

Calculus II Notes

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1 Integration Techniques

1.1 Fundamental Theorem of Calculus

Theorem 1.1.1 (Fundamental Theorem of Calculus): Let f be a function defined on an open interval I that contains a . If f is continuous on I , then the function F defined by

$$F(x) = \int_a^x f(t) \, dt$$

is uniformly continuous on I , differentiable on the open interval, and

$$F'(x) = f(x)$$

for all x in the open interval.

1.2 Common Differentiation and Integration Formulae

Derivative	Integral
$\frac{d}{dx}x = 1$, $\frac{d}{dx}c = 0$	$\int c \, dx = cx + c$
$\frac{d}{dx}x^n$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln(x) + c$
$\frac{d}{dx}e^{mx} = me^{mx}$	$\int e^{mx} \, dx = \frac{1}{m}e^{mx} + c$
$\frac{d}{dx}a^x = a^x \ln(a)$	$\int a^x \, dx = \frac{1}{\ln(a)}a^x + c$
$\frac{d}{dx}\sin(mx) = m \cos(mx)$	$\int \cos(mx) \, dx = \frac{1}{m}\sin(mx) + c$
$\frac{d}{dx}\cos(mx) = -m \sin(mx)$	$\int \sin(mx) \, dx = -\frac{1}{m}\cos(mx) + c$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\int \sec^2(x) \, dx = \tan(x) + c$
$\frac{d}{dx}\cot(x) = -\csc^2(x)$	$\int \csc^2(x) \, dx = -\cot(x) + c$
$\frac{d}{dx}\sec(x) = \sec(x) \tan(x)$	$\int \sec(x) \tan(x) \, dx = \sec(x) + c$
$\frac{d}{dx}\csc(x) = -\csc(x) \cot(x)$	$\int \csc(x) \cot(x) \, dx = -\csc(x) + c$
$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + c$
$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \tan^{-1}(x) + c$
$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1}(x) + c$
$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = -\cos^{-1}(x) + c$
$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = -\cot^{-1}(x) + c$
$\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = -\csc^{-1}(x) + c$
$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x} + c$

Table 1: Common Differentiation and Integration Formulae

1.3 More Formulae

1. $\int \tan(x) \, dx = \ln|\sec(x)| + c$
2. $\int \csc(x) \, dx = \ln|\tan \frac{x}{2}| + c$
3. $\int \sec(x) \, dx = \ln|\sec(x) + \tan(x)| + c$
4. $\int \sec(x) \, dx = \ln|\tan(\frac{\pi}{4} + \frac{x}{2})|$
5. $\int \csc(x) \, dx = -\ln|\csc(x) + \cot(x)| + c$
6. $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + c$
7. $\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \ln|\frac{a+x}{a-x}| + c$
8. $\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \ln|\frac{a-x}{a+x}| + c$
9. $\int \frac{1}{\sqrt{x^2+a^2}} \, dx = \ln|x + \sqrt{x^2+a^2}| + c$
10. $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}(\frac{x}{a}) + c$
11. $\int \sqrt{a^2-x^2} \, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$
12. $\int e^x [f(x) + f'(x)] \, dx = e^x f(x) + c$

1.4 Integration by Parts

Theorem 1.4.1 (Integration by Parts): Let u and v be differentiable functions of x . Then,

$$\int uv \, dx = u \int v \, dx - \int \left[\frac{du}{dx} \int v \, dx \right] dx$$

Or,

$$\int u \, dv = uv - \int v \, du$$

Proof:

Let $u = u(x)$ and $w = w(x)$. Then,

$$\frac{d(uw)}{dx} = u \frac{dw}{dx} + w \frac{du}{dx}$$

Integrating both sides, we get

$$\int \frac{d(uw)}{dx} \, dx = \int u \frac{dw}{dx} \, dx + \int w \frac{du}{dx} \, dx$$

Or,

$$uw = \int u \frac{dw}{dx} \, dx + \int w \frac{du}{dx} \, dx$$

Rearranging, we get

$$\int u \frac{dw}{dx} \, dx = uw + c - \int w \frac{du}{dx} \, dx$$

Let $v = \frac{dw}{dx}$, then $w = \int v \, dx$. Hence,

$$\int uv \, dx = u \int v \, dx - \int \left[\frac{du}{dx} \int v \, dx \right] dx$$

□

1.5 Method of Substitution

- A. $\int \frac{1}{(ax+b)\sqrt{cx+d}} dx$ Let $cx + d = z^2$
- B. $\int \frac{1}{\sin^m x \cos^m x} dx$ If $m + n = p$ is even, multiply and divide by $\sec^p x$ and let $\tan x = z$.
- C. $\int \frac{1}{\sin^m x + \cos^m x} dx$ If m is even, multiply and divide by $\sec^m x$.
- D. $\int \frac{\cos x}{a \cos x + b \sin x} dx$ Write $nom = l \times (denom) + m \times (denom)'$, then determine l and m .
- E. $\int \frac{\cos x}{a \cos x + b \sin x} dx + c$ Write $\sin x$ and $\cos x$ as $\tan \frac{x}{2}$.
- F. $\int \frac{1}{\sqrt{x^2+a^2}} dx$ Let $x = a \tan \theta$
- G. $\int \frac{1}{\sqrt{x^2-a^2}} dx$ Let $x = a \sec \theta$
- H. $\int \sqrt{a^2 - x^2} dx$ Let $x = a \sin \theta$

1.6 Trigonometric Integrals

Form	Looks like	Substitution	Limit Assumption
$\sqrt{b^2x^2 - a^2}$	$\sec^2 \theta - 1 = \tan^2 \theta$	$x = \frac{a}{b} \sec \theta$	$0 \leq \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \leq \pi$
$\sqrt{a^2 - b^2x^2}$	$1 - \sin^2 \theta = \cos^2 \theta$	$x = \frac{a}{b} \sin \theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2 + b^2x^2}$	$\tan^2 \theta + 1 = \sec^2 \theta$	$x = \frac{a}{b} \tan \theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Table 2: Trigonometric Integral Substitution

1.7 Partial Fractions

Factor in denominator	Term in partial fraction decomposition
$ax + b$	$\frac{A}{ax+b}$
$(ax + b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+b}{ax^2+bx+c}$
$(ax^2 + bx + c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Table 3: Partial Fraction Decomposition

1.8 Improper Integrals

Definition 1.8.1: Improper Integrals

An integral is said to be **improper** if one of the following conditions is met:

1. The interval of integration is infinite.
2. The integrand is discontinuous at one or more points in the interval of integration.

