Calculus III Notes

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1 3-Dimensional Space

The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise, the 2-D coordinate system is denoted by $\mathbb{R}^2$, and the 1-D coordinate system is denoted by $\mathbb{R}$.

1.1 Equations of Lines

Vector form

If $\vec{a}$ and $\vec{v}$ are parallel vectors, then $\vec{a} = t\vec{v}$ for some scalar t.
Now if we have a vector $\vec{r}$ as follows

$$\vec{r} = \vec{r_0} + \vec{a}$$

Then we can write

$$\vec{r} = \vec{r_0} + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, b \rangle$$

This is called the **vector form of the equation of a line**.

Parametric form

We can rewrite the vector form as

$$\vec{r} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$
$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

In other words

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

This set of equations is called the **parametric form of the equation of a line**.

Symmetric Equations of a Line

If we assume that a, b, and c are non-zero numbers, then we can solve each of the parametric equations for t. This gives us

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 1.1: Find the Equations of lines:
1. Through the points $(7, -3, 1)$ **and** $(-2, 1, 4)$
2. Through the point $(1, -5, 0)$ **and parallel to the line given by** $\vec{r}(t) = \langle 8 - 3t, -10 + 9t, -1 - t \rangle$
3. Through the point $(-7, 2, 4)$ **and orthogonal to both** $\vec{v} = \langle 0, -9, 1 \rangle$ **and** $\vec{w} = 3\hat{i} + \hat{j} - 4\hat{k}$
1.
Direction vector $\vec{d} = \langle -2 - 7, 1 + 3, 4 - 1 \rangle = \langle -9, 4, 3 \rangle$
Now, the vector form of the line is

$$\vec{r} = \langle 7, -3, 1 \rangle + t\langle -9, 4, 3 \rangle$$

The parametric form is

$$x = 7 - 9t, \quad y = -3 + 4t, \quad z = 1 + 3t$$

The symmetric form is

$$\frac{x-7}{-9} = \frac{y+3}{4} = \frac{z-1}{3}$$

2.
The direction vector is $\vec{d} = \langle 3, 9, -1 \rangle$
Hence, the vector form of the line is

$$\vec{r} = \langle 1, -5, 0 \rangle + t\langle 3, 9, -1 \rangle$$

The parametric form is

$$x = 1 + 3t, \quad y = -5 + 9t, \quad z = -t$$

And the symmetric form is

$$\frac{x-1}{3} = \frac{y+5}{9} = -z$$

3.
Direction vector

$$\vec{d} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -9 & 1 \\ 3 & 1 & -4 \end{vmatrix} = \langle 35, 3, 27 \rangle$$

Hence, the vector form of the line is

$$\vec{r} = \langle -7, 2, 4 \rangle + t\langle 35, 3, 27 \rangle$$

The parametric form is

$$x = -7 + 35t, \quad y = 2 + 3t, \quad z = 4 + 27t$$

The symmetric form is

$$\frac{x+7}{35} = \frac{y-2}{3} = \frac{z-4}{27}$$

Example 1.2: Determine if the two lines are parallel, orthogonal, or neither:
1. The line given by $\vec{r}(t) = \langle 4-7t, -10+5t, 21-4t \rangle$ **and the line given by** $\vec{r}(t) = \langle -2+3t, 7+5t, 5+t \rangle$
2. The line given by $x = 29, y = -3-6t, z = 12-t$ **and the line given by** $\vec{r}(t) = \langle 12-14t, 2+7t, -10+3t \rangle$

1.
The direction vectors are

$$\vec{d_1} = \langle -7, 5, -4 \rangle, \quad \vec{d_2} = \langle 3, 5, 1 \rangle$$

To check if they are parallel, we can check:

$$\frac{-7}{3} \neq \frac{5}{5} \neq \frac{-4}{1}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{d_1} \cdot \vec{d_2} = -7(3) + 5(5) + (-4)(1) = -21 + 25 - 4 = 0$$

Hence, they are orthogonal.
2.
The direction vectors are

$$\vec{d_1} = \langle 0, -6, -1 \rangle, \quad \vec{d_2} = \langle -14, 7, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{0}{-14} \neq \frac{-6}{7} \neq \frac{-1}{3}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{d_1} \cdot \vec{d_2} = 0(-14) + (-6)(7) + (-1)(3) = -42 - 3 = -45 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 1.3: Determine the intersection point of the two lines or show that they don't not intersect:
1. The line passing through the point $(0, -9, -1)$ **and** $(1, 6, -3)$ **and the line given by** $\vec{r}(t) = \langle -9 - 4t, 10 + 6t, 1 - 2t \rangle$
2. The line given by $x = 1 + 6t, t = -1 - 3t, z = 4 + 12t$ **and the line given by** $x = 4 + t, y = -10 - 8t, z = 3 - 5t$
1.
The direction vector of the first line is

$$\vec{d_1} = \langle 1 - 0, 6 + 9, -3 + 1 \rangle = \langle 1, 15, -2 \rangle$$

We can write the parametric equations of the first line as:

$$x = s, y = -9 + 15s, z = -1 - 2s$$

And the parametric equations of the second line as:

$$x = -9 - 4t, y = 10 + 6t, z = 1 - 2t$$

Setting them equal to each other we get,

$$\begin{aligned} 0 + t &= -9 - 4s \\ -9 + 15t &= 10 + 6s \\ -1 - 2t &= 1 - 2s \end{aligned}$$

Solving the first two equations, we get

$$t = -\frac{7}{3}, \quad s = \frac{1}{3}$$

Now, verifying the third equation, we get

$$\begin{aligned} -1 - 2(-\frac{7}{3}) &= 1 - 2(\frac{1}{3}) \\ -1 + \frac{14}{3} &= 1 - \frac{2}{3} \\ \frac{11}{3} &\neq \frac{1}{3} \end{aligned}$$

Since the third equation is not satisfied, the two lines do not intersect.
2.
The lines are given in parametric form.
Setting them equal to each other we get,

$$\begin{aligned} 1 + 6s &= 4 + t \\ -1 - 3s &= -10 - 8t \\ 4 + 12s &= 3 - 5t \end{aligned}$$

Solving the first two equations, we get

$$s = \frac{1}{3}, \quad t = -1$$

Now, verifying the third equation, we get

$$\begin{aligned} 4 + 12\left(\frac{1}{3}\right) &= 3 - 5\langle -1 \rangle \\ 8 &= 8 \end{aligned}$$

That means, the lines intersect. Substituting the values in the parametric equation, we get

$$\begin{aligned} x &= 1 + 6\left(\frac{1}{3}\right) = 3 \\ y &= -1 - 3\left(\frac{1}{3}\right) = -2 \\ z &= 4 + 12\left(\frac{1}{3}\right) = 8 \end{aligned}$$

Hence, the intersection point is $(3, -2, 8)$.

