Calculus II Notes

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Contents

1 Integration Techniques

1.1 Fundamental Theorem of Calculus

Theorem 1.1.1* (Fundamental Theorem of Calculus)*: *Let f be a function defined on an open interval I that contains a. If f is continuous on I, then the function F defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is uniformly continuous on I, differentiable on the open interval, and

$$F'(x) = f(x)$$

for all x in the open interval.

1.2 Common Differentiation and Integration Formulae

Derivative	**Integral**
$\frac{\mathrm{d}}{\mathrm{d}x}x = 1\ , \quad \frac{\mathrm{d}}{\mathrm{d}x}c = 0$	$\int c\,\mathrm{d}x = cx + c$
$\frac{\mathrm{d}}{\mathrm{d}x}x^n$	$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}$	$\int \frac{1}{x}\,\mathrm{d}x = \ln\lvert x\rvert + c$
$\frac{\mathrm{d}}{\mathrm{d}x}e^{mx} = me^{mx}$	$\int e^{mx}\,\mathrm{d}x = \frac{1}{m}e^{mx} + c$
$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x\ln(a)$	$\int a^x\,\mathrm{d}x = \frac{1}{\ln(a)}a^x + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sin(mx) = m\cos(mx)$	$\int \cos(mx)\,\mathrm{d}x = \frac{1}{m}\sin(mx) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cos(mx) = -m\sin(mx)$	$\int \sin(mx)\,\mathrm{d}x = -\frac{1}{m}\cos(mx) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$	$\int \sec^2(x)\,\mathrm{d}x = \tan(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot(x) = -\csc^2(x)$	$\int \csc^2(x)\,\mathrm{d}x = -\cot(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec(x) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,\mathrm{d}x = \sec(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc(x) = -\csc(x)\cot(x)$	$\int \csc(x)\cot(x)\,\mathrm{d}x = -\csc(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\cos^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}(x) = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,\mathrm{d}x = -\cot^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = \sec^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = -\csc^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x} + c$

Table 1: Common Differentiation and Integration Formulae

1.3 More Formulae

1. $\int \tan(x)\,\mathrm{d}x = \ln|\sec(x)| + c$
2. $\int \csc(x)\,\mathrm{d}x = \ln|\tan\frac{x}{2}| + c$
3. $\int \sec(x)\,\mathrm{d}x = \ln|\sec(x) + \tan(x)| + c$
4. $\int \sec(x)\,\mathrm{d}x = \ln|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)|$
5. $\int \csc(x)\,\mathrm{d}x = -\ln|\csc(x) + \cot(x)| + c$
6. $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$
7. $\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln|\frac{a+x}{a-x}| + c$
8. $\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln|\frac{a-x}{a+x}| + c$
9. $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln|x + \sqrt{x^2+a^2}| + c$
10. $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}(\frac{x}{a}) + c$
11. $\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$
12. $\int e^x\left[f(x) + f'(x)\right]\mathrm{d}x = e^x f(x) + c$

1.4 Integration by Parts

***Theorem 1.4.1* (Integration by Parts):** *Let u and v be differentiable functions of x. Then,*

$$\int uv\,\mathrm{d}x = u\int v\,\mathrm{d}x - \int\left[\frac{\mathrm{d}u}{\mathrm{d}x}\int v\,\mathrm{d}x\right]\mathrm{d}x$$

Or,

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

Proof:

Let $u = u(x)$ and $w = w(x)$. Then,

$$\frac{\mathrm{d}(uw)}{\mathrm{d}x} = u\frac{\mathrm{d}w}{\mathrm{d}x} + w\frac{\mathrm{d}u}{\mathrm{d}x}$$

Integrating both sides, we get

$$\int \frac{\mathrm{d}(uw)}{\mathrm{d}x}\,\mathrm{d}x = \int u\frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x + \int w\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Or,

$$uw = \int u\frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x + \int w\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Rearranging, we get

$$\int u\frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x = uw + c - \int w\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Let $v = \frac{\mathrm{d}w}{\mathrm{d}x}$, then $w = \int v\,\mathrm{d}x$. Hence,

$$\int uv\,\mathrm{d}x = u\int v\,\mathrm{d}x - \int\left[\frac{\mathrm{d}u}{\mathrm{d}x}\int v\,\mathrm{d}x\right]\mathrm{d}x$$

$\square$

1.5 Reduction Formulas

Reduction formulas are recursive formulas that express an integral in terms of a simpler integral of the same form. They are particularly useful for integrating powers of functions, as they reduce the power step by step until reaching a base case that can be integrated directly.

1.5.1 Trigonometric Reduction Formulas

Trigonometric Reduction Formulas

Powers of Sine

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

Powers of Cosine

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Powers of Tangent

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$$

Powers of Secant

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

where $n \geq 2$.

Derivation:
Let $I_n = \int \sec^n x \, dx$. We can write

$$I_n = \int \sec^{n-2} x \sec^2 x \, dx$$

Using integration by parts, let $u = \sec^{n-2} x$ and $dv = \sec^2 x \, dx$. Then:

$$du = (n-2) \sec^{n-3} x \cdot \sec x \tan x \, dx = (n-2) \sec^{n-2} x \tan x \, dx$$

$$v = \tan x$$

Applying integration by parts:

$$\begin{aligned} I_n &= \sec^{n-2} x \tan x - \int \tan x \cdot (n-2) \sec^{n-2} x \tan x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx \end{aligned}$$

Using the identity $\tan^2 x = \sec^2 x - 1$:

$$\begin{aligned} I_n &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2} \end{aligned}$$

Solving for I_n:

$$\begin{aligned} I_n + (n-2)I_n &= \sec^{n-2} x \tan x + (n-2)I_{n-2} \\ (n-1)I_n &= \sec^{n-2} x \tan x + (n-2)I_{n-2} \\ I_n &= \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2} \end{aligned}$$

□

Products of Sine and Cosine

For $\int \sin^m x \cos^n x \, \mathrm{d}x$ where $m, n \geq 1$:

- If m is odd: Let $u = \cos x$, use $\sin^2 x = 1 - \cos^2 x$
- If n is odd: Let $u = \sin x$, use $\cos^2 x = 1 - \sin^2 x$
- If both m and n are even: Use half-angle formulas or the reduction formula:

$$\int \sin^m x \cos^n x \, \mathrm{d}x = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, \mathrm{d}x$$

1.5.2 Polynomial with Exponential or Trigonometric Functions

Polynomial with Exponential or Trigonometric Functions

Polynomial Times Exponential

$$\int x^n e^{ax} \, \mathrm{d}x = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, \mathrm{d}x$$

Polynomial Times Sine

$$\int x^n \sin(ax) \, \mathrm{d}x = -\frac{1}{a} x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) \, \mathrm{d}x$$

Polynomial Times Cosine

$$\int x^n \cos(ax) \, \mathrm{d}x = \frac{1}{a} x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) \, \mathrm{d}x$$

where $n \geq 1$ and $a \neq 0$.

Note:-

For polynomial times sine or cosine, the reduction formulas alternate between sine and cosine. Continue applying the formulas until the power of x reduces to zero.

1.5.3 Other Common Reduction Formulas

Other Common Reduction Formulas

Powers of $(x^2 + a^2)$

$$\int (x^2 + a^2)^n \, \mathrm{d}x = \frac{x(x^2 + a^2)^n}{2n+1} + \frac{2na^2}{2n+1} \int (x^2 + a^2)^{n-1} \, \mathrm{d}x$$

where $n \geq 1$.

Reciprocal Powers of $(x^2 + a^2)$

$$\int \frac{1}{(x^2+a^2)^n}\,\mathrm{d}x = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{1}{(x^2+a^2)^{n-1}}\,\mathrm{d}x$$

where $n \geq 2$ and the base case is $\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$.

1.6 Method of Substitution

A. $\int \frac{1}{(ax+b)\sqrt{cx+d}}\,\mathrm{d}x$ Let $cx + d = z^2$

B. $\int \frac{1}{\sin^m x \cos^m x}\,\mathrm{d}x$ If $m + n = p$ is even, multiply and divide by $\sec^p x$ and let $\tan x = z$.

C. $\int \frac{1}{\sin^m x + \cos^m x}\,\mathrm{d}x$ If m is even, mulpliply and divide by $\sec^m x$.

D. $\int \frac{\cos x}{a\cos x + b\sin x}\,\mathrm{d}x$ Write $nom = l \times (denom) + m \times (denom)'$, then determine l and m.

E. $\int \frac{\cos x}{a\cos x + b\sin x}\,\mathrm{d}x + c$ Write $\sin x$ and $\cos x$ as $\tan \frac{x}{2}$.

F. $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x$ Let $x = a\tan\theta$

G. $\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x$ Let $x = a\sec\theta$

H. $\int \sqrt{a^2 - x^2}\,\mathrm{d}x$ Let $x = a\sin\theta$

1.7 Trigonometric Integrals

Form	**Looks like**	**Substitution**	**Limit Assumption**
$\sqrt{b^2x^2 - a^2}$	$\sec^2\theta - 1 = \tan^2\theta$	$x = \frac{a}{b}\sec\theta$	$0 \leq \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \leq \pi$
$\sqrt{a^2 - b^2x^2}$	$1 - \sin^2\theta = \cos^2\theta$	$x = \frac{a}{b}\sin\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2 + b^2x^2}$	$\tan^2\theta + 1 = \sec^2\theta$	$x = \frac{a}{b}\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Table 2: Trigonometric Integral Substitution

1.8 Partial Fractions

Factor in denominator	**Term in partial fraction decomposition**
$ax + b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+b}{ax^2+bx+c}$
$(ax^2 + bx + c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Table 3: Partial Fraction Decomposition

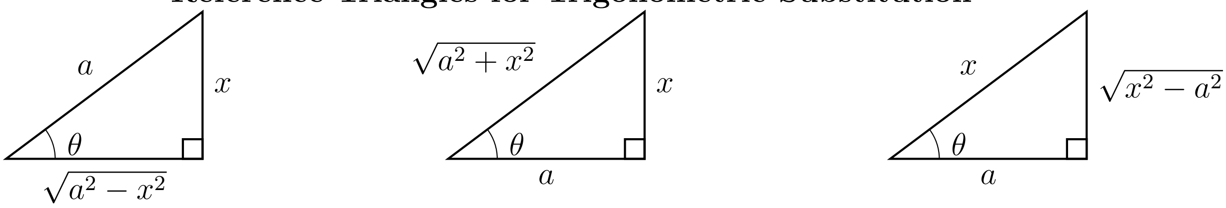
1.9 Integrals Involving Roots

For integrals involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$, use trigonometric substitution:

Expression	Substitution	Identity Used
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Table 4: Trigonometric Substitutions for Roots

Reference Triangles for Trigonometric Substitution



For $\sqrt{a^2-x^2}$: $x = a\sin\theta$ For $\sqrt{a^2+x^2}$: $x = a\tan\theta$ For $\sqrt{x^2-a^2}$: $x = a\sec\theta$

> **Example 1.1: Evaluate** $\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x$
> Here $a = 3$, so let $x = 3\sin\theta$, thus $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$.
>
> $$\begin{aligned}\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x &= \int \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\,\mathrm{d}\theta \\ &= \int \frac{3\cos\theta}{3\cos\theta}\,\mathrm{d}\theta \\ &= \int 1\,\mathrm{d}\theta \\ &= \theta + c \\ &= \sin^{-1}\left(\frac{x}{3}\right) + c\end{aligned}$$

1.10 Integrals Involving Quadratics

For integrals involving ax^2+bx+c, complete the square first:

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$$

Then use substitution $u = x + \frac{b}{2a}$ to convert to standard forms.

