Calculus III Notes

Turja Roy

Contents

1 3-Dimensional Space

The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise, the 2-D coordinate system is denoted by $\mathbb{R}^2$, and the 1-D coordinate system is denoted by $\mathbb{R}$.

1.1 Equations of Lines

Vector form

If $\vec{a}$ and $\vec{v}$ are parallel vectors, then $\vec{a} = t\vec{v}$ for some scalar t.
Now if we have a vector $\vec{r}$ as follows

$$\vec{r} = \vec{r_0} + \vec{a}$$

Then we can write

$$\vec{r} = \vec{r_0} + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, b \rangle$$

This is called the **vector form of the equation of a line**.

Parametric form

We can rewrite the vector form as

$$\vec{r} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$
$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

In other words

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

This set of equations is called the **parametric form of the equation of a line**.

Symmetric Equations of a Line

If we assume that a, b, and c are non-zero numbers, then we can solve each of the parametric equations for t. This gives us

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 1.1: Find the Equations of lines:
1. Through the points $(7, -3, 1)$ **and** $(-2, 1, 4)$
2. Through the point $(1, -5, 0)$ **and parallel to the line given by** $\vec{r}(t) = \langle 8 - 3t, -10 + 9t, -1 - t \rangle$
3. Through the point $(-7, 2, 4)$ **and orthogonal to both** $\vec{v} = \langle 0, -9, 1 \rangle$ **and** $\vec{w} = 3\hat{i} + \hat{j} - 4\hat{k}$
1.
Direction vector $\vec{d} = \langle -2 - 7, 1 + 3, 4 - 1 \rangle = \langle -9, 4, 3 \rangle$
Now, the vector form of the line is

$$\vec{r} = \langle 7, -3, 1 \rangle + t\langle -9, 4, 3 \rangle$$

The parametric form is

$$x = 7 - 9t, \quad y = -3 + 4t, \quad z = 1 + 3t$$

The symmetric form is

$$\frac{x-7}{-9}=\frac{y+3}{4}=\frac{z-1}{3}$$

2.
The direction vector is $\vec{d}=\langle 3,9,-1\rangle$
Hence, the vector form of the line is

$$\vec{r}=\langle 1,-5,0\rangle+t\langle 3,9,-1\rangle$$

The parametric form is

$$x=1+3t,\quad y=-5+9t,\quad z=-t$$

And the symmetric form is

$$\frac{x-1}{3}=\frac{y+5}{9}=-z$$

3.
Direction vector

$$\vec{d}=\vec{v}\times\vec{w}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & -9 & 1\\ 3 & 1 & -4\end{vmatrix}=\langle 35,3,27\rangle$$

Hence, the vector form of the line is

$$\vec{r}=\langle -7,2,4\rangle+t\langle 35,3,27\rangle$$

The parametric form is

$$x=-7+35t,\quad y=2+3t,\quad z=4+27t$$

The symmetric form is

$$\frac{x+7}{35}=\frac{y-2}{3}=\frac{z-4}{27}$$

Example 1.2: Determine if the two lines are parallel, orthogonal, or neither:
1. The line given by $\vec{r}(t)=\langle 4-7t,-10+5t,21-4t\rangle$ **and the line given by** $\vec{r}(t)=\langle -2+3t,7+5t,5+t\rangle$
2. The line given by $x=29, y=-3-6t, z=12-t$ **and the line given by** $\vec{r}(t)=\langle 12-14t,2+7t,-10+3t\rangle$
1.
The direction vectors are

$$\vec{d_1}=\langle -7,5,-4\rangle,\quad \vec{d_2}=\langle 3,5,1\rangle$$

To check if they are parallel, we can check:

$$\frac{-7}{3}\neq\frac{5}{5}\neq\frac{-4}{1}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{d_1}\cdot\vec{d_2}=-7(3)+5(5)+(-4)(1)=-21+25-4=0$$

Hence, they are orthogonal.
2.
The direction vectors are

$$\vec{d_1} = \langle 0, -6, -1 \rangle, \quad \vec{d_2} = \langle -14, 7, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{0}{-14} \neq \frac{-6}{7} \neq \frac{-1}{3}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{d_1} \cdot \vec{d_2} = 0(-14) + (-6)(7) + (-1)(3) = -42 - 3 = -45 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 1.3: Determine the intersection point of the two lines or show that they don't not intersect:
1. The line passing through the point $(0, -9, -1)$ **and** $(1, 6, -3)$ **and the line given by** $\vec{r}(t) = \langle -9 - 4t, 10 + 6t, 1 - 2t \rangle$
2. The line given by $x = 1 + 6t, t = -1 - 3t, z = 4 + 12t$ **and the line given by** $x = 4 + t, y = -10 - 8t, z = 3 - 5t$
1.
The direction vector of the first line is

$$\vec{d_1} = \langle 1 - 0, 6 + 9, -3 + 1 \rangle = \langle 1, 15, -2 \rangle$$

We can write the parametric equations of the first line as:

$$x = s, y = -9 + 15s, z = -1 - 2s$$

And the parametric equations of the second line as:

$$x = -9 - 4t, y = 10 + 6t, z = 1 - 2t$$

Setting them equal to each other we get,

$$\begin{aligned} 0 + t &= -9 - 4s \\ -9 + 15t &= 10 + 6s \\ -1 - 2t &= 1 - 2s \end{aligned}$$