Example 1.4: Which of the three coordinate planes does the line given by $x = 16t, y = -4 - 9t, z = 34$ intersect?

To intersect the xy-plane, we need $z = 0$. But here $z = 34$ is constant. Hence, the line does not intersect the xy-plane.
To intersect the yz-plane, we need $x = 0$. Hence,

$$16t = 0 \implies t = 0$$

And the intersection point is $(0, -4 - 9 \times 0, 34)$ or $(0, -4, 34)$.
To intersect the xz-plane, we need $y = 0$. Hence,

$$-4 - 9t = 0 \implies t = -\frac{4}{9}$$

And the intersection point is $\left(16\left(-\frac{4}{9}\right), 0, 43\right)$ or $\left(-\frac{64}{9}, 0, 34\right)$.

1.2 Equations of Planes

Vector form

Let's assume $\vec{r_0} = \langle x_0, y_0, z_0\rangle$ and $\vec{r} = \langle x, y, z\rangle$ are two position vectors and $\vec{r} - \vec{r_0}$ is a vector in the plane.
If $\vec{n} = \langle a, b, c\rangle$ is a normal to the plane (which means it's orthogonal to the vector $\vec{r} - \vec{r_0}$), then we can write

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \implies \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

This is called the **vector form of the equation of a plane**.

Scalar form

If we expand the vector equation in the following way,

$$\begin{aligned} \vec{n} \cdot (\vec{r} - \vec{r_0}) &= 0 \\ \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle &= 0 \end{aligned}$$

Computing the dot product, we get

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the **scalar form of the equation of a plane**.
This equation can also be written as

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

Example 1.5: Find the equation of the plane:
1. Through the point $(6, -3, 1)$**,** $(5, -4, 1)$**, and** $(3, -4, 0)$
2. The plane containing the point $(1, -5, 8)$ **and orthogonal to the line given by** $x = -3 + 15t, y = 14 - t, z = 9 - 3t$
3. The plane containing the point $(-8, 3, 7)$ **and parallel to the plane given by** $4x + 8y - 2z = 45$
4. The plane containing the two lines given by $\vec{r}(t) = \langle 7 + 5t, 2 + t, 6t\rangle$ **and** $\vec{r}(t) = \langle 7 - 6t, 2 - 2t, 10t\rangle$

1.
The given points are

$$A(6, -3, 1), B(5, -4, 1), C(3, -4, 0)$$

Two vectors in the plane are

$$\begin{aligned} \vec{AB} &= \langle 5 - 6, -4 + 3, 1 - 1\rangle = \langle -1, -1, 0\rangle \\ \vec{BC} &= \langle 3 - 5, -4 + 4, 0 - 1\rangle = \langle -2, 0, -1\rangle \end{aligned}$$

Normal vector on the place:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 0 \\ -2 & 0 & -1 \end{vmatrix} = \hat{i} - \hat{j} - 2\hat{k}$$

Now, using the point A, we can write the equation of the plane as

$$\begin{aligned}(x-6)-(y+3)-2(z-1)&=0\\ x-y-2z&=7\end{aligned}$$

2.
The normal vector is

$$\vec{n}=\langle 15,-1,-3\rangle$$

Using the point $(1,-5,8)$, the equation of the plane is

$$\begin{aligned}15(x-1)-(y+5)-3(z-8)&=0\\ 15x-y-3z&=15+5-24\\ 15x-y-3z+4&=0\end{aligned}$$

3.
The normal vector is

$$\vec{n}=\langle 4,8,-2\rangle$$

Using the point $(-8,3,7)$, the equation of the plane is

$$\begin{aligned}4(x+8)+8(y-3)-2(z-7)&=0\\ 4x+8y-2z&=-32+24+14\\ 4x+8y-2z+6&=0\end{aligned}$$

4.
The direction vectors of the two lines are

$$\vec{d_1}=\langle 5,1,6\rangle,\quad \vec{d_2}=\langle -6,-2,10\rangle$$

The normal vector is

$$\vec{n}=\vec{d_1}\times\vec{d_2}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ 5&1&6\\ -6&-2&10\end{vmatrix}=\langle 22,-86,-4\rangle$$

Using the point $A(7,2,0)$, the equation of the plane is

$$\begin{aligned}22(x-7)-86(y-2)-4(z-0)&=0\\ 22x-86y-4z-154+172&=0\\ 22x-86y-4z+18&=0\end{aligned}$$

Example 1.6: Determine if the two planes are parallel, orthogonal, or neither: The plane given by $3x+9y+7z=-1$ and the plane containing the points $(1,-1,9),(4,-1,2),(-2,3,4)$

The normal vector of the first plane is

$$\vec{n_1}=\langle 3,9,7\rangle$$

Let the points be

$$A(1,-1,9), B(4,-1,2), C(-2,3,4)$$

Two vectors in the plane are

$$\begin{aligned}\vec{AB} &= \langle 4-1, -1+1, 2-9\rangle = \langle 3,0,-7\rangle \\ \vec{AC} &= \langle -2-1, 3+1, 4-9\rangle = \langle -3,4,-5\rangle\end{aligned}$$

The normal vector of the second plane is

$$\vec{n_2} = \vec{AB}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -7 \\ -3 & 4 & -5 \end{vmatrix} = \langle 28,36,12\rangle = \langle 7,9,3\rangle$$

To check if they are parallel, we can check:

$$\frac{3}{7} \neq \frac{9}{9} \neq \frac{7}{3}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{n_1}\cdot\vec{n_2} = 3(7)+9(9)+7(3) = 21+81+21 = 123 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 1.7: Find the intersection of the plane given by $4x+y+10z=-2$ and the plane given by $-8x+2y+3z=-8$

The two planes are

$$\begin{aligned} 4x+y+10z &= -2 \\ -8x+2y+3z &= -8 \end{aligned}$$

Multiplying the first equation by 2 and adding it to the second equation, we get

$$4y+23z=-12 \quad\Longrightarrow\quad y=-3-\frac{23}{4}z$$

Substituting the value of y in the first equation, we get

$$16x-3-\frac{23}{4}z+10z=-2 \quad\Longrightarrow\quad x=\frac{1}{4}-\frac{17}{16}z$$

Let $z=t$ (a parameter). Then we get

$$\begin{aligned} x &= \frac{1}{4}-\frac{17}{16}t \\ y &= -3-\frac{23}{4}t \\ z &= t \end{aligned}$$

This is the parametric form of the line of intersection.
We can also write it in vector form as

$$\vec{r} = \langle\frac{1}{4},-3,0\rangle + t\langle -\frac{17}{16},-\frac{23}{4},1\rangle$$

1.3 Quadratic Surfaces

General form

The general form of a quadratic surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where $A, B, C, D, E, F, G, H, I, J$ are constants.