> **Example 1.2: Evaluate** $\int \frac{1}{x^2+4x+13}\,\mathrm{d}x$
> Complete the square:
>
> $$x^2+4x+13 = (x+2)^2+9 = (x+2)^2+3^2$$

Let $u = x + 2$, so $\mathrm{d}u = \mathrm{d}x$:

$$\begin{aligned}\int \frac{1}{x^2+4x+13}\,\mathrm{d}x &= \int \frac{1}{u^2+9}\,\mathrm{d}u \\ &= \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + c \\ &= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c\end{aligned}$$

1.11 Integration Strategy

When faced with an integral, use the following strategy:

1. **Simplify the integrand**: Expand, factor, or use algebraic manipulation
2. **Look for obvious substitutions**: If $u' = g'(x)$ appears, try $u = g(x)$
3. **Classify by type**:
    - Trigonometric integrals $\to$ use trig identities
    - Rational functions $\to$ use partial fractions
    - Products $\to$ try integration by parts
    - Roots of quadratics $\to$ complete the square or trig substitution
4. **Try different techniques**: If one method fails, try another
5. **Use tables or computer algebra systems**: For complex integrals

1.12 Approximating Definite Integrals

When an antiderivative cannot be found in closed form, use numerical approximation methods.

1.12.1 Midpoint Rule

Divide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$. Let $\bar{x}_i$ be the midpoint of the i-th subinterval.

$$\int_a^b f(x)\,\mathrm{d}x \approx M_n = \Delta x \sum_{i=1}^{n} f(\bar{x}_i)$$

1.12.2 Trapezoidal Rule

$$\int_a^b f(x)\,\mathrm{d}x \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

1.12.3 Simpson's Rule

Requires n to be even. Approximates the function with parabolas instead of lines:

$$\int_a^b f(x)\,\mathrm{d}x \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Note:-

Simpson's Rule is generally more accurate than the Trapezoidal Rule, which is more accurate than the Midpoint Rule for the same number of subintervals.

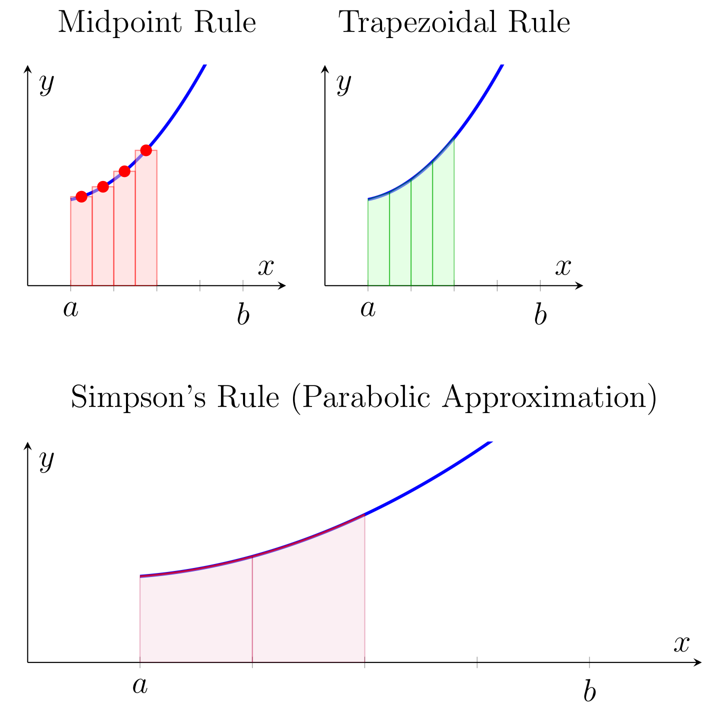


Figure 1.1: Comparison of Numerical Integration Methods for $\int_a^b f(x)\,\mathrm{d}x$

1.13 Improper Integrals

Definition 1.13.1: Improper Integrals

An integral is said to be **improper** if one of the following conditions is met:

1. The interval of integration is infinite.
2. The integrand is discontinuous at one or more points in the interval of integration.

The integral is said to **converge** if the limit of the integral exists, and **diverge** otherwise.

Type-1: If $\int_a^t f(x)\,\mathrm{d}x$ exists for all $t > a$, then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x$$

provided the limit exists and is finite.

Type-2: If $\int_t^b f(x)\,\mathrm{d}x$ exists for all $t < b$, then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\,\mathrm{d}x$$

provided the limit exists and is finite.

Type-3: If $\int_{-\infty}^{c} f(x)\,\mathrm{d}x$ and $\int_{c}^{\infty} f(x)\,\mathrm{d}x$ are both convergent, then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{c} f(x)\,\mathrm{d}x + \int_{c}^{\infty} f(x)\,\mathrm{d}x$$

Type-4: $\int_{a}^{b} f(x)\,\mathrm{d}x$ If $f(x)$ is discontinuous at $x = c$, then

$$\int_{a}^{b} f(x)\,\mathrm{d}x = \lim_{\epsilon \to 0}\left[\int_{a}^{c-\epsilon} f(x)\,\mathrm{d}x + \int_{c+\epsilon}^{b} f(x)\,\mathrm{d}x\right]$$

Note:-

If $a > 0$ then

$$\int_{a}^{\infty} \frac{1}{x^p}\,\mathrm{d}x$$

converges if $p > 1$ and diverges if $p \leq 1$.

1.13.1 Comparison Test

Theorem 1.13.2* (Comparison Theorem)*: *If $f(x) \geq g(x) \geq 0$ on the interval $[a, \infty)$, then*

1. *If $\int_{a}^{\infty} f(x)\,\mathrm{d}x$ converges, then so does $\int_{a}^{\infty} g(x)\,\mathrm{d}x$.*
2. *If $\int_{a}^{\infty} g(x)\,\mathrm{d}x$ diverges, then so does $\int_{a}^{\infty} f(x)\,\mathrm{d}x$.*

Example 1.3: Determine if the following integral is convergent or divergent:

$$\int_{2}^{\infty} \frac{\cos^2 x}{x^2}\,\mathrm{d}x$$

Notice that the numerator is bounded since

$$0 \leq \cos^2 x \leq 1$$

Hence, it's likely that the denominator will determine the convergence of the integral. Since $p = 2 > 1$,

$$\int_{2}^{\infty} \frac{1}{x^2}\,\mathrm{d}x$$

is convergent. Since

$$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

and $\int_{2}^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ is convergent, by the comparison test,

$$\int_{2}^{\infty} \frac{\cos^2 x}{x^2}\,\mathrm{d}x$$

is convergent.

Example 1.4: Determine if the following integral is convergent or divergent:

$$\int_3^\infty \frac{1}{x+e^x}\,\mathrm{d}x$$

In this case, the denominator determines the convergence of the integral. If we can find a larger function that converges, then the integral will converge. Notice that

$$\frac{1}{x+e^x} < \frac{1}{e^x} = e^{-x}$$

Also,

$$\begin{aligned}\int_3^\infty e^{-x}\,\mathrm{d}x &= \lim_{t\to\infty}\int_3^t e^{-x}\,\mathrm{d}x \\ &= \lim_{t\to\infty}\left(-e^{-t}+e^{-3}\right) \\ &= e^{-3}\end{aligned}$$

So, $\int_3^\infty e^{-x}\,\mathrm{d}x$ is convergent. Therefore, by the Comparison test,

$$\int_3^\infty \frac{1}{x+e^x}\,\mathrm{d}x$$

is also convergent.

2 Applications of Integrals

2.1 Arc Length

Consider a curve $y = f(x)$. We want to find the length of the curve from $x = a$ to $x = b$. We can approximate the curve by a series of line segments. The length of each line segment is given by the Pythagorean theorem:

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

The total length of the curve is given by the sum of the lengths of the line segments:

$$L = \int \mathrm{d}s = \int \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Similarly, for a curve $x = h(y)$, the length of the curve from $y = c$ to $y = d$ is given by:

$$L = \int \mathrm{d}s = \int \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

Arc Length Formula

$$L = \int \mathrm{d}s$$

where,

$$ds = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \quad \text{for } y = f(x), a \le x \le b$$
$$ds = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y \quad \text{for } x = h(y), c \le y \le d$$

2.2 Surface Area

Consider a curve $y = f(x)$ rotated about the x-axis. We want to find the surface area of the resulting surface. We can approximate the surface by a series of frustums. The surface area of each frustum is given by:

$$\mathrm{d}S = 2\pi y\,\mathrm{d}s = 2\pi f(x)\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

The total surface area of the surface is given by the sum of the surface areas of the frustums:

$$S = \int \mathrm{d}S = \int 2\pi y\,\mathrm{d}s = \int 2\pi f(x)\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Similarly, for a curve $x = h(y)$ rotated about the y-axis, the surface area of the resulting surface is given by:

$$A = \int 2\pi x\,\mathrm{d}s = \int 2\pi h(y)\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

Surface Area Formula

$$
\begin{aligned}
S &= \int \mathrm{d}S \\
&= \int 2\pi y \,\mathrm{d}s \quad \text{rotation about } x\text{–axis} \\
&= \int 2\pi x \,\mathrm{d}s \quad \text{rotation about } y\text{–axis}
\end{aligned}
$$

where,

$$
\begin{aligned}
dS &= 2\pi y \,\mathrm{d}s = 2\pi f(x)\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \quad \text{for } y = f(x), a \le x \le b \\
dS &= 2\pi x \,\mathrm{d}s = 2\pi h(y)\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y \quad \text{for } x = h(y), c \le y \le d
\end{aligned}
$$

2.3 Center of Mass

Suppose we want to find the center of mass of a region bounded by two curves $f(x)$ and $g(x)$ on the interval $[a, b]$.
The mass is

$$M = \rho \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x$$

Next, we need the **moments** of the region. There are two moments:

$$
\begin{aligned}
M_x &= \rho \int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right]\mathrm{d}x \\
M_y &= \rho \int_a^b x\left[f(x) - g(x)\right]\mathrm{d}x
\end{aligned}
$$

The coordinates of the center of mass, $(\bar{x}, \bar{y})$, are given by:

Center of Mass Formula

$$
\begin{aligned}
\bar{x} &= \frac{M_y}{M} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]\mathrm{d}x \\
\bar{y} &= \frac{M_x}{M} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right]\mathrm{d}x
\end{aligned}
$$

where,

$$A = \int_a^b \left[f(x) - g(x)\right]\mathrm{d}x$$

2.4 Probability

Every continuous random variable X, has a probability density function $f(x)$. Probability density functions satisfy the following conditions:

1. $f(x) \ge 0$ for all x.
2. $\displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$

Probability density finctions can be used to determine the probability that a continuous random variable lies between two values, say a and b. This probability is denoted by $P(a \leq X \leq b)$.