Solving the first two equations, we get

$$t = -\frac{7}{3}, \quad s = \frac{1}{3}$$

Now, verifying the third equation, we get

$$\begin{aligned} -1 - 2(-\frac{7}{3}) &= 1 - 2(\frac{1}{3}) \\ -1 + \frac{14}{3} &= 1 - \frac{2}{3} \\ \frac{11}{3} &\neq \frac{1}{3} \end{aligned}$$

Since the third equation is not satisfied, the two lines do not intersect.
2.
The lines are given in parametric form.
Setting them equal to each other we get,

$$\begin{aligned} 1 + 6s &= 4 + t \\ -1 - 3s &= -10 - 8t \\ 4 + 12s &= 3 - 5t \end{aligned}$$

Solving the first two equations, we get

$$s = \frac{1}{3}, \quad t = -1$$

Now, verifying the third equation, we get

$$\begin{aligned} 4 + 12\left(\frac{1}{3}\right) &= 3 - 5\langle -1 \rangle \\ 8 &= 8 \end{aligned}$$

That means, the lines intersect. Substituting the values in the parametric equation, we get

$$\begin{aligned} x &= 1 + 6\left(\frac{1}{3}\right) = 3 \\ y &= -1 - 3\left(\frac{1}{3}\right) = -2 \\ z &= 4 + 12\left(\frac{1}{3}\right) = 8 \end{aligned}$$

Hence, the intersection point is $(3, -2, 8)$.

Example 1.4: Which of the three coordinate planes does the line given by $x = 16t, y = -4 - 9t, z = 34$ intersect?

To intersect the xy-plane, we need $z = 0$. But here $z = 34$ is constant. Hence, the line does not intersect the xy-plane.
To intersect the yz-plane, we need $x = 0$. Hence,

$$16t = 0 \implies t = 0$$

And the intersection point is $(0, -4 - 9 \times 0, 34)$ or $(0, -4, 34)$.
To intersect the xz-plane, we need $y = 0$. Hence,

$$-4 - 9t = 0 \implies t = -\frac{4}{9}$$

And the intersection point is $\left(16\left(-\frac{4}{9}\right), 0, 43\right)$ or $\left(-\frac{64}{9}, 0, 34\right)$.

1.2 Equations of Planes

Vector form

Let's assume $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ and $\vec{r} = \langle x, y, z \rangle$ are two position vectors and $\vec{r} - \vec{r_0}$ is a vector in the plane.
If $\vec{n} = \langle a, b, c \rangle$ is a normal to the plane (which means it's orthogonal to the vector $\vec{r} - \vec{r_0}$), then we can write

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \implies \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

This is called the **vector form of the equation of a plane**.

Scalar form

If we expand the vector equation in the following way,

$$\begin{aligned} \vec{n} \cdot (\vec{r} - \vec{r_0}) &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \end{aligned}$$

Computing the dot product, we get

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the **scalar form of the equation of a plane**.
This equation can also be written as

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

Example 1.5: Find the equation of the plane:
1. Through the point $(6, -3, 1)$**,** $(5, -4, 1)$**, and** $(3, -4, 0)$
2. The plane containing the point $(1, -5, 8)$ **and orthogonal to the line given by** $x = -3 + 15t, y = 14 - t, z = 9 - 3t$
3. The plane containing the point $(-8, 3, 7)$ **and parallel to the plane given by** $4x + 8y - 2z = 45$
4. The plane containing the two lines given by $\vec{r}(t) = \langle 7 + 5t, 2 + t, 6t \rangle$ **and** $\vec{r}(t) = \langle 7 - 6t, 2 - 2t, 10t \rangle$

1.
The given points are

$$A(6, -3, 1), B(5, -4, 1), C(3, -4, 0)$$

Two vectors in the plane are

$$\begin{aligned} \vec{AB} &= \langle 5 - 6, -4 + 3, 1 - 1 \rangle = \langle -1, -1, 0 \rangle \\ \vec{BC} &= \langle 3 - 5, -4 + 4, 0 - 1 \rangle = \langle -2, 0, -1 \rangle \end{aligned}$$

Normal vector on the place:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 0 \\ -2 & 0 & -1 \end{vmatrix} = \hat{i} - \hat{j} - 2\hat{k}$$

Now, using the point A, we can write the equation of the plane as

$$\begin{aligned}(x-6)-(y+3)-2(z-1)&=0\\ x-y-2z&=7\end{aligned}$$

2.
The normal vector is

$$\vec{n}=\langle 15,-1,-3\rangle$$

Using the point $(1,-5,8)$, the equation of the plane is

$$\begin{aligned}15(x-1)-(y+5)-3(z-8)&=0\\ 15x-y-3z&=15+5-24\\ 15x-y-3z+4&=0\end{aligned}$$

3.
The normal vector is

$$\vec{n}=\langle 4,8,-2\rangle$$

Using the point $(-8,3,7)$, the equation of the plane is

$$\begin{aligned}4(x+8)+8(y-3)-2(z-7)&=0\\ 4x+8y-2z&=-32+24+14\\ 4x+8y-2z+6&=0\end{aligned}$$

