

Embedding Stacked Polytopes on a Polynomial-Size Grid

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Abstract

A stacking operation adds a d -simplex on top of a facet of a simplicial d -polytope while maintaining the convexity of the polytope. A stacked d -polytope is a polytope that is obtained from a d -simplex and a series of stacking operations. We show that for a fixed d every stacked d -polytope with n vertices can be realized with nonnegative integer coordinates. The coordinates are bounded by $O(n^{2+2\log(d-1)})$, except for one axis, where the coordinates are bounded by $O(n^{3+3\log(d-1)})$. The described realization can be computed with an easy algorithm.

The realization of the polytopes is obtained with a lifting technique which produces an embedding on a large grid. We establish a rounding scheme that places the vertices on a sparser grid, while maintaining the convexity of the embedding.

1 Introduction

Steinitz’s Theorem [Ste22, Zie95] states that the graphs of 3-polytopes¹ are exactly the planar 3-connected graphs. In particular, every planar 3-connected graph can be realized as a 3-polytope. The original proof is constructive, transforming the graph by a sequence of local operations down to a tetrahedron. Unfortunately, the resulting polytope construction requires exponentially many bits of accuracy for each vertex coordinate. Stated another way, this construction can place the n vertices on an integer grid, but that grid may have dimensions doubly exponential in n [OS94]. The situation in higher dimensions is even worse. Already in dimension 4, there are polytopes that cannot be realized with rational coordinates, and a 4-polytope that can be realized on the grid might require coordinates that are doubly exponential in the number of its vertices [Zie95]. Moreover, it is NP-hard to decide whether a lattice is a face lattice of a 4-polytope [RG96, RGZ95].

How large an integer grid do we need to embed a given planar 3-connected graph with n vertices as a polytope? This question goes back at least eighteen years as Problem 4.16 in Günter M. Ziegler’s book [Zie95]; he wrote that “it is quite possible that there is a quadratic upper bound” on the length of the maximum dimension. The best bound so far is exponential in n , namely $O(2^{7 \cdot 21^n})$ [BS10, MRS11]; see below for the long history. The central question is whether a polynomial grid suffices, that is, whether Steinitz’s Theorem can be made efficient. For comparison, a planar graph can be embedded in the plane with strictly convex faces using a polynomial-size grid [BR06]. In this paper, we give the first nontrivial subexponential upper bound for a large class of 3-polytopes. Moreover, our construction generalizes to higher dimensions and we show that a nontrivial class of d -polytopes can be realized with integers coordinates, which are bounded by a polynomial in n .

A d -dimensional stacked polytope is a polytope that is constructed by a sequence of “stacking operations” applied to a d -simplex. A *stacking operation* glues a d -simplex Δ atop a simplicial facet f of polytope, by identifying f with a face of Δ , while maintaining the convexity of the polytope. Thus a stacking operation removes one facet f and adds d new facets having a new

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¹In our terminology a *polytope* is always understood as a *convex* polytope.

common vertex. We call this new vertex *stacked on f*. Stacked 3-polytopes seem a natural class to study, because the stacking operation is a special case (perhaps the simplest) of the operations in Steinitz’s proof. Thus our solution for stacked 3-polytopes has potential to be generalized to general 3-polytopes. The graphs of stacked d -polytopes are d -trees, that is, maximal graphs of treewidth d .

Our results. We present an algorithm that realizes a stacked polytope on a grid whose dimensions are polynomial in n . Our main result is the following:

Theorem 1 *Every d -dimensional stacked polytope can be realized on an integer grid, such that all coordinates have size at most $10(d+1)! \cdot R^2$, except for one axis, where the coordinates have size at most $6 \cdot R^3$, for $R \leq dn^{\log(2d-2)}$.*

As a corollary of Theorem 1 we obtain that every stacked 3-polytope can be embedded on the grid of dimensions $2160n^4 \times 2160n^4 \times 162n^6$.

Related work. Several algorithms have been developed to realize a given graph as a 3-polytope. Most of these algorithms are based on the following two-stage approach. The first stage computes a plane (*flat*) embedding. To extend the plane drawing to a 3-polytope, the plane drawing must fulfill a criterion which can be phrased as an “equilibrium stress condition”. Roughly speaking, replacing every edge of the graph with a spring, the resulting system of springs must be in a stable state for the plane embedding. Plane drawings that fulfill this criterion for the interior vertices can be computed as barycentric embeddings, i.e., by Tutte’s method [Tut60, Tut63]. The main difficulty is to guarantee the equilibrium condition for the boundary vertices as well, because in general this goal is achievable only for certain locations of the outer face. The second stage computes a 3-polytope by assigning every vertex a height expressed in terms of the spring constants of the system of springs.

The two-stage approach finds application in a series of algorithms [CGT96, EG95, HK92, OS94, MRS11, RG96, Sch11]. The first result that improves the induced grid embedding of Steinitz’s construction is due to Onn and Sturmfels [OS94]; they achieved a grid size of $O(n^{160n^3})$. Richter-Gebert’s algorithm [RG96] uses a grid of size $O(2^{18n^2})$ for general 3-polytopes, and a grid of size $O(2^{5.43n})$ if the graph of the polytope contains at least one triangle. These bounds were improved by Ribó [Rib06] and later on by Ribó, Rote, and Schulz [MRS11]. The last paper expresses an upper bound for the grid size in terms of the number of spanning trees of the graph. Using the recent bounds of Buchin and Schulz [BS10] on the number of spanning trees, this approach gives an upper bound on the grid size of $O(2^{7.21n})$ for general 3-polytopes and $O(2^{4.83n})$ for 3-polytopes with at least one triangular face. These bounds are the best known to date for the general case. Very recently, Pak and Wilson proved that every simplicial² 3-polytope can be embedded on a grid of size $4n^3 \times 8n^5 \times (500n^8)^n$ [PW13].

Zickfeld showed in his PhD thesis [Zic07] that it is possible to embed very special cases of stacked 3-polytopes on a grid polynomial in n . First, if each stacking operation takes place on one of the three faces that were just created by the previous stacking operation (what might be called a *serpentine stacked polytope*), then there is an embedding on the $n \times n \times 3n^4$ grid. Second, if we perform the stacking in rounds, and in every round we stack on every face simultaneously (what might be called the *balanced stacked polytope*), then there is an embedding on a $\frac{4}{3}n \times \frac{4}{3}n \times O(n^{2.47})$ grid. Zickfeld’s embedding algorithm for balanced stacked 3-polytopes constructs a barycentric embedding. Because of the special structure of the underlying graph the plane embedding remarkably fits on a small grid.

Every stacked polytope can be extended to a balanced stacked polytope at the expense of adding an exponential number of vertices. By doing so, Zickfeld’s grid embedding for the balanced case induces a $O(2^{3.91n})$ grid embedding for general stacked 3-polytopes. From our experience, the most difficult-to-embed stacked polytopes are “almost balanced”. To construct such a polytope, we perform the stacking operations in rounds. In each round, we stack a vertex on two of the three faces that share a vertex that was introduced in the previous round.

²A polytope is *simplicial* if all its faces are simplices.

Little is known about the lower bound of the grid size for embeddings of 3-polytopes. An integral convex embedding of an n -gon in the plane needs an area of $\Omega(n^3)$ [AŽ95, And61, Thi91]. Therefore, realizing a 3-polytope with an $(n-1)$ -gonal face requires at least one dimension of size $\Omega(n^{3/2})$. For simplicial polytopes (and hence stacked polytopes), this lower-bound argument does not apply. Because every orthogonal projection of a 3-polytope decomposes into two noncrossing drawings, the projections into the xy and yz plane have to contain a noncrossing drawing of at least $n/2$ vertices. Thus we know that the coordinates must be at least $\Omega(n)$.

Alternative approaches for realizing general 3-polytopes come from the original proof of Steinitz’s theorem, as well as the Koebe–Andreev–Thurston circle-packing theorem, which induces a particular polytope realization called the *canonical polytope* [Zie95]. Das and Goodrich [DG97] essentially perform many inverted edge contractions on many independent vertices in one step, resulting in a singly exponential bound on the grid size. The proof of the Koebe–Andreev–Thurston circle-packing theorem relies on nonlinear methods and makes the features of the 3-dimensional embedding obtained from a circle packing intractable; see [Sch91] for an overview. Lovász studied a method for realizing 3-polytopes using a vector of the nullspace of a “Colin de Verdière matrix” of rank 3 [Lov00]. It is easy to construct these matrices for stacked polytopes; however, without additional requirements, the computed grid embedding might again need an exponential-size grid.

Our contribution. At a high level, we follow the popular two-stage approach: we compute a flat embedding and then lift it to a d -polytope. Although this two stage approach is well known for realizing 3-polytopes, it cannot be easily extended to higher dimensions. There exists a generalization of equilibrium stresses for polyhedral complexes in higher dimensions by Rybnikov [Ryb99]. However, it is not straightforward to operate with the formalism used by Rybnikov for our purposes. To overcome the difficulties of handling the more complex behavior of the higher-dimensional polytopes and to keep our presentation self-contained, we develop our own specialized methods to study liftings of *stacked* polytopes. Notice that our methods coincide with the approach of Rybnikov, however, we omit the proof, since it is not required for this presentation.