Note:-

$$P(a \leq X \leq b) = \int_a^b f(x)\,\mathrm{d}x$$

Probability density functions can also be used to determine the mean of a continuous random variable. The mean is given by:

Mean of a Continuous Random Variable

$$\mu = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$$

3 Parametric Equations and Polar Coordinates

There are a great many curves out there that cannot be expressed in a single equation in terms of only x and y. To deal with such problems, we introduce **parametric equations**. Instead of defining y in terms of $x(y = f(x))$ or x in terms of $y(x = h(y))$, we define both x and y in terms of a third variable called a parameter as follows:

$$x = f(t) \quad y = g(t)$$

This third variable is usually denoted by t. Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called a **parametric curve**.

3.1 Parametric Equations and Curves

Unlike graphs of functions $y = f(x)$, parametric curves can trace out shapes that fail the vertical line test. For instance, a circle cannot be written as a single function $y = f(x)$, but can be easily expressed parametrically.

Example 3.1: Sketch the parametric curve $x = t^2$, $y = 2t - 1$ for $-2 \leq t \leq 2$.

We can create a table of values:

t	$x = t^2$	$y = 2t - 1$
-2	4	-5
-1	1	-3
0	0	-1
1	1	1
2	4	3

Plotting these points (x, y) and connecting them traces out a parabola opening to the right.
To eliminate the parameter, note that $x = t^2$, so $t = \pm\sqrt{x}$. Then:

$$y = 2t - 1 = \pm 2\sqrt{x} - 1$$

Since the parameter t ranges from -2 to 2, we get both branches. However, note that $x \geq 0$ always (since $x = t^2$).

Example 3.2: Sketch the parametric curve $x = \cos(t)$, $y = \sin(t)$ for $0 \leq t \leq 2\pi$.

This is the unit circle! At $t = 0$, we're at $(1, 0)$. At $t = \pi/2$, we're at $(0, 1)$. At $t = \pi$, we're at $(-1, 0)$. At $t = 3\pi/2$, we're at $(0, -1)$, and at $t = 2\pi$ we return to $(1, 0)$. To eliminate the parameter, use the Pythagorean identity:

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$$

So the curve traces the circle $x^2 + y^2 = 1$, starting at $(1, 0)$ and moving counterclockwise.

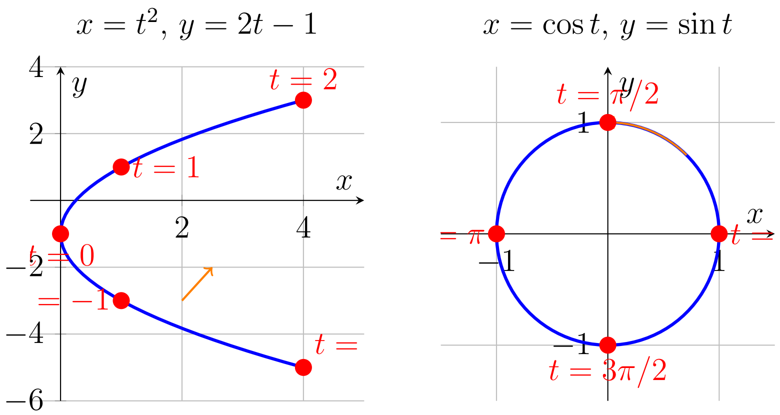


Figure 3.1: Parametric Curves with Direction of Motion

3.2 Tangents with Parametric Equations

Derivative for Parametric Equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \quad \text{provided } \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

Tangents for Parametric Equations

Horizontal Tangent:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0, \quad \text{provided } \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

Vertical Tangent:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0, \quad \text{provided } \frac{\mathrm{d}y}{\mathrm{d}t} \neq 0$$

Second Derivative for Parametric Equations

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

3.3 Area with Parametric Equations

Area with Parametric Equations

For the area between the parametric curve and the x-axis:

$$A = \int_{t_1}^{t_2} y(t)\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

Use this formula when the curve is traced vertically (from bottom to top or top to bottom).
For the area between the parametric curve and the y-axis:

$$A = \int_{t_1}^{t_2} x(t)\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t$$

Use this formula when the curve is traced horizontally (from left to right or right to left).

3.4 Arc Length with Parametric Equations

The arc length of a curve is given by

$$L = \int \mathrm{d}s$$

where,

$$\begin{aligned} \mathrm{d}s &= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \qquad \text{if } y = f(x), a \le x \le b \\ \mathrm{d}s &= \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y \qquad \text{if } x = h(y), c \le y \le d \end{aligned}$$

Using the first $\mathrm{d}s$, we can write

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

Then the arc length formula becomes,

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}\right)^2}\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_\alpha^\beta \frac{1}{\left|\frac{\mathrm{d}x}{\mathrm{d}t}\right|}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \end{aligned}$$

Arc Length with Parametric Equations

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

3.5 Surface Area with Parametric Equations

Surface Area with Parametric Equations

$$\begin{aligned} S &= \int 2\pi y\,\mathrm{d}s \qquad \text{rotation about } x\text{–axis} \\ S &= \int 2\pi x\,\mathrm{d}s \qquad \text{rotation about } y\text{–axis} \end{aligned}$$

where,

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

3.6 Polar Coordinates

In polar coordinates, each point is determined by a distance r from the origin and an angle θ measured counterclockwise from the positive x-axis. This provides an alternative way to describe points in the plane.

Polar to Cartesian Conversion

To convert from polar coordinates (r, θ) to Cartesian coordinates (x, y):

$$x = r\cos\theta \quad y = r\sin\theta$$

Cartesian to Polar Conversion

To convert from Cartesian coordinates (x, y) to polar coordinates (r, θ):

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Note: The formula for θ requires care with quadrants. The functions $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ are the fundamental relationships.

3.6.1 Common Polar Curves

Below are some classic curves that are most naturally expressed in polar coordinates:

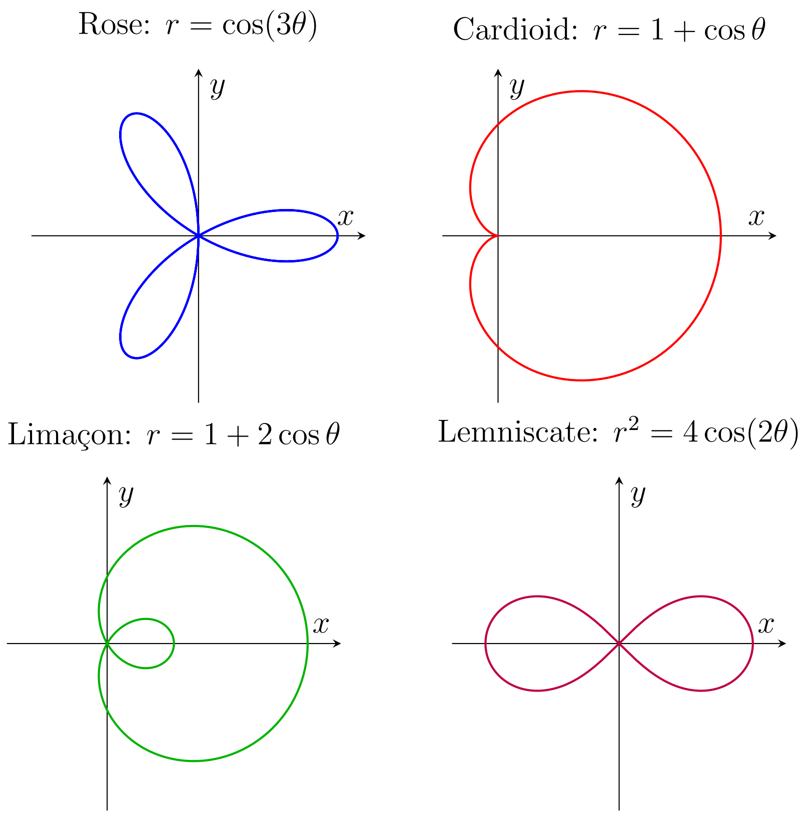


Figure 3.2: Common Polar Curves

3.7 Tangents with Polar Coordinates

To find $\frac{dy}{dx}$ for a polar curve $r = f(\theta)$, we use the parametric derivatives. Since $x = r\cos\theta$ and $y = r\sin\theta$, applying the product rule gives:

$$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta \quad \frac{dy}{d\theta} = r'\sin\theta + r\cos\theta$$

Tangents with Polar Coordinates

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{r' \sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

where $r' = \frac{\mathrm{d}r}{\mathrm{d}\theta}$.

3.8 Area with Polar Coordinates

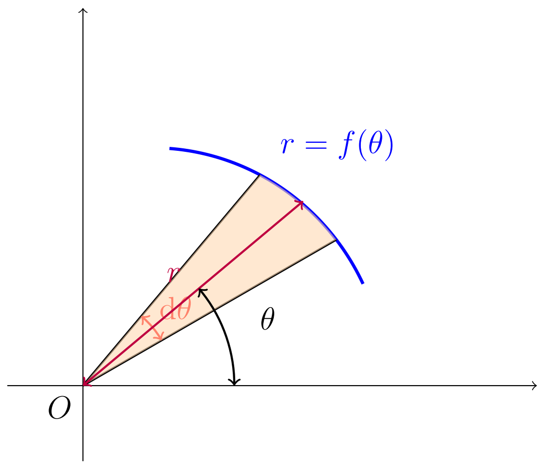
Area with Polar Coordinates

For the area between two polar curves $r = r_o(\theta)$ (outer radius) and $r = r_i(\theta)$ (inner radius):

$$A = \frac{1}{2}\int_\alpha^\beta \left(r_o^2 - r_i^2\right) \mathrm{d}\theta$$

For the area enclosed by a single polar curve $r = f(\theta)$, set $r_i = 0$:

$$A = \frac{1}{2}\int_\alpha^\beta r^2 \,\mathrm{d}\theta$$



Area element: $\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta$

Figure 3.3: Polar Area Element: A small sector with angle $\mathrm{d}\theta$ and radius r

3.9 Arc Length with Polar Coordinates

Let

$$r = f(\theta) \quad \alpha \leq \theta \leq \beta$$

$$\begin{aligned} x &= r\cos(\theta) \\ x &= f(\theta)\cos\theta \\ \frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta \end{aligned} \qquad \begin{aligned} y &= r\sin(\theta) \\ y &= f(\theta)\sin\theta \\ \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta \end{aligned}$$

Now,

$$
\begin{aligned}
\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta\right)^2 \\
&= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \cos^2\theta - 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\
&+ \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \sin^2\theta + 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\
&= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2
\end{aligned}
$$

Arc Length with Polar Coordinates

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

3.10 Surface Area with Polar Coordinates

Let

$$r = f(\theta) \quad \alpha \le \theta \le \beta$$

$$
\begin{aligned}
x &= r\cos(\theta) & y &= r\sin(\theta) \\
x &= f(\theta)\cos\theta & y &= f(\theta)\sin\theta
\end{aligned}
$$