4.
The direction vectors of the two lines are

$$\vec{d_1}=\langle 5,1,6\rangle,\quad \vec{d_2}=\langle -6,-2,10\rangle$$

The normal vector is

$$\vec{n}=\vec{d_1}\times\vec{d_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & 1 & 6\\ -6 & -2 & 10\end{vmatrix}=\langle 22,-86,-4\rangle$$

Using the point $A(7,2,0)$, the equation of the plane is

$$\begin{aligned}22(x-7)-86(y-2)-4(z-0)&=0\\ 22x-86y-4z-154+172&=0\\ 22x-86y-4z+18&=0\end{aligned}$$

Example 1.6: Determine if the two planes are parallel, orthogonal, or neither: The plane given by $3x+9y+7z=-1$ and the plane containing the points $(1,-1,9),(4,-1,2),(-2,3,4)$

The normal vector of the first plane is

$$\vec{n_1}=\langle 3,9,7\rangle$$

Let the points be

$$A(1,-1,9), B(4,-1,2), C(-2,3,4)$$

Two vectors in the plane are

$$\vec{AB} = \langle 4-1, -1+1, 2-9\rangle = \langle 3,0,-7\rangle$$
$$\vec{AC} = \langle -2-1, 3+1, 4-9\rangle = \langle -3,4,-5\rangle$$

The normal vector of the second plane is

$$\vec{n_2} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -7 \\ -3 & 4 & -5 \end{vmatrix} = \langle 28, 36, 12\rangle = \langle 7,9,3\rangle$$

To check if they are parallel, we can check:

$$\frac{3}{7} \neq \frac{9}{9} \neq \frac{7}{3}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{n_1} \cdot \vec{n_2} = 3(7) + 9(9) + 7(3) = 21 + 81 + 21 = 123 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 1.7: Find the intersection of the plane given by $4x + y + 10z = -2$ **and the plane given by** $-8x + 2y + 3z = -8$

The two planes are

$$4x + y + 10z = -2$$
$$-8x + 2y + 3z = -8$$

Multiplying the first equation by 2 and adding it to the second equation, we get

$$4y + 23z = -12 \quad \Longrightarrow \quad y = -3 - \frac{23}{4}z$$

Substituting the value of y in the first equation, we get

$$16x - 3 - \frac{23}{4}z + 10z = -2 \quad \Longrightarrow \quad x = \frac{1}{4} - \frac{17}{16}z$$

Let $z = t$ (a parameter). Then we get

$$x = \frac{1}{4} - \frac{17}{16}t$$
$$y = -3 - \frac{23}{4}t$$
$$z = t$$

This is the parametric form of the line of intersection.
We can also write it in vector form as

$$\vec{r} = \langle \frac{1}{4}, -3, 0\rangle + t\langle -\frac{17}{16}, -\frac{23}{4}, 1\rangle$$

1.3 Quadratic Surfaces

General form

The general form of a quadratic surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where $A, B, C, D, E, F, G, H, I, J$ are constants.

Ellipsoid

The general equation of an ellipsoid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the ellipsoid and a, b, c are the semi-axis lengths.
If $a = b = c$, we get a sphere.

Cone

The general equation of a cone that opens along the z-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{(z-l)^2}{c^2}$$

where (h, k, l) is the center of the cone and a, b, c are the semi-axis lengths.

Cylinder

The general equation of a cylinder that opens along the z-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where (h, k) is the center of the cylinder and a, b are the semi-axis lengths.
If $a = b$, we get a circular cylinder.

Hyperboloid of One Sheet

The general equation of a hyperboloid of one sheet is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

Hyperboloid of Two Sheets

The general equation of a hyperboloid of two sheets is

$$-\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

Elliptic Paraboloid

The general equation of an elliptic paraboloid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

Hyperbolic Paraboloid

The general equation of a hyperbolic paraboloid is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

1.4 Calculus with Vector Functions

Let

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Note:-

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$

Note:-

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{u} + \vec{v}) = \vec{u}' + \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(c\vec{u}) = c\vec{u}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{u} \cdot \vec{v}) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{u} \times \vec{v}) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\vec{u}f(t)\big) = f'(t)\vec{u}'(f(t))$$

Note:-

$$\int \vec{r}(t)\,\mathrm{d}t = \langle \int f(t)\,\mathrm{d}t\,, \int g(t)\,\mathrm{d}t\,, \int h(t)\,\mathrm{d}t \rangle$$

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \left\langle \int_a^b f(t)\,\mathrm{d}t + \int_a^b g(t)\,\mathrm{d}t + \int_a^b h(t)\,\mathrm{d}t \right\rangle$$

1.5 Tangent, Normal, and Binormal Vectors

Unit Tangent vector

Given the vector function $\vec{r}(t)$, we call $\vec{r}'(t)$ the **tangent vector** The unit tangent vector is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Unit Normal vector

If $\vec{T}(t)$ is the unit tangent vector, then the **unit normal vector** is given by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Note:-

If $\vec{r}(t)$ is a vector such that $\|\vec{r}(t)\| = c$ for all t, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$

Binormal vector

The **binormal vector** is given by

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The binormal vector is orthogonal to both the tangent and normal vectors.

