Integration Techniques

Integration by Parts: $\int u\,dv = uv - \int v\,du$
LIATE Rule: Choose u in order: Logarithmic, Inverse trig, Algebraic, Trig, Exponential
Common Integrals:
- $\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$
- $\int \cot x\,dx = \ln|\sin x| + C$
- $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
- $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln|\frac{a+x}{a-x}| + C$
- $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln|\frac{x-a}{x+a}| + C$
- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln|x + \sqrt{x^2 + a^2}| + C$
- $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln|x + \sqrt{x^2 - a^2}| + C$
- $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}(\frac{x}{a}) + C$
- $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$

Trig Reduction Formulas:
- $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$
- $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$
- $\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$
- $\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$
- $\int \sin^m x\cos^n x\,dx = -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx$

Polynomial Reduction:
- $\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$
- $\int x^n \sin(ax)\,dx = -\frac{1}{a}x^n\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)\,dx$
- $\int x^n \cos(ax)\,dx = \frac{1}{a}x^n\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\,dx$

Trig Substitution:
- $\sqrt{a^2 - x^2}$: use $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$
- $\sqrt{a^2 + x^2}$: use $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$
- $\sqrt{x^2 - a^2}$: use $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\,d\theta$

Numerical Integration ($\Delta x = \frac{b-a}{n}$):
- Midpoint: $M_n = \Delta x \sum_{i=1}^{n} f(\bar{x}_i)$ where $\bar{x}_i = \frac{x_{i-1}+x_i}{2}$
- Trapezoid: $T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$
- Simpson's: $S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + f(x_n)]$ (n even)

Improper Integrals:
- Type 1: $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
- Type 2: $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$
- Type 3: $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$
- Discontinuity at $c \in [a,b]$: $\int_a^b = \lim_{\epsilon\to 0^+}[\int_a^{c-\epsilon} + \int_{c+\epsilon}^b]$

$\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$, diverges if $p \le 1$
Comparison Test for Improper Integrals: If $f(x) \ge g(x) \ge 0$:
- $\int_a^\infty f(x)\,dx$ converges $\implies \int_a^\infty g(x)\,dx$ converges
- $\int_a^\infty g(x)\,dx$ diverges $\implies \int_a^\infty f(x)\,dx$ diverges

Applications of Integrals

Arc Length:
- $y = f(x)$, $a \le x \le b$: $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
- $x = h(y)$, $c \le y \le d$: $L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$

Surface Area:
- Rotation about x-axis: $S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
- Rotation about y-axis: $S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$

Center of Mass/Centroid: $\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$, $\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)^2 - g(x)^2]\,dx$ where $A = \int_a^b [f(x) - g(x)]\,dx$
Probability: $P(a \le X \le b) = \int_a^b f(x)\,dx$ where $\int_{-\infty}^\infty f(x)\,dx = 1$
Mean: $\mu = \int_{-\infty}^\infty x f(x)\,dx$

Parametric Equations

Derivative: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ (provided $dx/dt \ne 0$)
Second Derivative: $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) / \frac{dx}{dt}$

Tangent Lines:
- Horizontal: $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \ne 0$
- Vertical: $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \ne 0$

Arc Length: $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Surface Area:
- About x-axis: $S = \int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
- About y-axis: $S = \int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Area: $A = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$ (enclosed by curve and x-axis)

Polar Coordinates

Conversion: $x = r\cos\theta$, $y = r\sin\theta$ $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$, $r = \sqrt{x^2 + y^2}$
Derivative: $\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$ where $r' = \frac{dr}{d\theta}$
Area:
- Single curve: $A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$
- Between curves: $A = \frac{1}{2}\int_\alpha^\beta (r_{outer}^2 - r_{inner}^2)\,d\theta$

Arc Length: $L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$
Surface Area:
- About x-axis: $S = \int_\alpha^\beta 2\pi y\sqrt{r^2 + (r')^2}\,d\theta = \int 2\pi r\sin\theta\sqrt{r^2 + (r')^2}\,d\theta$
- About y-axis: $S = \int_\alpha^\beta 2\pi x\sqrt{r^2 + (r')^2}\,d\theta = \int 2\pi r\cos\theta\sqrt{r^2 + (r')^2}\,d\theta$