Instead of specifying the stress and then compute the barycentric embedding we construct the “stress” and the flat embedding in parallel. To specify the stress we define the heights of the lifting (actually, the vertical movement of the vertices as induced by the stacking operation). In our presentation the concept of stress is reduced to a certificate for the convexity of the lifting, but it is not defining the flat embedding. On the other hand we still use barycentric coordinates to determine the flat embedding. A crucial step in our algorithm is the construction of a balanced set of barycentric coordinates, which corresponds to face volumes in the flat embedding. Initially, all faces have the same volume, but to prevent large heights in the lifting, we increase the volumes of the small faces. To see which faces must be blown up, we make use of a decomposition technique from data structural analysis called *heavy-light edge decomposition* [Tar83]. Based on this decomposition, we subdivide the stacked polytope into a hierarchy of (serpentine) stacked polytopes, which we use to define the barycentric coordinates. At this stage the lifting of the flat embedding would result in a grid embedding with exponential coordinates. But since we have balanced the volume assignments, we can allow a small perturbation of the embedding, while maintaining its convexity. Analyzing the size of the feasible perturbations shows that we can round to points on a polynomially sized grid.

A preliminary version of this work was presented at the 22nd ACM-SIAM Symposium on Discrete Algorithms (SODA) in San Francisco [DS11]. In the preliminary version we were focused on the more prominent 3-dimensional case and did not present any bounds for higher-dimensional polytopes. For the sake of a unified presentation we changed the construction of the lifting slightly. In the preliminary version we defined the constructed stress as a linear combination of stresses defined on certain K_4 s. In this paper we specify the “movement” for every stacked vertex (its vertical shift) directly. This can be considered as the dual definition of the lifting. Another difference is the more careful analysis of the size of the z -coordinates in the final embedding. In contrast to the preliminary version, where we presented bound of $224,000n^{18}$ for the z -coordinates, we present a different method for bounding the height of the lifting, which yields an upper bound of $162n^6$ instead. In a paper that followed the preliminary version Igamberdiev and Schulz introduced a duality transformation for 3-polytopes that allows to control the grid size of the dual polytope [IS13]. By this our results for stacked polytopes can be transferred to their dual polytopes (truncated

3-polytopes), which shows that also this class can be realized on a polynomial-sized grid.

2 Specifying the geometry of stacked polytopes

In this section we develop the necessary tools for defining an embedding of a stacked d -polytope with the two-stage approach.

2.1 Matrices, determinants, simplices

We start our presentation with introducing some notation. Assume that $S = (\mathbf{s}_1, \dots, \mathbf{s}_k)$ is a sequence of $(k-1)$ -dimensional vectors. Then we denote by $(\mathbf{s}_1, \dots, \mathbf{s}_k)$ the matrix whose row vectors (in order) are $\mathbf{s}_1, \dots, \mathbf{s}_k$. We define as

$$[S] = \det \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \dots & \mathbf{s}_k \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

Notice that $[S]$ equals $(k-1)!$ times the volume of the simplex spanned by $\mathbf{s}_1, \dots, \mathbf{s}_k$. When working with sequences we use the binary \circ operator to concatenate two sequences. If there is no danger of confusion we identify an element with the singleton set that contains this element. If f is a function on \mathcal{U} we extend f in the natural way to act on subsets (or sequences) on \mathcal{U} , i.e., we set $f(X) := \bigcup_{x \in X} f(x)$.

Throughout the paper we denote for a sequence X of affinely independent points the simplex spanned by X by Δ_X . If this simplex is the realization of some face of an embedding of a polytope we denote this face by f_X .

By convention we understand as \mathbf{R}^k the space spanned by the first k standard basis vectors. In this sense we consider the subspace \mathbf{R}^{k-1} as space embedded inside \mathbf{R}^k . Furthermore, in all our construction we assume that no hyperplane in \mathbf{R}^k is orthogonal with respect to \mathbf{R}^{k-1} .

2.2 Liftings

Our algorithm for realizing stacked polytopes is based on the *lifting technique*. In order to generate a realization we first construct a *flat embedding* of the stacked d -polytope. A flat embedding, is a projection of a realization of the stacked polytope into a $(d-1)$ -dimensional subspace. The flat embedding will be generated by repeated subdivisions carried out in this subspace starting from a $(d-1)$ -simplex. The assignment of an additional coordinate to every vertex is called a *lifting*. For the assignment of the new coordinate we use the function $\sigma: \mathbf{R}^{d-1} \times \mathbf{R} \mapsto \mathbf{R}^d$, which maps the point $(x_1, \dots, x_{d-1})^T$ and some value z to $(x_1, \dots, x_{d-1}, z)^T$. The function σ is complemented by the projection function $\pi: \mathbf{R}^d \mapsto \mathbf{R}^{d-1}$ that deletes the d -th coordinate. As shortcut notation we write $\llbracket S \rrbracket$ for the expression $[\pi(S)]$. In the following we refer to the d -th coordinate of a point in \mathbf{R}^d as its z -coordinate.

Let $\mathbf{p} = (p_1, p_2, \dots, p_{d-1})^T$ be a point in \mathbf{R}^{d-1} . We denote by $\bar{\mathbf{p}}$ the point $(p_1, p_2, \dots, p_{d-1}, 1)^T \in \mathbf{R}^d$. A hyperplane $h \subset \mathbf{R}^d$, is characterized by a function $z_h: \mathbf{R}^{d-1} \mapsto \mathbf{R}$ that assigns every point $\mathbf{p} \in \mathbf{R}^{d-1}$ a new coordinate, such that $\sigma(\mathbf{p}, z_h(\mathbf{p}))$ lies on h . We denote by $h(S)$ the hyperplane spanned by S and use as shortcut notation z_S for $z_{h(S)}$. The function z_h can be expressed as

$$z_h(\mathbf{p}) := \langle \mathbf{a}, \bar{\mathbf{p}} \rangle, \tag{1}$$

where $\mathbf{a} \in \mathbf{R}^d$ is a vector that depends on h . If we know a set S that spans h we can derive an alternative expression for z_h given by the following lemma.

Lemma 1 *Let S be a set of d affinely independent points from \mathbf{R}^d . For every point $\mathbf{p} \in \mathbf{R}^{d-1}$ we have*

$$z_S(\mathbf{p}) = \frac{[S \circ \sigma(\mathbf{p}, 0)]}{\llbracket S \rrbracket}.$$

Proof. Let $\mathbf{a} = (a_1, \dots, a_d)^T$ be the vector corresponding to expression (1) for the plane h spanned by S . For every $\mathbf{s}_i \in S$ let $\mathbf{q}_i = \pi(\mathbf{s}_i)$. Since \mathbf{s}_i lies on h we have that for all $i \leq d$

$$z_S(\mathbf{q}_i) = \langle \mathbf{a}, \bar{\mathbf{q}}_i \rangle = z_i,$$

for z_i being the z -coordinate of \mathbf{s}_i . These d constraints can be summarized as

$$Q^T \cdot \mathbf{a}^T = (z_1, \dots, z_d)^T := \mathbf{z}, \quad (2)$$

where $Q := (\bar{\mathbf{q}}_1, \dots, \bar{\mathbf{q}}_d)$. We apply Cramer's rule to solve Equation (2) for \mathbf{a} and obtain

$$a_i = \frac{\det Q(i)}{\det Q},$$

where $Q(i)$ denotes the matrix Q^T , with column number i replaced by \mathbf{z} . Plugging the last expression into Equation (1) yields

$$\begin{aligned} z_S(\mathbf{p}) &= \sum_{i=1}^d a_i \bar{p}_i = \frac{1}{\det Q} \left(0 \cdot \det Q + \sum_{i=1}^d \bar{p}_i \det Q(i) \right) \\ &= \frac{(-1)^d}{\llbracket S \rrbracket} \det \begin{pmatrix} p_1 & & & \\ p_2 & \mathbf{q}_1 & \dots & \mathbf{q}_d \\ \vdots & & & \\ 0 & z_1 & \dots & z_d \\ 1 & 1 & \dots & 1 \end{pmatrix} = (-1)^d \frac{[\sigma(\mathbf{p}, 0) \circ S]}{\llbracket S \rrbracket} = \frac{[S \circ \sigma(\mathbf{p}, 0)]}{\llbracket S \rrbracket}. \end{aligned}$$

□

2.3 Creases and stresses

Let f, g be two hyperplanes in \mathbf{R}^d . Furthermore, let S , resp. T , be a sequence of d affinely independent points of f , resp. g , such that deleting the last element in S and T gives the same subsequence X . It follows that f and g intersect in a flat of dimension $d - 2$ that contains the simplex Δ_X . Furthermore, we denote by \mathbf{s}_d the last point in the sequence S , that is the point not in T , and similarly we denote by \mathbf{t}_d the point in T that is not in S . We also set $\mathbf{r} = \pi(\mathbf{s}_d)$. The situation is depicted in Fig. 1. We define as *creasing* on X

$$c(f, g, X) := \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket}. \quad (3)$$

Roughly speaking, the creasing is a measure for the intersection angle of f and g with respect to the volume of Δ_X .

The following lemma shows that the function c is well-defined.