Surface Area with Polar Coordinates

$$
\begin{aligned}
S &= \int 2\pi y\,\mathrm{d}s \qquad \text{rotation about } x\text{–axis} \\
S &= \int 2\pi x\,\mathrm{d}s \qquad \text{rotation about } y\text{–axis}
\end{aligned}
$$

where,

$$\mathrm{d}s = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \qquad r = f(\theta), \alpha \le \theta \le \beta$$

4 Sequences

4.1 Definition

Definition 4.1.1: Sequence

A **sequence** is a function whose domain is the set of natural numbers $\mathbb{N}$. The sequence is denoted by $\{a_n\}$ and the value of the function at n (the n-th term) is denoted by a_n. Various ways of representing a sequence are:

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

4.2 Precise Definition of Limit of a Sequence

Precise Definition of Limit

1. We say that
$$\lim_{n\to\infty} a_n = L$$
if
$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - L| < \epsilon$$
2. We say that
$$\lim_{n\to\infty} a_n = \infty$$
if
$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n > M$$
3. We say that
$$\lim_{n\to\infty} a_n = -\infty$$
if
$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n < M$$

4.3 Convergence of Sequences

***Theorem 4.3.1* (Convergence of Sequences):** *A sequence $\{a_n\}$ is said to be **convergent** if there exists a real number L such that*

$$\lim_{n\to\infty} a_n = L$$

***Theorem 4.3.2* (Uniqueness of Limits):** *If a sequence $\{a_n\}$ converges, then its limit is unique.*

Theorem 4.3.3: *Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.*

Properties of Convergent Sequences

If $\{a_n\}$ and $\{b_n\}$ are both convergent sequences, then:

1.
$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

2.
$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$

3.
$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$

4.
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \qquad \text{provided } \lim_{n\to\infty} b_n \neq 0$$

5.
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p, \qquad \text{provided } a_n \geq 0$$

Theorem 4.3.4* (Squeeze Theorem for Sequences)*: *If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences such that $a_n \leq b_n \leq c_n$ for all $n \geq N$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$ then $\lim\limits_{n\to\infty} b_n = L$.*

Theorem 4.3.5:
$$\text{If } \lim_{n\to\infty} |a_n| = 0, \text{ then } \lim_{n\to\infty} a_n = 0$$

Proof:
The main thing to this proof is to note that,
$$-|a_n| \leq a_n \leq |a_n|$$
Then note that,
$$\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = 0$$
We then have that,
$$0 \leq \lim_{n\to\infty} a_n \leq 0$$
and so by the Squeeze Theorem,
$$\lim_{n\to\infty} a_n = 0$$

Theorem 4.3.6* (Convergent Sequences are Bounded)*: *If a sequence $\{a_n\}$ is convergent, then it is bounded.*

Example 4.1: Determine if the following sequences converge or diverge:

1 $\left\{\dfrac{n^2}{2n^2+1}\right\}$

2 $\left\{\dfrac{(-1)^n}{n}\right\}$

3 $\left\{\dfrac{n!}{n^n}\right\}$

1. We can use the theorem about converting sequences to functions. Let $f(x) = \dfrac{x^2}{2x^2+1}$. Then
$$\lim_{x\to\infty} \frac{x^2}{2x^2+1} = \lim_{x\to\infty} \frac{1}{2+1/x^2} = \frac{1}{2}$$
Therefore, $\displaystyle\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$ and the sequence converges.

2. Note that $\left|\dfrac{(-1)^n}{n}\right| = \dfrac{1}{n} \to 0$ as $n \to \infty$. By the theorem about absolute values, $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ and the sequence converges.

3. For large n, $n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$ grows much slower than $n^n = n\cdot n\cdot n\cdots n\cdot n$. We have
$$0 < \frac{n!}{n^n} = \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{2}{n}\cdot\frac{1}{n} \le \frac{n}{n}\cdot\frac{n}{n}\cdot\frac{1}{n} = \frac{1}{n}$$
Since $\displaystyle\lim_{n\to\infty} 0 = 0$ and $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$, by the Squeeze Theorem, $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0$ and the sequence converges.

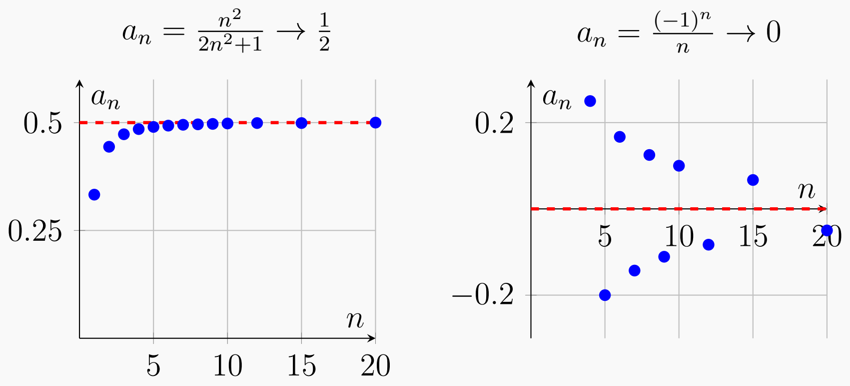


Figure 4.1: Sequence Convergence: Terms approach their limits as $n \to \infty$

Theorem 4.3.7: *For the sequence $\{a_n\}$ if both $\displaystyle\lim_{n\to\infty} a_{2n} = L$ and $\displaystyle\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\displaystyle\lim_{n\to\infty} a_n = L$.*

Theorem 4.3.8: *The sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \le 1$ and diverges for all other values of r. Also,*
$$\lim_{n\to\infty} r^n \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \end{cases}$$

Theorem 4.3.9: *For the sequence $\{a_n\}$ if both $\displaystyle\lim_{n\to\infty} a_{2n} = L$ and $\displaystyle\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\displaystyle\lim_{n\to\infty} a_n = L$.*

Proof:

Let $\epsilon > 0$. Then since $\lim_{n\to\infty} a_{2n} = L$,

$$\exists N_1 \in \mathbb{N} : \forall n \geq N_1, |a_{2n} - L| < \epsilon$$

Similarly, because $\lim_{n\to\infty} a_{2n+1} = L$,

$$\exists N_2 \in \mathbb{N} : \forall n \geq N_2, |a_{2n+1} - L| < \epsilon$$

Now, let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. Then either $a_n = a_{2k}$ for some $k > N_1$ or $a_n = a_{2k+1}$ for some $k > N_2$, And so in either case we have that

$$|a_n - L| < \epsilon$$

Therefore, $\lim_{n\to\infty} a_n = L$ and so $\{a_n\}$ is convergent.

4.4 Bounded and Monotonic Sequences

Definition 4.4.1: Bounded Sequence

A sequence $\{a_n\}$ is **bounded** if

$$\exists M \in \mathbb{R} : \forall n \in \mathbb{N}, |a_n| \leq M$$

Upper and Lower Bounds

If

$$\exists m \in \mathbb{R} : \forall n \in \mathbb{N}, m \leq a_n$$

the sequence $\{a_n\}$ is said to be **bounded below** and m is a **lower bound** of the sequence. Similarly, if

$$\exists M \in \mathbb{R} : \forall n \in \mathbb{N}, M \geq a_n$$

the sequence $\{a_n\}$ is said to be **bounded above** and M is an **upper bound** of the sequence.

Theorem 4.4.2* (Bounded Sequence)*: *If a sequence $\{a_n\}$ is both bounded above and below, then it is bounded. That is, if*

$$\exists m, M \in \mathbb{R} : \forall n \in \mathbb{N}, m \leq a_n \leq M$$

then $\{a_n\}$ is bounded.

Example 4.2: Determine if the following sequences are bounded:

1 $\left\{\dfrac{\sin(n)}{n}\right\}$

2 $\{(-1)^n n\}$

3 $\left\{\dfrac{2n}{n+1}\right\}$

1. Since $-1 \leq \sin(n) \leq 1$ for all n, and $n > 0$ for all $n \in \mathbb{N}$, we have

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

For $n \geq 1$, this gives us $-1 \leq \frac{\sin(n)}{n} \leq 1$. Therefore, the sequence is bounded with $m = -1$ and $M = 1$.

2. The terms of this sequence alternate: $-1, 2, -3, 4, -5, 6, \ldots$. As n increases, $|(-1)^n n| = n$ grows without bound. Therefore, the sequence is unbounded.

3. We can rewrite $\dfrac{2n}{n+1} = \dfrac{2n+2-2}{n+1} = \dfrac{2(n+1)-2}{n+1} = 2 - \dfrac{2}{n+1}$. Since $n \geq 1$, we have $\dfrac{2}{n+1} \leq \dfrac{2}{2} = 1$, so $a_n \geq 2 - 1 = 1$. Also, since $\dfrac{2}{n+1} > 0$, we have $a_n < 2$. Therefore, $1 \leq a_n < 2$ for all n, and the sequence is bounded.

Definition 4.4.3: Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if for all $n \in \mathbb{N}$

$$a_{n+1} \geq a_n \quad \text{or} \quad a_{n+1} \leq a_n$$

Theorem 4.4.4: *If a sequence $\{a_n\}$ is both bounded and monotonic, then it is convergent.*

Example 4.3: Determine if the following sequences are monotonic:

1 $\left\{\dfrac{n}{n+1}\right\}$

2 $\left\{\dfrac{n+3}{n^2}\right\}$

1. Consider $a_n = \dfrac{n}{n+1}$. To check if the sequence is monotonic, we can check if $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$. We have

$$a_{n+1} = \frac{n+1}{n+2} \qquad \text{and} \qquad a_n = \frac{n}{n+1}$$

Comparing:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{n^2+2n+1-n^2-2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0$$

Since $a_{n+1} - a_n > 0$, we have $a_{n+1} > a_n$ for all n, so the sequence is monotonically increasing.

2. Consider $a_n = \dfrac{n+3}{n^2}$. Let's check the first few terms: $a_1 = 4$, $a_2 = \frac{5}{4}$, $a_3 = \frac{6}{9} = \frac{2}{3}$, $a_4 = \frac{7}{16}$. The sequence appears to be decreasing, but let's verify. We have

$$a_{n+1} - a_n = \frac{n+4}{(n+1)^2} - \frac{n+3}{n^2} = \frac{n^2(n+4) - (n+3)(n+1)^2}{n^2(n+1)^2}$$

Expanding the numerator:

$$n^3+4n^2-(n+3)(n^2+2n+1) = n^3+4n^2-n^3-2n^2-n-3n^2-6n-3 = -n^2-7n-3 < 0$$

Since $a_{n+1} - a_n < 0$, we have $a_{n+1} < a_n$ for all n, so the sequence is monotonically decreasing.

5 Series

Definition 5.0.1: Series

A **series** is the sum of the terms of a sequence. Given a sequence $\{a_n\}$, the series is denoted by

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

The **n-th partial sum** of the series is defined as

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

5.1 Convergence of Series

Theorem 5.1.1* (Convergence of Series)*: *The series $\sum a_n$ converges if and only if the sequence of partial sums $\{s_n\}$ is convergent. That is,*

$$\sum a_n \text{ converges} \iff \lim_{n\to\infty} s_n \text{ exists}$$

If the series $\sum a_n$ converges, then

$$\lim_{n\to\infty} s_n = s$$

where s is the sum of the series.