1.6 Arc Length with Vector Functions

Note:-

The arc length of a vector function $\vec{r}(t)$ from $t = a$ to $t = b$ is given by

$$L = \int_a^b \|\vec{r}'(t)\| \, \mathrm{d}t$$

Or,

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2} \, \mathrm{d}t$$

1.7 Curvature

Curvature of a curve in 3-D space

The curvature of a curve in 3-D space is given by

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

where $\vec{T}(t)$ is the unit tangent vector and $\vec{r}(t)$ is the position vector.
This can also be written as

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

2 Partial Derivatives

2.1 First Order Partial Derivatives

Definition 2.1.1: First Order Partial Derivative

The **first order partial derivative** of a function $f(x, y)$ is the derivative of f with respect to one variable while treating the other variable as a constant. The partial derivative of f wrt x is denoted by:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

And the partial derivative of f wrt y is denoted by:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

They can also be written in the following notations:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\big(f(x, y)\big) = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\big(f(x, y)\big) = D_y f$$

2.2 Interpretations of Partial Derivatives

Much like the first derivative of a function of one variable, the first order partial derivatives of a function of multiple variables can be interpreted as the slope of the tangent line to the surface defined by the function at a point.

Slopes of Traces

Partial derivatives are the slopes of traces. The partial derivative $f_x(a, b)$ is the slope of the trace of $f(x, y)$ for the plane $y = b$ at the point (a, b). Likewise, the partial derivative $f_y(x, y)$ is the slope of the trace of $f(x, y)$ for the plane $x = a$ at the point (a, b)

Example 2.1: Determine if $f(x, y) = \frac{x^2}{y^3}$ is increasing or decreasing at $(2, 5)$, if:
(a) we allow x to vary and hold y fixed,
(b) we allow y to vary and hold x fixed.

(a) To find the partial derivative with respect to x, we treat y as a constant:

$$f_x(x, y) = \frac{2x}{y^3} \qquad \Longrightarrow \qquad f_x(2, 5) = \frac{4}{125} > 0$$

This means that f is increasing in the x direction at the point $(2, 5)$.
(b) To find the partial derivative with respect to y, we treat x as a constant:

$$f_y(x, y) = -\frac{3x^2}{y^4} \qquad \Longrightarrow \qquad f_y(2, 5) = -\frac{12}{625} < 0$$

This means that f is decreasing in the y direction at the point $(2, 5)$.

Partial derivatives can also be interpreted as the slope of the tangent plane to the surface defined by the function at a point. The tangent plane is a linear approximation of the surface at that point.

Example 2.2: Find the slopes of the traces to $z = 10 - 4x^2 - y^2$ at the point $(1, 2)$.
The partial derivative with respect to x is:

$$f_x(x, y) = -8x \qquad \implies \qquad f_x(1, 2) = -8$$

The partial derivative with respect to y is:

$$f_y(x, y) = -2y \qquad \implies \qquad f_y(1, 2) = -4$$

Thus, the slope of the trace in the x direction at $(1, 2)$ is -8, and the slope of the trace in the y direction at $(1, 2)$ is -4.

We can also use partial derivatives to find the equations of the tangent lines to the traces of a surface at a point.

Example 2.3: Find the vector equations of the tangent lines to the traces to $z = 10 - 4x^2 - y^2$ at the point $(1, 2)$
The point on the trace is

$$(1, 2, f(1, 2)) = (1, 2, 10 - 4(1)^2 - (2)^2) = (1, 2, 2)$$

Hence, the equation of the tangent line to the trace for the plane $y = 2$ is:

$$\vec{r_x}(t) = \langle 1, 2, 2\rangle + t\langle 1, 0, -8\rangle = \langle 1 + t, 2, 2 - 8t\rangle$$

And the equation of the tangent line to the trace for the plane $x = 1$ is:

$$\vec{r_y}(t) = \langle 1, 2, 2\rangle + t\langle 0, 1, -4\rangle = \langle 1, 2 + t, 2 - 4t\rangle$$

Example 2.4: Find the vector equations of the tangent lines to the traces for $f(x, y) = \sin x \cos y$ at $\left(\frac{\pi}{3}, \frac{-\pi}{4}\right)$
The point on the trace is

$$\left(\frac{\pi}{3}, -\frac{\pi}{4}, f\left(\frac{\pi}{3}, -\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{3}, -\frac{\pi}{4}, \sin\left(\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4}\right)$$

Hence, the equation of the tangent line to the trace for the plane $y = -\frac{\pi}{4}$ is:

$$\begin{aligned}
\vec{r_x}(t) &= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4}\right\rangle + t\left\langle 1, 0, f_x(x, y)\right\rangle \\
&= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4}\right\rangle + t\left\langle 1, 0, \cos\left(\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{4}\right)\right\rangle \\
&= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4}\right\rangle + t\left\langle 1, 0, \frac{1}{2\sqrt{2}}\right\rangle
\end{aligned}$$

And the equation of the tangent line to the trace for the plane $x = \frac{\pi}{3}$ is:

$$\begin{aligned}
\vec{r_y}(t) &= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4} \right\rangle + t\left\langle 0, 1, f_y(x,y) \right\rangle \\
&= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4} \right\rangle + t\left\langle 0, 1, -\sin\left(\frac{\pi}{3}\right)\sin\left(-\frac{\pi}{4}\right) \right\rangle \\
&= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4} \right\rangle + t\left\langle 0, 1, \frac{\sqrt{6}}{4} \right\rangle
\end{aligned}$$

