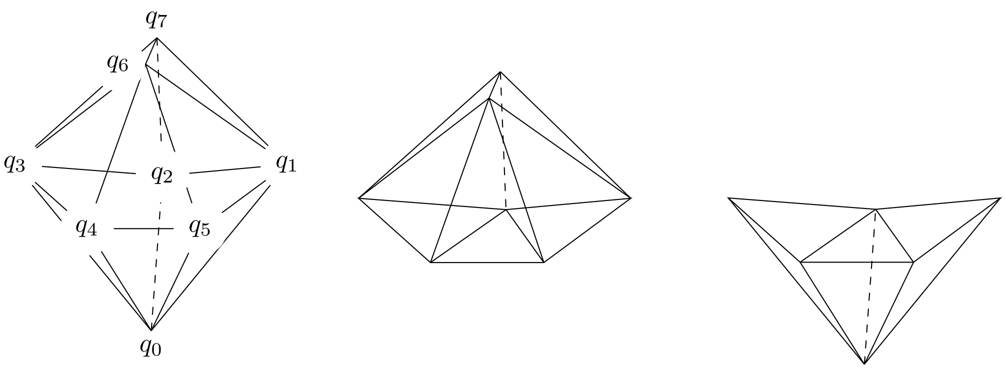
5.2 A Dual Transform for Simplicial Polytopes

In this subsection we prove the following most general theorem of the paper.

Theorem 8. *Let G be a triangulation and let $\mathcal{P} = (q_i)$ be a realization of G as a convex polytope with integer coordinates. Then there exists a realization $(\phi_f)_{f\in F(G)}$ of the dual graph G^* as a convex polytope with integer coordinates bounded by*

$$|\phi_f| < \max |q_i|^{O(\Delta_G)}.$$

Proof. The proof of the theorem consists of two big steps and one smaller final remark. First we analyze the original polytope and decompose it into two parts equipped with CDV matrices: (i) a nonintegral flat embedding and (ii) an integral pyramid. On the second step we perturb the flat part so that the coordinates of the embedding and the CDV matrix become integral. In the end we glue the two parts together to form a cone-convex embedding of the original polytope and apply the construction of Theorem 1.



(a) $\mathcal{P}$ (b) $\mathcal{P}^{\uparrow}$ (c) $\mathcal{P}^{\downarrow}$

Figure 4: The original polytope (a) decomposed into the upper part (b) the lower part (c). In the figure we have $v_{12} = v_{23} = v_7$ and $v_{34} = v_{45} = v_{51} = v_6$.

Step 1. As every triangulation, G has either a vertex of degree 3, 4 or 5. We consider the later case, the others can be handled in a similar way. We denote the degree-5 vertex by v_0 and its neighbors in counterclockwise order by $v_1, v_2, \ldots, v_5$ (Figure 4a).

We temporarily remove the vertex q_0 and consider the polytope $\mathcal{P}^{\uparrow} := \mathrm{Conv}(q_1, \ldots, q_n)$ formed by all the others vertices (Figure 4b). Clearly, its graph $G^{\uparrow}$ is the graph G of the original polytope with removed vertex v_0 and a triangulated new 5-gonal face $(v_1, \ldots, v_5)$ If $q_1, \ldots, q_5$ were not in general position, this face would not be triangulated, we then triangulate it arbitrarily, allowing additional edges to be flat in $\mathcal{P}^{\uparrow}$.

Without loss of generality, we assume that the 5-gonal face $(v_1, \ldots, v_5)$ is triangulated with additional edges (v_2v_4) and (v_2v_5). We also assume that the new triangular face $(q_2q_4q_5)$ of $\mathcal{P}^{\uparrow}$ lies in the plane $\{z = C_0\}$, that q_0 lies in the half-space $\{z < C_0\}$ and all other vertices of $\mathcal{P}$ in $\{z \geq C_0\}$ for some positive integer C_0. The triangulated 5-gon $q_1, \ldots, q_5$ then divides $\mathcal{P}$ onto two polytopes, the upper one $\mathcal{P}^{\uparrow}$ and the lower one (possibly non convex, Figure 4c) that we denote by $\mathcal{P}^{\downarrow} := \mathrm{Cl}(\mathcal{P} \setminus \mathcal{P}^{\uparrow})$ with graphs $G^{\uparrow}$ and $G^{\downarrow}$ correspondingly.