Lemma 2 *Let $c(f, g, X)$ be defined as above, i.e., S spans f , T spans g , such that $S = X \circ \mathbf{s}_d$, and $T = X \circ \mathbf{t}_d$.*

(a) *It holds that $c(f, g, X) = -c(g, f, X)$.*

(b) *The value of $c(f, g, X)$ is independent from the choice of \mathbf{s}_d and \mathbf{t}_d .*

Proof. Let T be the sequence $(\mathbf{t}_1, \dots, \mathbf{t}_d)$. We use Lemma 1 to rephrase $c(f, g, X)$ and obtain

$$\begin{aligned} \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} &= \frac{[T \circ \sigma(\mathbf{r}, 0)] / \llbracket T \rrbracket - z_S(\mathbf{r})}{\llbracket S \rrbracket} \\ &= \frac{\sum_{i=1}^d (-1)^{i+d} z_T(\pi(\mathbf{t}_i)) (T \circ \sigma(\mathbf{r}, 0))_{d,i} - z_S(\mathbf{r}) \llbracket T \rrbracket}{\llbracket T \rrbracket \llbracket S \rrbracket}, \end{aligned}$$

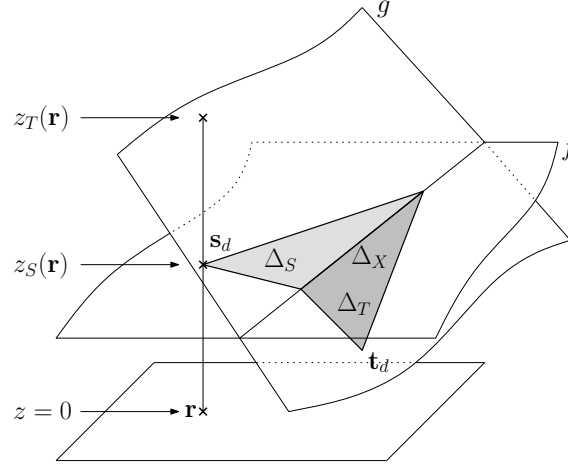


Figure 1: The creasing on X is defined in terms of the objects depicted in the figure (for $d = 3$).

where $(T \circ \sigma(\mathbf{r}, 0))_{i,j}$ denotes the (i, j) -cofactor of $(T \circ \sigma(\mathbf{r}, 0))$. Since $\llbracket T \rrbracket = (T \circ \sigma(\mathbf{r}, 0))_{d,d+1}$ it follows that

$$c(f, g, X) = \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} = \frac{[T \circ \sigma(\mathbf{r}, z_S(\mathbf{r}))]}{\llbracket T \rrbracket \llbracket S \rrbracket} = \frac{[T \circ \mathbf{s}_d]}{\llbracket T \rrbracket \llbracket S \rrbracket}. \quad (4)$$

Notice that $[T \circ \mathbf{s}_d] = -[S \circ \mathbf{t}_d]$, since both matrices differ by one column swap (the sequence of the first $d-1$ members of S and T coincide). As a consequence, exchanging S and T in the left hand side of (4) changes only the sign, which proves statement (a).

To prove statement (b) we argue as follows. Assume we have replaced \mathbf{t}_d in T (that is the only point in T not in X) by some other point on $h(T)$. The replacement of \mathbf{t}_d does not change $z_T(\mathbf{r})$ in Equation (3), and all other parts of the equation only depend on S . Therefore, the expression given in (3) does not depend on \mathbf{t}_d . It remains to show, that also changing the last point in S , does not change the creasing of X . However, this follows easily since by statement (a) we can interchange the roles of S and T (this results in a sign change), then we apply the above argument, and then we change S and T back (which cancels the sign change). \square

Based on the concept of creasing we are now ready to define the crucial concept of stress. Assume that we want to realize the simplicial polytope P and we have some canonical convex realization of P that contains the origin $\mathbf{0}$ in its interior. Furthermore, for every $(d-2)$ -face of P , we fix some order of the vertices and assume from now on that whenever a sequence of vertices of a $(d-2)$ -face is considered then this sequence respects this order. Suppose that f is a (simplicial) facet of P that contains a $(d-2)$ -face with vertex set X . Let \mathbf{y} be the vertex of f that is not in X . We call f *right of* X , if $[X \circ \mathbf{y}] > 0$. If on the other hand $[X \circ \mathbf{y}] < 0$, we say that f is *left of* X . Here the expressions $[\cdot]$ are with respect to the canonical realization of P . Notice that every $(d-2)$ -face f_X has exactly one face left of it, and one face right of it.

Definition 1 (stress) Let P be d -polytope embedded in \mathbf{R}^d and let X be a sequence of points which defines a simplicial $(d-2)$ -face. We define as stress on X

$$\omega_X := c(h_l, h_r, X), \quad (5)$$

where h_r contains the face right of X and h_l contains the face left of X .

We can immediately deduce the following property for the stress along X .

Lemma 3 The stress on X is not affected by the order of the elements in X .

Proof. When reordering the elements of X , only the sign of a creasing along X might change. When it flips then our notion of left and right face will also be exchanged, which means that the left face becomes the right face with respect to X and vice versa. Thus, if we reorder X such that the sign of the creasing is preserved, the corresponding stress is also preserved.

Let \bar{X} be a reordering of X , such that the creasing along X changes its sign and the right/left position of the faces is swapped. Suppose that f contains the face left of \bar{X} and right of X , and that g contains the other face incident to X . Then we have by (3) and Lemma 2

$$\omega_{\bar{X}} = c(f, g, \bar{X}) = -c(f, g, X) = c(g, f, X) = \omega_X.$$

□

2.4 Convexity of barycentric subdivisions

An important intermediate step in the embedding algorithm, is the construction of a flat embedding, which is a simplicial complex constructed by repeated weighted barycentric subdivisions of a $(d-1)$ -simplex Δ_B . The combinatorial structure of the flat embedding represents the face lattice of the stacked polytope. Let Δ_S be a $(d-1)$ -simplex and let $\Delta_1, \Delta_2, \dots, \Delta_d$ be the $(d-2)$ -simplices that define its boundary. A *barycentric subdivision* adds a new point \mathbf{p} in the interior of Δ_S and splits Δ_S into d $(d-1)$ -simplices. For every simplex Δ_i we obtain a new simplex Δ'_i spanned by $\Delta_i \cup \{\mathbf{p}\}$. If the volume of all Δ'_i s are given such that they sum up to the volume of Δ_S , then there is one unique vertex \mathbf{p} , such that the subdivision respects this volume assignment. We call the volumes the Δ'_i s the *barycentric coordinates* of \mathbf{p} with respect to Δ_S .

We apply repeated barycentric subdivisions to create (a flat) embedding of a stacked polytope. In particular, the subdivisions carry out the stacking operations geometrically (projected in the $z = 0$ hyperplane). We start the subdivision process with two copies of a $(d-1)$ -simplex in the $z = 0$ hyperplane glued along the boundary. One of these “initial” simplices will be the *base face* f_B , which remains unaltered during further modifications. The other face will be repeatedly subdivided such that the combinatorial structure of the stacked polytope is obtained. The constructed realization is still flat, which means that it lies in the $z = 0$ hyperplane. To complete our construction we assign every vertex that is not on the base face a positive z -coordinate.

The difficult part for the height assignment is to choose the heights such that the resulting lifting gives a *convex* realization of the simplicial complex. To guarantee the convexity of the final embedding we use the stresses induced by the lifting. By knowing the signs of all stresses we can determine if the selected z -coordinates produce a convex embedding with help of the following lemma.

Lemma 4 *Let \mathcal{C} be the realization of a polyhedral surface (not self-intersecting) such that it contains one face spanned by B which induces a $(d-1)$ -simplex Δ_B in the $z = 0$ hyperplane. We assume that $\pi(\mathcal{C}) = \pi(\Delta_B)$ and that every two facets in $\pi(\mathcal{C} \setminus \Delta_B)$ have disjoint interiors. If all $(d-2)$ -simplices of \mathcal{C} on the boundary of B have a negative stress and all other stresses are positive then \mathcal{C} encloses a convex body.*

Proof. As a preliminary step we show that \mathcal{C} is *locally convex*. By this we mean that for every facet f_T spanned by $T \neq B$ of \mathcal{C} , all adjacent facets lie on the same side of the hyperplane that contains T . Let S be the vertices of the face adjacent to f_T in \mathcal{C} . The sequences S and T are ordered such that they coincide on the first $d-2$ elements. We order X such that f_S is left of X . Assume that $\llbracket T \rrbracket > 0$.

Case 1 ($S \neq B$): We denote by \mathbf{r} the vertex in S that is not in T projected in the $z = 0$ hyperplane. Since $\omega_X > 0$, we know that $c(f_S, f_T, X) > 0$. Since $\pi(S)$ and $\pi(T)$ have a disjoint interior, we can deduce that $\llbracket S \rrbracket$ and $\llbracket T \rrbracket$ have a different sign, and hence $\llbracket S \rrbracket < 0$. This leads to

$$\omega_X > 0 \iff c(f_S, f_T, X) > 0 \iff \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} > 0 \iff z_T(\mathbf{r}) < z_S(\mathbf{r}).$$

Hence the interior of f_S lies “above” $h(T)$.

Case 2 ($S = B$): The only difference here is that by assumption $\omega_X < 0$ and that $\llbracket S \rrbracket$ and $\llbracket T \rrbracket$ have the same sign, i.e., $\llbracket S \rrbracket > 0$. Therefore,

$$\omega_X < 0 \iff c(f_S, f_T, X) < 0 \iff \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} < 0 \iff z_T(\mathbf{r}) > z_S(\mathbf{r}).$$

Again, the interior of f_S lies “above” $h(T)$.

For both cases we have assumed that $\llbracket T \rrbracket > 0$. If $\llbracket T \rrbracket$ would be negative instead, then in both cases we would have by the above arguments that the interior of S lies always “below” the hyperplane spanned by T .