Example 5.1:

$$\begin{aligned}
&\lim_{n\to\infty} n = \infty && \text{(diverges)}\\
&\lim_{n\to\infty} \frac{1}{n^2-1} = 0 && \text{(converges)}\\
&\lim_{n\to\infty} (-1)^n \text{ does not exist} && \text{(diverges)}\\
&\lim_{n\to\infty} \frac{1}{3^{n-1}} = 0 && \text{(converges)}
\end{aligned}$$

Theorem 5.1.2: *If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Properties of Convergent Series

If $\sum a_n$ and $\sum b_n$ are both convergent series, then:

1. $\sum ca_n$, where c is a constant, is also convergent and

$$\sum ca_n = c \sum a_n$$

2. $\sum (a_n \pm b_n)$ is also convergent and

$$\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$$

5.2 Divergence Test

***Theorem 5.2.1* (Divergence Test):** *If* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum a_n$ *diverges.*

Note:-

The Divergence Test is the contrapositive of the previous theorem. This is often the first test to try when testing a series for convergence, as it is quick to apply. However, if $\lim_{n\to\infty} a_n = 0$, the test is **inconclusive** and we must use other tests.

5.3 Special Series

5.3.1 Geometric Series

Definition 5.3.1: Geometric Series

A series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

is called a **geometric series**. The sum of the series is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r| < 1$$

5.3.2 Telescoping Series

Definition 5.3.2: Telescoping Series

A series of the form

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - a_2 + a_2 - a_3 + \cdots$$

is called a **telescoping series**. The sum of the series is

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_{n+1}$$

Example 5.2: Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution: First, we use partial fraction decomposition:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

Multiplying both sides by $n(n+1)$ gives $1 = A(n+1) + Bn$. Setting $n = 0$ gives $A = 1$, and setting $n = -1$ gives $B = -1$. Thus:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Now we can write the partial sum:

$$\begin{aligned} s_N &= \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1} \right) \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

Taking the limit as $N \to \infty$:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1} \right) = 1$$

5.3.3 Harmonic Series

Definition 5.3.3: Harmonic Series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is called the **harmonic series**. The harmonic series diverges.

5.4 Integral Test

***Theorem 5.4.1* (Integral Test):** *Let $f(x)$ be a continuous, positive, and decreasing function for $x \geq 1$. Then the series*

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x)\,dx$$

converges. That is,

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

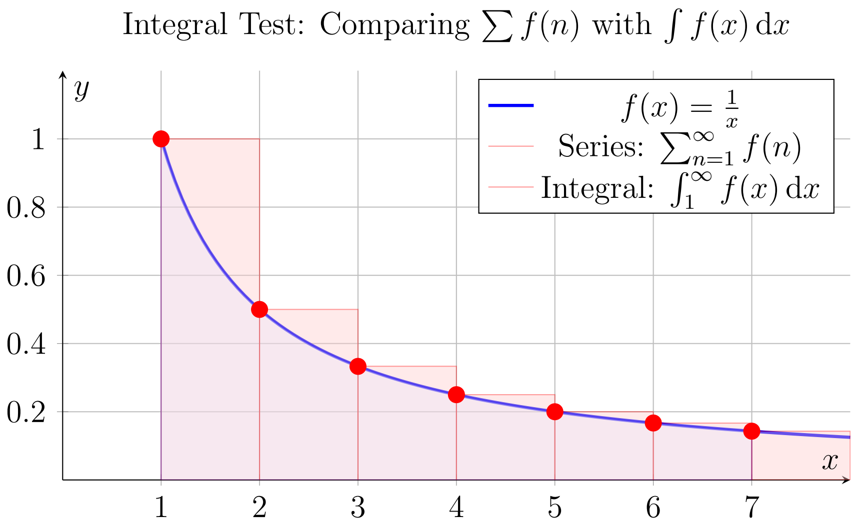


Figure 5.1: The series $\sum f(n)$ (rectangles) compared with $\int f(x)\,\mathrm{d}x$ (shaded area)

Proof:
Let $s_n = f(1) + f(2) + \cdots + f(n)$ and $s_{n+1} = f(1) + f(2) + \cdots + f(n) + f(n+1)$. Then

$$s_{n+1} - s_n = f(n+1) \geq 0$$

and

$$\int_n^{n+1} f(x)\,dx \leq f(n) \leq \int_{n-1}^{n} f(x)\,dx$$

Summing from 1 to n gives

$$s_{n+1} - s_1 \leq \int_1^n f(x)\,dx \leq s_n$$

Taking the limit as $n \to \infty$ gives

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1} = s$$

which implies that the series converges if and only if the integral converges. □

5.4.1 The p-series Test

Theorem 5.4.2* (*p*-series Test)*: *The series*

$$\sum_{n=k}^{\infty} \frac{1}{n^p} \qquad \text{where } k \in \mathbb{N}$$

converges if $p > 1$ *and diverges if* $p \leq 1$.

Proof:
If $p > 1$, then the integral

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^b = \frac{1}{p-1}$$

converges. If $p \leq 1$, then the integral diverges. □

5.5 Comparison Test/Limit Comparison Test

Theorem 5.5.1* (Comparison Test)*: *Let $\sum a_n$ and $\sum b_n$ be two series with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then:*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*
2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Proof:
Let the partial sums be

$$s_n = \sum_{i=1}^{n} a_i \quad \text{and} \quad t_n = \sum_{i=1}^{n} b_i$$

Since $a_n, b_n \geq 0$, we know that

$$\begin{aligned} s_n \leq s_n + a_{n+1} &= \sum_{i=1}^{n} a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = s_{n+1} && \implies s_n \leq s_{n+1} \\ t_n \leq t_n + b_{n+1} &= \sum_{i=1}^{n} b_i + b_{n+1} = \sum_{i=1}^{n+1} b_i = t_{n+1} && \implies t_n \leq t_{n+1} \end{aligned}$$

Also, since $a_n \leq b_n$, we have

$$s_n = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i = t_n$$

If $\sum b_n$ converges, then $\{t_n\}$ is bounded and increasing, so it converges. Since $s_n \leq t_n$, $\{s_n\}$ is also bounded and increasing, so it converges. □

Theorem 5.5.2* (Limit Comparison Test)*: *Let $\sum a_n$ and $\sum b_n$ be two series with $a_n, b_n > 0$ for all n. If*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where $c \in \mathbb{R}_+$ and $c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Proof:
Since $0 < c < \infty$, there exists two positive finite numbers m and M such that $m < c < M$. Now we have

$$m < \frac{a_n}{b_n} < M$$
$$mb_n < a_n < Mb_n$$

Now, if $\sum b_n$ converges, then so does $\sum Mb_n$, and since $a_n < Mb_n$, for all n by the Comparison Test, $\sum a_n$ also converges.
Similarly, if $\sum b_n$ diverges, then so does $\sum mb_n$, and since $mb_n < a_n$, for all n by the Comparison Test, $\sum a_n$ also diverges. □

5.6 Alternating Series Test

Theorem 5.6.1* (Alternating Series Test)*: *Let $\sum a_n$ be an alternating series, that is, $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n > 0$ for all n. Then if:*

1. $\lim\limits_{n \to \infty} b_n = 0$
2. $\forall n, b_{n+1} \leq b_n$

the series $\sum a_n$ is convergent.

Proof:
Let $s_n = \sum_{i=1}^{n} a_i$. Since b_n is decreasing and $\lim_{n\to\infty} b_n = 0$, we can say

$$\forall n, b_n - b_{n+1} \geq 0$$

Now, we have

$$\begin{aligned} s_{2n} &= b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots + b_{2n-1} - b_{2n} \\ &= b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \end{aligned}$$

Since b_n is decreasing, s_{2n} is increasing and bounded above by b_1.
Let's assume that its limit is s, that is

$$\lim_{n\to\infty} s_{2n} = s$$

Then

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} (s_{2n} + b_{2n+1}) = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} = s + 0 = s$$

So, we know that both $\{s_{2n}\}$ and $\{s_{2n+1}\}$ are convergent sequences and they both have the same limit.
We also know that $\{s_n\}$ is a convergent sequence with a limit of s. This in turn implies that the series $\sum a_n$ is convergent □

5.7 Absolute and Conditional Convergence

Definition 5.7.1: Absolute and Conditional Convergence

A series $\sum a_n$ is said to be **absolutely convergent** if $\sum |a_n|$ converges. A series that converges but not absolutely is said to be **conditionally convergent**.

Theorem 5.7.2: *If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.*

Note:-
The converse is not true: a series can be convergent without being absolutely convergent. Such a series is called conditionally convergent.

Theorem 5.7.3* (Riemann Rearrangement Theorem)*: *Given the series $\sum a_n$:*

1. *If $\sum a_n$ is absolutely convergent with sum s, then any rearrangement of $\sum a_n$ also converges to s.*
2. *If $\sum a_n$ is conditionally convergent, then for any real number r (including $\pm\infty$), there exists a rearrangement of $\sum a_n$ that converges to r or diverges to $\pm\infty$.*

Note:-
This remarkable theorem shows that conditionally convergent series are very sensitive to the order of their terms. Rearranging the terms can produce any desired sum or even cause divergence!

5.8 Ratio Test

***Theorem 5.8.1* (Ratio Test):** *Let $\sum a_n$ be a series and let*

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then:

1. *If $L < 1$, then $\sum a_n$ is absolutely convergent.*
2. *If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.*
3. *If $L = 1$, then the test is inconclusive.*

Proof:

Let $L < 1$ and $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then there exists a number r such that $L < r < 1$. Since $L < r$, there exists a number N such that for all $n > N$,

$$\left| \frac{a_{n+1}}{a_n} \right| < r$$
$$|a_{n+1}| < r\,|a_n|$$
$$|a_{n+1}| < r\,|a_n| < r^2\,|a_{n-1}| < \cdots < r^{n-N}\,|a_N|$$
$$|a_{n+1}| < r^{n-N}\,|a_N|$$

Since $r < 1$, the series $\sum r^{n-N}\,|a_N|$ converges by the geometric series test. By the Comparison Test, $\sum a_n$ also converges. $\square$

5.9 Root Test

***Theorem 5.9.1* (Root Test):** *Let $\sum a_n$ be a series and let*

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then:

1. *If $L < 1$, then $\sum a_n$ is absolutely convergent.*
2. *If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.*
3. *If $L = 1$, then the test is inconclusive.*

Note:-

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Proof:

Let $L < 1$ and $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$. Then there exists a number r such that $L < r < 1$. Since $L < r$, there exists a number N such that for all $n > N$,

$$\sqrt[n]{|a_n|} < r$$
$$|a_n| < r^n$$

Since $r < 1$, the series $\sum r^n$ converges by the geometric series test. By the Comparison Test, $\sum a_n$ also converges. $\square$

5.10 Strategies for Series Test

Divergence Test If $\lim_{n\to\infty} a_n \neq 0$.

Geometric Series Test If $a_n = ar^n$ or $a_n = ar^{n-1}$

Integral Test If $a_n = f(n)$ and $f(x)$ is continuous, positive, and decreasing

p-series Test If $a_n = \frac{1}{n^p}$

Comparison Test If a_n is hard to work with, but b_n is easy to work with

Limit Comparison Test If a_n is a rational expression involving only polynomials.