Common Polar Curves:
- Circle: $r = a$ (centered at origin), $r = 2a\cos\theta$ (tangent to y-axis)
- Line: $\theta = \alpha$ (through origin), $r = \frac{c}{\cos\theta - \sin\theta}$ (general)
- Cardioid: $r = a(1 \pm \cos\theta)$ or $r = a(1 \pm \sin\theta)$
- Limaçon: $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$
- Rose: $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$ (n odd: n petals; n even: $2n$ petals)
- Lemniscate: $r^2 = a^2\cos(2\theta)$ or $r^2 = a^2\sin(2\theta)$
- Spiral: $r = a\theta$ (Archimedean), $r = ae^{b\theta}$ (logarithmic)

Sequences

Definition: $\{a_n\}_{n=1}^\infty$ where $a_n = f(n)$ for $n \in \mathbb{N}$
Limit: $\lim_{n\to\infty} a_n = L$ if $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \ge N, |a_n - L| < \epsilon$
Limit Properties: If $\lim a_n = A$ and $\lim b_n = B$, then:
- $\lim(a_n \pm b_n) = A \pm B$
- $\lim(ca_n) = cA$
- $\lim(a_n b_n) = AB$
- $\lim \frac{a_n}{b_n} = \frac{A}{B}$ if $B \ne 0$
- $\lim(a_n^p) = A^p$ if $a_n \ge 0$

Squeeze Theorem: If $a_n \le b_n \le c_n$ and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$
Important: If $\lim |a_n| = 0$, then $\lim a_n = 0$
Bounded: $\exists M : \forall n, |a_n| \le M$
Monotonic: Increasing: $a_{n+1} \ge a_n$ for all n; Decreasing: $a_{n+1} \le a_n$ for all n
Monotonic Convergence Theorem: Bounded & monotonic $\implies$ convergent
Convergent sequences are bounded
Special: $\{r^n\}$ converges if $|r| \le 1$; $\lim r^n = 0$ if $|r| < 1$, $= 1$ if $r = 1$

Series

Partial Sum: $s_n = \sum_{i=1}^n a_i$
Convergence: $\sum_{n=1}^\infty a_n$ converges if $\lim_{n\to\infty} s_n$ exists
If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ (Contrapositive: Divergence Test)
Geometric Series: $\sum_{n=0}^\infty ar^n = \frac{a}{1-r}$ if $|r| < 1$; diverges if $|r| \ge 1$
Telescoping Series: $\sum_{n=1}^\infty (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_{n+1}$
Harmonic Series: $\sum_{n=1}^\infty \frac{1}{n}$ diverges
p-Series: $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \le 1$
Series Properties: If $\sum a_n$ and $\sum b_n$ converge:
- $\sum ca_n = c\sum a_n$
- $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$

Convergence Tests:
1. **Divergence Test:** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges. If $\lim a_n = 0$, test is inconclusive.
2. **Integral Test:** If f is continuous, positive, decreasing for $x \geq 1$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.
3. **Comparison Test:** If $0 \leq a_n \leq b_n$ for all n:
   - $\sum b_n$ converges $\implies \sum a_n$ converges
   - $\sum a_n$ diverges $\implies \sum b_n$ diverges
4. **Limit Comparison Test:** If $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
5. **Alternating Series Test:** If $b_n > 0$, $b_{n+1} \leq b_n$ for all n, and $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n b_n$ converges. **Error bound:** $|R_n| = |s - s_n| \leq b_{n+1}$
6. **Ratio Test:** Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then:
   - $L < 1 \implies$ converges absolutely
   - $L > 1$ or $L = \infty \implies$ diverges
   - $L = 1 \implies$ inconclusive
7. **Root Test:** Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$. Then:
   - $L < 1 \implies$ converges absolutely
   - $L > 1$ or $L = \infty \implies$ diverges
   - $L = 1 \implies$ inconclusive