We have shown that \mathcal{C} is locally convex, but this is not sufficient to prove that the complex encloses a convex body. However, as shown in [MNS⁺99], if the polyhedral surface is locally convex and star-shaped, then it is convex. We remove the bounding hyperplane $h(B)$ of the face f_B from the polyhedral surface \mathcal{C} and get a new polyhedral complex \mathcal{C}' . Note that if \mathcal{C}' encloses a convex body, then so does \mathcal{C} . Every vertical ray emanating from a point in \mathcal{C}' in direction towards $z = -\infty$ does not intersect \mathcal{C}' again. Consider now a small projective transformation that projects the point at infinity to a point inside the convex hull of \mathcal{C}' . The image of the transformation is star-shaped, and hence encloses a convex body. Since convexity is preserved under projective transformations, \mathcal{C}' encloses also a star-shaped and therefore a convex body. \square

We conclude this section with a lemma which proves the (in some sense) reverse direction of Lemma 4.

Lemma 5 *Let \mathcal{C} be a simplicial polytope realized in \mathbf{R}^d with nonnegative z -coordinates and with a designated face f_B lying in the $z = 0$ hyperplane such that $\pi(\Delta_B) = \pi(\mathcal{C})$. If $\llbracket B \rrbracket$ is positive then \mathcal{C} has a stress that is negative on all $(d - 2)$ -faces on f_B and positive otherwise.*

Proof. Since \mathcal{C} is convex, it is locally convex in the sense of the proof of Lemma 4. As a consequence, by the same arguments as given in the proof of Lemma 4, the stress of two $(d - 2)$ -faces lying on a common facet have the same sign, with the only exception when one of the $(d - 2)$ -faces lies on f_B and the other does not. In this case the stress signs differ. Hence, the induced stress has either the desired sign pattern, or it has the desired sign pattern with flipped signs on every edge. Due to our assumptions in the lemma, the latter case cannot occur. To see this assume that the facet f_S is adjacent to f_B via f_X . Since $\llbracket B \rrbracket > 0$ we also have $\llbracket S \rrbracket > 0$. Let \mathbf{p} be a point of f_S that is not on f_X . For $\mathbf{r} = \pi(\mathbf{p})$ we have

$$z_S(\mathbf{r}) > z_B(\mathbf{r}) \iff z_S(\mathbf{r}) - z_B(\mathbf{r}) > 0 \iff \frac{z_B(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} < 0 \iff \omega_X < 0.$$

\square

In the remainder of the paper we assume that we have ordered the vertex set of the face f_B such that $\llbracket B \rrbracket$ is positive. As we will see later our computed realizations have nonnegative z -coordinates.

2.5 Keeping track of the stresses while stacking

We discuss next how a stacking operation changes the associated stresses in a lifted barycentric subdivision. Let Δ_D be a $(d - 1)$ -simplex of a simplicial complex embedded in \mathbf{R}^d . We stack the new vertex \mathbf{p} on top of D . To determine the stacking geometrically, we describe the location of $p := \pi(\mathbf{p})$ inside $\pi(\Delta_D)$ with barycentric coordinates, that is we specify the absolute areas of the $(d - 1)$ -simplices containing p in the projection. Additionally, we describe how far the z -coordinate $z_{\mathbf{p}}$ of \mathbf{p} lies above $h(D)$. This will be denoted by $\zeta := z_{\mathbf{p}} - z_D$. We refer to ζ as the *vertical shift*. For convenience we assume that the stacking process increases the z -coordinate of the stacked point, since this will always be the case in the following. In particular, all face normals of faces different from f_B point always “upwards”. However, the following observations can be easily generalized for stacking operations that decrease the z -coordinates of the stacked point.

The addition of \mathbf{p} has two effects. First of all, by stacking \mathbf{p} we create new $(d - 2)$ -faces, furthermore, the existing $(d - 2)$ -faces at the boundary f_D are now incident to new facets and hence the corresponding stress is altered. We refer to the newly introduced $(d - 2)$ -faces as the *interior faces*, and to the $(d - 2)$ -faces on the boundary of f_D as the *exterior faces* of the stacking operation.

We discuss first how the stresses on the exterior faces are altered.

Lemma 6 Assume that we have stacked a vertex on a convex polytope realized in \mathbf{R}^d with stress ω as described in Lemma 5. Let X be the point set of an exterior face f_X (as defined above). After the stacking the new stress $\hat{\omega}$ equals

$$\hat{\omega}_X := \omega_X - \frac{\zeta}{\llbracket S \rrbracket},$$

where $S = X \circ \mathbf{p}$ and ζ is the vertical shift of the stacking.

Proof. Let T be the ordered point set, such that f_T and f_S are adjacent after stacking \mathbf{p} , and f_X is the intersection of f_S and f_T . We denote as h the hyperplane that contains the face f_D on which \mathbf{p} was stacked onto; see Fig. 2.

Case 1 (f_X is not on f_B): By Lemma 3 we can assume that the order of X implies that f_S is left of f_T . This gives for $p = \pi(\mathbf{p})$

$$\hat{\omega}_X = c(f_S, f_T, X) = \frac{z_T(p) - z_S(p)}{\llbracket S \rrbracket} = \frac{z_T(p) - (z_h(p) + \zeta)}{\llbracket S \rrbracket} = \omega_X - \frac{\zeta}{\llbracket S \rrbracket}. \quad (6)$$

Due to our conventions we have by Lemma 5 that $\omega_X > 0$. Since $\omega_X = (z_T(p) - z_h(p))/\llbracket S \rrbracket$ is positive, as well as $z_T(p) - z_h(p)$ we have that $\llbracket S \rrbracket > 0$ and the statement of the lemma follows. Fig. 2(a) depicts the situation.

Case 2 (f_X is on f_B): Again we can assume that f_S is left of f_T . Observe that (6) also holds in case 2. However, this time we have by Lemma 5 that $\omega_X = (z_T(p) - z_h(p))/\llbracket S \rrbracket < 0$. But since $(z_T(p) - z_h(p))$ is negative $\llbracket S \rrbracket$ is again positive, as shown in Fig. 2(b), which proves the lemma. \square

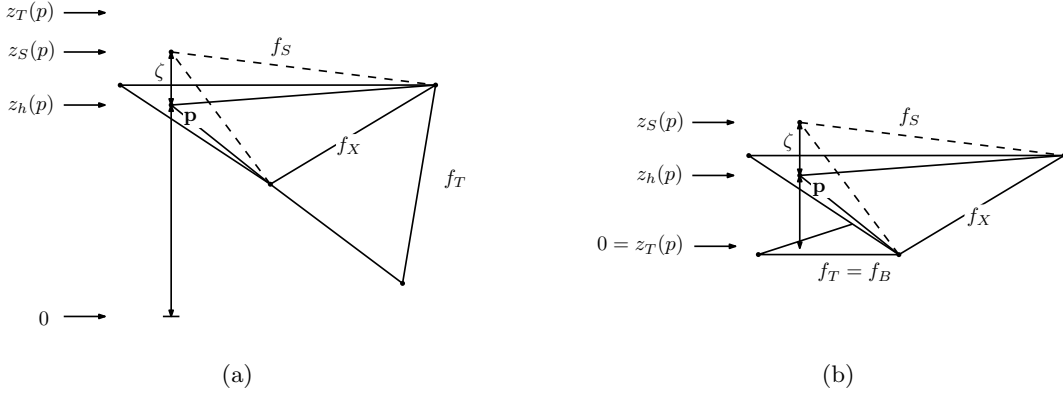


Figure 2: The situation (for $d = 3$) as discussed in Lemma 7: the effect of stacking the vertex \mathbf{p} for an exterior face not on f_B (a) and on f_B (b).

Lemma 7 Assume we have stacked a vertex \mathbf{p} on top of some face f_D . Let X be the point set of an interior face f_X as defined above. Furthermore, let $S := X \circ \mathbf{s}_d$ and $T := X \circ \mathbf{t}_d$ be the sequence of points of the faces separated by X . After the stacking the stress on X equals the positive number

$$\omega_X := \zeta \frac{\llbracket D \rrbracket}{\llbracket S \rrbracket \llbracket T \rrbracket}.$$

Proof. Fig. 3 depicts the situation described in the lemma. By Lemma 3 we can assume that f_S is left of f_T . Then we have due to Equation (4) that

$$\omega_X = c(f_S, f_T, X) = \frac{[T \circ \mathbf{s}_d]}{\llbracket T \rrbracket \llbracket S \rrbracket}.$$

Assume we have doubled f_D and stack with an ε -small ζ at the “top side” of the induced complex. This will surely generate a convex d -simplex, and by Lemma 5 the sign of the induced ω_X is positive. Clearly, the sign of ω_X does not change if we increase the vertical shift.

We know that $||T \circ s_d||$ is $(d-1)!$ times the volume of the corresponding simplex. This volume on the other hand can be expressed as $\zeta \cdot |\text{volume}(\pi(\Delta_D))|/(d-1)$, and $||D||$ equals $(d-2)!$ times the volume of $\pi(\Delta_D)$. Hence we have $||T \circ s_d|| = \zeta \cdot ||D||$ and the statement of the lemma follows. \square

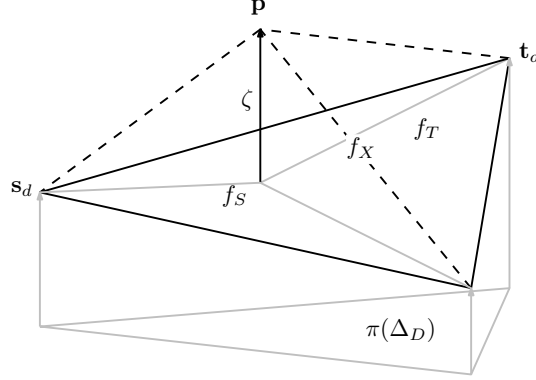


Figure 3: The situation (for $d = 3$) for an interior face as discussed in Lemma 7.