Ellipsoid

The general equation of an ellipsoid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the ellipsoid and a, b, c are the semi-axis lengths.
If $a = b = c$, we get a sphere.

Cone

The general equation of a cone that opens along the z-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{(z-l)^2}{c^2}$$

where (h, k, l) is the center of the cone and a, b, c are the semi-axis lengths.

Cylinder

The general equation of a cylinder that opens along the z-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where (h, k) is the center of the cylinder and a, b are the semi-axis lengths.
If $a = b$, we get a circular cylinder.

Hyperboloid of One Sheet

The general equation of a hyperboloid of one sheet is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

Hyperboloid of Two Sheets

The general equation of a hyperboloid of two sheets is

$$-\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

Elliptic Paraboloid

The general equation of an elliptic paraboloid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

Hyperbolic Paraboloid

The general equation of a hyperbolic paraboloid is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

1.4 Calculus with Vector Functions

Let

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Note:-

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$

Note:-

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{u} + \vec{v}) = \vec{u}' + \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(c\vec{u}) = c\vec{u}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{u} \cdot \vec{v}) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{u} \times \vec{v}) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\vec{u}f(t)\big) = f'(t)\vec{u}'(f(t))$$

Note:-

$$\int \vec{r}(t)\,\mathrm{d}t = \langle \int f(t)\,\mathrm{d}t\,, \int g(t)\,\mathrm{d}t\,, \int h(t)\,\mathrm{d}t \rangle$$

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \left\langle \int_a^b f(t)\,\mathrm{d}t + \int_a^b g(t)\,\mathrm{d}t + \int_a^b h(t)\,\mathrm{d}t \right\rangle$$

1.5 Tangent, Normal, and Binormal Vectors

Unit Tangent vector

Given the vector function $\vec{r}(t)$, we call $\vec{r}'(t)$ the **tangent vector** The unit tangent vector is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Unit Normal vector

If $\vec{T}(t)$ is the unit tangent vector, then the **unit normal vector** is given by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Note:-

If $\vec{r}(t)$ is a vector such that $\|\vec{r}(t)\| = c$ for all t, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$

Binormal vector

The **binormal vector** is given by

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The binormal vector is orthogonal to both the tangent and normal vectors.

1.6 Arc Length with Vector Functions

Note:-

The arc length of a vector function $\vec{r}(t)$ from $t = a$ to $t = b$ is given by

$$L = \int_a^b \|\vec{r}'(t)\| \,\mathrm{d}t$$

Or,

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2} \,\mathrm{d}t$$

1.7 Curvature

Curvature of a curve in 3-D space

The curvature of a curve in 3-D space is given by

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

where $\vec{T}(t)$ is the unit tangent vector and $\vec{r}(t)$ is the position vector.
This can also be written as

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

2 Partial Derivatives

2.1 First Order Partial Derivatives

Definition 2.1.1: First Order Partial Derivative

The **first order partial derivative** of a function $f(x, y)$ is the derivative of f with respect to one variable while treating the other variable as a constant. The partial derivative of f wrt x is denoted by:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

And the partial derivative of f wrt y is denoted by:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

They can also be written in the following notations:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\big(f(x, y)\big) = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\big(f(x, y)\big) = D_y f$$

2.2 Interpretations of Partial Derivatives

Much like the first derivative of a function of one variable, the first order partial derivatives of a function of multiple variables can be interpreted as the slope of the tangent line to the surface defined by the function at a point.

Slopes of Traces

Partial derivatives are the slopes of traces. The partial derivative $f_x(a, b)$ is the slope of the trace of $f(x, y)$ for the plane $y = b$ at the point (a, b). Likewise, the partial derivative $f_y(x, y)$ is the slope of the trace of $f(x, y)$ for the plane $x = a$ at the point (a, b)

Example 2.1: Determine if $f(x, y) = \frac{x^2}{y^3}$ is increasing or decreasing at $(2, 5)$, if:
(a) we allow x to vary and hold y fixed,
(b) we allow y to vary and hold x fixed.

(a) To find the partial derivative with respect to x, we treat y as a constant:

$$f_x(x, y) = \frac{2x}{y^3} \qquad \Longrightarrow \qquad f_x(2, 5) = \frac{4}{125} > 0$$

This means that f is increasing in the x direction at the point $(2, 5)$.
(b) To find the partial derivative with respect to y, we treat x as a constant:

$$f_y(x, y) = -\frac{3x^2}{y^4} \qquad \Longrightarrow \qquad f_y(2, 5) = -\frac{12}{625} < 0$$

This means that f is decreasing in the y direction at the point $(2, 5)$.

Partial derivatives can also be interpreted as the slope of the tangent plane to the surface defined by the function at a point. The tangent plane is a linear approximation of the surface at that point.

Example 2.2: Find the slopes of the traces to $z = 10 - 4x^2 - y^2$ at the point $(1, 2)$.
The partial derivative with respect to x is:

$$f_x(x, y) = -8x \qquad \Longrightarrow \qquad f_x(1, 2) = -8$$

The partial derivative with respect to y is:

$$f_y(x, y) = -2y \qquad \Longrightarrow \qquad f_y(1, 2) = -4$$

Thus, the slope of the trace in the x direction at $(1, 2)$ is -8, and the slope of the trace in the y direction at $(1, 2)$ is -4.

We can also use partial derivatives to find the equations of the tangent lines to the traces of a surface at a point.

Example 2.3: Find the vector equations of the tangent lines to the traces to $z = 10 - 4x^2 - y^2$ at the point $(1, 2)$
The point on the trace is

$$(1, 2, f(1, 2)) = (1, 2, 10 - 4(1)^2 - (2)^2) = (1, 2, 2)$$

Hence, the equation of the tangent line to the trace for the plane $y = 2$ is:

$$\vec{r_x}(t) = \langle 1, 2, 2\rangle + t\langle 1, 0, -8\rangle = \langle 1 + t, 2, 2 - 8t\rangle$$

And the equation of the tangent line to the trace for the plane $x = 1$ is:

$$\vec{r_y}(t) = \langle 1, 2, 2\rangle + t\langle 0, 1, -4\rangle = \langle 1, 2 + t, 2 - 4t\rangle$$

Example 2.4: Find the vector equations of the tangent lines to the traces for $f(x, y) = \sin x \cos y$ at $\left(\frac{\pi}{3}, \frac{-\pi}{4}\right)$
The point on the trace is

$$\left(\frac{\pi}{3}, -\frac{\pi}{4}, f\left(\frac{\pi}{3}, -\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{3}, -\frac{\pi}{4}, \sin\left(\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4}\right)$$