Consider the half-spaces H_{125} and H_{234} bounded by the faces $(q_1q_2q_5)$ and $(q_2q_3q_4)$ of $\mathcal{P}^{\uparrow}$ and containing this polytope. The interior of $\mathcal{P}^{\downarrow} \cap (H_{125}^c \cap H_{234}^c)$ is nonempty and all the bounding planes pass through integral vertices of $\mathcal{P}$.[4] So, after scaling with an appropriate

[4] For a set $A \subset \mathbb{R}^n$ we denote by A^c its complement.

polynomial in $\max|q_i|$, the interior of $\mathcal{P}^{\downarrow} \cap (H^c_{125} \cap H^c_{234})$ also contains integral points. Without loss of generality we assume that $(0,0,0)^T \in \mathcal{P}^{\downarrow} \cap (H^c_{125} \cap H^c_{234})$.

The realizations $\mathcal{P}^{\uparrow}$ and $\mathcal{P}^{\downarrow}$ of $G^{\uparrow}$ and $G^{\downarrow}$ possess the canonical CDV matrices $[M^{\uparrow}_{ij}]$ and $[M^{\downarrow}_{ij}]$ correspondingly. These matrices sum up to $M = M^{\uparrow} + M^{\downarrow}$, which is the canonical CDV matrix for the original embedding $\mathcal{P}$ of G. In particular, weights on the additional edges cancel, i.e., $M^{\uparrow}_{24} + M^{\downarrow}_{24} = M^{\uparrow}_{25} + M^{\downarrow}_{25} = 0$, and $M = M^{\uparrow} + M^{\downarrow}$ is a positive CDV matrix as a canonical CDV matrix for a convex polytope $\mathcal{P}$ containing $(0,0,0)^T$ in its interior.

Denote the faces of G adjacent to the edges $(v_1v_2), (v_2v_3), \dots, (v_5v_1)$ not containing v_0 by $(v_1v_2v_{12}), (v_2v_3v_{23}), \dots, (v_5v_1v_{51})$ (it might be that some of the points $v_{12}, \dots, v_{51}$ coincide). We consider the subset

$$\mathrm{V_s} := \{v_1, v_2, \dots, v_5, v_{12}, \dots, v_{51}\}$$

of vertices of G and call them *stable*, since they will be kept in position throughout some parts of our algorithm. We project $q_1, \dots, q_n$ to the plane $\{z = \prod_{v_i \in \mathrm{V_s}} z_i\}$ along the rays emanating from $(0,0,0)^T$, where z_i is the z-coordinate of q_i (Figure 5a). We denote the projection of q_i by $\bar{q}_i$. The projection plane is chosen so that the stable vertices $\bar{q}_i$, $v_i \in \mathrm{V_s}$, have integer coordinates. Since $(0,0,0)^T \in \mathcal{P}^{\downarrow} \cap (H^c_{125} \cap H^c_{234})$ the projection lies within the 5-gon $\bar{q}_1, \dots, \bar{q}_5$ and this 5-gon is convex.