Absolute vs Conditional Convergence:
- Absolutely convergent: $\sum |a_n|$ converges
- If absolutely convergent, then convergent
- Conditionally convergent: $\sum a_n$ converges but $\sum |a_n|$ diverges

Power Series

Form: $\sum_{n=0}^{\infty} c_n(x-a)^n$ where a is center
Radius of Convergence R: Series converges absolutely for $|x-a| < R$, diverges for $|x-a| > R$
Finding R:
- Ratio Test: $\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$
- Root Test: $\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|c_n|}$

Interval of Convergence: Test endpoints $x = a \pm R$ separately
Term-by-Term Operations: If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with radius R:
- $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ for $|x-a| < R$ (same R)
- $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$ for $|x-a| < R$ (same R)
- $f(x) \pm g(x) = \sum_{n=0}^{\infty} (c_n \pm d_n)(x-a)^n$

Taylor & Maclaurin Series

Taylor Series about $x = a$: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Taylor Polynomial: $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$
Taylor Remainder: $R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x

Error Bound: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$
Series converges to $f(x)$ if and only if $\lim_{n\to\infty} R_n(x) = 0$
Maclaurin Series (Taylor at $a = 0$):
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for all x
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for all x
- $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all x
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for $-1 < x \leq 1$
- $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $|x| \leq 1$
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ for $|x| < 1$
- $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$ for $|x| < 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots$ for $|x| < 1$

Binomial Coefficient: $\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$, $\binom{k}{0} = 1$

3D Space - Vectors

Vector: $\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

Magnitude: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Unit Vector: $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$

Dot Product: $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{a}||\vec{b}|\cos\theta$
Properties:
- $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
- Commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- Distributive: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Cross Product: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= (a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$
Properties:
- $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$
- $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ (right-hand rule)
- Anti-commutative: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times \vec{a} = \vec{0}$
- $\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0}$
- Area of parallelogram: $|\vec{a} \times \vec{b}|$

Scalar Triple Product: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Volume of parallelepiped: $|\vec{a} \cdot (\vec{b} \times \vec{c})|$

Projection: $\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\hat{b}$

Component: $\text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

3D Space - Lines

Vector Form: $\vec{r}(t) = \vec{r_0} + t\vec{v}$ where $\vec{v}$ is direction vector

Parametric: $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$

Symmetric: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ (if $a, b, c \neq 0$)

Two Points: Direction vector $\vec{v} = P_2 - P_1 = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

Parallel Lines: $\vec{d_1} = k\vec{d_2}$ for some scalar k (or $\vec{d_1} \times \vec{d_2} = \vec{0}$)

Orthogonal Lines: $\vec{d_1} \cdot \vec{d_2} = 0$

Angle Between Lines: $\cos\theta = \frac{|\vec{d_1} \cdot \vec{d_2}|}{|\vec{d_1}||\vec{d_2}|}$

Skew Lines: Not parallel and do not intersect

3D Space - Planes

Vector Form: $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$ where $\vec{n}$ is normal vector

Scalar Form: $ax + by + cz = d$ where $\vec{n} = \langle a, b, c \rangle$

Point-Normal Form: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Three Points: $\vec{n} = \vec{AB} \times \vec{AC}$, then use point-normal form

Parallel Planes: $\vec{n_1} = k\vec{n_2}$ or $\vec{n_1} \times \vec{n_2} = \vec{0}$

Orthogonal Planes: $\vec{n_1} \cdot \vec{n_2} = 0$

Angle Between Planes: $\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}$

Line-Plane Intersection: Substitute parametric line equations into plane equation

Plane-Plane Intersection: Solve system of two equations (result is a line)

Distance Formulas

Point to Point: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Point to Line: $d = \frac{|\vec{P_0 P_1} \times \vec{d}|}{|\vec{d}|}$ where P_0 on line, P_1 is point, $\vec{d}$ is direction

Point to Plane: $d = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ where plane: $ax + by + cz = d$, point: (x_0, y_0, z_0)