2.3 Higher Order Partial Derivatives

Second Order Partial Derivatives

The **Second order partial derivatives** of a function $f(x, y)$ are the partial derivatives of the first order partial derivatives. The second order partial derivatives are denoted by:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y^2}x$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x^2}y$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Clairaut's Theorem

If the second order partial derivatives f_{xy} and f_{yx} are continuous at a point, then they are equal at that point:

$$f_{xy} = f_{yx}$$

Like second order derivatives, there are higher order partial derivatives as well. The third order partial derivatives are denoted by:

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3}$$

$$f_{xyx} = (f_{xy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial x\,\partial y\,\partial x}$$

$$f_{yxx} = (f_{yx})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\,\partial y}\right) = \frac{\partial^2 f}{\partial x^2\,\partial y}$$

This also applies to functions of more than two variables. For example,

$$f_{xz}(x, y, z) = f_{zx}(x, y, z)$$

Extension of Clairaut's Theorem

In general, we can extend Clairaut's theorem to any function and mixed partial derivatives.

That means:

$$f_{ssrtsrr} = f_{trsrssr} = f_{rrssst} = \dots$$

Example 2.5: Find all the second order partial derivatives of function $Q(u,v,w) = u^4 \sin w^2 - \frac{2v}{u^4} + \ln\big(v^2 w\big)$

To find the second order partial derivatives, we first find the first order partial derivatives:

$$Q_u = 4u^3 \sin w^2 + \frac{8v}{u^5}$$
$$Q_v = \frac{-2}{u^4} + \frac{2vw}{v^2 w} = -\frac{2}{u^4} + \frac{2}{v}$$
$$Q_w = 2u^4 w \cos w^2 + \frac{1}{w}$$

Now we can find the second order partial derivatives:

$$Q_{uu} = 12u^2 \sin w^2 - \frac{40v}{u^6}$$
$$Q_{uv} = Q_{vu} = \frac{8}{u^5}$$
$$Q_{uw} = 8u^3 w \cos w^2$$
$$Q_{vv} = -\frac{2}{v^2}$$
$$Q_{vw} = Q_{wv} = 0$$
$$Q_{ww} = 2u^4 \cos w^2 - 4u^4 w^2 \sin w^2 - \frac{1}{w^2}$$

Example 2.6: Given $w = \ln\left(\frac{xy}{z}\right) + 8x^4 y^3 \sqrt{z}, find \frac{\partial^5 w}{\partial x\, \partial z^2\, \partial y\, \partial x}$

Using Clairaut's theorem,

$$\frac{\partial^5 w}{\partial x\, \partial z^2\, \partial y\, \partial x} = \frac{\partial^5 w}{\partial x^2\, \partial y\, \partial z^2}$$

Now,

$$\frac{\partial w}{\partial x} = \frac{y}{z} \cdot \frac{z}{xy} + 32x^3 y^3 \sqrt{z} = \frac{1}{x} + 32x^3 y^3 \sqrt{z}$$
$$\frac{\partial^2 w}{\partial x^2} = -\frac{1}{x^2} + 96x^2 y^3 \sqrt{z}$$
$$\frac{\partial^3 w}{\partial x^2\, \partial y} = 288x^2 y^2 \sqrt{z}$$
$$\frac{\partial^4 w}{\partial x^2\, \partial y\, \partial z} = \frac{144x^2 y^2}{\sqrt{z}}$$
$$\frac{\partial^5 w}{\partial x^2\, \partial y\, \partial z^2} = -72x^2 y^2 z^{-3/2}$$

Example 2.7: Given $f(x,y) = \frac{x^6}{1+6y} - \cos\big(x^2\big) + 6e^x \sin(y)$**, find** f_{xxyxyx}

$$
\begin{aligned}
f_x &= \frac{6x^5}{1+6y} + 2x\sin\left(x^2\right) + 6e^x\sin(y) \\
f_{xx} &= \frac{30x^4}{1+6y} + 2\sin\left(x^2\right) + 4x^2\cos\left(x^2\right) + 6e^x\sin(y) \\
f_{xxx} &= \frac{120x^3}{1+6y} + 12x\cos\left(x^2\right) - 8x^3\sin\left(x^2\right) + 6e^x\sin(y) \\
f_{xxxx} &= \frac{360x^2}{1+6y} + 12\cos\left(x^2\right) - 48x^2\sin\left(x^2\right) - 16x^4\cos\left(x^2\right) + 6e^x\sin(y) \\
f_{xxxxy} &= -\frac{2160x^2}{(1+6y)^2} + 6e^x\cos(y) \\
f_{xxxxyy} &= \frac{25920x^2}{(1+6y)^3} - 6e^x\sin(y)
\end{aligned}
$$