The integral is said to **converge** if the limit of the integral exists, and **diverge** otherwise.

Type-1: If $\int_a^t f(x) dx$ exists for all $t > a$, then

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

provided the limit exists and is finite.

Type-2: If $\int_t^b f(x) dx$ exists for all $t < b$, then

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

provided the limit exists and is finite.

Type-3: If $\int_{-\infty}^c f(x) dx$ and $\int_c^\infty f(x) dx$ are both convergent, then

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx$$

Type-4: $\int_a^b f(x) dx$ If $f(x)$ is discontinuous at $x = c$, then

$$\int_a^b f(x) dx = \lim_{\epsilon \rightarrow 0} \left[\int_a^{c-\epsilon} f(x) dx + \int_{c+\epsilon}^b f(x) dx \right]$$

Note:-

If $a > 0$ then

$$\int_a^\infty \frac{1}{x^p} dx$$

converges if $p > 1$ and diverges if $p \leq 1$.

1.8.1 Comparison Test

Theorem 1.8.2 (Comparison Theorem): If $f(x) \geq g(x) \geq 0$ on the interval $[a, \infty)$, then

1. If $\int_a^\infty f(x) dx$ converges, then so does $\int_a^\infty g(x) dx$.
2. If $\int_a^\infty g(x) dx$ diverges, then so does $\int_a^\infty f(x) dx$.

Example 1.1: Determine if the following integral is convergent or divergent:

$$\int_2^{\infty} \frac{\cos^2 x}{x^2} dx$$

Notice that the numerator is bounded since

$$0 \leq \cos^2 x \leq 1$$

Hence, it's likely that the denominator will determine the convergence of the integral. Since $p = 2 > 1$,

$$\int_2^{\infty} \frac{1}{x^2} dx$$

is convergent. Since

$$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

and $\int_2^{\infty} \frac{1}{x^2} dx$ is convergent, by the comparison test,

$$\int_2^{\infty} \frac{\cos^2 x}{x^2} dx$$

is convergent.

Example 1.2: Determine if the following integral is convergent or divergent:

$$\int_3^{\infty} \frac{1}{x + e^x} dx$$

In this case, the denominator determines the convergence of the integral. If we can find a larger function that converges, then the integral will converge. Notice that

$$\frac{1}{x + e^x} < \frac{1}{e^x} = e^{-x}$$

Also,

$$\begin{aligned} \int_3^{\infty} e^{-x} dx &= \lim_{t \rightarrow \infty} \int_3^t e^{-x} dx \\ &= \lim_{t \rightarrow \infty} (-e^{-t} + e^{-3}) \\ &= e^{-3} \end{aligned}$$

So, $\int_3^{\infty} e^{-x} dx$ is convergent. Therefore, by the Comparison test,

$$\int_3^{\infty} \frac{1}{x + e^x} dx$$

is also convergent.

2 Applications of Integrals

2.1 Arc Length

Consider a curve $y = f(x)$. We want to find the length of the curve from $x = a$ to $x = b$. We can approximate the curve by a series of line segments. The length of each line segment is given by the Pythagorean theorem:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

The total length of the curve is given by the sum of the lengths of the line segments:

$$L = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Similarly, for a curve $x = h(y)$, the length of the curve from $y = c$ to $y = d$ is given by:

$$L = \int ds = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Arc Length Formula

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{for } y = f(x), a \leq x \leq b$$
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \quad \text{for } x = h(y), c \leq y \leq d$$

2.2 Surface Area

Consider a curve $y = f(x)$ rotated about the x -axis. We want to find the surface area of the resulting surface. We can approximate the surface by a series of frustums. The surface area of each frustum is given by:

$$dS = 2\pi y ds = 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

The total surface area of the surface is given by the sum of the surface areas of the frustums:

$$S = \int dS = \int 2\pi y ds = \int 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Similarly, for a curve $x = h(y)$ rotated about the y -axis, the surface area of the resulting surface is given by:

$$A = \int 2\pi x ds = \int 2\pi h(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Surface Area Formula

$$\begin{aligned} S &= \int dS \\ &= \int 2\pi y \, ds \quad \text{rotation about } x\text{-axis} \\ &= \int 2\pi x \, ds \quad \text{rotation about } y\text{-axis} \end{aligned}$$

where,

$$\begin{aligned} dS &= 2\pi y \, ds = 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{for } y = f(x), a \leq x \leq b \\ dS &= 2\pi x \, ds = 2\pi h(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \quad \text{for } x = h(y), c \leq y \leq d \end{aligned}$$

2.3 Center of Mass

Suppose we want to find the center of mass of a region bounded by two curves $f(x)$ and $g(x)$ on the interval $[a, b]$.

The mass is

$$M = \rho \int_a^b (f(x) - g(x)) \, dx$$

Next, we need the **moments** of the region. There are two moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2} [f(x)^2 - g(x)^2] \, dx \\ M_y &= \rho \int_a^b x [f(x) - g(x)] \, dx \end{aligned}$$

The coordinates of the center of mass, (\bar{x}, \bar{y}) , are given by:

Center of Mass Formula

$$\begin{aligned} \bar{x} &= \frac{M_y}{M} = \frac{1}{A} \int_a^b x [f(x) - g(x)] \, dx \\ \bar{y} &= \frac{M_x}{M} = \frac{1}{A} \int_a^b \frac{1}{2} [f(x)^2 - g(x)^2] \, dx \end{aligned}$$

where,

$$A = \int_a^b [f(x) - g(x)] \, dx$$

2.4 Probability

Every continuous random variable X , has a probability density function $f(x)$. Probability density functions satisfy the following conditions:

1. $f(x) \geq 0$ for all x .
2. $\int_{-\infty}^{\infty} f(x) \, dx = 1$

Probability density functions can be used to determine the probability that a continuous random variable lies between two values, say a and b . This probability is denoted by $P(a \leq X \leq b)$.

Note:-

$$P(a \leq X \leq b) = \int_a^b f(x) \, dx$$

Probability density functions can also be used to determine the mean of a continuous random variable. The mean is given by:

Mean of a Continuous Random Variable

$$\mu = \int_{-\infty}^{\infty} xf(x) \, dx$$

3 Parametric Equations and Polar Coordinates

There are great many curves out there that cannot be expressed in a single equation in terms of only x and y . To deal with such problems, we introduce **parametric equations**. Instead of defining y in terms of x ($y = f(x)$) or x in terms of y ($x = h(y)$), we define both x and y in terms of a third variable called a parameter as follows:

$$x = f(t) \quad y = g(t)$$

This third variable is usually denoted by t . Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called a **parametric curve**.