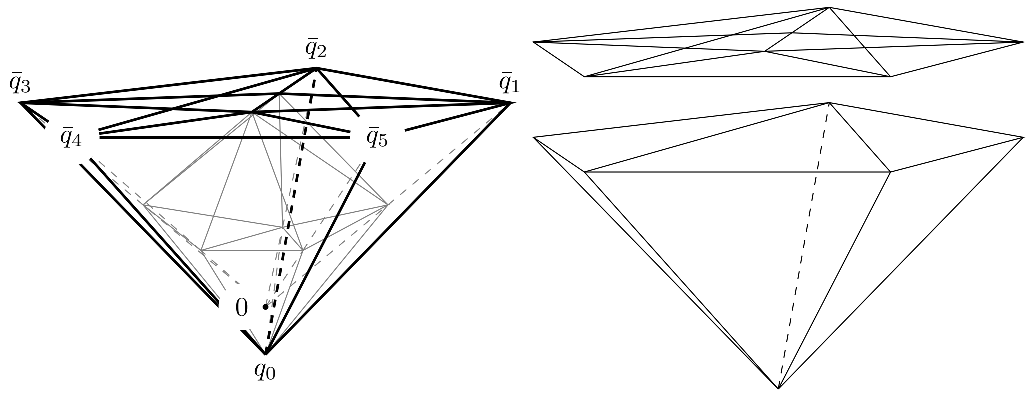
The projection changes the canonical CDV matrices $[M^{\uparrow,\downarrow}_{ij}]$ to canonical CDV matrices $[\frac{1}{\lambda_i\lambda_j} M^{\uparrow,\downarrow}_{ij}]$ where $\bar{q}_i = \lambda_i q_i$. We additionally multiply these matrices with $C_M = \max|q_i|^{O(1)}$, which gives

$$\bar{M}^{\uparrow,\downarrow}_{ij} := C_M \frac{1}{\lambda_i \lambda_j} M^{\uparrow,\downarrow}_{ij},$$

so that $\bar{M}^{\uparrow,\downarrow}_{ij}$ is integer for every edge $(v_iv_j) \in \mathrm{E}^{\uparrow}_{\mathrm{s}} \cup \mathrm{E}^{\downarrow}_{\mathrm{s}}$ where

$$\begin{aligned}
\mathrm{E}^{\downarrow}_{\mathrm{s}} &:= \{(v_0v_1), (v_0v_2), \dots, (v_0v_5)\}, \\
\mathrm{E}^{\uparrow}_{\mathrm{s}} &:= \{(v_1v_2), (v_2v_3), \dots, (v_5v_1)\} \cup \{(v_2v_4), (v_2v_5)\}.
\end{aligned}$$

Similarly to $\mathrm{V_s}$ we call these edges *stable*. We will use the integrality of $M^{\uparrow,\downarrow}_{ij}$ for the stable edges on the following steps of the proof.



(a) Projecting of $\mathcal{P}$ (gray) to $\bar{\mathcal{P}}$ (bold black) along the rays emanating from $(0,0,0)^T$ (dashed gray).

(b) Decomposition of $\bar{\mathcal{P}}$ into $\bar{\mathcal{P}}^{\uparrow}$ (top) and $\bar{\mathcal{P}}^{\downarrow}$ (bottom).

Figure 5

As a result of the first step we have two embeddings (Figure 5b)

$$\bar{\mathcal{P}}^{\uparrow} := (\bar{q}_1, \dots, \bar{q}_n) \text{ and } \bar{\mathcal{P}}^{\downarrow} := (q_0, \bar{q}_1, \dots, \bar{q}_5)$$

of graphs $G^\uparrow$ and $G^\downarrow$ equipped with CDV matrices $[\bar{M}^\uparrow_{ij}]$ and $[\bar{M}^\downarrow_{ij}]$ such that the embedding $\bar{\mathcal{P}}^\uparrow$ is flat, the coordinates of all the stable vertices $v_i \in \mathrm{V_s}$ as well as the stresses $\bar{M}^{\uparrow,\downarrow}_{ij}$ for all the stable edges $(v_i v_j) \in \mathrm{E}^\uparrow_\mathrm{s} \cup \mathrm{E}^\downarrow_\mathrm{s}$ are integers and $\bar{M} := \bar{M}^\uparrow + \bar{M}^\downarrow$ is a positive CDV matrix for the embedding $\bar{\mathcal{P}} := \bar{\mathcal{P}}^\uparrow \cup \bar{\mathcal{P}}^\downarrow$ of the original graph G. Note that the stresses on the additional edges still cancel since $\bar{M}_{24} + \bar{M}_{24} = \frac{1}{z_2 z_4}(M_{24} + M_{24}) = 0$ and $\bar{M}_{25} + \bar{M}_{25} = \frac{1}{z_2 z_5}(M_{25} + M_{25}) = 0$.