Hence, the equation of the tangent line to the trace for the plane $y = -\frac{\pi}{4}$ is:

$$\begin{aligned}
\vec{r_x}(t) &= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4}\right\rangle + t\left\langle 1, 0, f_x(x, y)\right\rangle \\
&= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4}\right\rangle + t\left\langle 1, 0, \cos\left(\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{4}\right)\right\rangle \\
&= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4}\right\rangle + t\left\langle 1, 0, \frac{1}{2\sqrt{2}}\right\rangle
\end{aligned}$$

And the equation of the tangent line to the trace for the plane $x = \frac{\pi}{3}$ is:

$$\begin{aligned}
\vec{r_y}(t) &= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4} \right\rangle + t\left\langle 0, 1, f_y(x, y) \right\rangle \\
&= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4} \right\rangle + t\left\langle 0, 1, -\sin\left(\frac{\pi}{3}\right)\sin\left(-\frac{\pi}{4}\right) \right\rangle \\
&= \left\langle \frac{\pi}{3}, -\frac{\pi}{4}, \frac{\sqrt{6}}{4} \right\rangle + t\left\langle 0, 1, \frac{\sqrt{6}}{4} \right\rangle
\end{aligned}$$

2.3 Higher Order Partial Derivatives

Second Order Partial Derivatives

The **Second order partial derivatives** of a function $f(x, y)$ are the partial derivatives of the first order partial derivatives. The second order partial derivatives are denoted by:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y^2}x$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x^2}y$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Clairaut's Theorem

If the second order partial derivatives f_{xy} and f_{yx} are continuous at a point, then they are equal at that point:

$$f_{xy} = f_{yx}$$

Like second order derivatives, there are higher order partial derivatives as well. The third order partial derivatives are denoted by:

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3}$$

$$f_{xyx} = (f_{xy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial x\,\partial y\,\partial x}$$

$$f_{yxx} = (f_{yx})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\,\partial y}\right) = \frac{\partial^2 f}{\partial x^2\,\partial y}$$

This also applies to functions of more than two variables. For example,

$$f_{xz}(x, y, z) = f_{zx}(x, y, z)$$

Extension of Clairaut's Theorem

In general, we can extend Clairaut's theorem to any function and mixed partial derivatives.

That means:

$$f_{ssrtsrr} = f_{trsrssr} = f_{rrssst} = \dots$$

Example 2.5: Find all the second order partial derivatives of function $Q(u, v, w) = u^4 \sin w^2 - \frac{2v}{u^4} + \ln\bigl(v^2 w\bigr)$

To find the second order partial derivatives, we first find the first order partial derivatives:

$$\begin{aligned} Q_u &= 4u^3 \sin w^2 + \frac{8v}{u^5} \\ Q_v &= \frac{-2}{u^4} + \frac{2vw}{v^2 w} = -\frac{2}{u^4} + \frac{2}{v} \\ Q_w &= 2u^4 w \cos w^2 + \frac{1}{w} \end{aligned}$$

Now we can find the second order partial derivatives:

$$\begin{aligned} Q_{uu} &= 12u^2 \sin w^2 - \frac{40v}{u^6} \\ Q_{uv} &= Q_{vu} = \frac{8}{u^5} \\ Q_{uw} &= 8u^3 w \cos w^2 \\ Q_{vv} &= -\frac{2}{v^2} \\ Q_{vw} &= Q_{wv} = 0 \\ Q_{ww} &= 2u^4 \cos w^2 - 4u^4 w^2 \sin w^2 - \frac{1}{w^2} \end{aligned}$$

Example 2.6: Given $w = \ln\left(\frac{xy}{z}\right) + 8x^4 y^3 \sqrt{z}, find \frac{\partial^5 w}{\partial x\, \partial z^2\, \partial y\, \partial x}$

Using Clairaut's theorem,

$$\frac{\partial^5 w}{\partial x\, \partial z^2\, \partial y\, \partial x} = \frac{\partial^5 w}{\partial x^2\, \partial y\, \partial z^2}$$

Now,

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{y}{z} \cdot \frac{z}{xy} + 32x^3 y^3 \sqrt{z} = \frac{1}{x} + 32x^3 y^3 \sqrt{z} \\ \frac{\partial^2 w}{\partial x^2} &= -\frac{1}{x^2} + 96x^2 y^3 \sqrt{z} \\ \frac{\partial^3 w}{\partial x^2\, \partial y} &= 288x^2 y^2 \sqrt{z} \\ \frac{\partial^4 w}{\partial x^2\, \partial y\, \partial z} &= \frac{144x^2 y^2}{\sqrt{z}} \\ \frac{\partial^5 w}{\partial x^2\, \partial y\, \partial z^2} &= -72x^2 y^2 z^{-3/2} \end{aligned}$$

Example 2.7: Given $f(x, y) = \frac{x^6}{1+6y} - \cos\bigl(x^2\bigr) + 6e^x \sin(y)$**, find** f_{xxyxyx}

$$
\begin{aligned}
f_x &= \frac{6x^5}{1+6y} + 2x\sin\left(x^2\right) + 6e^x\sin(y) \\
f_{xx} &= \frac{30x^4}{1+6y} + 2\sin\left(x^2\right) + 4x^2\cos\left(x^2\right) + 6e^x\sin(y) \\
f_{xxx} &= \frac{120x^3}{1+6y} + 12x\cos\left(x^2\right) - 8x^3\sin\left(x^2\right) + 6e^x\sin(y) \\
f_{xxxx} &= \frac{360x^2}{1+6y} + 12\cos\left(x^2\right) - 48x^2\sin\left(x^2\right) - 16x^4\cos\left(x^2\right) + 6e^x\sin(y) \\
f_{xxxxy} &= -\frac{2160x^2}{(1+6y)^2} + 6e^x\cos(y) \\
f_{xxxxyy} &= \frac{25920x^2}{(1+6y)^3} - 6e^x\sin(y)
\end{aligned}
$$

2.4 Differentials

Differentials

The **differential** of a function $f(x,y)$ is a linear approximation of the change in the function at a point.
The differential of f is denoted by:

$$\mathrm{d}f = f_x\,\mathrm{d}x + f_y\,\mathrm{d}y$$

where dx and dy are small changes in x and y, respectively.
For a given function $w = g(x,y,z)$, the differential is given by:

$$\mathrm{d}w = g_x\,\mathrm{d}x + g_y\,\mathrm{d}y + g_z\,\mathrm{d}z$$

Example 2.8: Compute the differential for $u = \frac{t^3 r^6}{s^2}$

$$\mathrm{d}u = \frac{3t^2r^6}{s^2}\,\mathrm{d}t + \frac{6t^3r^5}{s^2}\,\mathrm{d}r - \frac{2t^3r^6}{s^3}\,\mathrm{d}s$$