3.1 Tangents with Parametric Equations

Derivative for Parametric Equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{provided } \frac{dx}{dt} \neq 0$$

Tangents for Parametric Equations

Horizontal Tangent:

$$\frac{dy}{dt} = 0, \quad \text{provided } \frac{dx}{dt} \neq 0$$

Vertical Tangent:

$$\frac{dx}{dt} = 0, \quad \text{provided } \frac{dy}{dt} \neq 0$$

Second Derivative for Parametric Equations

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$$

3.2 Area with Parametric Equations

Area with Parametric Equations

or

$$A = \int_{t_1}^{t_2} y(t) \frac{dx}{dt} dt$$

$$A = \int_{t_1}^{t_2} x(t) \frac{dy}{dt} dt$$

3.3 Arc Length with Parametric Equations

The arc length of a curve is given by

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{if } y = f(x), a \leq x \leq b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \quad \text{if } x = h(y), c \leq y \leq d$$

Using the first ds , we can write

$$dx = \frac{dx}{dt} dt$$

Then the arc length formula becomes,

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \frac{dx}{dt} dt$$

$$= \int_{\alpha}^{\beta} \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \frac{dx}{dt} dt$$

Arc Length with Parametric Equations

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

3.4 Surface Area with Parametric Equations

Surface Area with Parametric Equations

$$S = \int 2\pi y ds \quad \text{rotation about } x\text{-axis}$$

$$S = \int 2\pi x ds \quad \text{rotation about } y\text{-axis}$$

where,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

3.5 Polar Coordinates

Polar to Cartesian Conversion

$$x = r \cos \theta \quad y = r \sin \theta$$

Cartesian to Polar Conversion

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$\tan \theta = \frac{y}{x}$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

3.6 Tangents with Polar Coordinates

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta$$

$$\frac{dr}{d\theta} = \frac{dr}{dx} \frac{dx}{d\theta} + \frac{dr}{dy} \frac{dy}{d\theta}$$

Tangents with Polar Coordinates

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$$

3.7 Area with Polar Coordinates

Area with Polar Coordinates

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_o^2 - r_i^2) d\theta$$

3.8 Arc Length with Polar Coordinates

Let

$$r = f(\theta) \quad \alpha \leq \theta \leq \beta$$

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

$$x = f(\theta) \cos \theta$$

$$y = f(\theta) \sin \theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

Now,

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta} \cos \theta - r \sin \theta\right)^2 + \left(\frac{dr}{d\theta} \sin \theta + r \cos \theta\right)^2 \\ &= \left(\frac{dr}{d\theta}\right)^2 \cos^2 \theta - 2r \frac{dr}{d\theta} \cos \theta \sin \theta + r^2 \sin^2 \theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2 \theta + 2r \frac{dr}{d\theta} \sin \theta \cos \theta + r^2 \cos^2 \theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2 \end{aligned}$$

Arc Length with Polar Coordinates

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

3.9 Surface Area with Polar Coordinates

Let

$$r = f(\theta) \quad \alpha \leq \theta \leq \beta$$

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

$$x = f(\theta) \cos \theta$$

$$y = f(\theta) \sin \theta$$

• Surface Area with Polar Coordinates •

$$S = \int 2\pi y \, ds \quad \text{rotation about } x\text{-axis}$$

$$S = \int 2\pi x \, ds \quad \text{rotation about } y\text{-axis}$$

where,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad r = f(\theta), \alpha \leq \theta \leq \beta$$

4 Sequences

4.1 Definition

Definition 4.1.1: Sequence

A **sequence** is a function whose domain is the set of natural numbers \mathbb{N} . The function is denoted by a_n and the value of the function at n is denoted by a_n .

Various ways of representing a sequence are:

$$\{a_1, a_2, \dots, a_n, a_{n+1}, \dots\} \quad \{a_n\} \quad \{a_n\}_{n=1}^{\infty}$$

4.2 Precise Definition of Limit of a Sequence

Precise Definition of Limit

1. We say that

$$\lim_{n \rightarrow \infty} a_n = L$$

if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - L| < \epsilon$$

2. We say that

$$\lim_{n \rightarrow \infty} a_n = \infty$$

if

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n > M$$

3. We say that

$$\lim_{n \rightarrow \infty} a_n = -\infty$$

if

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n < M$$

4.3 Convergence of Sequences

Theorem 4.3.1 (Convergence of Sequences): A sequence $\{a_n\}$ is said to be **convergent** if there exists a real number L such that

$$\lim_{n \rightarrow \infty} a_n = L$$

Theorem 4.3.2: Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim_{x \rightarrow \infty} f(x) = L$ then $\lim_{n \rightarrow \infty} a_n = L$.

Properties of Convergent Sequences

If $\{a_n\}$ and $\{b_n\}$ are both convergent sequences, then:

- 1.

$$\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$$

- 2.

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n$$

3.

$$\lim_{n \rightarrow \infty} (a_n b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) \left(\lim_{n \rightarrow \infty} b_n \right)$$

4.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}, \quad \text{provided } \lim_{n \rightarrow \infty} b_n \neq 0$$

5.

$$\lim_{n \rightarrow \infty} a_n^p = \left[\lim_{n \rightarrow \infty} a_n \right]^p, \quad \text{provided } a_n \geq 0$$

Theorem 4.3.3 (Squeeze Theorem for Sequences): If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences such that $a_n \leq b_n \leq c_n$ for all $n \geq N$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ then $\lim_{n \rightarrow \infty} b_n = L$.

Theorem 4.3.4:

$$\text{If } \lim_{n \rightarrow \infty} |a_n| = 0, \text{ then } \lim_{n \rightarrow \infty} a_n = 0$$

Proof:

The main thing to this proof is to note that,

$$-|a_n| \leq a_n \leq |a_n|$$

Then note that,

$$\lim_{n \rightarrow \infty} -|a_n| = - \lim_{n \rightarrow \infty} |a_n| = 0$$

We then have that,

$$0 \leq \lim_{n \rightarrow \infty} a_n \leq 0$$

and so by the Squeeze Theorem,

$$\lim_{n \rightarrow \infty} a_n = 0$$

Theorem 4.3.5: The sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \leq 1$ and diverges for all other values of r . Also,

$$\lim_{n \rightarrow \infty} r^n \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \end{cases}$$

Theorem 4.3.6: For the sequence $\{a_n\}$ if both $\lim_{n \rightarrow \infty} a_{2n} = L$ and $\lim_{n \rightarrow \infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n \rightarrow \infty} a_n = L$.