2.4 Differentials

Differentials

The **differential** of a function $f(x, y)$ is a linear approximation of the change in the function at a point.
The differential of f is denoted by:

$$\mathrm{d}f = f_x\,\mathrm{d}x + f_y\,\mathrm{d}y$$

where dx and dy are small changes in x and y, respectively.
For a given function $w = g(x, y, z)$, the differential is given by:

$$\mathrm{d}w = g_x\,\mathrm{d}x + g_y\,\mathrm{d}y + g_z\,\mathrm{d}z$$

Example 2.8: Compute the differential for $u = \frac{t^3 r^6}{s^2}$

$$\mathrm{d}u = \frac{3t^2r^6}{s^2}\,\mathrm{d}t + \frac{6t^3r^5}{s^2}\,\mathrm{d}r - \frac{2t^3r^6}{s^3}\,\mathrm{d}s$$

2.5 Chain Rule

Case 1: If $z = f(x, y)$, $x = g(t)$, and $y = h(t)$, then the chain rule states that:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\text{Or,}\quad \frac{\mathrm{d}z}{\mathrm{d}t} = f_x\frac{\mathrm{d}x}{\mathrm{d}t} + f_y\frac{\mathrm{d}y}{\mathrm{d}t}$$

Example 2.9: Compute $\frac{\mathrm{d}z}{\mathrm{d}t}$ for $z = xe^{xy}$, $x = t^2$, $y = t^{-1}$

$$
\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\
&= (e^{xy} + yxe^{xy})(2t) + x^2e^{xy}(-t^{-2}) \\
&= 2t(e^{xy} + xye^{xy}) - x^2e^{xy}t^{-2} \\
&= 2t(e^t + te^t) - t^4e^tt^{-2} \\
&= 2te^t + t^2e^t
\end{aligned}
$$

<u>Case 2:</u> If $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, then the chain rule states that:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\text{Or,}\quad \frac{\partial z}{\partial s} = f_x\frac{\partial x}{\partial s} + f_y\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\text{Or,}\quad \frac{\partial z}{\partial t} = f_x\frac{\partial x}{\partial t} + f_y\frac{\partial y}{\partial t}$$

Example 2.10: Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = e^{2r}\sin(3\theta)$, $r = st - t^2$, $\theta = \sqrt{s^2 + t^2}$

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial s} \\
&= (2e^{2r}\sin(3\theta))(t + 0) + (3e^{2r}\cos(3\theta))\left(\frac{s}{\sqrt{s^2 + t^2}}\right) \\
&= 2te^{2r}\sin(3\theta) + 3e^{2r}\cos(3\theta)\frac{s}{\sqrt{s^2 + t^2}} \\
&= 2te^{2(st-t^2)}\sin\left(3\sqrt{s^2 + t^2}\right) + 3e^{2(st-t^2)}\cos\left(3\sqrt{s^2 + t^2}\right)\frac{s}{\sqrt{s^2 + t^2}}
\end{aligned}
$$

And,

$$
\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial t} \\
&= (2e^{2r}\sin(3\theta))(s - 2t) + (3e^{2r}\cos(3\theta))\left(\frac{t}{\sqrt{s^2 + t^2}}\right) \\
&= 2(s - 2t)e^{2r}\sin(3\theta) + 3e^{2r}\cos(3\theta)\frac{t}{\sqrt{s^2 + t^2}} \\
&= 2(s - 2t)e^{2(st-t^2)}\sin\left(3\sqrt{s^2 + t^2}\right) + 3e^{2(st-t^2)}\cos\left(3\sqrt{s^2 + t^2}\right)\frac{t}{\sqrt{s^2 + t^2}}
\end{aligned}
$$

Chain Rule

Given the following conditions:

(i) $z = f(x_1, x_2, \ldots, x_n)$ is a function of n variables,

(ii) Each variable $x_i(t_1, t_2, \ldots, t_m)$ is a function of m variables, Then for any variable t_i ($i = 1, 2, \ldots, m$), we have the following chain rule:

$$\boxed{\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}}$$

Example 2.11: Compute $\frac{\partial^2 f}{\partial \theta^2}$ for $f(x, y)$ if $x = r\cos\theta$ and $y = r\sin\theta$

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\ &= -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}\end{aligned}$$

Now, we know the second derivative is

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}\right)$$

Now, we can separately compute $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$:

$$\begin{aligned}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) &= -r\sin\theta\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + r\cos\theta\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ &= -r\sin\theta\frac{\partial^2 f}{\partial x^2} + r\cos\theta\frac{\partial^2 f}{\partial y\,\partial x} \\ \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right) &= -r\sin\theta\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + r\cos\theta + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \\ &= -r\sin\theta\frac{\partial^2 f}{\partial x\,\partial y} + r\cos\theta\frac{\partial^2 f}{\partial y^2}\end{aligned}$$

Finally, we can substitute these into the second derivative:

$$\begin{aligned}\frac{\partial^2 f}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}\right) \\ &= -r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\left(-r\sin\theta\frac{\partial^2 f}{\partial x^2} + r\cos\theta\frac{\partial^2 f}{\partial y\,\partial x}\right) \\ &\quad - r\sin\theta\frac{\partial f}{\partial y} + r\cos\theta\left(-r\sin\theta\frac{\partial^2 f}{\partial x\,\partial y} + r\cos\theta\frac{\partial^2 f}{\partial y^2}\right) \\ &= -r\cos\theta\frac{\partial f}{\partial x} + r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2} - r^2\sin\theta\cos\theta\frac{\partial^2 f}{\partial y\,\partial x} \\ &\quad - r\sin\theta\frac{\partial f}{\partial y} - r^2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\,\partial y} + r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2} \\ &= -r\left(\cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}\right) \\ &\quad + r^2\left(\sin^2\theta\frac{\partial^2 f}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\,\partial y} + \cos^2\theta\frac{\partial^2 f}{\partial y^2}\right)\end{aligned}$$