2.6 The tree-representation of a stacked polytope

The combinatorial structure of a polytope is typically encoded by its face lattice. If the polytope is a stacked polytope, we can also describe its structure by *recording* in which way we have carried out the stacking operations. The most natural way to keep track of the stacking operations is an ordered rooted tree. Let P be a stacked polytope, then the *tree-representation* of P is a d -ary tree $\mathcal{T}(P)$, which is defined as follows: The leaves of the trees are in one-to-one correspondence to the facets of P , with the exception of the face f_B , which is not present in the tree. Interior nodes correspond to d -cycles in the graph of P that used to be facets at some point during the stacking process.³ The root represents the initial copy of f_B in the beginning. The children of a node v represent the faces that were introduced by stacking a vertex onto the face associated with v . To reproduce the combinatorial structure from $\mathcal{T}(P)$ we fix an ordering for the edges emanating from an interior node v , such that it is possible to map the children of v to the faces generated by stacking onto v 's face in a unique way. So we assume we have ordered the edges such that the mapping between the faces and nodes can be reproduced.

We use the tree-representation of P to specify the geometry of the flat embedding of P . This can be achieved by assigning for every leaf v of $\mathcal{T}(P)$ a rational weight $a(v)$, which corresponds to the volume of v 's face in the flat embedding of P . More precisely, $a(r)$ specifies the volume of v 's face times $(d-1)!$. To emphasize this relationship, we call the weights $a(v)$ *face-weights*. We extend the face-weight assignments for the interior nodes by summing for every node the face-weights of its children recursively up. After we have determined the location of f_B , such that its volume is $(d-1)!$ times the face-weight of the root of $\mathcal{T}(P)$, the coordinates of the whole flat embedding are determined. In particular, for every stacking operation, the (normalized) face-weights specify the location of the stacked vertex since they denote its barycentric coordinates. Thus, we can traverse $\mathcal{T}(P)$ and reproduce the flat embedding incrementally in a unique way.

3 The embedding algorithm

We assume that the combinatorial structure of the stacked polytope is given in form of its tree-representation $\mathcal{T}(P)$. The embedding algorithm works in three steps. First we generate the face-weights for $\mathcal{T}(P)$ and fix the coordinates for the face f_B . This will give us a flat embedding

³ When considering an intermediate configuration in the stacking process, we refer to the d -cycles of P that are faces in the intermediate polytope also as *faces*, although they are not necessarily faces of P .

of the polytope. In the next step we “lift” the polytope, by defining for every vertex v_i the *vertical shift* ζ_i . Assume that we have stacked v_i onto the face f . By construction we pick always positive vertical shifts. By carefully choosing the right vertical shifts we obtain an embedding of P as *convex* polytope, however this embedding does not fit on a polynomial grid. To obtain a small grid embedding we round the points to appropriate grid points, while maintaining convexity.

3.1 Balancing face-weights

We apply a technique from data structure analysis called the *heavy edge tree-decomposition*, hetd for short (see Tarjan [Tar83]). Roughly speaking, the hetd decomposes a tree into paths, such that the induced hierarchical structure of the decomposition is balanced. We continue with a brief review of the hetd. Let u be a non-leaf of a rooted tree T with root r . We denote by T_u the subtree of T rooted at u . Let v be the child of u such that T_v has the largest number of nodes (compared to the subtrees of the other children of u), breaking ties arbitrarily. We call the edge (u, v) a *heavy edge*, and the edges to the other children of u *light edges*. The heavy edges induce a decomposition of T into (maximal) paths, called *heavy paths*, and light edges; see Fig. 4. We call a heavy path with its incident light edges (ignoring the possible edge from the top most vertex of the heavy path to its parent) a *heavy caterpillar*. The hetd decomposes the edges of T into heavy caterpillars. We say that two heavy caterpillars are adjacent if their graphs would be adjacent subgraphs in T . This adjacency relation induces a hierarchy, which we represent as a rooted tree $\mathcal{H}(T)$. The nodes in $\mathcal{H}(T)$ are the heavy caterpillars, and its edges represent the adjacency relation between caterpillars. The root of $\mathcal{H}(T)$ is the heavy caterpillar that includes the root of T . By construction, every root-leaf path in T visits at most $\lceil \log n \rceil$ many light edges, and thus the height of $\mathcal{H}(T)$ is at most $\lceil \log n \rceil - 1 \leq \lceil \log n \rceil$. Fig. 4 shows an example of a tree-representation and its associated hierarchy as a tree.

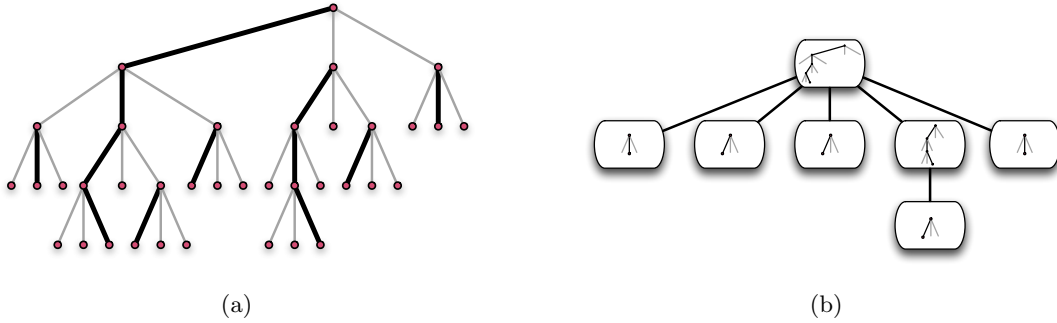


Figure 4: A tree-representation \mathcal{T} of a stacked 3-polytope (a) and the corresponding hierarchy based on the caterpillars as tree $\mathcal{H}(\mathcal{T})$ (b).

The assignment of the face-weight is carried out in the hetd of $\mathcal{T}(P)$. We call a face-weight assignment for a node v *balanced* if the weights of all children of v connected via light edges are the same. If every node is balanced we call the face-weight assignment balanced.

Lemma 8 *We can find a balanced set of face-weights for $\mathcal{T}(P)$ such that the face-weight associated with the root is at most $d \cdot n^{\log(2d-2)}$ and no face-weight is less than 1.*

Proof. Let $H = \mathcal{H}(\mathcal{T}(P))$ be the tree representing the hierarchy of the heavy caterpillars. We say that a node h in H has height k , if the longest path from h to a leaf of H has length k . We start with setting the face-weight of all leaves in $\mathcal{T}(P)$ to 1 and update the face-weight for the interior vertices accordingly. This guarantees that the nodes in H with height zero are balanced, however, H might not be balanced. At this moment, the face-weight of the root equals the number of leaves in $\mathcal{T}(P)$, which is given by the number of facets of P minus 1. Therefore, the root has at this point a face-weight of $1 + (n - d)(d - 1) < nd$.

Let us now consider a node with height 1 in H and its associated caterpillar C . For every node on the heavy path of C we do the following. We check whether the face-weight of its (light edge) children are the same. If this is not the case we increase the face-weights of the lighter children such that all face-weights are equal to the face-weight of a maximal (light edge) children. Notice that if we increase the face-weight of an interior node we have to propagate this change throughout the tree (upwards and downwards). However this can be done without changing too many weights as follows. Let u be a node in T for which we increase the face-weight by δ and let the parent of u be v . In our setting (u, v) is always a light edge. To keep the face-weights consistent we take the heavy path that contains u as root and increase the weights of every vertex on this path by δ . This makes the subtree rooted at u consistent with the new face-weight of u . Now we are left with propagating the face-weight “upwards”. This is achieved by increasing all face-weights along the path from v to the root of $\mathcal{T}(P)$. Although we have increased many face-weights, the total face-weight of the root went only up by δ . Notice that if the subtree of u was balanced, it is still balanced after the adjustments.

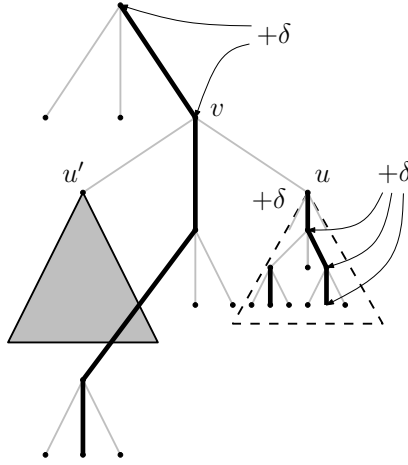


Figure 5: Balancing the tree at node u . The weight at u has to be increased by δ which is smaller than the weight of u' . In order to keep the weights consistent we have to add δ to every node on u 's heavy path and at all nodes on the path from v to the root. If u 's subtree was balanced, it remains balanced after the modifications.