Step 2. In the second step we consider only the upper part $\bar{\mathcal{P}}^\uparrow$ of the polytope with its CDV matrix $\bar{M}^\uparrow_{ij}$. The goal of this step is to perturb the nonstable vertices of $\bar{\mathcal{P}}^\uparrow$ as well as the stresses $\bar{M}^\uparrow_{ij}$ on nonstable edges to integers. These two objectives will be handled separately.

Vertex perturbation. We consider vertices of $\bar{\mathcal{P}}^\uparrow$. To make all the coordinates of $\bar{\mathcal{P}}^\uparrow$ integral we lift them to 3d (I), scale and perturb (II), and project back (III).

(I) $\bar{\mathcal{P}}^\uparrow$ is a flat embedding, so (Lemma 3) its CDV matrix $\bar{M}^\uparrow$ defines an equilibrium stress and we use it to construct the Maxwell-Cremona lifting of $\bar{\mathcal{P}}^\uparrow$ that we denote by

$$\bar{\mathcal{P}}^{\uparrow,L} = (\bar{q}^L_i)_{1 \le i \le n}$$

with the face $(\bar{q}^L_2 \bar{q}^L_4 \bar{q}^L_5)$ fixed in the initial plane $\{z = \prod_{v_i \in \mathrm{V_s}} z_i\}$. Since all $\bar{M}_{ij}$ for $(v_i v_j) \in \mathrm{E}^\uparrow_\mathrm{s}$ as well as all $\bar{q}_i$ for $v_i \in \mathrm{V_s}$ are integral, and the face $(\bar{q}^L_2 \bar{q}^L_4 \bar{q}^L_5)$ is fixed in a horizontal plane, all the liftings $\bar{q}^L_i$ for $v_i \in \mathrm{V_s}$ are integral as well.

(II) To make all the vertices of $\bar{\mathcal{P}}^{\uparrow,L}$ integral, we round them to the nearest integral point. To preserve the convexity of the edges of $\bar{\mathcal{P}}^{\uparrow,L}$ and the combinatorics of its projection $\bar{\mathcal{P}}^\uparrow$, we first scale with a large but polynomial factor: we set

$$\dot{\mathcal{P}}^{\uparrow,L} := (\dot{q}^L_i)_{1 \le i \le n}, \quad \dot{q}^L_i := [C \cdot \bar{q}^L_i]$$

where $[q]$ is the nearest integral point to the point q and

$$C := \max\left(1, \left\lceil \frac{3 \max |[\bar{q}_i \bar{q}_j]|}{\min |[\bar{q}_i \bar{q}_j \bar{q}_k]|} \right\rceil, \left\lceil \frac{4 \max |[\bar{q}_i \bar{q}_j \bar{q}_k]|}{\min |[\bar{q}^L_i \bar{q}^L_j \bar{q}^L_k \bar{q}^L_l]|} \right\rceil \right).$$

Extrema are taken over all the doubles, triples and quadruples of points under consideration. The rounding does not affect $\bar{q}^L_i$ for $v_i \in \mathrm{V_s}$ since they are initially integral, hence we have $\dot{q}^L_i = C \cdot \bar{q}^L_i$ for $v_i \in \mathrm{V_s}$.