2.5.1 Implicit Differentiation

Implicit Differentiation

If $F(x, y) = 0$ is a function where $y = y(x)$, then we can use implicit differentiation to find the derivative of y with respect to x. The chain rule gives us:

$$F_x + F_y \frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

This can be extended to functions of more than two variables. We can start by assuming that $z = f(x, y)$ and we want to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
To find $\frac{\partial z}{\partial x}$, we differentiate both sides wrt x:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, we get:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example 2.12: Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $x\cos(3y) + x^3y^5 = 3x - e^{xy}$
First, we rearrange the equation in the form $F(x, y) = 0$:

$$x\cos(3y) + x^3y^5 - 3x + e^{xy} = 0$$

Now, the derivative is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = -\frac{\cos(3y) + 3x^2y^5 - 3 + ye^{xy}}{-3x\sin(3y) + 5x^3y^4 + xe^{xy}}$$

Example 2.13: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^2\sin(2y - 5z) = 1 + y\cos(6zx)$
First, let's rearrange the equation in the form $F(x, y, z) = 0$:

$$x^2\sin(2y - 5z) - 1 - y\cos(6zx) = 0$$

Now, the derivatives are:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x\sin(2y - 5z) + 6yz\sin(6zx)}{-5x^2\cos(2y - 5z) + 6xy\sin(6zx)}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2x^2\cos(2y - 5z) - \cos(6zx)}{-5x^2\cos(2y - 5z) + 6xy\sin(6zx)}$$

2.6 Directional Derivatives

Definition 2.6.1: Directional Derivative

The rate of change of $f(x, y)$ in the direction of the unit vector $\vec{u} = \langle a, b\rangle$ is called the **directional derivative** of f and is denoted by $D_{\vec{u}}f(x, y)$. The definition of the directional derivative is:

$$D_{\vec{u}}f(x, y) = \lim_{h\to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h}$$

Now, in practice, finding this limit can be difficult. We can derive an equivalent formula for taking directional derivatives.
Let's define a new function of one variable:

$$g(z) = f(x_0 + az, y_0 + bz)$$

where x_0, y_0, a, b are constants. Then, by the definition of the derivative, we have

$$g'(z) = \lim_{h\to 0} \frac{g(z + h) - g(z)}{h}$$

For $z = 0$, we have:

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\vec{u}}f(x_0, y_0)$$

Thus, we have the following relationship:

$$g'(0) = D_{\vec{u}f(x_0,y_0)}$$

Now, let's rewrite $g(z)$ as follows:

$$g(z) = f(x, y) \quad \text{where } x = x_0 + az \text{ and } y = y_0 + bz$$

We can now apply the chain rule to find $g'(z)$:

$$g'(z) = \frac{\mathrm{d}g}{\mathrm{d}z} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}z} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}z} = f_x(x, y)a + f_x(x, y)b$$

If we take $z = 0$, we get $x = x_0$ and $y = y_0$, and finally we have:

$$D_{\vec{u}}f(x_0, y_0) = g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Note:-

$$D_{\vec{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$
$$D_{\vec{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

Example 2.14: Find each of the directional derivatives:
(a) $D_{\vec{u}}f(8, 1, 2)$ **where** $f(x, y, z) = \ln\frac{x}{z} + \ln\frac{z}{y} + xy^2$ **in the direction of** $\vec{v} = \langle 1, 5, 2\rangle$
(b) $D_{\vec{u}}f(x, y)$ **where** $f(x, y) = xe^{xy} + y$ **and** $\vec{u}$ **is the unit vector in the direction of** $\theta = \frac{2\pi}{3}$

(a) First, the unit vector in the direction of $\vec{v}$ is:

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{\langle 1,5,2\rangle}{\sqrt{1^2+5^2+2^2}} = \frac{\langle 1,5,2\rangle}{\sqrt{30}}$$

Simplifying the function, we have:

$$f(x,y,z) = \ln(x) - \ln(z) + \ln(z) - \ln(y) + xy^2 = \ln(x) - \ln(y) + xy^2$$

The directional derivative is then,

$$\begin{aligned}
D_{\vec{u}} f(x,y,z) &= \frac{1}{\sqrt{30}}\left[f_x(x,y,z) + 5f_y(x,y,z) + 2f_z(x,y,z)\right] \\
D_{\vec{u}} f(8,1,2) &= \frac{1}{\sqrt{30}}\left[1\left(\frac{1}{x}+y^2\right) + 5\left(-\frac{1}{y}+2xy\right) + 2\cdot 0\right]_{(8,1,2)} \\
&= \frac{1}{\sqrt{30}}\left(\frac{1}{8}+1-5+80\right) \\
&= \frac{609}{8\sqrt{30}}
\end{aligned}$$

(b) The unit vector in the direction of $\theta = \frac{2\pi}{3}$ is:

$$\vec{u} = \left\langle \cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right)\right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$$