2.5 Chain Rule

Case 1: If $z = f(x,y)$, $x = g(t)$, and $y = h(t)$, then the chain rule states that:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\text{Or,}\quad \frac{\mathrm{d}z}{\mathrm{d}t} = f_x\frac{\mathrm{d}x}{\mathrm{d}t} + f_y\frac{\mathrm{d}y}{\mathrm{d}t}$$

Example 2.9: Compute $\frac{\mathrm{d}z}{\mathrm{d}t}$ for $z = xe^{xy}$, $x = t^2$, $y = t^{-1}$

$$
\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\
&= (e^{xy} + yxe^{xy})(2t) + x^2e^{xy}(-t^{-2}) \\
&= 2t(e^{xy} + xye^{xy}) - x^2e^{xy}t^{-2} \\
&= 2t(e^t + te^t) - t^4e^tt^{-2} \\
&= 2te^t + t^2e^t
\end{aligned}
$$

Case 2: If $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, then the chain rule states that:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\text{Or,}\quad \frac{\partial z}{\partial s} = f_x\frac{\partial x}{\partial s} + f_y\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\text{Or,}\quad \frac{\partial z}{\partial t} = f_x\frac{\partial x}{\partial t} + f_y\frac{\partial y}{\partial t}$$

Example 2.10: Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = e^{2r}\sin(3\theta)$, $r = st - t^2$, $\theta = \sqrt{s^2 + t^2}$

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial s} \\
&= (2e^{2r}\sin(3\theta))(t + 0) + (3e^{2r}\cos(3\theta))\left(\frac{s}{\sqrt{s^2 + t^2}}\right) \\
&= 2te^{2r}\sin(3\theta) + 3e^{2r}\cos(3\theta)\frac{s}{\sqrt{s^2 + t^2}} \\
&= 2te^{2(st-t^2)}\sin\left(3\sqrt{s^2 + t^2}\right) + 3e^{2(st-t^2)}\cos\left(3\sqrt{s^2 + t^2}\right)\frac{s}{\sqrt{s^2 + t^2}}
\end{aligned}
$$

And,

$$
\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial t} \\
&= (2e^{2r}\sin(3\theta))(s - 2t) + (3e^{2r}\cos(3\theta))\left(\frac{t}{\sqrt{s^2 + t^2}}\right) \\
&= 2(s - 2t)e^{2r}\sin(3\theta) + 3e^{2r}\cos(3\theta)\frac{t}{\sqrt{s^2 + t^2}} \\
&= 2(s - 2t)e^{2(st-t^2)}\sin\left(3\sqrt{s^2 + t^2}\right) + 3e^{2(st-t^2)}\cos\left(3\sqrt{s^2 + t^2}\right)\frac{t}{\sqrt{s^2 + t^2}}
\end{aligned}
$$

Chain Rule

Given the following conditions:

(i) $z = f(x_1, x_2, \ldots, x_n)$ is a function of n variables,

(ii) Each variable $x_i(t_1, t_2, \ldots, t_m)$ is a function of m variables, Then for any variable t_i $(i = 1, 2, \ldots, m)$, we have the following chain rule:

$$\boxed{\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}}$$

Example 2.11: Compute $\frac{\partial^2 f}{\partial \theta^2}$ for $f(x, y)$ if $x = r\cos\theta$ and $y = r\sin\theta$

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\ &= -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}\end{aligned}$$

Now, we know the second derivative is

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}\right)$$

Now, we can separately compute $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$:

$$\begin{aligned}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) &= -r\sin\theta\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + r\cos\theta\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ &= -r\sin\theta\frac{\partial^2 f}{\partial x^2} + r\cos\theta\frac{\partial^2 f}{\partial y\,\partial x} \\ \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right) &= -r\sin\theta\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + r\cos\theta + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \\ &= -r\sin\theta\frac{\partial^2 f}{\partial x\,\partial y} + r\cos\theta\frac{\partial^2 f}{\partial y^2}\end{aligned}$$

Finally, we can substitute these into the second derivative:

$$\begin{aligned}\frac{\partial^2 f}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}\right) \\ &= -r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\left(-r\sin\theta\frac{\partial^2 f}{\partial x^2} + r\cos\theta\frac{\partial^2 f}{\partial y\,\partial x}\right) \\ &\quad - r\sin\theta\frac{\partial f}{\partial y} + r\cos\theta\left(-r\sin\theta\frac{\partial^2 f}{\partial x\,\partial y} + r\cos\theta\frac{\partial^2 f}{\partial y^2}\right) \\ &= -r\cos\theta\frac{\partial f}{\partial x} + r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2} - r^2\sin\theta\cos\theta\frac{\partial^2 f}{\partial y\,\partial x} \\ &\quad - r\sin\theta\frac{\partial f}{\partial y} - r^2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\,\partial y} + r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2} \\ &= -r\left(\cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}\right) \\ &\quad + r^2\left(\sin^2\theta\frac{\partial^2 f}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\,\partial y} + \cos^2\theta\frac{\partial^2 f}{\partial y^2}\right)\end{aligned}$$

2.5.1 Implicit Differentiation

Implicit Differentiation

If $F(x, y) = 0$ is a function where $y = y(x)$, then we can use implicit differentiation to find the derivative of y with respect to x. The chain rule gives us:

$$F_x + F_y \frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

This can be extended to functions of more than two variables. We can start by assuming that $z = f(x, y)$ and we want to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
To find $\frac{\partial z}{\partial x}$, we differentiate both sides wrt x:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, we get:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example 2.12: Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $x\cos(3y) + x^3y^5 = 3x - e^{xy}$
First, we rearrange the equation in the form $F(x, y) = 0$:

$$x\cos(3y) + x^3y^5 - 3x + e^{xy} = 0$$

Now, the derivative is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = -\frac{\cos(3y) + 3x^2y^5 - 3 + ye^{xy}}{-3x\sin(3y) + 5x^3y^4 + xe^{xy}}$$

Example 2.13: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^2\sin(2y - 5z) = 1 + y\cos(6zx)$
First, let's rearrange the equation in the form $F(x, y, z) = 0$:

$$x^2\sin(2y - 5z) - 1 - y\cos(6zx) = 0$$

Now, the derivatives are:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x\sin(2y - 5z) + 6yz\sin(6zx)}{-5x^2\cos(2y - 5z) + 6xy\sin(6zx)}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2x^2\cos(2y - 5z) - \cos(6zx)}{-5x^2\cos(2y - 5z) + 6xy\sin(6zx)}$$