The factor C is chosen so that after rounding no volume and no area participating in the decomposition (3), $\bar{M}^\uparrow_{ij} = \frac{[\bar{q}^L_i \bar{q}^L_j \bar{q}^L_k \bar{q}^L_l]}{[\bar{q}_i \bar{q}_j \bar{q}_k][\bar{q}_l \bar{q}_j \bar{q}_i]}$, changes its sign. This guarantees that the combinatorics of the flat embedding $\bar{\mathcal{P}}^\uparrow$ as well as the convexity of all the edges of the 3d embedding $\bar{\mathcal{P}}^{\uparrow,L}$ are preserved. To accurately bound C we notice that $|[\bar{q}_i \bar{q}_j]|$, $|[\bar{q}_i \bar{q}_j \bar{q}_k]|$ and $\bar{M}^\uparrow_{ij}$ are bounded from below and from above by $\max |q_i|^{-C_b}$ and $\max |q_i|^{C_a}$ with some universal constants C_a and C_b. Since $\bar{M}^\uparrow_{ij} = \frac{[\bar{q}^L_i \bar{q}^L_j \bar{q}^L_k \bar{q}^L_l]}{[\bar{q}_i \bar{q}_j \bar{q}_k][\bar{q}_l \bar{q}_j \bar{q}_i]}$, we have $\max |q_i|^{-C'_b} \le [\bar{q}^L_i \bar{q}^L_j \bar{q}^L_k \bar{q}^L_l] \le \max |q_i|^{C'_a}$ for some universal constants C'_a and C'_b as well. All this guarantees that $C \le \max |q_i|^{O(1)}$.

(III) We orthogonally project $(\dot{q}^L_i)$ back to the plane containing the face $(\dot{q}^L_2 \dot{q}^L_4 \dot{q}^L_5)$, $\{z = C \cdot \prod_{v_i \in \mathrm{V_s}} z_i\}$, and get an embedding that we denote by

$$\dot{\mathcal{P}}^\uparrow = (\dot{q}_i)_{1 \le i \le n}.$$

This is an integer plane embedding of $G^\uparrow$ combinatorially equivalent to $(\bar{q}_i)$ and it possesses an equilibrium stress

$$\dot{M}^\uparrow_{ij} := \frac{[\dot{q}^L_i \dot{q}^L_j \dot{q}^L_k \dot{q}^L_l]}{[\dot{q}_i \dot{q}_j \dot{q}_k][\dot{q}_l \dot{q}_j \dot{q}_i]}$$

with $\operatorname{sign}(\dot{M}_{ij}^{\uparrow}) = \operatorname{sign}(\bar{M}_{ij}^{\uparrow})$ for every edge $(v_iv_j) \in E(G^{\uparrow})$. Moreover,

$$\dot{M}_{ij}^{\uparrow} = \frac{[\dot{q}_i^L \dot{q}_j^L \dot{q}_k^L \dot{q}_l^L]}{[\dot{q}_i \dot{q}_j \dot{q}_k][\dot{q}_l \dot{q}_j \dot{q}_i]} = \frac{C^3[\bar{q}_i^L \bar{q}_j^L \bar{q}_k^L \bar{q}_l^L]}{C^2[\bar{q}_i \bar{q}_j \bar{q}_k]C^2[\bar{q}_l \bar{q}_j \bar{q}_i]} = \frac{1}{C}\bar{M}_{ij}^{\uparrow}$$

for every stable edge $(v_iv_j) \in \mathrm{E}_\mathrm{s}^{\uparrow}$ and $\dot{q}_i = C\bar{q}_i$ for all the stable vertices $v_i \in \mathrm{V}_\mathrm{s}$.

Stress perturbation. In the next step we scale the stresses $\dot{M}_{ij}^{\uparrow}$ (I),(II) and round them to integers (III). The Wheel-decomposition Theorem asserts that

$$\dot{M}_{ij}^{\uparrow} = \sum \alpha_k \Omega_{ij}^a(W_k),$$

where $|\Omega_{ij}^a(W_k)| \leq \max|q_i|^{O(\Delta_\mathrm{G})}$ are integers since they are products of $\deg(v_k) - 1$ integral triangle areas and $\alpha_k = \frac{\dot{z}_k}{D_k}$ where $D_k = \prod_{1 \leq t \leq \deg(v_k)}[\dot{q}_{k_t}\dot{q}_{k_{t+1}}\dot{q}_k]$ is the product of areas of all the triangle faces of $G^{\uparrow}$ adjacent to $\dot{q}_k$, and $\dot{z}_k$ is the z-coordinate of $\dot{q}_k^L$.