So, the directional derivative is:

$$\begin{aligned}
D_{\vec{u}} f(x,y) &= \left(-\frac{1}{2}\right)(e^{xy} + xye^{xy}) + \left(\frac{\sqrt{3}}{2}\right)\left(x^2e^{xy}+1\right) \\
D_{\vec{u}} f(2,0) &= \left(-\frac{1}{2}\right)(1) + \left(\frac{\sqrt{3}}{2}\right)(5) \\
&= \frac{5\sqrt{3}-1}{2}
\end{aligned}$$

Notice, the directional derivative can also be written in the following way:

$$\begin{aligned}
D_{\vec{u}} f(x,y,z) &= f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z) \\
&= \langle f_x, f_y, f_z\rangle \cdot \langle a,b,c\rangle
\end{aligned}$$

In other words, the directional derivative is the dot product of the gradient vector and the unit vector in the direction of interest.

Gradient Vector

The **gradient vector** of a function $f(x,y)$ is denoted by ∇f and is defined as:

$$\nabla f = \langle f_x, f_y\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle$$

For a function $f(x,y,z)$, the gradient vector is:

$$\nabla f = \langle f_x, f_y, f_z\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle$$

Directional Derivative

The directional derivative can also be expressed in terms of the gradient vector:

$$D_{\vec{u}} f(\vec{x}) = \nabla f \cdot \vec{u}$$

where $\vec{x} = \langle x,y,z\rangle$ or $\vec{x} = \langle x,y\rangle$ depending on the function and $\vec{u}$ is the unit vector in the direction of interest.

Example 2.15: Find the directional derivative

$$D_{\vec{u}} f(\vec{x}) \text{ for } f(x,y,z) = \sin(yz) + \ln\left(x^2\right) \text{ at } (1,1,\pi) \text{ in the direction of } \vec{v} = \langle 1,1,-1\rangle$$

The gradient vector is:

$$\begin{aligned}\nabla f(x,y,z) &= \left\langle \frac{2}{x}, z\cos(yz), y\cos(yz)\right\rangle \\ \nabla f(1,1,\pi) &= \left\langle \frac{2}{1}, \pi\cos(\pi), \cos(\pi)\right\rangle = \langle 2, -\pi, -1\rangle\end{aligned}$$

The unit vector in the direction of $\vec{v}$ is:

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{\langle 1,1,-1\rangle}{\sqrt{3}} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\rangle$$

Hence, the directional derivative is:

$$\begin{aligned}D_{\vec{u}} f(1,1,\pi) &= \nabla f \cdot \vec{u} \\ &= \left\langle 2, -\pi, -1\right\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\rangle \\ &= 2\cdot\frac{1}{\sqrt{3}} - \pi\cdot\frac{1}{\sqrt{3}} + 1\cdot\frac{1}{\sqrt{3}} = \frac{3-\pi}{\sqrt{3}}\end{aligned}$$

Theorem 2.6.2: *The maximum value of $D_{\vec{u}} f(\vec{x})$ (and hence then the maximum rate of change of the function $f(\vec{x})$) is given by $||\nabla f(\vec{x})||$ and will occur in the direction given by $\nabla f(\vec{x})$.*

Proof:

We can use a nice fact about dot products as well as the fact that $\vec{u}$ is a unit vector to proof this theorem:

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = ||\nabla f||\,||\vec{u}||\cos\theta = ||\nabla f||\cos\theta$$

where θ is the angle between the gradient and $\vec{u}$.

Now, the largest possible value of $\cos\theta$ is 1, which occurs at $\theta = 0$. Therefore, the maximum value of $D_{\vec{u}}f(\vec{x})$ is $||\nabla f(\vec{x})||$. Also, the maximum value occurs when the angle between the gradient and $\vec{u}$ is zero, or in other words, when $\vec{u}$ is pointing in the same direction as the gradient.

Note:-

The gradient vector $\nabla f(x_0, y_0)$ is orthogonal (or perpendicular) to the level curve/contour curve $f(x, y) = k$ at the point (x_0, y_0). Likewise, the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0).

Proof:

Let S be the level surface given by $f(x, y, z) = k$ and let $P(x_0, y_0, z_0)$ be a point on the surface S.

Now, let C be any curve on the surface S that contains the point P. Let $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ be the vector equation for C and suppose that t_0 is the value of t such that $\vec{r}(t_0) = \langle x_0, y_0, z_0\rangle$. In other words, t_0 is the value of t that gives P.

Since C lies on S, we know that the points on C must satisfy the equation for S. That is

$$f(x(t), y(t), z(t)) = k$$

Using the chain rule, we get:

$$\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

Notice that $\nabla f = \langle f_x, f_y, f_z\rangle$ and $\vec{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ so this becomes:

$$\nabla f \cdot \vec{r}'(t) = 0$$

At $t = t_0$,

$$\nabla f(x_0, y_0, z_0) \cdot \vec{r}'(t_0) = 0$$

This then tells us that the gradient vector at P (i.e. $\nabla f(x_0, y_0, z_0)$) is orthogonal to the tangent vector $\vec{r}'(t_0)$ to any curve C that passes through P and on the surface S and su must also be orthogonal to the surface S.