Proof:

Let $\epsilon > 0$. Then since $\lim_{n \rightarrow \infty} a_{2n} = L$,

$$\exists N_1 \in \mathbb{N} : \forall n \geq N_1, |a_{2n} - L| < \epsilon$$

Similarly, because $\lim_{n \rightarrow \infty} a_{2n+1} = L$,

$$\exists N_2 \in \mathbb{N} : \forall n \geq N_2, |a_{2n+1} - L| < \epsilon$$

Now, let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. Then either $a_n = a_{2k}$ for some $k > N_1$ or $a_n = a_{2k+1}$ for some $k > N_2$. And so in either case we have that

$$|a_n - L| < \epsilon$$

Therefore, $\lim_{n \rightarrow \infty} a_n = L$ and so $\{a_n\}$ is convergent.

4.4 Bounded and Monotonic Sequences

Definition 4.4.1: Bounded Sequence

A sequence $\{a_n\}$ is **bounded** if

$$\exists M \in \mathbb{R} : \forall n \in \mathbb{N}, |a_n| \leq M$$

Upper and Lower Bounds

If

$$\forall n, \exists m : m \leq a_n$$

the sequence $\{a_n\}$ is said to be **bounded below** and m is a **lower bound** of the sequence. Similarly, if

$$\forall n, \exists M : M \geq a_n$$

the sequence $\{a_n\}$ is said to be **bounded above** and M is an **upper bound** of the sequence.

Theorem 4.4.2 (Bounded Sequence): If a sequence $\{a_n\}$ is both bounded above and below, then it is bounded. That is, if

$$\exists m, M \in \mathbb{R} : \forall n \in \mathbb{N}, m \leq a_n \leq M$$

then $\{a_n\}$ is bounded.

Definition 4.4.3: Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if for all $n \in \mathbb{N}$

$$a_{n+1} \geq a_n \quad \text{or} \quad a_{n+1} \leq a_n$$

Theorem 4.4.4: If a sequence $\{a_n\}$ is both bounded and monotonic, then it is convergent.

5 Series

Properties of Convergent Series

If $\sum a_n$ and $\sum b_n$ are both convergent series, then:

1. $\sum ca_n$, where c is a constant, is also convergent and

$$\sum ca_n = c \sum a_n$$

2. $\sum(a_n \pm b_n)$ is also convergent and

$$\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$$

5.1 Convergence of Series

Theorem 5.1.1 (Convergence of Series): The series $\sum a_n$ converges if and only if the sequence of partial sums $\{s_n\}$ is convergent. That is,

$$\sum a_n \text{ converges} \iff \lim_{n \rightarrow \infty} s_n \text{ exists}$$

If the series $\sum a_n$ converges, then

$$\lim_{n \rightarrow \infty} s_n = s$$

where s is the sum of the series.

Example 5.1:

$$\lim_{n \rightarrow \infty} n = \infty \quad (\text{diverges})$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2 - 1} = 0 \quad (\text{converges})$$

$$\lim_{n \rightarrow \infty} (-1)^n \text{ does not exist} \quad (\text{diverges})$$

$$\lim_{n \rightarrow \infty} \frac{1}{3^{n-1}} = 0 \quad (\text{converges})$$

Theorem 5.1.2: If $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

5.2 Absolute and Conditional Convergence

Definition 5.2.1: Absolute and Conditional Convergence

A series $\sum a_n$ is said to be **absolutely convergent** if $\sum |a_n|$ converges. A series that converges but not absolutely is said to be **conditionally convergent**.

Note:-

If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is also convergent. But if $\sum a_n$ is conditionally convergent, then $\sum a_n$ is not necessarily absolutely convergent.

Note:-

Given the series $\sum a_n$:

1. If $\sum a_n$ is absolutely convergent and its value is s then any rearrangement of $\sum a_n$ will also have a value of s .
2. If $\sum a_n$ is conditionally convergent and r is any real number then there is a rearrangement of $\sum a_n$, say $\sum b_n$, such that $\sum b_n = r$.

5.3 Special Series

5.3.1 Geometric Series

Definition 5.3.1: Geometric Series

A series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

is called a **geometric series**. The sum of the series is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r| < 1$$

5.3.2 Telescoping Series

Definition 5.3.2: Telescoping Series

A series of the form

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - a_2 + a_2 - a_3 + \cdots$$

is called a **telescoping series**. The sum of the series is

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n \rightarrow \infty} a_{n+1}$$

5.3.3 Harmonic Series

Definition 5.3.3: Harmonic Series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is called the **harmonic series**. The harmonic series diverges.

5.4 Divergence Test

Theorem 5.4.1 (Divergence Test): If $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum a_n$ diverges.

5.5 Integral Test

Theorem 5.5.1 (Integral Test): Let $f(x)$ be a continuous, positive, and decreasing function for $x \geq 1$. Then the series

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x) dx$$

converges. That is,

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x) dx \text{ converges}$$

Proof:

Let $s_n = f(1) + f(2) + \cdots + f(n)$ and $s_{n+1} = f(1) + f(2) + \cdots + f(n) + f(n+1)$. Then

$$s_{n+1} - s_n = f(n+1) \geq 0$$

and

$$\int_n^{n+1} f(x) dx \leq f(n) \leq \int_{n-1}^n f(x) dx$$

Summing from 1 to n gives

$$s_{n+1} - s_1 \leq \int_0^n f(x) dx \leq s_n$$

Taking the limit as $n \rightarrow \infty$ gives

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} s_{n+1} = s$$

which implies that the series converges if and only if the integral converges. □

5.5.1 The p -series Test

Theorem 5.5.2 (p -series Test): The series

$$\sum_{n=k}^{\infty} \frac{1}{n^p} \quad \text{where } k > 0$$

converges if $p > 1$ and diverges if $p \leq 1$.