(I) We scale coefficients $\dot{\alpha}_k$ to $\ddot{\alpha}_k := \dot{C}\dot{\alpha}_k$ with

$$\dot{C} := \prod_{k \in \mathrm{V}_\mathrm{s}} D_k \leq \max|q_i|^{O(\Delta_\mathrm{G})}.$$

That gives

$$\ddot{M}_{ij}^{\uparrow} := \dot{C}\dot{M}_{ij}^{\uparrow} = \sum_k \ddot{\alpha}_k \Omega_{ij}^a(W_k)$$

with $\ddot{\alpha}_k = \dot{C}\dot{\alpha}_k = \frac{z_k}{D_k} \cdot \prod_{v_i \in \mathrm{V}_\mathrm{s}} D_i$ which is an integer for every $k \in \mathrm{V}_\mathrm{s}$.

(II) We define

$$\ddot{C} := \max(1, \lceil 4 \max_{i,j,k} |\Omega_{ij}^a(W_k)| / \min_{ij}(|\ddot{M}_{ij}^{\uparrow}|) \rceil) = \max|q_i|^{O(\Delta_\mathrm{G})},$$

multiply the coefficients again and round to the nearest integer

$$\dddot{\alpha} := [\ddot{C}\ddot{\alpha}_k],$$

and construct the final equilibrium stress

$$\dddot{M}^{\uparrow} := \sum \dddot{\alpha}_k \Omega_{ij}^a(W_k).$$

This stress is integer and the definition of $\ddot{C}$ guarantees that the perturbation did not affect the signs of the stresses, that is $\operatorname{sign}(\dddot{M}_{ij}^{\uparrow}) = \operatorname{sign}(\ddot{M}_{ij}^{\uparrow})$ for every edge $(v_iv_j) \in E(G^{\uparrow})$. Moreover $\ddot{\alpha}_k \in \mathbb{Z}$ and so $\dddot{\alpha}_k = \ddot{C}\ddot{\alpha}_k$ for every $v_k \in \mathrm{V}_\mathrm{s}$. Only the stable vertices $v_k \in \mathrm{V}_\mathrm{s}$ participate in the decompositions $\dddot{M}_{ij}^{\uparrow} = \sum \dddot{\alpha}_k \Omega_{ij}^a(W_k)$ of weights on the stable edges $(v_iv_j) \in \mathrm{E}_\mathrm{s}^{\uparrow}$. So,

$$\dddot{M}_{ij}^{\uparrow} = \ddot{C}\sum \ddot{\alpha}_k \Omega_{ij}^a(W_k) = \ddot{C}\dot{C}\sum \dot{\alpha}_k \Omega_{ij}^a(W_k) = \ddot{C}\dot{C}\dot{M}_{ij}^{\uparrow} = \frac{\dot{C}\ddot{C}}{C}\bar{M}_{ij}$$

for every stable edge $(v_iv_j) \in \mathrm{E}_\mathrm{s}^{\uparrow}$.

The final step. In the final step we combine the constructed embedding $\dot{\mathcal{P}}^{\uparrow}$ of $G^{\uparrow}$ equipped with the CDV matrix $[\dddot{M}_{ij}^{\uparrow}]$ and the embedding $\bar{\mathcal{P}}^{\downarrow}$ of $G^{\downarrow}$ equipped with the CDV matrix $[\bar{M}_{ij}^{\downarrow}]$ to form a cone-convex embedding $\dot{\mathcal{P}}$ of G with a positive CDV matrix $[\dot{M}_{ij}]$.

First, we scale $\bar{\mathcal{P}}^{\downarrow}$ so that it fits with $\dot{\mathcal{P}}^{\uparrow}$ and define

$$\dot{\mathcal{P}} := \dot{\mathcal{P}}^{\uparrow} \cup C \cdot \bar{\mathcal{P}}^{\downarrow}.$$