Following this approach we can now balance all nodes in caterpillars that belong to nodes of height 1. Let us now analyze by how much we have increased the face-weights. For one single node v on C let u_+ be the child with the largest weight δ_v before the adjustment. We had to increase the face-weight of all siblings of u_+ by at most δ_v . Since there are $d - 2$ (light) siblings the increase was at most $(d - 2)\delta_v$. This is done for every node on C 's heavy path. Notice that the charge δ_v can be mapped to the face-weights of leaves in the subtree rooted at u_+ . For any other vertex v' on C 's heavy path the charge $\delta_{v'}$ is mapped to the leaves of the subtree rooted at v' . These subtrees, however, are disjoint. This is not only true for the nodes on C 's heavy paths, but for all nodes on heavy paths on caterpillar that belong to nodes of height 1 in H . As a consequence, we have increased the previous total weight face-weight ρ (that is the sum of the face-weights of all leaves, or equivalently the face-weight of the root) by at most $(d - 2)\rho$, or put differently, we have increased ρ by a multiplicative factor of at most $(d - 1)$.

We repeat this process and rebalance the nodes in the caterpillars with larger heights. Each time we multiply the total weight by at most $(d - 1)$. Since the maximal height of a node in H is $\lfloor \log n \rfloor$, we have increased the total face-weight by a factor of at most $(d - 1)^{\lfloor \log n \rfloor} \leq n^{\log(d-1)}$. Therefore, the face-weight of the root is at most $dn^{1+\log(d-1)} = dn^{\log(2d-2)}$. \square

In the following we denote the (final) face-weight of the root as R . To finish the definition of the flat embedding we fix the shape of f_B as follows: One of the vertices of B lies at the origin. We set $L = \sqrt[d-1]{R}$ and place the other vertices of B at $L \cdot \mathbf{e}_i$, for \mathbf{e}_i being a vector of the standard basis of \mathbf{R}^{d-1} , such that B spans the $(d - 1)$ -simplex Δ_B . By construction the volume of the

simplex equals $1/(d-1)!$ times the face-weight of the root of $\mathcal{T}(P)$. It follows that

$$|\llbracket B \rrbracket| = R \leq d \cdot n^{\log(2d-2)}.$$

By our choice of f_B the volume of every face in the flat embedding equals $(d-1)!$ times the volume of its face-weight. An example of a balanced tree-representation with its induced flat embedding is depicted in Fig. 6.

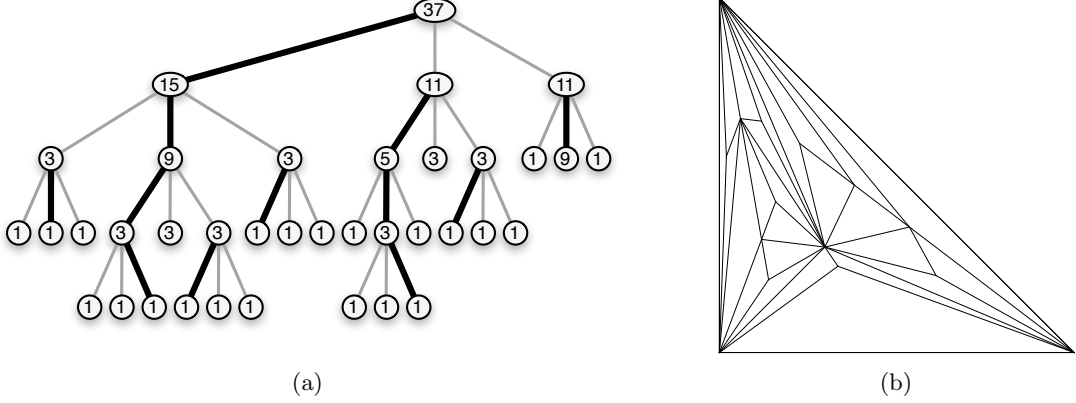


Figure 6: The tree-representation of Fig. 4 with balanced face-weights, denoted in the nodes (a), and the corresponding flat embedding (b).

3.2 Assigning heights

After we have assigned the face-weights we specify the heights of the lifting by determining the vertical shift ζ_i for every vertex v_i based on the face-weights. Let us assume vertex v_i was stacked as \mathbf{p} onto some face f_D . Let the boundary of f_D be formed by the $(d-2)$ -faces f_{X_1}, \dots, f_{X_d} . The stacking introduces the new facets f_{Y_1}, \dots, f_{Y_d} , where for $1 \leq j \leq d$ we have $Y_j := X_j \circ \mathbf{p}$. The nodes of the faces f_{Y_j} in $\mathcal{T}(P)$ that are connected to the node of f_D via a light edge have all the same face-weight. Let this weight be B_i . The only (possible) other face-weight for a node associated with one of the faces f_{Y_j} is denoted by A_i . We set as the vertical shift for the vertex v_i

$$\zeta_i := A_i \cdot B_i. \quad (7)$$

Lemma 9 *The embedding induced by the face-weights and the vertical shifts ζ_i guarantees:*

- 1.) For every $(d-2)$ -face f_X not on f_B we have $\omega_X > 1$.
- 2.) For every $(d-2)$ -face f_X on f_B we have $0 > \omega_X > -R \leq -d \cdot n^{\log(2d-2)}$.

Proof. We first bound the stresses on faces not on f_B . In this proof volumes of faces are referring to the volumes in the flat embedding. We study how the stress on ω_X evolves during the stacking process. The initial value of ω_X is assigned by some stacking operation that introduced f_X . We assume that this stacking operation stacked a vertex on the face f_D . The face-weights of the new facets introduced by the stacking are all the same, namely B_i , except for one possible larger face-weight, namely A_i . Let f_S and f_T be the two facets incident to f_X such that $\llbracket S \rrbracket \geq \llbracket T \rrbracket$. By Lemma 7, we have that at this moment

$$\omega_X = \zeta_i \left| \frac{\llbracket D \rrbracket}{\llbracket S \rrbracket \llbracket T \rrbracket} \right| = A_i B_i \left| \frac{\llbracket D \rrbracket}{\llbracket S \rrbracket \llbracket T \rrbracket} \right| \geq |\llbracket D \rrbracket|,$$

since $\llbracket S \rrbracket \leq A_i$ and $\llbracket T \rrbracket = B_i$.

This positive initial stress decreases when stacking on a face that has f_X on the boundary. So assume we stacked \mathbf{p}_k on such face. Let $C_X(k)$ denote the amount of the decrement due to this stacking. By Lemma 6 we have $C_X(k) = \zeta_k / \|\llbracket S_k \rrbracket\|$, where $S_k = X \circ \mathbf{p}_k$. Recall that $\zeta_k = A_k B_k$, where A_k and B_k are the two different values of face-weights of faces introduced by stacking \mathbf{p}_k . It follows that $\|\llbracket S_k \rrbracket\| \in \{A_k, B_k\}$, and therefore $C_X(k)$ equals either A_k or B_k . Thus, the value of $C_X(k)$ can be attributed to a face-weight of a face f_{Y_k} that was introduced by stacking \mathbf{p}_k . Notice that f_{S_k} is different from f_{Y_k} . The following stacking operations that decrease ω_X further, stack either onto f_{S_k} , or on the opposite facet incident to f_X (see Fig. 7). As a consequence, the faces $\{f_{Y_j} \mid \text{stacking of } \mathbf{p}_j \text{ decrements } \omega_X\}$ have disjoint interiors and are properly contained inside $\Delta_{\pi(D)}$. Since it is not possible to cover $\Delta_{\pi(D)}$ with these faces completely, we have that $\sum_j C_X(j) < \|\llbracket D \rrbracket\|$ and since all face-weight are integers, the difference is at least one. Hence, after all stacking operations the stress ω_X remains a positive number.

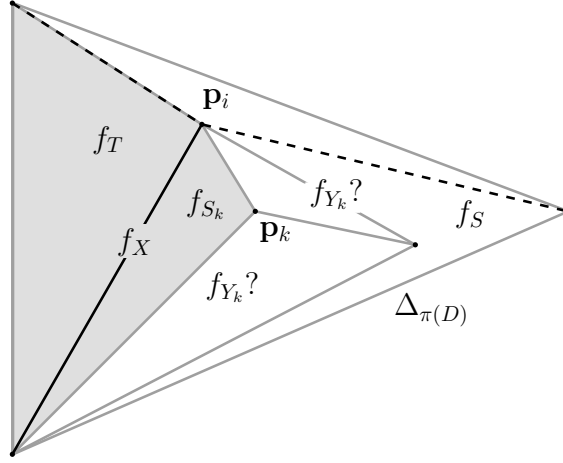


Figure 7: When stacking \mathbf{p}_k the stress on X is decreased by at most the face-weight attributed to f_{Y_j} . We have two candidates for f_{Y_k} . All further negative charges will be attributed to “volumes” contained inside S_k and T .