Proof:

If $p > 1$, then the integral

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b = \frac{1}{p-1}$$

converges. If $p \leq 1$, then the integral diverges. □

5.6 Comparison Test/Limit Comparison Test

Theorem 5.6.1 (Comparison Test): Let $\sum a_n$ and $\sum b_n$ are two series with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n . Then:

1. If $\sum b_n$ converges, then $\sum a_n$ converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Proof:

Let the partial sums be

$$s_n = \sum_{i=1}^n a_i \quad \text{and} \quad t_n = \sum_{i=1}^n b_i$$

Since $a_n, b_n \geq 0$, we know that

$$\begin{aligned} s_n \leq s_n + a_{n+1} &= \sum_{i=1}^n a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = s_{n+1} && \implies s_n \leq s_{n+1} \\ t_n \leq t_n + b_{n+1} &= \sum_{i=1}^n b_i + b_{n+1} = \sum_{i=1}^{n+1} b_i = t_{n+1} && \implies t_n \leq t_{n+1} \end{aligned}$$

Also, since $a_n \leq b_n$, we have

$$s_n = \sum_{i=1}^n a_i \leq \sum_{i=1}^n b_i = t_n$$

If $\sum b_n$ converges, then $\{t_n\}$ is bounded and increasing, so it converges. Since $s_n \leq t_n$, $\{s_n\}$ is also bounded and increasing, so it converges. \square

Theorem 5.6.2 (Limit Comparison Test): Let $\sum a_n$ and $\sum b_n$ be two series with $a_n, b_n > 0$ for all n . If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where $c \in \mathbb{R}_+$ and $c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Proof:

Since $0 < c < \infty$, there exists two positive finite numbers m and M such that $m < c < M$.

Now we have

$$\begin{aligned} m &< \frac{a_n}{b_n} < M \\ mb_n &< a_n < Mb_n \end{aligned}$$

Now, if $\sum b_n$ converges, then so does $\sum Mb_n$, and since $a_n < Mb_n$, for all n by the Comparison Test, $\sum a_n$ also converges.

Similarly, if $\sum b_n$ diverges, then so does $\sum mb_n$, and since $mb_n < a_n$, for all n by the Comparison Test, $\sum a_n$ also diverges. \square

5.7 Alternating Series Test

Theorem 5.7.1 (Alternating Series Test): Let $\sum a_n$ be an alternating series, that is, $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n > 0$ for all n . Then if:

1. $\lim_{n \rightarrow \infty} b_n = 0$

2. $\forall n, b_{n+1} \leq b_n$

the series $\sum a_n$ is convergent.

Proof:

Let $s_n = \sum_{i=1}^n a_i$. Since b_n is decreasing and $\lim_{n \rightarrow \infty} b_n = 0$, we can say

$$\forall n, b_n - b_{n+1} \geq 0$$

Now, we have

$$\begin{aligned} s_{2n} &= b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots + b_{2n-1} - b_{2n} \\ &= b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \end{aligned}$$

Since b_n is decreasing, s_{2n} is increasing and bounded above by b_1 .

Let's assume that its limit is s , that is

$$\lim_{n \rightarrow \infty} s_{2n} = s$$

Then

$$\lim_{n \rightarrow \infty} s_{2n+1} = \lim_{n \rightarrow \infty} (s_{2n} + b_{2n+1}) = \lim_{n \rightarrow \infty} s_{2n} + \lim_{n \rightarrow \infty} b_{2n+1} = s + 0 = s$$

So, we know that both $\{s_{2n}\}$ and $\{s_{2n+1}\}$ are convergent sequences and they both have the same limit.

We also know that $\{s_n\}$ is a convergent sequence with a limit of s . This in turn implies that the series $\sum a_n$ is convergent □

5.8 Ratio Test

Theorem 5.8.1 (Ratio Test): Let $\sum a_n$ be a series and let

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then:

1. If $L < 1$, then $\sum a_n$ is absolutely convergent.
2. If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.
3. If $L = 1$, then the test is inconclusive.

Proof:

Let $L < 1$ and $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then there exists a number r such that $L < r < 1$. Since $L < r$, there exists a number N such that for all $n > N$,

$$\left| \frac{a_{n+1}}{a_n} \right| < r$$

$$\begin{aligned}
|a_{n+1}| &< r |a_n| \\
|a_{n+1}| &< r |a_n| < r^2 |a_{n-1}| < \cdots < r^{n-N} |a_N| \\
|a_{n+1}| &< r^{n-N} |a_N|
\end{aligned}$$

Since $r < 1$, the series $\sum r^{n-N} |a_N|$ converges by the p -series test. By the Comparison Test, $\sum a_n$ also converges. \square

5.9 Root Test

Theorem 5.9.1 (Root Test): Let $\sum a_n$ be a series and let

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$

Then:

1. If $L < 1$, then $\sum a_n$ is absolutely convergent.
2. If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.
3. If $L = 1$, then the test is inconclusive.

Note:-

$$\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

Proof:

Let $L < 1$ and $L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$. Then there exists a number r such that $L < r < 1$. Since $L < r$, there exists a number N such that for all $n > N$,

$$\begin{aligned}
\sqrt[n]{|a_n|} &< r \\
|a_n| &< r^n
\end{aligned}$$

Since $r < 1$, the series $\sum r^n$ converges by the geometric series test. By the Comparison Test, $\sum a_n$ also converges. \square

5.10 Strategies for Series Test

Divergence Test If $\lim_{n \rightarrow \infty} a_n \neq 0$.

Geometric Series Test If $a_n = ar^n$ or $a_n = ar^{n-1}$

Integral Test If $a_n = f(n)$ and $f(x)$ is continuous, positive, and decreasing

p -series Test If $a_n = \frac{1}{n^p}$

Comparison Test If a_n is hard to work with, but b_n is easy to work with

Limit Comparison Test If a_n is a rational expression involving only polynomials.

Alternating Series Test If $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$

Ratio Test If a_n is a product of terms

Root Test If a_n is a power of terms

5.11 Estimating the Value of a Series

5.11.1 Integral Test

Integral Test

$$s_n + \int_{n+1}^{\infty} f(x) \, dx \leq s \leq s_n + \int_n^{\infty} f(x) \, dx$$

5.11.2 Comparison Test

Given a series $\sum a_n$, let's assume that we've used the comparison test to show that it's convergent. Therefore we found a second series $\sum b_n$ that converged and $a_n \leq b_n$ for all n .

Now, let

$$R_n = \sum_{k=n+1}^{\infty} a_k \quad \text{and} \quad T_n = \sum_{k=n+1}^{\infty} b_k$$

Since, $a_n \leq b_n$ we also know that $R_n \leq T_n$.

Comparison Test

$$R_n \leq T_n \leq \int_n^{\infty} g(x) \, dx, \quad \text{where } g(n) = b_n$$

5.11.3 Alternating Series Test

Alternating Series Test

$$|R_n| = |s - s_n| \leq b_{n+1}$$

5.11.4 Ratio Test

Ratio Test

To get an estimate of the remainder, let's first define the following sequence,

$$r_n = \frac{a_{n+1}}{a_n}$$

We now have two possible cases:

1. If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, then

$$R_n \leq \frac{a_{n+1}}{1 - r_{n+1}}$$

2. If $\{r_n\}$ is an increasing sequence, then

$$R_n \leq \frac{a_{n+1}}{1 - L}, \quad \text{where } L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

5.12 Power Series

Definition 5.12.1: Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where c_n are constants and called coefficients and a is a fixed number. The number a is called the **center** of the power series.