Second we scale $\dddot{M}^{\uparrow}_{ij}$ and $\bar{M}^{\downarrow}$ to define

$$\dot{M} := C\dddot{M}^{\uparrow} + \dot{C}\ddot{C}\bar{M}^{\downarrow}.$$

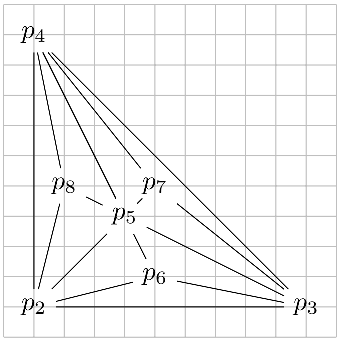
Since $\dot{M}_{ij} = C\frac{\dot{C}\ddot{C}}{C}\bar{M}^{\uparrow}_{ij} + \dot{C}\ddot{C}\bar{M}^{\downarrow}_{ij} = \dot{C}\ddot{C}(\bar{M}^{\uparrow}_{ij} + \bar{M}^{\downarrow}_{ij})$ for every stable edge $(v_iv_j) \in \mathrm{E}^{\uparrow}_{\mathrm{s}}$, stresses on the additional edges (v_2v_4) and (v_2v_5) cancel: $\dot{M}_{24} = \dot{C}\ddot{C}(\bar{M}^{\uparrow}_{24} + \bar{M}^{\downarrow}_{24}) = 0$ and $\dot{M}_{25} = \dot{C}\ddot{C}(\bar{M}^{\uparrow}_{25} + \bar{M}^{\downarrow}_{25}) = 0$. Thus, $\dot{M}$ becomes an integer CDV matrix for $\dot{\mathcal{P}}$, and $\dot{\mathcal{P}}$ is cone-convex. Moreover, $\dot{M}$ is positive: $\dot{M}_{ij} = \dot{C}\ddot{C}\bar{M}_{ij} > 0$ for non-canceled edges $(v_iv_j) \in \mathrm{E}^{\uparrow}_{\mathrm{s}}$ and for all other edges of G the signs of $\bar{M}_{ij}$ were preserved positive during the construction.

To complete the proof we follow the construction presented in the Theorem 1.

□

6 Example

As an example for our embedding algorithm for truncated polytopes we show how to embed the *truncated tetrahedron.* This polytope is obtained from the tetrahedron by truncating all of its four vertices. The graph G^* of this polytope and its dual G are depicted in Figure 2. We start with the planar embedding of $G_{\uparrow} = G[v_2, \ldots, v_8]$ that is defined on Figure 6.



$p_2 = (-3,-3)^T$
$p_3 = (6,-3)^T$
$p_4 = (-3,6)^T$
$p_5 = (0,0)^T$
$p_6 = (1,-2)^T$
$p_7 = (1,1)^T$
$p_8 = (-2,1)^T$

Figure 6: The original plane embedding of $G_{\uparrow}$.

The embedding has the integer stress $\omega_{52} = \omega_{53} = \omega_{54} = 1$, $\omega_{23} = \omega_{34} = \omega_{42} = -2$, and all other stresses have value 3.

Following Theorem 2, we embed the drawing of $G_{\uparrow}$ onto the plane $\{z = 1\}$ and construct a CDV matrix $[M'_{ij}]$ (see Equation (4)). We add the point $q_1 := (0,0,-1)^T$ and compute as an equilibrium stress for the embedding of K_5 formed by $\{q_0 = (0,0,0)^T, q_1, q_1, q_3, q_4\}$:

$$\begin{aligned}
&\omega''_{12} = \omega''_{13} = \omega''_{14} = 3,\\
&\omega''_{23} = \omega''_{34} = \omega''_{42} = 1,\\
&\omega''_{02} = \omega''_{03} = \omega''_{04} = -6,\\
&\omega''_{01} = -18.
\end{aligned}$$

The corresponding CDV matrices $[M''_{ij}]_{1\le ij\le 4}$ and $[M']_{2\le ij\le 8}$ are:

$$M'' = \begin{pmatrix} 9 & 3 & 3 & 3 \\ 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \end{pmatrix}, \text{ and } \quad M' = \begin{pmatrix} -3 & -2 & -2 & 1 & 3 & 0 & 3 \\ -2 & -3 & -2 & 1 & 3 & 3 & 0 \\ -2 & -2 & -3 & 1 & 0 & 3 & 3 \\ 1 & 1 & 1 & -12 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & -9 & 0 & 0 \\ 0 & 3 & 3 & 3 & 0 & -9 & 0 \\ 3 & 0 & 3 & 3 & 0 & 0 & -9 \end{pmatrix}. \tag{4}$$