For the stresses on the boundary we can argue as follows. The stress on X is a combination of several negative charges, but without having an initial positive charge. By the above argument, the negative charges are attributed to faces with disjoint interiors. All these faces are contained inside Δ_B and hence ω_X is at most $-\|\llbracket B \rrbracket\|$. By Lemma 8 we have that $\|\llbracket B \rrbracket\| = R \leq d \cdot n^{\log(2d-2)}$, and the lemma follows. \square

3.3 Rounding to grid points

Let us wrap up what we have constructed so far. Based on the tree-representation we have defined weights for each face. This gave rise to a realization of a projection of the stacked polytope. As a next step we have computed a lifting based on the face-weights. This construction produces a convex realization of the desired stacked polytope, we even know that the stresses are polynomially related. However, the realization does not lie on a polynomial grid yet. To obtain an integer realization we round the coordinates of the points down. The rounding will be carried out in two steps. First we perturb all coordinates such that they are multiples of some parameter $1/\text{pert}$, resp. $1/\text{pert}_z$ for the z -coordinates. In a second step we scale the perturbed embedding by multiplying all coordinates with pert , resp. with pert_z , for the z -coordinates. The details for the rounding procedure are little bit more subtle: We start with rounding the coordinates in the flat embedding, then we update the vertical shifts ζ_i slightly. The z -coordinates are rounded with respect to the lifting defined by the modified vertical shifts.

Since projected volumes play an important role in our construction we discuss as a first step how the rounding will effect these volumes.

Lemma 10 *If we round the coordinates of the flat embedding down, such that every coordinate is a multiple of $1/\text{pert}$ we have that for every facet f_X*

$$|\llbracket X \rrbracket - \llbracket X' \rrbracket| \leq \frac{(d+1)!}{\text{pert}} L^{d-2},$$

where X' denotes the points X after the rounding.

Proof. Remember that the points X are contained inside the simplex Δ_B , which is spanned by the standard basis vectors of \mathbf{R}^{d-1} scaled by $L = \sqrt[d-1]{R}$ and the origin. We prove a slightly stronger statement of the lemma, namely that for every sequence X_k of $k \leq d$ points from X with removed $d-k$ first coordinates we have that $|\llbracket X_k \rrbracket - \llbracket X'_k \rrbracket| \leq (k+1)!L^{k-2}/\text{pert}$. The proof goes via induction over k . Let $E(k) = \max_{X_k} |\llbracket X_k \rrbracket - \llbracket X'_k \rrbracket|$ be the maximal change of the differences of the determinants we want to bound. We denote as $x_i \in \mathbf{R}$ the first coordinate (the x -coordinates) of the i th point in X in order of their appearance. Moreover, for a point sequence P we denote by P^{-i} the set with removed x -coordinates and without the i -th point of P . The rounding changes the coordinates at most by an additive value of $\pm 1/\text{pert}$. Let the change of the coordinates x_i be ε_i . We assume that $\llbracket X_k \rrbracket > 0$ and consider some X_k that realizes the maximum change. We estimate $E(k)$ using the Laplace expansion for the determinants by

$$\begin{aligned} E(k) = |\llbracket X_k \rrbracket - \llbracket X'_k \rrbracket| &\leq \left| \sum_{i=1}^k (-1)^k x_i \llbracket X_k^{-i} \rrbracket - \sum_{i=1}^k (-1)^k (x_i + \varepsilon_i) \llbracket X'_k{}^{-i} \rrbracket \right| \\ &\leq \left| \sum_{i=1}^k (-1)^k x_i \llbracket X_k^{-i} \rrbracket - \sum_{i=1}^k (-1)^k (x_i + \varepsilon_i) (\llbracket X_k^{-i} \rrbracket \pm E(k-1)) \right| \\ &\leq \sum_{i=1}^k (|\llbracket X_k^{-i} \rrbracket|/\text{pert} + (x_i + \varepsilon_i)(E(k-1))) \\ &\leq k[L^{k-2}/\text{pert} + L \cdot E(k-1)]. \end{aligned}$$

For the last transition we used that $x_i + \varepsilon_i$ is at most L , since we have rounded down and the maximal coordinate before rounding was L . We also upper bounded the determinants $\llbracket X_k^{-i} \rrbracket$ by L^{k-2} . This bound follows since the maximal volume of a k -simplex in Δ_B is spanned by $(k+1)$ points of $B \setminus \mathbf{0}$. Such a point set would also maximize $\llbracket X_k^{-i} \rrbracket$ and hence this determinant is at most L^{k-2} .

Using as base case $E(2) = 2/\text{pert}$ we obtain as solution for the above recurrence

$$E(k) \leq \left(\sum_{s=2}^k \frac{k!}{s!} \right) \frac{L^{k-2}}{\text{pert}}.$$

To get a simpler expression for $E(d)$ we bound $\sum_{s=2}^d k!/s!$ from above by $(d+1)!$ and the statement of the lemma follows. \square

In the following we use $\llbracket X' \rrbracket$ and similar expressions to denote the corresponding determinants after rounding.

Corollary 1 *If we round the coordinates of the flat embedding down, such that every coordinate is a multiple of $1/\text{pert}$, we have for every face f_X in the flat embedding*

$$1 - \frac{(d+1)!L^{d-2}}{\text{pert}} \leq \frac{\llbracket X' \rrbracket}{\llbracket X \rrbracket} \leq 1 + \frac{(d+1)!L^{d-2}}{\text{pert}}.$$

Proof. The statement follows from Lemma 10 and the observation that for any face f_X in the flat embedding we have $1 \leq |\llbracket X \rrbracket|$. \square

As a next step we discuss, how to set the parameter pert , such that the perturbed flat embedding will still have a positive stress. To describe the lifting (stress) we defined for every vertex in the flat embedding a vertical shift ζ_i , as given in (7). The definition of ζ_i was based on the

face-weights obtained from the balanced tree-representation. We adjust the vertical shifts after the perturbation slightly. When stacking \mathbf{p}_i we introduced d new faces. Let the faces f_{A_i} and f_{B_i} be the two faces out of the d newly introduced faces with the larger volume. We define

$$\zeta'_i := |\llbracket A'_i \rrbracket \llbracket B'_i \rrbracket|.$$

Lemma 11 *When we pick as the perturbation parameter $\text{pert} = 10(d+1)!L^{d-2}R$ then the perturbed flat embedding with the vertical shifts $(\zeta'_i)_{i \leq n}$ induces an embedding, whose interior stresses are at least $4/5$, and whose boundary stresses are negative and larger than $-2R$.*

Proof. We mimic the strategy of the proof of Lemma 9. Again, all faces in this proof are considered as projected into the $z = 0$ hyperplane. Every stress ω_X is a combination of a positive charge and several negative charges. Let ω_X^+ be the positive charge, that is the initial nonzero stress with respect to the stacking sequence, and let ω_X^- the absolute value of the sum of all negative charges, such that $\omega_X = \omega_X^+ - \omega_X^-$. To bound ω_X we derive bounds for ω_X^+ and ω_X^- . We start with the bound on the positive charge. Assume ω_X^+ was introduced by stacking \mathbf{p}_i at some face f_D , such that due to Lemma 7 we have

$$\omega_X^+ = \zeta'_i \left| \frac{\llbracket D' \rrbracket}{\llbracket S' \rrbracket \llbracket T' \rrbracket} \right|.$$

Here, f_S and f_T are the two faces introduced by stacking \mathbf{p}_i that contain f_X . The height ζ'_i is defined as the product of two face-weights corresponding to f_{A_i} and f_{B_i} . Assume that $\llbracket A'_i \rrbracket \geq \llbracket B'_i \rrbracket$ and that $\llbracket S' \rrbracket \geq \llbracket T' \rrbracket$. By the definition of ζ'_i we have that $\llbracket A'_i \rrbracket \geq \llbracket S' \rrbracket$ and $\llbracket B'_i \rrbracket \geq \llbracket T' \rrbracket$. Therefore, $\omega_X \geq |\llbracket D' \rrbracket|$.

The value of ω_X^- is composed of several charges. Whenever we stack inside a face that contains f_X we increase ω_X^- . Let us study now one of these situations. Assume we stack \mathbf{p}_k inside a face that contains f_X . Let f_{S_k} be the new face that contains f_X . By Lemma 6 we increase ω_X^- by $|\zeta'_j / \llbracket S'_k \rrbracket| =: \text{inc}_{X,k}$. Due to the balanced face-weights we had in the unperturbed setting only two different new “face volume values” when stacking a vertex. Hence for $\zeta'_i := |\llbracket A'_i \rrbracket \llbracket B'_i \rrbracket|$, we had either $\llbracket S_k \rrbracket = \llbracket A_k \rrbracket$, or $\llbracket S_k \rrbracket = \llbracket B_k \rrbracket$. Assume it was $\llbracket S_k \rrbracket = \llbracket A_k \rrbracket$. We set $\delta_+ := 1 + \frac{(d+1)!L^{d-2}}{\text{pert}}$ and $\delta_- = 1 - \frac{(d+1)!L^{d-2}}{\text{pert}}$. Then we have according to Corollary 1

$$\text{inc}_{X,k} = \left| \frac{\llbracket A'_k \rrbracket}{\llbracket S'_k \rrbracket} \llbracket B'_k \rrbracket \right| \leq \left| \frac{\delta_+ \llbracket A_k \rrbracket}{\delta_- \llbracket S_k \rrbracket} \llbracket B'_k \rrbracket \right| = \frac{\delta_+}{\delta_-} |\llbracket B'_k \rrbracket|.$$

Let K be the set of vertex indices, whose stacking contributed to ω_X^- . As noticed in Lemma 9, for any two distinct $s, t \in K$ we have that f_{B_s} and f_{B_t} have disjoint interiors, and furthermore the set $\bigcup_{k \in K} f_{B_k}$ is contained in the perturbed simplex $\pi(\Delta_D)$, but “misses” at least one face. By Corollary 1 the volume of the missing face is at least δ_- . Therefore,

$$\omega_X^- = \sum_{k \in K} \text{inc}_{X,k} = \sum_{k \in K} \frac{\delta_+}{\delta_-} |\llbracket B'_k \rrbracket| = \frac{\delta_+}{\delta_-} \sum_{k \in K} |\llbracket B'_k \rrbracket| \leq \frac{\delta_+}{\delta_-} (|\llbracket D' \rrbracket| - \delta_-) = \frac{\delta_+}{\delta_-} |\llbracket D' \rrbracket| - \delta_+.$$