Theorem 5.12.2 (Convergence of Power Series): Given the power series $\sum c_n(x-a)^n$, the **radius of convergence** is given by

$$R = \frac{1}{L}$$

where

$$L = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$$

The power series converges absolutely for all x such that $|x-a| < R$ and diverges for all x such that $|x-a| > R, x \neq a$.

N.B: the series always converges at $x = a$.

5.13 Power Series and Functions

Theorem 5.13.1 (Power Series and Functions): Let $f(x) = \sum c_n(x-a)^n$ be a power series with radius of convergence R . Then:

1. $f(x)$ is continuous and differentiable on the interval $(-R, R)$

2. $f'(x) = \sum n c_n (x-a)^{n-1}$

3. $\int f(x) dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

4. The radius of convergence of $f'(x)$ and $\int f(x) dx$ are also R .

5.14 Taylor Series

Definition 5.14.1: Taylor Series

The **Taylor series** of a function $f(x)$ about $x = a$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

where $f^{(n)}(a)$ is the n th derivative of $f(x)$ evaluated at $x = a$.

To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the **n-th degree Taylor polynomial** of $f(x)$ as

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i$$

Notice that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

the n-th degree Taylor polynomial is just the partial sum for the series. The **remainder** is defined to be

$$R_n(x) = f(x) - T_n(x)$$

So, the remainder is just the *error* between the function $f(x)$ and the n-th degree Taylor polynomial $T_n(x)$ for a given n .

With this definition, we can then write the function as

$$f(x) = T_n(x) + R_n(x)$$

Theorem 5.14.2: Suppose that $f(x) = T_n(x) + R_n(x)$. Then if

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

for $|x-a| < R$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

on $|x-a| < R$.

5.14.1 Maclaurin Series

Definition 5.14.3: Maclaurin Series

The **Maclaurin series** of a function $f(x)$ is the Taylor series of $f(x)$ about $x = 0$. That is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Note:-

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

5.15 Binomial Series

Theorem 5.15.1 (Binomial Theorem): If n is a positive integer, then

$$\begin{aligned}(a+b)^n &= \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n\end{aligned}$$

where,

$$\begin{aligned}\binom{n}{i} &= \frac{n!}{i!(n-i)!} = \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!} \\ \binom{n}{0} &= 1\end{aligned}$$

Binomial Series

If k is a number and $|x| < 1$, then

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \binom{k}{n}x^n\end{aligned}$$

where,

$$\begin{aligned}\binom{k}{n} &= \frac{k!}{n!(k-n)!} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \\ \binom{k}{0} &= 1\end{aligned}$$

6 3-Dimensional Space

The 3-D coordinate system is often denoted by \mathbb{R}^3 . Likewise, the 2-D coordinate system is denoted by \mathbb{R}^2 , and the 1-D coordinate system is denoted by \mathbb{R} .

6.1 Equations of Lines

Vector form

If \vec{a} and \vec{v} are parallel vectors, then $\vec{a} = t\vec{v}$ for some scalar t .
Now if we have a vector \vec{r} as follows

$$\vec{r} = \vec{r}_0 + \vec{a}$$

Then we can write

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

This is called the **vector form of the equation of a line**.

Parametric form

We can rewrite the vector form as

$$\begin{aligned}\vec{r} &= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \\ \langle x, y, z \rangle &= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle\end{aligned}$$

In other words

$$\begin{aligned}x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc\end{aligned}$$

This set of equations is called the **parametric form of the equation of a line**.

Symmetric Equations of a Line

If we assume that a, b , and c are non-zero numbers, then we can solve each of the parametric equations for t . This gives us

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 6.1: Find the Equations of lines:

1. Through the points $(7, -3, 1)$ and $(-2, 1, 4)$
2. Through the point $(1, -5, 0)$ and parallel to the line given by $\vec{r}(t) = \langle 8 - 3t, -10 + 9t, -1 - t \rangle$
3. Through the point $(-7, 2, 4)$ and orthogonal to both $\vec{v} = \langle 0, -9, 1 \rangle$ and $\vec{w} = 3\hat{i} + \hat{j} - 4\hat{k}$

1.

Direction vector $\vec{d} = \langle -2 - 7, 1 + 3, 4 - 1 \rangle = \langle -9, 4, 3 \rangle$

Now, the vector form of the line is

$$\vec{r} = \langle 7, -3, 1 \rangle + t\langle -9, 4, 3 \rangle$$

The parametric form is

$$x = 7 - 9t, \quad y = -3 + 4t, \quad z = 1 + 3t$$

The symmetric form is

$$\frac{x-7}{-9} = \frac{y+3}{4} = \frac{z-1}{3}$$

2.

The direction vector is $\vec{d} = \langle 3, 9, -1 \rangle$

Hence, the vector form of the line is

$$\vec{r} = \langle 1, -5, 0 \rangle + t\langle 3, 9, -1 \rangle$$

The parametric form is

$$x = 1 + 3t, \quad y = -5 + 9t, \quad z = -t$$

And the symmetric form is

$$\frac{x-1}{3} = \frac{y+5}{9} = -z$$

3.

Direction vector

$$\vec{d} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -9 & 1 \\ 3 & 1 & -4 \end{vmatrix} = \langle 35, 3, 27 \rangle$$

Hence, the vector form of the line is

$$\vec{r} = \langle -7, 2, 4 \rangle + t\langle 35, 3, 27 \rangle$$

The parametric form is

$$x = -7 + 35t, \quad y = 2 + 3t, \quad z = 4 + 27t$$

The symmetric form is

$$\frac{x+7}{35} = \frac{y-2}{3} = \frac{z-4}{27}$$

Example 6.2: Determine if the two lines are parallel, orthogonal, or neither:

1. The line given by $\vec{r}(t) = \langle 4 - 7t, -10 + 5t, 21 - 4t \rangle$ and the line given by $\vec{r}(t) = \langle -2 + 3t, 7 + 5t, 5 + t \rangle$

2. The line given by $x = 29, y = -3 - 6t, z = 12 - t$ and the line given by $\vec{r}(t) = \langle 12 - 14t, 2 + 7t, -10 + 3t \rangle$

1.