We extend M' and M'' to the whole G and form the final CDV matrix $M = M' + 3M''$:

$$[M]_{1\le ij\le 8} = \begin{pmatrix} 27 & 9 & 9 & 9 & 0 & 0 & 0 & 0 \\ 9 & 0 & 1 & 1 & 1 & 3 & 0 & 3 \\ 9 & 1 & 0 & 1 & 1 & 3 & 3 & 0 \\ 9 & 1 & 1 & 0 & 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & 1 & -12 & 3 & 3 & 3 \\ 0 & 3 & 3 & 0 & 3 & -9 & 0 & 0 \\ 0 & 0 & 3 & 3 & 3 & 0 & -9 & 0 \\ 0 & 3 & 0 & 3 & 3 & 0 & 0 & -9 \end{pmatrix}.$$

We can now apply Theorem 1 and compute the vectors (ϕ_f). We first assign $\phi_{(236)} = (0, -18, 27)^T$. The remaining vectors are then iteratively computed with (2). We obtain as a result:

$$\begin{aligned}
& & \phi_{(236)} &= (0,-18,27)^T \\
\phi_{(265)} - \phi_{(236)} &= M_{26}(u_2 \times u_6) = (-3, 12, 27)^T & \phi_{(265)} &= (-3,-6,54)^T \\
\phi_{(258)} - \phi_{(265)} &= M_{25}(u_2 \times u_5) = (-3, 3, 0)^T & \phi_{(258)} &= (-6,-3,54)^T \\
\phi_{(284)} - \phi_{(258)} &= M_{28}(u_2 \times u_8) = (-12, 3, -27)^T & \phi_{(284)} &= (-18,0,27)^T \\
\phi_{(485)} - \phi_{(284)} &= M_{48}(u_4 \times u_8) = (15, 3, 27)^T & \phi_{(485)} &= (-3,3,54)^T \\
\phi_{(457)} - \phi_{(485)} &= M_{45}(u_4 \times u_5) = (6, 3, 0)^T & \phi_{(457)} &= (3,6,54)^T \\
\phi_{(473)} - \phi_{(457)} &= M_{47}(u_4 \times u_7) = (15, 12, -27)^T & \phi_{(473)} &= (18,18,27)^T \\
\phi_{(375)} - \phi_{(473)} &= M_{37}(u_3 \times u_7) = (-12, -15, 27)^T & \phi_{(375)} &= (6,3,54)^T \\
\phi_{(356)} - \phi_{(375)} &= M_{35}(u_3 \times u_5) = (-3, -6, 0)^T & \phi_{(356)} &= (3,-3,54)^T \\
\phi_{(362)} - \phi_{(356)} &= M_{36}(u_3 \times u_6) = (-3, -15, -27)^T & \phi_{(362)} &= (0,-18,27)^T \\
\phi_{(321)} - \phi_{(362)} &= M_{32}(u_3 \times u_2) = (0, -9, -27)^T & \phi_{(321)} &= (0,-27,0)^T \\
\phi_{(124)} - \phi_{(321)} &= M_{12}(u_1 \times u_2) = (-27, 27, 0)^T & \phi_{(124)} &= (-27,0,0)^T \\
\phi_{(143)} - \phi_{(124)} &= M_{14}(u_1 \times u_4) = (54, 27, 0)^T & \phi_{(143)} &= (27,27,0)^T.
\end{aligned}$$

The final result is depicted in Figure 7. The embedding requires a $54 \times 54 \times 54$ grid.

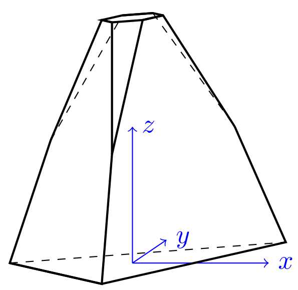


Figure 7: The final embedding of the truncated tetrahedron.

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