Alternating Series Test If $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$

Ratio Test If a_n is a product of terms

Root Test If a_n is a power of terms

5.11 Estimating the Value of a Series

5.11.1 Integral Test

Integral Test

$$s_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq s \leq s_n + \int_{n}^{\infty} f(x)\,\mathrm{d}x$$

5.11.2 Comparison Test

Given a series $\sum a_n$, let's assume that we've used the comparison test to show that it's convergent. Therefore we found a second series $\sum b_n$ that converged and $a_n \leq b_n$ for all n.
Now, let

$$R_n = \sum_{k=n+1}^{\infty} a_k \quad \text{and} \quad T_n = \sum_{k=n+1}^{\infty} b_k$$

Since, $a_n \leq b_n$ we also know that $R_n \leq T_n$.

Comparison Test

$$R_n \leq T_n \leq \int_{n}^{\infty} g(x)\,\mathrm{d}x\,, \qquad \text{where } g(n) = b_n$$

5.11.3 Alternating Series Test

Alternating Series Test

$$|R_n| = |s - s_n| \leq b_{n+1}$$

5.11.4 Ratio Test

Ratio Test

To get an estimate of the remainder, let's first define the following sequence,

$$r_n = \frac{a_{n+1}}{a_n}$$

We now have two possible cases:

1. If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, then

$$R_n \leq \frac{a_{n+1}}{1 - r_{n+1}}$$

2. If $\{r_n\}$ is an increasing sequence, then

$$R_n \leq \frac{a_{n+1}}{1 - L}, \qquad \text{where } L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

5.12 Power Series

Definition 5.12.1: Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where c_n are constants and called coefficients and a is a fixed number. The number a is called the **center** of the power series.

Theorem 5.12.2* (Convergence of Power Series)*: *Given the power series $\sum c_n(x-a)^n$, there exists a number R called the* ***radius of convergence*** *such that:*

1. *The series converges absolutely for all x with $|x-a| < R$*
2. *The series diverges for all x with $|x-a| > R$*

To find R, apply the ***Ratio Test*** *or* ***Root Test****:*

- ***Ratio Test:*** *Compute $L = \lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right|$. The series converges when $L < 1$, which gives $|x-a| < R$.*
- ***Root Test:*** *Compute $L = \lim_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|}$. The series converges when $L < 1$, which gives $|x-a| < R$.*

N.B: *the series always converges at $x = a$.*

Interval vs Radius of Convergence

The **radius of convergence** R is a non-negative number (or ∞) that tells us how far from the center a the series converges. The **interval of convergence** is the actual set of x-values for which the series converges.

Given radius R and center a:

- The series converges absolutely on the open interval $(a - R, a + R)$
- The series diverges outside the closed interval $[a - R, a + R]$
- At the endpoints $x = a - R$ and $x = a + R$, convergence must be tested separately using other tests (p-series, alternating series, etc.)

- The interval of convergence is one of: $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$

5.13 Power Series and Functions

Theorem 5.13.1 **(Power Series and Functions)*:*** *Let $f(x) = \sum c_n(x-a)^n$ be a power series with radius of convergence R. Then:*

1. *$f(x)$ is continuous and differentiable on the interval $(a-R, a+R)$*
2. $f'(x) = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
3. $\displaystyle\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}$
4. *The radius of convergence of $f'(x)$ and $\int f(x)\,\mathrm{d}x$ are also R.*

5.14 Properties of Power Series

Theorem 5.14.1: *Suppose $f(x) = \lim\limits_{n=k\to\infty} a_n x^n$ and $g(x) = \lim\limits_{n=k\to\infty} b_n x^n$ on an interval I and fix some $c \neq 0$. Then:*

1. ***Addition/Subtraction:***

$$f(x) \pm g(x) = \lim_{n=k\to\infty} (a_n \pm b_n)x^n \text{ for } x \in I$$

2. ***Multiplication:*** *If $k + m \geq 0$ then*

$$x^m f(x) = \lim_{n=k\to\infty} a_n x^{n+m} \quad \forall x \in I, x \neq 0$$

with

$$\lim_{n=k\to\infty} a_n x^{n+m} = \lim_{x\to 0} x^m f(x) \text{ when } x = 0.$$

3. ***Composition:***

$$f(cx^m) = \lim_{n=k\to\infty} a_n (cx^m)^n \text{ for } cx^m \in I$$

Theorem 5.14.2 **(Term-by-Term Differentiation and Integration)*:*** *Let $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series with radius of convergence $R > 0$. Then F is differentiable on $(c-R, c+R)$ and:*

1. ***Differentiation:***

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \text{ for } |x-a| < R$$

2. ***Integration:***

$$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} \text{ for } |x-a| < R$$

Theorem 5.14.3* (Abel's Theorem)*: *If a power series $F(x) = \lim_{n=0\to\infty} a_n(x-c)^n$ has a radius of convergence $R \in (0,\infty)$ and F converges at an endpoint, then F is continuous at that endpoint.*

5.15 Taylor Series

Definition 5.15.1: Taylor Series

The **Taylor series** of a function $f(x)$ about $x = a$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $f^{(n)}(a)$ is the nth derivative of $f(x)$ evaluated at $x = a$.

To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the **n-th degree Taylor polynomial** of $f(x)$ as

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Notice that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the n-th degree Taylor polynomial is just the partial sum for the series.
The **remainder** is defined to be

$$R_n(x) = f(x) - T_n(x)$$

So, the remainder is just the *error* between the function $f(x)$ and the n-th degree Taylor polynomial $T_n(x)$ for a given n.
With this definition, we can then write the function as

$$f(x) = T_n(x) + R_n(x)$$

Theorem 5.15.2: *Suppose that $f(x) = T_n(x) + R_n(x)$. Then if*

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

on $|x-a| < R$.

***Theorem 5.15.3* (Taylor's Remainder Theorem):** *If $f(x)$ has $n+1$ continuous derivatives on the interval containing a and x, then there exists a number c between a and x such that*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

*This is called the **Lagrange form** of the remainder.*

***Theorem 5.15.4* (Lagrange Error Bound):** *If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then for $|x-a| \leq d$,*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \leq \frac{M}{(n+1)!}d^{n+1}$$

5.15.1 Maclaurin Series

Definition 5.15.5: Maclaurin Series

The **Maclaurin series** of a function $f(x)$ is the Taylor series of $f(x)$ about $x = 0$. That is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Common Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for all } x$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{for all } x$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{for all } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for } -1 \leq x \leq 1$$

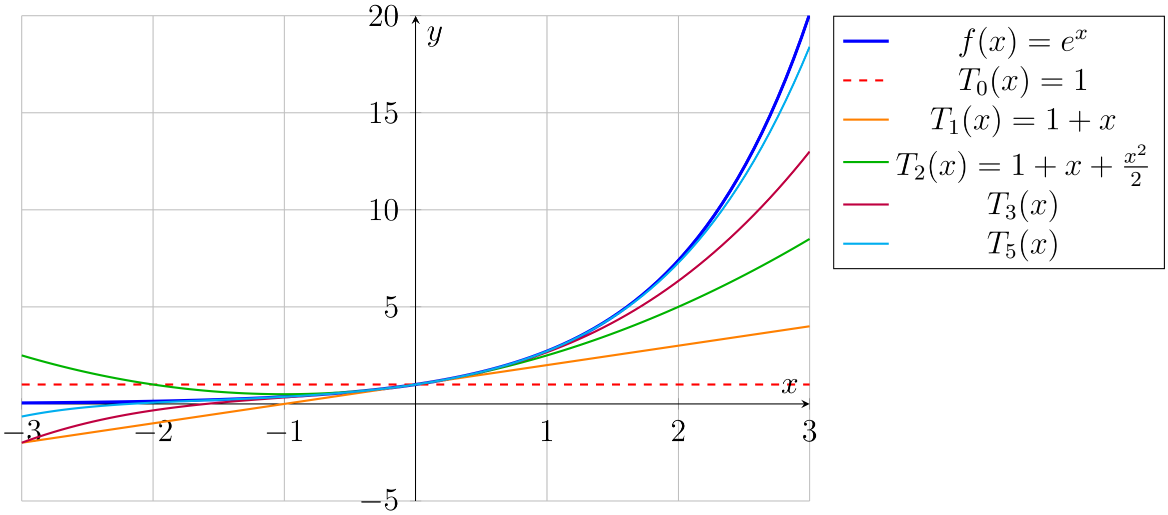


Figure 5.2: Taylor Polynomial Approximations of e^x centered at $x = 0$

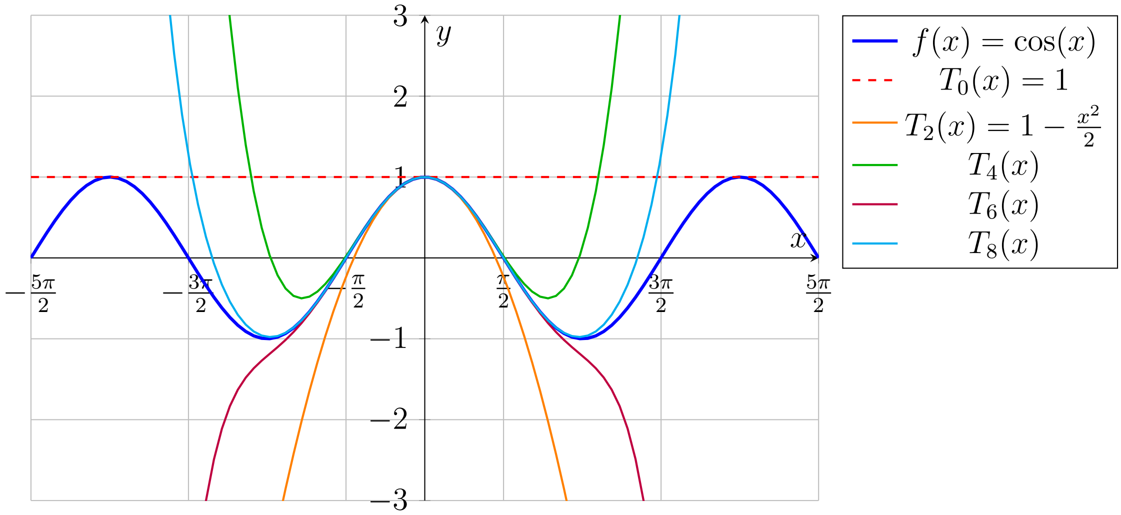


Figure 5.3: Taylor Polynomial Approximations of $\cos(x)$ centered at $x = 0$

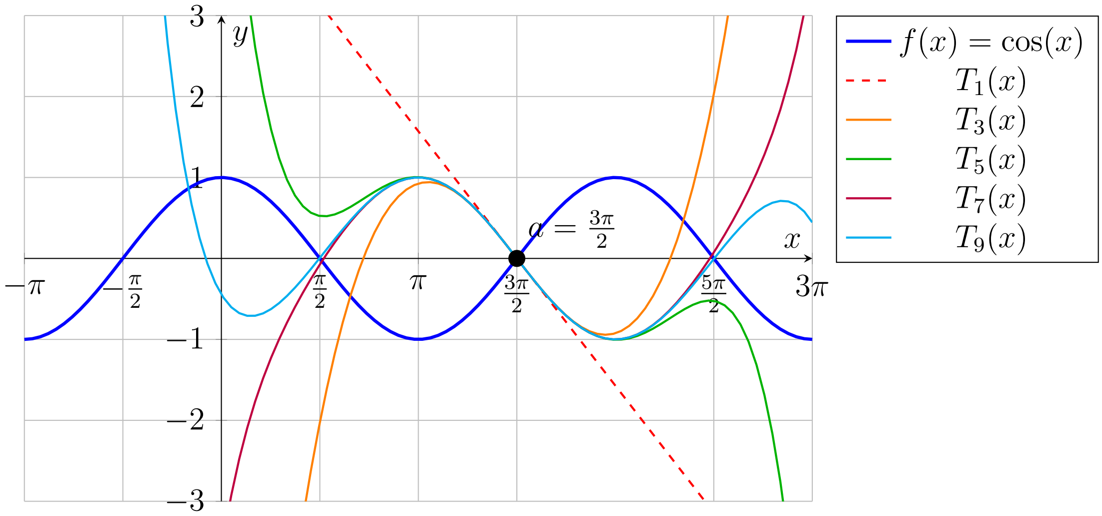


Figure 5.4: Taylor Polynomial Approximations of $\cos(x)$ centered at $x = \frac{3\pi}{2}$