2.6 Directional Derivatives

Definition 2.6.1: Directional Derivative

The rate of change of $f(x, y)$ in the direction of the unit vector $\vec{u} = \langle a, b\rangle$ is called the **directional derivative** of f and is denoted by $D_{\vec{u}}f(x, y)$. The definition of the directional derivative is:

$$D_{\vec{u}}f(x, y) = \lim_{h\to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h}$$

Now, in practice, finding this limit can be difficult. We can derive an equivalent formula for taking directional derivatives.
Let's define a new function of one variable:

$$g(z) = f(x_0 + az, y_0 + bz)$$

where x_0, y_0, a, b are constants. Then, by the definition of the derivative, we have

$$g'(z) = \lim_{h\to 0} \frac{g(z + h) - g(z)}{h}$$

For $z = 0$, we have:

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\vec{u}}f(x_0, y_0)$$

Thus, we have the following relationship:

$$g'(0) = D_{\vec{u}f(x_0,y_0)}$$

Now, let's rewrite $g(z)$ as follows:

$$g(z) = f(x, y) \quad \text{where } x = x_0 + az \text{ and } y = y_0 + bz$$

We can now apply the chain rule to find $g'(z)$:

$$g'(z) = \frac{\mathrm{d}g}{\mathrm{d}z} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}z} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}z} = f_x(x, y)a + f_x(x, y)b$$

If we take $z = 0$, we get $x = x_0$ and $y = y_0$, and finally we have:

$$D_{\vec{u}}f(x_0, y_0) = g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Note:-

$$D_{\vec{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$
$$D_{\vec{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

Example 2.14: Find each of the directional derivatives:
(a) $D_{\vec{u}}f(8, 1, 2)$ **where** $f(x, y, z) = \ln\frac{x}{z} + \ln\frac{z}{y} + xy^2$ **in the direction of** $\vec{v} = \langle 1, 5, 2\rangle$
(b) $D_{\vec{u}}f(x, y)$ **where** $f(x, y) = xe^{xy} + y$ **and** $\vec{u}$ **is the unit vector in the direction of** $\theta = \frac{2\pi}{3}$

(a) First, the unit vector in the direction of $\vec{v}$ is:

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{\langle 1, 5, 2\rangle}{\sqrt{1^2 + 5^2 + 2^2}} = \frac{\langle 1, 5, 2\rangle}{\sqrt{30}}$$

Simplifying the function, we have:

$$f(x, y, z) = \ln(x) - \ln(z) + \ln(z) - \ln(y) + xy^2 = \ln(x) - \ln(y) + xy^2$$

The directional derivative is then,

$$\begin{aligned}
D_{\vec{u}} f(x, y, z) &= \frac{1}{\sqrt{30}} \left[f_x(x, y, z) + 5 f_y(x, y, z) + 2 f_z(x, y, z)\right] \\
D_{\vec{u}} f(8, 1, 2) &= \frac{1}{\sqrt{30}} \left[1\left(\frac{1}{x} + y^2\right) + 5\left(-\frac{1}{y} + 2xy\right) + 2 \cdot 0\right]_{(8,1,2)} \\
&= \frac{1}{\sqrt{30}} \left(\frac{1}{8} + 1 - 5 + 80\right) \\
&= \frac{609}{8\sqrt{30}}
\end{aligned}$$

(b) The unit vector in the direction of $\theta = \frac{2\pi}{3}$ is:

$$\vec{u} = \left\langle \cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right)\right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$$

So, the directional derivative is:

$$\begin{aligned}
D_{\vec{u}} f(x, y) &= \left(-\frac{1}{2}\right)(e^{xy} + xye^{xy}) + \left(\frac{\sqrt{3}}{2}\right)\left(x^2 e^{xy} + 1\right) \\
D_{\vec{u}} f(2, 0) &= \left(-\frac{1}{2}\right)(1) + \left(\frac{\sqrt{3}}{2}\right)(5) \\
&= \frac{5\sqrt{3} - 1}{2}
\end{aligned}$$

Notice, the directional derivative can also be written in the following way:

$$\begin{aligned}
D_{\vec{u}} f(x, y, z) &= f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z) \\
&= \langle f_x, f_y, f_z\rangle \cdot \langle a, b, c\rangle
\end{aligned}$$

In other words, the directional derivative is the dot product of the gradient vector and the unit vector in the direction of interest.

Gradient Vector

The **gradient vector** of a function $f(x, y)$ is denoted by ∇f and is defined as:

$$\nabla f = \langle f_x, f_y\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle$$

For a function $f(x,y,z)$, the gradient vector is:

$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Directional Derivative

The directional derivative can also be expressed in terms of the gradient vector:

$$D_{\vec{u}} f(\vec{x}) = \nabla f \cdot \vec{u}$$

where $\vec{x} = \langle x, y, z \rangle$ or $\vec{x} = \langle x, y \rangle$ depending on the function and $\vec{u}$ is the unit vector in the direction of interest.

Example 2.15: Find the directional derivative

$$D_{\vec{u}} f(\vec{x}) \textbf{ for } f(x,y,z) = \sin(yz) + \ln\left(x^2\right) \textbf{ at } (1,1,\pi)$$

in the direction of $\vec{v} = \langle 1, 1, -1 \rangle$

The gradient vector is:

$$\nabla f(x,y,z) = \left\langle \frac{2}{x}, z\cos(yz), y\cos(yz) \right\rangle$$
$$\nabla f(1,1,\pi) = \left\langle \frac{2}{1}, \pi\cos(\pi), \cos(\pi) \right\rangle = \langle 2, -\pi, -1 \rangle$$

The unit vector in the direction of $\vec{v}$ is:

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{\langle 1, 1, -1 \rangle}{\sqrt{3}} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$$

Hence, the directional derivative is:

$$\begin{aligned} D_{\vec{u}} f(1,1,\pi) &= \nabla f \cdot \vec{u} \\ &= \left\langle 2, -\pi, -1 \right\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle \\ &= 2 \cdot \frac{1}{\sqrt{3}} - \pi \cdot \frac{1}{\sqrt{3}} + 1 \cdot \frac{1}{\sqrt{3}} = \frac{3-\pi}{\sqrt{3}} \end{aligned}$$

Theorem 2.6.2: *The maximum value of $D_{\vec{u}} f(\vec{x})$ (and hence then the maximum rate of change of the function $f(\vec{x})$) is given by $||\nabla f(\vec{x})||$ and will occur in the direction given by $\nabla f(\vec{x})$.*

Proof:

We can use a nice fact about dot products as well as the fact that $\vec{u}$ is a unit vector to proof this theorem:

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = ||\nabla f||\,||\vec{u}|| \cos\theta = ||\nabla f|| \cos\theta$$

where θ is the angle between the gradient and $\vec{u}$.
Now, the largest possible value of $\cos\theta$ is 1, which occurs at $\theta = 0$. Therefore, the maximum

value of $D_{\vec{u}}f(\vec{x})$ is $||\nabla f(\vec{x})||$. Also, the maximum value occurs when the angle between the gradient and $\vec{u}$ is zero, or in other words, when $\vec{u}$ is pointing in the same direction as the gradient.