We finish the proof by combining the bounds for ω_X^+ and ω_X^- . When we pick $\text{pert} = 10(d+1)!L^{d-1}R$ as specified in the lemma then we get $\delta_+ = 1 + \frac{1}{10R}$ and $\delta_- = 1 - \frac{1}{10R}$. We can now obtain the following bound for ω_X (for simplicity we assume that D is ordered such that $\llbracket D \rrbracket$ is

positive) when X is an interior face:

$$\begin{aligned}
\omega_X &\geq \omega_X^+ - \omega_X^- \\
&\geq \llbracket D' \rrbracket - \left(\frac{\delta_+}{\delta_-} \llbracket D' \rrbracket - \delta_+ \right) \\
&= \llbracket D' \rrbracket \left(1 - \frac{\delta_+}{\delta_-} \right) + \delta_+ \\
&\geq \delta_+ \llbracket B \rrbracket \left(1 - \frac{\delta_+}{\delta_-} \right) + \delta_+ = \delta_+ R \left(1 - \frac{\delta_+}{\delta_-} \right) + \delta_+ \\
&= \frac{80R^2 - 2R - 1}{100R^2 - 10R} \\
&\geq \frac{4}{5}.
\end{aligned}$$

If X is a boundary face we have $\omega_X^+ = 0$. Notice that for our choice of pert we have $\delta_+/\delta_- \leq 2$, for $R \geq 3$. We conclude that for a boundary face we have

$$\omega_X \geq - \left(\frac{\delta_+}{\delta_-} \llbracket B' \rrbracket + \delta_+ \right) \geq - \frac{\delta_+}{\delta_-} \llbracket B' \rrbracket \geq -2 \llbracket B' \rrbracket \geq -2R.$$

□

The choice of the perturbation parameter $\text{pert} = 10(d+1)!L^{d-2}R$ ensures, that no volume of a face will flip its sign. In particular, by Lemma 10, the change of volume is less than $1/(10\llbracket B \rrbracket)$.

To construct an integer realization we need to round the z -coordinate as well. Similarly to the determinants we denote the rounded down coordinate z_i by z'_i . We round the z -coordinates such that every z -coordinate is a multiple of $1/\text{pert}_z$, where pert_z is some value to be determined later. The final analysis requires an upper bound for the maximal z -coordinate before rounding, which we give in the following lemma.

Lemma 12 *For z_{\max} being the maximal z -coordinate in the lifting of the perturbed flat embedding induced by the vertical shifts $(\zeta'_i)_i$ we have that*

$$0 < z_{\max} < 2R^2.$$

Proof. The z -coordinates of all vertices not on f_B are positive since all vertical shifts are positive. We consider the perturbed flat embedding. Take any boundary $(d-2)$ -face f_X of f_B . By Lemma 11 we have that $-\omega_X < 2R$. Let \mathbf{p}_i be the vertex with the highest z -coordinate z_{\max} and set $p_i = \pi(\mathbf{p}_i)$. Due to the convexity of the lifting we have that $z_Y(p_i) \geq z_{\max}$, where f_Y is the facet adjacent to f_B via f_X (we let Y and B coincide on the first $d-2$ vertices). By Equation (3) we have $\omega_X \llbracket Y \rrbracket = z_B(p_i) - z_Y(p_i) = -z_Y(p_i)$. Since f_Y is contained inside f_B it follows that

$$z_{\max} \leq z_Y(p_i) < -\omega_X \llbracket Y \rrbracket < 2R^2.$$

□

By rounding the z -coordinates we might violate the convexity of the lifting. The following lemma shows us how to carry out the rounding of the z -coordinates after we have rounded the coordinates of the flat embedding, such that the resulting embedding remains a convex polytope.

Lemma 13 *When setting $\text{pert}_z = 3R$ and rounding such that all z -coordinates are multiples of $1/\text{pert}_z$, then the lifting defined by $(\zeta'_i)_i$ and the perturbed flat embedding will remain an embedding of a convex polytope.*

Proof. Let z'_i be the coordinate z_i after rounding. We have that $z_i - z'_i \leq 1/\text{pert}_z$. By Lemma 11 the interior faces in the perturbed flat embedding have a stress that is at least $4/5$. Let f_X be any interior $(d-2)$ -face that is incident to the facets f_S and f_T in the flat embedding. By Equation (4) we can express ω'_X in terms of $\llbracket \mathcal{T}' \rrbracket$, $\llbracket S' \rrbracket$, and $\llbracket T' \rrbracket$, where \mathcal{T}' is formed by the point set $S' \cup T'$.

The value of $[\mathcal{T}']$ can be expressed as $\sum_{i \in I} z'_i \llbracket A'_i \rrbracket$, where I is the index set of the vertices of \mathcal{T}' and A'_i is given by $\mathcal{T}' \setminus \{\mathbf{p}'_i\}$ in some appropriate order. As usual f_B denotes the boundary face. Since we rounded the coordinates of the flat embedded down we have $\llbracket B' \rrbracket \leq \llbracket B \rrbracket$. Note that the projections of the simplices spanned by the sets A'_i double-cover the projection of \mathcal{T}' into the $z = 0$ hyperplane, and therefore $\sum_{i \in I} \llbracket A'_i \rrbracket \leq 2\llbracket B' \rrbracket \leq 2\llbracket B \rrbracket \leq 2R$. Moreover, by Lemma 10 and our choice of pert, we have that $|\llbracket S \rrbracket \llbracket T \rrbracket| \geq (1 - 1/(10R))^2$. The stress on an interior $(d-2)$ -face X after rounding (ω'_X) can be bounded as follows

$$\begin{aligned} \omega'_X &\geq \frac{[\mathcal{T}']}{\llbracket S' \rrbracket \llbracket T' \rrbracket} = \frac{\sum_{i \in I} (z_i - 1/\text{pert}_z) \llbracket A'_i \rrbracket}{\llbracket S' \rrbracket \llbracket T' \rrbracket} \\ &\geq \omega_X - \frac{\sum_{i \in I} \llbracket A'_i \rrbracket}{(1 - 1/(10R))^2 \text{pert}_z} \\ &\geq \frac{4}{5} - \frac{2R}{3(1 - 1/(10R))^2 R} \\ &= \frac{4}{5} - \frac{200R^2}{3(1 - 10R)^2} \end{aligned}$$

Note that the last expression is a monotone increasing function which is positive for $R \leq 3$.

We are left with checking the sign for the stresses on the $(d-2)$ -faces that define the boundary of f_B . The stresses for these faces have to remain negative. Note that this is certainly the case, if all z -coordinates after the rounding remain positive. Before rounding the z -coordinates, every z -coordinate of a vertex not on f_B was at least as large as the smallest vertical shift ζ'_i . Since the vertical shifts are defined as the sum of two face-weights, we have that the nonzero z -coordinates are at least $(1 - 1/(10R))^2$. As observed earlier, rounding the z -coordinates decreases the z -coordinates by at most $1/(3R)$. Therefore we have for every $\mathbf{p}_i \notin B$

$$z'_i \geq \left(1 - \frac{1}{10R}\right)^2 - \frac{1}{3R} > 1 - \frac{1}{5R} - \frac{1}{3R} > 0,$$

for every $R \geq 3$. Hence, after rounding the z -coordinates the sign pattern of the stresses verifies the convexity of the perturbed realization. \square

We now summarize our analysis and state the main theorem.

Theorem 1 *Every d -dimensional stacked polytope can be realized on an integer grid, such that all coordinates have size at most $10(d+1)! \cdot R^2$, except for one axis, where the coordinates have size at most $6 \cdot R^3$, for $R \leq dn^{\log(2d-2)}$.*

Proof. To get integer coordinates we multiply all coordinates after the rounding with pert, except the z -coordinates, which we multiply with pert_z . Since the maximal z -coordinate is by Lemma 12 at most $2R^2$, and we scale by $\text{pert}_z = 3R$ the bound for the z -coordinates in the theorem follows. All other coordinates are positive and smaller than L before rounding. Hence by scaling with $\text{pert} = 10(d+1)!L^{d-2}R$ we get that all coordinates are integers (Lemma 11) and the maximum coordinate has size $10(d+1)!L^{d-1}R$. Plugging in the definition of L gives as upper bound $10 \cdot (d+1)!R^2$ as asserted. \square

By expressing the quantity R in terms of n we can restate Theorem 1 as the following corollary.

Corollary 2 *For a fixed d , every d -dimensional stacked polytope can be realized on an integer grid polynomial in n . The size of the largest z -coordinate is bounded by $O(n^{3+3\log(d-1)})$, all other coordinates are bounded by $O(n^{2+2\log(d-1)})$.*

Table 1 lists the induced grid bounds for $d = 3, \dots, 10$.

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d	exponent largest non- z -coordinate	exponent largest z -coordinate
3	4	6
4	5.17	7.76
5	6.	9
6	6.65	9.97
7	7.17	10.76
8	7.62	11.43
9	8	12
10	8.34	12.51

Table 1: The induced grid bounds in terms of n up to dimension 10.

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