The direction vectors are

$$\vec{d}_1 = \langle -7, 5, -4 \rangle, \quad \vec{d}_2 = \langle 3, 5, 1 \rangle$$

To check if they are parallel, we can check:

$$\frac{-7}{3} \neq \frac{5}{5} \neq \frac{-4}{1}$$

which means they are not parallel.

To check if they are orthogonal, we can check:

$$\vec{d}_1 \cdot \vec{d}_2 = -7(3) + 5(5) + (-4)(1) = -21 + 25 - 4 = 0$$

Hence, they are orthogonal.

2.

The direction vectors are

$$\vec{d}_1 = \langle 0, -6, -1 \rangle, \quad \vec{d}_2 = \langle -14, 7, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{0}{-14} \neq \frac{-6}{7} \neq \frac{-1}{3}$$

which means they are not parallel.

To check if they are orthogonal, we can check:

$$\vec{d}_1 \cdot \vec{d}_2 = 0(-14) + (-6)(7) + (-1)(3) = -42 - 3 = -45 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 6.3: Determine the intersection point of the two lines or show that they don't intersect:

1. The line passing through the point $(0, -9, -1)$ and $(1, 6, -3)$ and the line given by $\vec{r}(t) = \langle -9 - 4t, 10 + 6t, 1 - 2t \rangle$

2. The line given by $x = 1 + 6t, t = -1 - 3t, z = 4 + 12t$ and the line given by $x = 4 + t, y = -10 - 8t, z = 3 - 5t$

1.

The direction vector of the first line is

$$\vec{d}_1 = \langle 1 - 0, 6 + 9, -3 + 1 \rangle = \langle 1, 15, -2 \rangle$$

We can write the parametric equations of the first line as:

$$x = s, y = -9 + 15s, z = -1 - 2s$$

And the parametric equations of the second line as:

$$x = -9 - 4t, y = 10 + 6t, z = 1 - 2t$$

Setting them equal to each other we get,

$$\begin{aligned} 0 + t &= -9 - 4s \\ -9 + 15t &= 10 + 6s \\ -1 - 2t &= 1 - 2s \end{aligned}$$

Solving the first two equations, we get

$$t = -\frac{7}{3}, \quad s = \frac{1}{3}$$

Now, verifying the third equation, we get

$$\begin{aligned} -1 - 2\left(-\frac{7}{3}\right) &= 1 - 2\left(\frac{1}{3}\right) \\ -1 + \frac{14}{3} &= 1 - \frac{2}{3} \\ \frac{11}{3} &\neq \frac{1}{3} \end{aligned}$$

Since the third equation is not satisfied, the two lines do not intersect.

2.

The lines are given in parametric form.

Setting them equal to each other we get,

$$\begin{aligned}1 + 6s &= 4 + t \\ -1 - 3s &= -10 - 8t \\ 4 + 12s &= 3 - 5t\end{aligned}$$

Solving the first two equations, we get

$$s = \frac{1}{3}, \quad t = -1$$

Now, verifying the third equation, we get

$$\begin{aligned}4 + 12\left(\frac{1}{3}\right) &= 3 - 5(-1) \\ 8 &= 8\end{aligned}$$

That means, the lines intersect. Substituting the values in the parametric equation, we get

$$\begin{aligned}x &= 1 + 6\left(\frac{1}{3}\right) = 3 \\ y &= -1 - 3\left(\frac{1}{3}\right) = -2 \\ z &= 4 + 12\left(\frac{1}{3}\right) = 8\end{aligned}$$

Hence, the intersection point is $(3, -2, 8)$.

Example 6.4: Which of the three coordinate planes does the line given by $x = 16t, y = -4 - 9t, z = 34$ intersect?

To intersect the xy -plane, we need $z = 0$. But here $z = 34$ is constant. Hence, the line does not intersect the xy -plane.

To intersect the yz -plane, we need $x = 0$. Hence,

$$16t = 0 \implies t = 0$$

And the intersection point is $(0, -4 - 9 \times 0, 34)$ or $(0, -4, 34)$.

To intersect the xz -plane, we need $y = 0$. Hence,

$$-4 - 9t = 0 \implies t = -\frac{4}{9}$$

And the intersection point is $\left(16\left(-\frac{4}{9}\right), 0, 34\right)$ or $\left(-\frac{64}{9}, 0, 34\right)$.

6.2 Equations of Planes

Vector form

Let's assume $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\vec{r} = \langle x, y, z \rangle$ are two position vectors and $\vec{r} - \vec{r}_0$ is a vector in the plane.

If $\vec{n} = \langle a, b, c \rangle$ is a normal to the plane (which means it's orthogonal to the vector $\vec{r} - \vec{r}_0$), then we can write

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \implies \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

This is called the **vector form of the equation of a plane**.

Scalar form

If we expand the vector equation in the following way,

$$\begin{aligned} \vec{n} \cdot (\vec{r} - \vec{r}_0) &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \end{aligned}$$

Computing the dot product, we get

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the **scalar form of the equation of a plane**.

This equation can also be written as

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

Example 6.5: Find the equation of the plane:

1. Through the point $(6, -3, 1)$, $(5, -4, 1)$, and $(3, -4, 0)$
2. The plane containing the point $(1, -5, 8)$ and orthogonal to the line given by $x = -3 + 15t, y = 14 - t, z = 9 - 3t$
3. The plane containing the point $(-8, 3, 7)$ and parallel to the plane given by $4x + 8y - 2z = 45$
4. The plane containing the two lines given by $\vec{r}(t) = \langle 7 + 5t, 2 + t, 6t \rangle$ and $\vec{r}(t) = \langle 7 - 6t, 2 - 2t, 10t \rangle$

1.

The given points are

$$A(6, -3, 1), B(5, -4, 1), C(3, -4, 0)$$

Two vectors in the plane are

$$\begin{aligned} \vec{AB} &= \langle 5 - 6, -4 + 3, 1 - 1 \rangle = \langle -1, -1, 0 \rangle \\ \vec{BC} &= \langle 3 - 5, -4 + 4, 0 - 1 \rangle = \langle -2, 0, -1 \rangle \end{aligned}$$

Normal vector on the plane:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 0 \\ -2 & 0 & -1 \end{vmatrix} = \hat{i} - \hat{j} - 2\hat{k}$$

Now, using the point A , we can write the equation of the plane as

$$\begin{aligned}(x - 6) - (y + 3) - 2(z - 1) &= 0 \\ x - y - 2z &= 7\end{aligned}$$

2.