Note:-

The gradient vector $\nabla f(x_0, y_0)$ is orthogonal (or perpendicular) to the level curve/contour curve $f(x,y) = k$ at the point (x_0, y_0). Likewise, the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x,y,z) = k$ at the point (x_0, y_0, z_0).

Proof:

Let S be the level surface given by $f(x,y,z) = k$ and let $P(x_0, y_0, z_0)$ be a point on the surface S.

Now, let C be any curve on the surface S that contains the point P. Let $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ be the vector equation for C and suppose that t_0 is the value of t such that $\vec{r}(t_0) = \langle x_0, y_0, z_0\rangle$. In other words, t_0 is the value of t that gives P.

Since C lies on S, we know that the points on C must satisfy the equation for S. That is

$$f(x(t), y(t), z(t)) = k$$

Using the chain rule, we get:

$$\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

Notice that $\nabla f = \langle f_x, f_y, f_z\rangle$ and $\vec{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ so this becomes:

$$\nabla f \cdot \vec{r}'(t) = 0$$

At $t = t_0$,

$$\nabla f(x_0, y_0, z_0) \cdot \vec{r}'(t_0) = 0$$

This then tells us that the gradient vector at P (i.e. $\nabla f(x_0, y_0, z_0)$) is orthogonal to the tangent vector $\vec{r}'(t_0)$ to any curve C that passes through P and on the surface S and su must also be orthogonal to the surface S.

3 Applications of Partial Derivatives

3.1 Tangent Planes and Linear Approximations

Let there be a point (x_0, y_0) on function $z = f(x, y)$ in $\mathbb{R}^3$. Let C_1 be the trace to $f(x, y)$ for plane $y = y_0$ and C_2 be the trace to $f(x, y)$ for plane $x = x_0$. Now, let L_1 and L_2 be the tangent lines to C_1 and C_2 at point (x_0, y_0), respectively. The tangent plane to $f(x, y)$ at point (x_0, y_0) is defined as the plane that contains both lines L_1 and L_2.

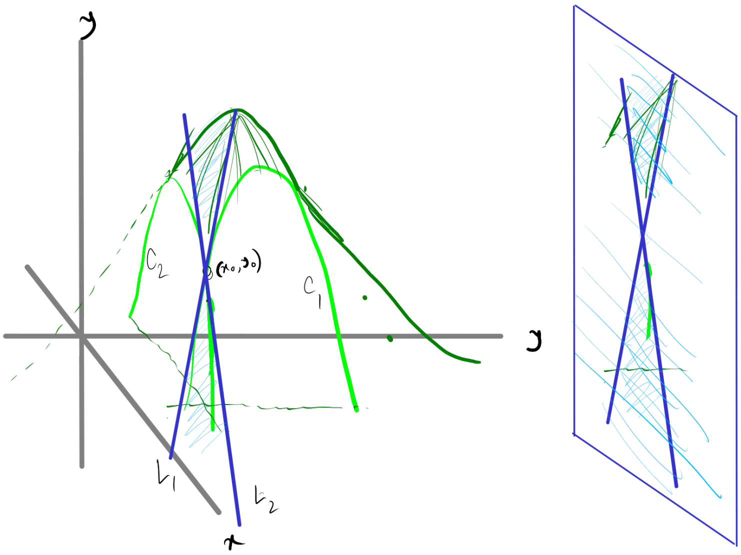

Figure 3.1.1: Tangent plane

Now, we need to find the equation of the tangent plane. The general equation of a plane is given by

$$a(x - x_0) + v(y - y_0) + c(z - z_0) = 0$$

where (x_0, y_0, z_0) is a point on the plane and a, b, and c are the coefficients of the plane. The equation can be rewritten as

$$z - z_0 = -\frac{a}{c}(x - x_0) - \frac{b}{c}(y - y_0)$$

Let's rename the constants as follows:

$$A = -\frac{a}{c}, \qquad B = -\frac{b}{c}$$

Thus we get

$$z - z_0 = A(x - x_0) + B(y - y_0)$$

Now, assuming $y = y_0$ (i.e., y is fixed) and $x = x_0$ (i.e., x is fixed) we get respectively

$$z - z_0 = A(x - x_0) \qquad \text{and} \qquad z - z_0 = B(y - y_0)$$

Note, these are the equations for the tangent lines L_1 and L_2 respectively, where the slopes are respectively $A = f_x(x_0, y_0)$ and $B = f_y(x_0, y_0)$. We also know $z_0 = f(x_0, y_0)$.
Hence, the equation of the tangent plane is given by

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

The linear approximation for a surface "near" the point (x_0, y_0) is given then

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 3.1: Find the linear approximation to $z = \frac{10x^2}{x-y}$ at $(4, -1)$

The linear approximation is given by

$$L(x, y) = z(4, -1) + z_x(4, -1)(x - 4) + z_y(4, -1)(y + 1)$$

Here,

$$\begin{aligned}
z(4, -1) &= \frac{10 \times 4^2}{4 - (-1)} \\
&= \frac{160}{5} = 32 \\
z_x(4, -1) &= \frac{\partial}{\partial x}\left(\frac{10x^2}{x - y}\right)\bigg|_{(4,-1)} \\
&= \frac{20x(x - y) - 10x^2}{(x - y)^2}\bigg|_{(4,-1)} \\
&= \frac{20 \times 4(4 - (-1)) - 10 \times 4^2}{(4 - (-1))^2} \\
&= \frac{80 \times 5 - 160}{25} = \frac{400 - 160}{25} = \frac{240}{25} = 9.6 \\
z_y(4, -1) &= \frac{\partial}{\partial y}\left(\frac{10x^2}{x - y}\right)\bigg|_{(4,-1)} \\
&= \frac{10x^2}{(x - y)^2}\bigg|_{(4,-1)} \\
&= \frac{10 \times 4^2}{(4 - (-1))^2} = \frac{160}{25} = 6.4
\end{aligned}$$

Hence, the linear approximation is given by

$$\boxed{L(x, y) = 32 + 9.6(x - 4) + 6.4(y + 1)}$$

3.2 Gradient Vector, Tangent Planes, and Normal Lines

Note:-

The gradient vector $\nabla f(x_0, y_0)$ is orthogonal to the level curve $f(x, y) = k$ at the point (x_0, y_0). Likewise, the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0).