Between Parallel Planes: $d = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$ where $ax + by + cz = d_1$ and $ax + by + cz = d_2$

Between Skew Lines: $d = \frac{|(\vec{P_1 P_2}) \cdot (\vec{d_1} \times \vec{d_2})|}{|\vec{d_1} \times \vec{d_2}|}$

Quadratic Surfaces

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (all positive, sphere if $a = b = c$)

Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ (double cone)

Cylinder: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (extends along z-axis)

Hyperboloid of 1 Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (one negative)

Hyperboloid of 2 Sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (two negatives)

Elliptic Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ (all same sign, bowl shape)

Hyperbolic Paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$ (saddle shape)

Vector Functions

Vector Function: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

Limit: $\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$

Derivative: $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Derivative Rules:
- $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$
- $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$
- $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
- $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$ (chain rule)

Integration: $\int \vec{r}(t)\,dt = \langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \rangle$

Arc Length: $L = \int_a^b |\vec{r}'(t)|\,dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$

Arc Length Function: $s(t) = \int_a^t |\vec{r}'(u)|\,du$

Unit Tangent Vector: $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

Unit Normal Vector: $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$

Binormal Vector: $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ (always unit vector)

Important: If $|\vec{r}(t)| = c$ (constant), then $\vec{r}(t) \perp \vec{r}'(t)$

Curvature:
- $\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$
- $\kappa = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$
- For $y = f(x)$: $\kappa = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$

Radius of Curvature: $\rho = \frac{1}{\kappa}$

Important Identities

Trigonometric:
- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$
- $\sin(2x) = 2\sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\sin^2 x = \frac{1 - \cos(2x)}{2}$, $\cos^2 x = \frac{1 + \cos(2x)}{2}$
- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$
- $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$

Hyperbolic:
- $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$
- $\tanh x = \frac{\sinh x}{\cosh x}$
- $\cosh^2 x - \sinh^2 x = 1$
- $1 - \tanh^2 x = \text{sech}^2 x$
- $\frac{d}{dx}\sinh x = \cosh x$, $\frac{d}{dx}\cosh x = \sinh x$
- $\frac{d}{dx}\tanh x = \text{sech}^2 x$

Logarithmic:
- $\ln(ab) = \ln a + \ln b$
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- $\ln\left(a^b\right) = b \ln a$
- $\log_b a = \frac{\ln a}{\ln b}$
- $a = e^{\ln a}$

Integration Techniques Summary

Products of $\sin^m x \cos^n x$:
- m odd: $u = \cos x$, use $\sin^2 x = 1 - \cos^2 x$
- n odd: $u = \sin x$, use $\cos^2 x = 1 - \sin^2 x$
- Both even: use half-angle formulas $\sin^2 x = \frac{1 - \cos 2x}{2}$, $\cos^2 x = \frac{1 + \cos 2x}{2}$

Products of $\sec^m x \tan^n x$:
- n odd: $u = \sec x$, use $\tan^2 x = \sec^2 x - 1$
- m even: $u = \tan x$, use $\sec^2 x = 1 + \tan^2 x$

Partial Fractions:
- Linear factor $(ax + b)$: $\frac{A}{ax+b}$
- Repeated linear $(ax + b)^k$: $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
- Irreducible quadratic $ax^2 + bx + c$: $\frac{Ax+B}{ax^2+bx+c}$
- Repeated quadratic $(ax^2 + bx + c)^k$: $\frac{A_1 x + B_1}{ax^2+bx+c} + \cdots + \frac{A_k x + B_k}{(ax^2+bx+c)^k}$

Completing the Square: $ax^2 + bx + c = a\left[(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a^2}\right]$
Then substitute $u = x + \frac{b}{2a}$

Strategy Guides

Integration Strategy:
1. Simplify (expand, factor, cancel)
2. Look for obvious substitutions ($u' = g'(x)$)
3. Classify:
   - Trig integrals → identities, reduction formulas
   - Rational functions → partial fractions
   - Products → integration by parts (LIATE)
   - Roots of $a^2 \pm x^2$ or $x^2 \pm a^2$ → trig substitution
   - Quadratic in denominator → complete square