The normal vector is

$$\vec{n} = \langle 15, -1, -3 \rangle$$

Using the point $(1, -5, 8)$, the equation of the plane is

$$\begin{aligned}15(x - 1) - (y + 5) - 3(z - 8) &= 0 \\ 15x - y - 3z &= 15 + 5 - 24 \\ 15x - y - 3z + 4 &= 0\end{aligned}$$

3.

The normal vector is

$$\vec{n} = \langle 4, 8, -2 \rangle$$

Using the point $(-8, 3, 7)$, the equation of the plane is

$$\begin{aligned}4(x + 8) + 8(y - 3) - 2(z - 7) &= 0 \\ 4x + 8y - 2z &= -32 + 24 + 14 \\ 4x + 8y - 2z + 6 &= 0\end{aligned}$$

4.

The direction vectors of the two lines are

$$\vec{d}_1 = \langle 5, 1, 6 \rangle, \quad \vec{d}_2 = \langle -6, -2, 10 \rangle$$

The normal vector is

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 1 & 6 \\ -6 & -2 & 10 \end{vmatrix} = \langle 22, -86, -4 \rangle$$

Using the point $A(7, 2, 0)$, the equation of the plane is

$$\begin{aligned}22(x - 7) - 86(y - 2) - 4(z - 0) &= 0 \\ 22x - 86y - 4z - 154 + 172 &= 0 \\ 22x - 86y - 4z + 18 &= 0\end{aligned}$$

Example 6.6: Determine if the two planes are parallel, orthogonal, or neither:
The plane given by $3x + 9y + 7z = -1$ and the plane containing the points $(1, -1, 9), (4, -1, 2), (-2, 3, 4)$

The normal vector of the first plane is

$$\vec{n}_1 = \langle 3, 9, 7 \rangle$$

Let the points be

$$A(1, -1, 9), B(4, -1, 2), C(-2, 3, 4)$$

Two vectors in the plane are

$$\begin{aligned}\vec{AB} &= \langle 4 - 1, -1 + 1, 2 - 9 \rangle = \langle 3, 0, -7 \rangle \\ \vec{AC} &= \langle -2 - 1, 3 + 1, 4 - 9 \rangle = \langle -3, 4, -5 \rangle\end{aligned}$$

The normal vector of the second plane is

$$\vec{n}_2 = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -7 \\ -3 & 4 & -5 \end{vmatrix} = \langle 28, 36, 12 \rangle = \langle 7, 9, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{3}{7} \neq \frac{9}{9} \neq \frac{7}{3}$$

which means they are not parallel.

To check if they are orthogonal, we can check:

$$\vec{n}_1 \cdot \vec{n}_2 = 3(7) + 9(9) + 7(3) = 21 + 81 + 21 = 123 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 6.7: Find the intersection of the plane given by $4x + y + 10z = -2$ and the plane given by $-8x + 2y + 3z = -8$

The two planes are

$$\begin{aligned}4x + y + 10z &= -2 \\ -8x + 2y + 3z &= -8\end{aligned}$$

Multiplying the first equation by 2 and adding it to the second equation, we get

$$4y + 23z = -12 \implies y = -3 - \frac{23}{4}z$$

Substituting the value of y in the first equation, we get

$$16x - 3 - \frac{23}{4}z + 10z = -2 \implies x = \frac{1}{4} - \frac{17}{16}z$$

Let $z = t$ (a parameter). Then we get

$$\begin{aligned}x &= \frac{1}{4} - \frac{17}{16}t \\ y &= -3 - \frac{23}{4}t \\ z &= t\end{aligned}$$

This is the parametric form of the line of intersection.

We can also write it in vector form as

$$\vec{r} = \langle \frac{1}{4}, -3, 0 \rangle + t \langle -\frac{17}{16}, -\frac{23}{4}, 1 \rangle$$

6.3 Quadratic Surfaces

General form

The general form of a quadratic surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where $A, B, C, D, E, F, G, H, I, J$ are constants.

Ellipsoid

The general equation of an ellipsoid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the ellipsoid and a, b, c are the semi-axis lengths. If $a = b = c$, we get a sphere.

Cone

The general equation of a cone that opens along the z -axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{(z-l)^2}{c^2}$$

where (h, k, l) is the center of the cone and a, b, c are the semi-axis lengths.

Cylinder

The general equation of a cylinder that opens along the z -axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where (h, k) is the center of the cylinder and a, b are the semi-axis lengths. If $a = b$, we get a circular cylinder.

Hyperboloid of One Sheet

The general equation of a hyperboloid of one sheet is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

Hyperboloid of Two Sheets

The general equation of a hyperboloid of two sheets is

$$-\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

•Elliptic Paraboloid•

The general equation of an elliptic paraboloid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

•Hyperbolic Paraboloid•

The general equation of a hyperbolic paraboloid is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

6.4 Calculus with Vector Functions

Let

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

•Note:-•

$$\lim_{t \rightarrow a} \vec{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle$$

•Note:-•

$$\frac{d}{dt} (\vec{u} + \vec{v}) = \vec{u}' + \vec{v}'$$

$$\frac{d}{dt} (c\vec{u}) = c\vec{u}'$$

$$\frac{d}{dt} (\vec{u} \cdot \vec{v}) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

$$\frac{d}{dt} (\vec{u} \times \vec{v}) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$$

$$\frac{d}{dt} (\vec{u} f(t)) = f'(t) \vec{u}'(f(t))$$

•Note:-•

$$\int \vec{r}(t) dt = \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle$$

$$\int_a^b \vec{r}(t) dt = \left\langle \int_a^b f(t) dt + \int_a^b g(t) dt + \int_a^b h(t) dt \right\rangle$$

6.5 Tangent, Normal, and Binormal Vectors

•Unit Tangent vector•

Given the vector function $\vec{r}(t)$, we call $\vec{r}'(t)$ the **tangent vector** The unit tangent vector is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Unit Normal vector

If $\vec{T}(t)$ is the unit tangent vector, then the **unit normal vector** is given by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Note:-

If $\vec{r}(t)$ is a vector such that $\|\vec{r}(t)\| = c$ for all t , then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$

Binormal vector

The **binormal vector** is given by

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The binormal vector is orthogonal to both the tangent and normal vectors.

6.6 Arc Length with Vector Functions

Note:-

The arc length of a vector function $\vec{r}(t)$ from $t = a$ to $t = b$ is given by

$$L = \int_a^b \|\vec{r}'(t)\| dt$$

Or,

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2} dt$$

6.7 Curvature

Curvature of a curve in 3-D space

The curvature of a curve in 3-D space is given by

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

where $\vec{T}(t)$ is the unit tangent vector and $\vec{r}(t)$ is the position vector.

This can also be written as

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$