5.16 Binomial Series

Theorem 5.16.1* (Binomial Theorem)*: *If n is a positive integer, then*

$$\begin{aligned}(a+b)^n &= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \cdots + nab^{n-1} + b^n\end{aligned}$$

where,

$$\begin{aligned}\binom{n}{i} &= \frac{n!}{i!(n-i)!} = \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!} \\ \binom{n}{0} &= 1\end{aligned}$$

Binomial Series

If k is any real number and $|x| < 1$, then

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= 1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots + \binom{k}{n} x^n + \cdots\end{aligned}$$

where,

$$\begin{aligned}\binom{k}{n} &= \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \\ \binom{k}{0} &= 1\end{aligned}$$

6 3-Dimensional Space

The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise, the 2-D coordinate system is denoted by $\mathbb{R}^2$, and the 1-D coordinate system is denoted by $\mathbb{R}$.

6.1 Equations of Lines

Vector form

If $\vec{a}$ and $\vec{v}$ are parallel vectors, then $\vec{a} = t\vec{v}$ for some scalar t.
Now if we have a vector $\vec{r}$ as follows

$$\vec{r} = \vec{r_0} + \vec{a}$$

Then we can write

$$\vec{r} = \vec{r_0} + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

This is called the **vector form of the equation of a line**.

Parametric form

We can rewrite the vector form as

$$\vec{r} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$
$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

In other words

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

This set of equations is called the **parametric form of the equation of a line**.

Symmetric Equations of a Line

If we assume that a, b, and c are non-zero numbers, then we can solve each of the parametric equations for t. This gives us

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 6.1: Find the Equations of lines:
1. Through the points $(7, -3, 1)$ **and** $(-2, 1, 4)$
2. Through the point $(1, -5, 0)$ **and parallel to the line given by** $\vec{r}(t) = \langle 8 - 3t, -10 + 9t, -1 - t \rangle$
3. Through the point $(-7, 2, 4)$ **and orthogonal to both** $\vec{v} = \langle 0, -9, 1 \rangle$ **and** $\vec{w} = 3\hat{\imath} + \hat{\jmath} - 4\hat{k}$
1.
Direction vector $\vec{d} = \langle -2 - 7, 1 + 3, 4 - 1 \rangle = \langle -9, 4, 3 \rangle$
Now, the vector form of the line is

$$\vec{r} = \langle 7, -3, 1 \rangle + t\langle -9, 4, 3 \rangle$$

The parametric form is

$$x = 7 - 9t, \quad y = -3 + 4t, \quad z = 1 + 3t$$

The symmetric form is
$$\frac{x-7}{-9} = \frac{y+3}{4} = \frac{z-1}{3}$$
2.
The direction vector is $\vec{d} = \langle -3, 9, -1 \rangle$
Hence, the vector form of the line is
$$\vec{r} = \langle 1, -5, 0 \rangle + t\langle -3, 9, -1 \rangle$$
The parametric form is
$$x = 1 + 3t, \quad y = -5 + 9t, \quad z = -t$$
And the symmetric form is
$$\frac{x-1}{3} = \frac{y+5}{9} = -z$$
3.
Direction vector
$$\vec{d} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -9 & 1 \\ 3 & 1 & -4 \end{vmatrix} = \langle 35, 3, 27 \rangle$$
Hence, the vector form of the line is
$$\vec{r} = \langle -7, 2, 4 \rangle + t\langle 35, 3, 27 \rangle$$
The parametric form is
$$x = -7 + 35t, \quad y = 2 + 3t, \quad z = 4 + 27t$$
The symmetric form is
$$\frac{x+7}{35} = \frac{y-2}{3} = \frac{z-4}{27}$$

Example 6.2: Determine if the two lines are parallel, orthogonal, or neither:
1. The line given by $\vec{r}(t) = \langle 4 - 7t, -10 + 5t, 21 - 4t \rangle$ **and the line given by** $\vec{r}(t) = \langle -2 + 3t, 7 + 5t, 5 + t \rangle$
2. The line given by $x = 29, y = -3 - 6t, z = 12 - t$ **and the line given by** $\vec{r}(t) = \langle 12 - 14t, 2 + 7t, -10 + 3t \rangle$
1.
The direction vectors are
$$\vec{d_1} = \langle -7, 5, -4 \rangle, \quad \vec{d_2} = \langle 3, 5, 1 \rangle$$
To check if they are parallel, we can check:
$$\frac{-7}{3} \neq \frac{5}{5} \neq \frac{-4}{1}$$
which means they are not parallel.
To check if they are orthogonal, we can check:
$$\vec{d_1} \cdot \vec{d_2} = -7(3) + 5(5) + (-4)(1) = -21 + 25 - 4 = 0$$

Hence, they are orthogonal.
2.
The direction vectors are

$$\vec{d_1} = \langle 0, -6, -1 \rangle, \quad \vec{d_2} = \langle -14, 7, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{0}{-14} \neq \frac{-6}{7} \neq \frac{-1}{3}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{d_1} \cdot \vec{d_2} = 0(-14) + (-6)(7) + (-1)(3) = -42 - 3 = -45 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 6.3: Determine the intersection point of the two lines or show that they don't not intersect:
1. The line passing through the point $(0, -9, -1)$ **and** $(1, 6, -3)$ **and the line given by** $\vec{r}(t) = \langle -9 - 4t, 10 + 6t, 1 - 2t \rangle$
2. The line given by $x = 1 + 6t, y = -1 - 3t, z = 4 + 12t$ **and the line given by** $x = 4 + t, y = -10 - 8t, z = 3 - 5t$
1.
The direction vector of the first line is

$$\vec{d_1} = \langle 1 - 0, 6 + 9, -3 + 1 \rangle = \langle 1, 15, -2 \rangle$$

We can write the parametric equations of the first line as:

$$x = s, y = -9 + 15s, z = -1 - 2s$$

And the parametric equations of the second line as:

$$x = -9 - 4t, y = 10 + 6t, z = 1 - 2t$$

Setting them equal to each other we get,

$$\begin{aligned} 0 + t &= -9 - 4s \\ -9 + 15t &= 10 + 6s \\ -1 - 2t &= 1 - 2s \end{aligned}$$

Solving the first two equations, we get

$$t = -\frac{7}{3}, \quad s = \frac{1}{3}$$

Now, verifying the third equation, we get

$$\begin{aligned} -1 - 2(-\frac{7}{3}) &= 1 - 2(\frac{1}{3}) \\ -1 + \frac{14}{3} &= 1 - \frac{2}{3} \\ \frac{11}{3} &\neq \frac{1}{3} \end{aligned}$$

Since the third equation is not satisfied, the two lines do not intersect.
2.
The lines are given in parametric form.
Setting them equal to each other we get,

$$\begin{aligned} 1 + 6s &= 4 + t \\ -1 - 3s &= -10 - 8t \\ 4 + 12s &= 3 - 5t \end{aligned}$$

Solving the first two equations, we get

$$s = \frac{1}{3}, \quad t = -1$$

Now, verifying the third equation, we get

$$\begin{aligned} 4 + 12\left(\frac{1}{3}\right) &= 3 - 5\langle -1 \rangle \\ 8 &= 8 \end{aligned}$$

That means, the lines intersect. Substituting the values in the parametric equation, we get

$$\begin{aligned} x &= 1 + 6\left(\frac{1}{3}\right) = 3 \\ y &= -1 - 3\left(\frac{1}{3}\right) = -2 \\ z &= 4 + 12\left(\frac{1}{3}\right) = 8 \end{aligned}$$

Hence, the intersection point is $(3, -2, 8)$.

Example 6.4: Which of the three coordinate planes does the line given by $x = 16t, y = -4 - 9t, z = 34$ intersect?

To intersect the xy-plane, we need $z = 0$. But here $z = 34$ is constant. Hence, the line does not intersect the xy-plane.
To intersect the yz-plane, we need $x = 0$. Hence,

$$16t = 0 \implies t = 0$$

And the intersection point is $(0, -4 - 9 \times 0, 34)$ or $(0, -4, 34)$.
To intersect the xz-plane, we need $y = 0$. Hence,

$$-4 - 9t = 0 \implies t = -\frac{4}{9}$$

And the intersection point is $\left(16\left(-\frac{4}{9}\right), 0, 43\right)$ or $\left(-\frac{64}{9}, 0, 34\right)$.

6.2 Equations of Planes

Vector form

Let's assume $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ and $\vec{r} = \langle x, y, z \rangle$ are two position vectors and $\vec{r} - \vec{r_0}$ is a vector in the plane.
If $\vec{n} = \langle a, b, c \rangle$ is a normal to the plane (which means it's orthogonal to the vector $\vec{r} - \vec{r_0}$), then we can write

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \Longrightarrow \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

This is called the **vector form of the equation of a plane**.