Series Test Selection:
1. Always try **Divergence Test** first
2. Recognize special forms:
   - Geometric: ar^n
   - p-series: $\frac{1}{n^p}$
   - Telescoping: $a_n - a_{n+1}$
3. For rational expressions: **Limit Comparison**
4. For alternating series: **Alternating Series Test**
5. For factorials or exponentials: **Ratio Test**
6. For n-th powers: **Root Test**
7. For comparisons: **Comparison Test**
8. If $a_n = f(n)$ where f is easy to integrate: **Integral Test**

Common Derivatives

- $\frac{d}{dx} x^n = n x^{n-1}$
- $\frac{d}{dx} e^x = e^x$, $\frac{d}{dx} a^x = a^x \ln a$
- $\frac{d}{dx} \ln x = \frac{1}{x}$, $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$
- $\frac{d}{dx} \sin x = \cos x$, $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = \sec^2 x$, $\frac{d}{dx} \cot x = -\csc^2 x$
- $\frac{d}{dx} \sec x = \sec x \tan x$, $\frac{d}{dx} \csc x = -\csc x \cot x$
- $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$, $\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$
- $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$, $\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$

Useful Limits

- $\lim_{n \to \infty} \frac{1}{n^p} = 0$ for $p > 0$
- $\lim_{n \to \infty} r^n = 0$ if $|r| < 1$
- $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$
- $\lim_{n \to \infty} n^{1/n} = 1$
- $\lim_{n \to \infty} \frac{\ln n}{n} = 0$
- $\lim_{n \to \infty} \frac{n^k}{a^n} = 0$ for $a > 1$, any k
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
- $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$
- $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

Key Theorems

Fundamental Theorem of Calculus:
- Part 1: $\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$
- Part 2: $\int_a^b f(x)\, dx = F(b) - F(a)$ where $F'(x) = f(x)$

Mean Value Theorem: If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Intermediate Value Theorem: If f continuous on $[a,b]$ and N is between $f(a)$ and $f(b)$, then $\exists c \in (a,b)$ such that $f(c) = N$

L'Hôpital's Rule: If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ (if limit exists)

Abel's Theorem: If power series $\sum c_n (x-a)^n$ has radius $R \in (0, \infty)$ and converges at an endpoint, then it's continuous at that endpoint.

Error Estimates

Integral Test: $s_n + \int_{n+1}^{\infty} f(x)\, dx \leq s \leq s_n + \int_n^{\infty} f(x)\, dx$

Alternating Series: $|R_n| = |s - s_n| \leq b_{n+1}$

Ratio Test: Let $r_n = \frac{a_{n+1}}{a_n}$
- If $\{r_n\}$ decreasing: $R_n \leq \frac{a_{n+1}}{1 - r_{n+1}}$
- If $\{r_n\}$ increasing: $R_n \leq \frac{a_{n+1}}{1 - L}$ where $L = \lim_{n \to \infty} r_n$

Comparison Test: If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $R_n \leq T_n = \sum_{k=n+1}^{\infty} b_k$

Special Values

Constants:
- $e \approx 2.71828$
- $\pi \approx 3.14159$
- $\ln 2 \approx 0.693$
- $\ln 10 \approx 2.303$
- $\sqrt{2} \approx 1.414$
- $\sqrt{3} \approx 1.732$
- Golden ratio: $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$

Common Angles:
- $\sin 0 = 0$, $\cos 0 = 1$, $\tan 0 = 0$
- $\sin \frac{\pi}{6} = \frac{1}{2}$, $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$
- $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\tan \frac{\pi}{4} = 1$
- $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos \frac{\pi}{3} = \frac{1}{2}$, $\tan \frac{\pi}{3} = \sqrt{3}$
- $\sin \frac{\pi}{2} = 1$, $\cos \frac{\pi}{2} = 0$, $\tan \frac{\pi}{2}$ undefined

Factorials: $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$

Calculus II Formula Sheet   Good luck on your final!