Scalar form

If we expand the vector equation in the following way,

$$\begin{aligned} \vec{n} \cdot (\vec{r} - \vec{r_0}) &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \end{aligned}$$

Computing the dot product, we get

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the **scalar form of the equation of a plane**.
This equation can also be written as

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

Example 6.5: Find the equation of the plane:
1. Through the point $(6, -3, 1)$**,** $(5, -4, 1)$**, and** $(3, -4, 0)$
2. The plane containing the point $(1, -5, 8)$ **and orthogonal to the line given by** $x = -3 + 15t, y = 14 - t, z = 9 - 3t$
3. The plane containing the point $(-8, 3, 7)$ **and parallel to the plane given by** $4x + 8y - 2z = 45$
4. The plane containing the two lines given by $\vec{r}(t) = \langle 7 + 5t, 2 + t, 6t \rangle$ **and** $\vec{r}(t) = \langle 7 - 6t, 2 - 2t, 10t \rangle$

1.
The given points are

$$A(6, -3, 1), B(5, -4, 1), C(3, -4, 0)$$

Two vectors in the plane are

$$\begin{aligned} \vec{AB} &= \langle 5 - 6, -4 + 3, 1 - 1 \rangle = \langle -1, -1, 0 \rangle \\ \vec{BC} &= \langle 3 - 5, -4 + 4, 0 - 1 \rangle = \langle -2, 0, -1 \rangle \end{aligned}$$

Normal vector on the place:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 0 \\ -2 & 0 & -1 \end{vmatrix} = \hat{i} - \hat{j} - 2\hat{k}$$

Now, using the point A, we can write the equation of the plane as

$$\begin{aligned}(x-6)-(y+3)-2(z-1)&=0\\ x-y-2z&=7\end{aligned}$$

2.
The normal vector is

$$\vec{n}=\langle 15,-1,-3\rangle$$

Using the point $(1,-5,8)$, the equation of the plane is

$$\begin{aligned}15(x-1)-(y+5)-3(z-8)&=0\\ 15x-y-3z&=15+5-24\\ 15x-y-3z+4&=0\end{aligned}$$

3.
The normal vector is

$$\vec{n}=\langle 4,8,-2\rangle$$

Using the point $(-8,3,7)$, the equation of the plane is

$$\begin{aligned}4(x+8)+8(y-3)-2(z-7)&=0\\ 4x+8y-2z&=-32+24+14\\ 4x+8y-2z+6&=0\end{aligned}$$

4.
The direction vectors of the two lines are

$$\vec{d_1}=\langle 5,1,6\rangle,\quad \vec{d_2}=\langle -6,-2,10\rangle$$

The normal vector is

$$\vec{n}=\vec{d_1}\times\vec{d_2}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ 5&1&6\\ -6&-2&10\end{vmatrix}=\langle 22,-86,-4\rangle$$

Using the point $A(7,2,0)$, the equation of the plane is

$$\begin{aligned}22(x-7)-86(y-2)-4(z-0)&=0\\ 22x-86y-4z-154+172&=0\\ 22x-86y-4z+18&=0\end{aligned}$$

Example 6.6: Determine if the two planes are parallel, orthogonal, or neither: The plane given by $3x+9y+7z=-1$ and the plane containing the points $(1,-1,9),(4,-1,2),(-2,3,4)$

The normal vector of the first plane is

$$\vec{n_1}=\langle 3,9,7\rangle$$

Let the points be

$$A(1, -1, 9), B(4, -1, 2), C(-2, 3, 4)$$

Two vectors in the plane are

$$\vec{AB} = \langle 4-1, -1+1, 2-9 \rangle = \langle 3, 0, -7 \rangle$$
$$\vec{AC} = \langle -2-1, 3+1, 4-9 \rangle = \langle -3, 4, -5 \rangle$$

The normal vector of the second plane is

$$\vec{n_2} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -7 \\ -3 & 4 & -5 \end{vmatrix} = \langle 28, 36, 12 \rangle = \langle 7, 9, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{3}{7} \neq \frac{9}{9} \neq \frac{7}{3}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{n_1} \cdot \vec{n_2} = 3(7) + 9(9) + 7(3) = 21 + 81 + 21 = 123 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 6.7: Find the intersection of the plane given by $4x + y + 10z = -2$ and the plane given by $-8x + 2y + 3z = -8$

The two planes are

$$4x + y + 10z = -2$$
$$-8x + 2y + 3z = -8$$

Multiplying the first equation by 2 and adding it to the second equation, we get

$$4y + 23z = -12 \implies y = -3 - \frac{23}{4}z$$

Substituting the value of y in the first equation, we get

$$16x - 3 - \frac{23}{4}z + 10z = -2 \implies x = \frac{1}{4} - \frac{17}{16}z$$

Let $z = t$ (a parameter). Then we get

$$x = \frac{1}{4} - \frac{17}{16}t$$
$$y = -3 - \frac{23}{4}t$$
$$z = t$$

This is the parametric form of the line of intersection.
We can also write it in vector form as

$$\vec{r} = \langle \frac{1}{4}, -3, 0 \rangle + t \langle -\frac{17}{16}, -\frac{23}{4}, 1 \rangle$$

6.3 Quadratic Surfaces

General form

The general form of a quadratic surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where $A, B, C, D, E, F, G, H, I, J$ are constants.

Ellipsoid

The general equation of an ellipsoid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the ellipsoid and a, b, c are the semi-axis lengths.
If $a = b = c$, we get a sphere.

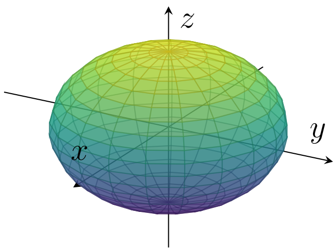


Figure 6.1: Ellipsoid: $\frac{x^2}{4} + \frac{y^2}{2.25} + z^2 = 1$

Cone

The general equation of a cone that opens along the z-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{(z-l)^2}{c^2}$$

where (h, k, l) is the center of the cone and a, b, c are the semi-axis lengths.

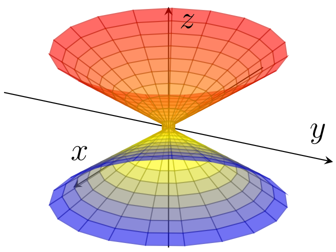


Figure 6.2: Cone: $x^2 + y^2 = z^2$

Cylinder

The general equation of a cylinder that opens along the z-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where (h, k) is the center of the cylinder and a, b are the semi-axis lengths.
If $a = b$, we get a circular cylinder.

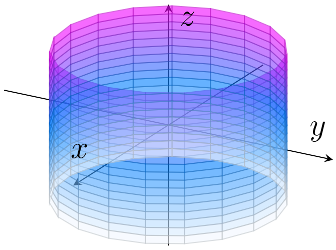


Figure 6.3: Circular Cylinder: $x^2 + y^2 = 1$

Hyperboloid of One Sheet

The general equation of a hyperboloid of one sheet is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

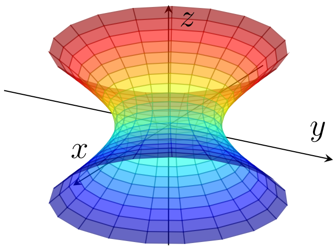


Figure 6.4: Hyperboloid of One Sheet: $x^2 + y^2 - z^2 = 1$

Hyperboloid of Two Sheets

The general equation of a hyperboloid of two sheets is

$$-\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

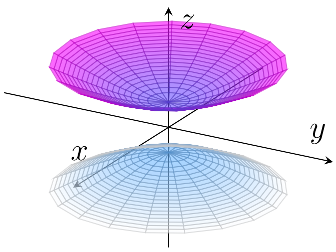


Figure 6.5: Hyperboloid of Two Sheets: $z^2 - x^2 - y^2 = 1$

Elliptic Paraboloid

The general equation of an elliptic paraboloid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

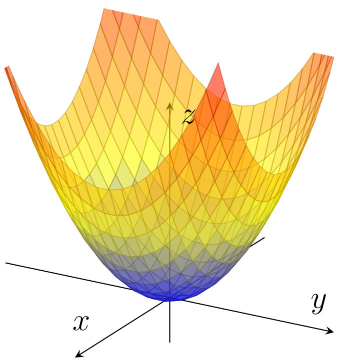


Figure 6.6: Elliptic Paraboloid: $z = x^2 + y^2$

Hyperbolic Paraboloid

The general equation of a hyperbolic paraboloid is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

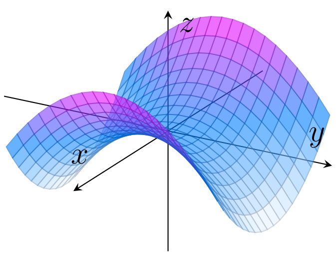


Figure 6.7: Hyperbolic Paraboloid (Saddle): $z = x^2 - y^2$

6.4 Calculus with Vector Functions

Let

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Note:-

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

Note:-

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{u} + \vec{v}) = \vec{u}' + \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(c\vec{u}) = c\vec{u}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{u} \cdot \vec{v}) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{u} \times \vec{v}) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\vec{u} f(t)\big) = f'(t)\vec{u}'(f(t))$$

Note:-

$$\int \vec{r}(t)\,\mathrm{dt} = \langle \int f(t)\,\mathrm{dt}\,, \int g(t)\,\mathrm{dt}\,, \int h(t)\,\mathrm{dt}\rangle$$

$$\int_a^b \vec{r}(t)\,\mathrm{dt} = \left\langle \int_a^b f(t)\,\mathrm{dt} + \int_a^b g(t)\,\mathrm{dt} + \int_a^b h(t)\,\mathrm{dt} \right\rangle$$

6.5 Tangent, Normal, and Binormal Vectors

Unit Tangent vector

Given the vector function $\vec{r}(t)$, we call $\vec{r}'(t)$ the **tangent vector** The unit tangent vector is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Unit Normal vector

If $\vec{T}(t)$ is the unit tangent vector, then the **unit normal vector** is given by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Note:-

If $\vec{r}(t)$ is a vector such that $\|\vec{r}(t)\| = c$ for all t, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$

Binormal vector

The **binormal vector** is given by

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The binormal vector is orthogonal to both the tangent and normal vectors.

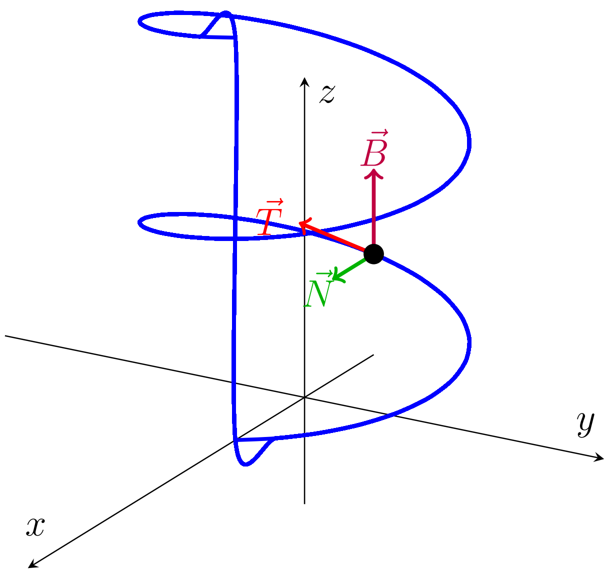


Figure 6.8: TNB Frame (Frenet-Serret Frame) on a helix $\vec{r}(t) = \langle \cos t, \sin t, 0.3t \rangle$

6.6 Arc Length with Vector Functions

Note:-

The arc length of a vector function $\vec{r}(t)$ from $t = a$ to $t = b$ is given by

$$L = \int_a^b \|\vec{r}'(t)\| \, \mathrm{d}t$$

Or,

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2} \, \mathrm{d}t$$

6.7 Curvature

Curvature of a curve in 3-D space

The curvature of a curve in 3-D space is given by

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

where $\vec{T}(t)$ is the unit tangent vector and $\vec{r}(t)$ is the position vector.
This can also be written as

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$