We know, the gradient vector is

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

So, the tangent plane to the surface $f(x, y, z) = k$ at (x_0, y_0, z_0) is given by the equation

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

This is a general form of the equation of a tangent plane than that of the previous section. However, we can also write the equation in the previous form. For that, we need to find the tangent plane to the surface given by $z = f(x, y)$ at the point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$. We can rewrite as

$$f(x, y) - z = 0$$

Now, defining a new function as

$$F(x, y, z) = f(x, y) - z$$

The surface given by $z = f(x, y)$ is identical to the surface given by $F(x, y, z) = 0$. Now, the gradient vector for F is

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle f_x, f_y, -1 \rangle$$

Note:

$$\begin{aligned} F_x &= \frac{\partial}{\partial x}\big[f(x, y) - z\big] = f_x \\ F_y &= \frac{\partial}{\partial y}\big[f(x, y) - z\big] = f_y \\ F_z &= \frac{\partial}{\partial z}\big[f(x, y) - z\big] = -1 \end{aligned}$$

Hence, the tangent plane is then

$$F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$$

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

We can also find the normal line to the surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0). The equation of line requires a point and a parallel vector, which is given by the gradient vector. Hence, the equation of the normal line is given by

$$\boxed{\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\nabla f(x_0, y_0, z_0)}$$

Tangent Plane

The tangent plane to the surface given by $F(x, y, z) = f(x, y) - z$ at the point (x_0, y_0, z_0) is given by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Normal Line

The normal line to the surface given by $F(x, y, z) = f(x, y) - z$ at the point (x_0, y_0, z_0) is given by

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\nabla F(x_0, y_0, z_0)$$

Example 3.2: Find the tangent plane and normal line to $9yz - \sqrt{x^2 - 8z} = xy^2 - 26$ **at** $(3, 1, -2)$

Rearranging the equation, we get

$$xy^2 + \sqrt{x^2 - 8z} - 9yz - 26 = 0$$

The gradient vector is given by

$$\begin{aligned}\nabla F &= \langle y^2 + \frac{x}{\sqrt{x^2-8z}}, 2xy - 9z, -\frac{4}{\sqrt{x^2-8z}} - 9y\rangle \\ \nabla F(3,1,-2) &= \langle 1 + \frac{3}{5}, 6+18, -\frac{4}{5} - 9\rangle \\ &= \langle \frac{8}{5}, 24, \frac{-49}{5}\rangle\end{aligned}$$

The tangent plane, hence, would be

$$\begin{aligned}\frac{8}{5}(x-3) + 24(y-1) - \frac{49}{5}(z+2) = 0 \\ 8x - 24 + 120y - 120 - 49z - 98 = 0 \\ 8x + 120y - 49z = 242\end{aligned}$$

And the normal line is given by the equation

$$\frac{x-3}{\frac{8}{5}} = \frac{y-1}{24} = \frac{z+2}{-\frac{49}{g}}$$

$$\frac{x-3}{8} = \frac{y-1}{120} = -\frac{z+2}{49}$$

Example 3.3: Find the point(s) on $6x^2 + y^2 - 3z^2 = 4$ where the tangent plane to the surface is parallel to the plane given by $2x + 7y - z = 6$.

The gradient vector of the surface is given by

$$\nabla F = \langle 12x, 2y, -6z\rangle$$

The normal vector to the parallel plane is

$$\vec{n} = \langle 2, 7, -1\rangle$$

Since the planes are parallel, the gradient vector must be a scalar multiple of the normal vector, i.e.,

$$\langle 12x, 2y, -6z\rangle = k\langle 2, 7, -1\rangle$$

This gives us the following equations:

$$12x = 2k \implies k = 6x$$

$$2y = 7k = 42x \implies y = 21x$$

$$-6z = -k \implies z = x$$

Now, substituting these values in the equation of the surface, we get

$$\begin{aligned}6x^2 + (21x)^2 - 3x^2 = 4 \\ 6x^2 + 441x^2 - 3x^2 = 4 \\ x^2 = \frac{4}{444}\end{aligned}$$

$$\therefore x = \pm\frac{1}{\sqrt{111}}, \qquad y = \pm\frac{21}{\sqrt{111}}, \qquad z = \pm\frac{1}{\sqrt{111}}$$

Hence, the points are

$$\left(\frac{1}{\sqrt{111}}, \frac{21}{\sqrt{111}}, \frac{1}{\sqrt{111}}\right) \text{ and } \left(-\frac{1}{\sqrt{111}}, -\frac{21}{\sqrt{111}}, -\frac{1}{\sqrt{111}}\right)$$

3.3 Relative Minimums and Maximums

Definition 3.3.1: Relative Minimum and Maximum

A function $f(x, y)$ has a **relative minimum** at the point (a, b) if $f(x, y) \geq f(a, b)$ for all points $x(, y)$ in some region around (a, b).

A function $f(x, y)$ has a **relative maximum** at the point (a, b) if $f(x, y) \leq f(a, b)$ for all points $x(, y)$ in some region around (a, b).

Definition 3.3.2: Critical Point

The point (a, b) is a **critical point** (or a **stationary point**) of the function $f(x, y)$ provided one of the following is true:

1. $\nabla f(a, b) = \vec{0}$ (i.e., $f_x(a, b) = 0$ and $f_y(a, b) = 0$)
2. $f_x(a, b)$ and/or $f_y(a, b)$ do not exist

Note:-

If the point (a, b) is a relative extrema of the function $f(x, y)$ and the first order derivative of $f(x, y)$ exists at (a, b), then (a, b) is a critical point of $f(x, y)$. However, the converse is not true. That is, a critical point need not be a relative extrema.

Proof:

Let $g(x) = f(x, y)$ and suppose that $f(x, y)$ has a relative extrema at (a, b). However, this also means that $g(x)$ also has a relative extrema at $x = a$. By Fermat's theorem, we know that $g'(a) = 0$. But we also know that $g'(a) = f_x(a, b)$ and so we have that $f_x(a, b) = 0$.
If we now define $h(y) = f(a, b)$, then we can similarly show that $f_y(a, b) = 0$.
So, putting all these together means that $\nabla f(a, b) = \vec{0}$ and so $f(x, y)$ has a critical point at (a, b).

Example of a point that is a critical point but not a relative extrema is the point $(0, 0)$ for the function $f(x, y) = x^2 - y^2$. In the x-axis, the function becomes $f(x, 0) = x^2$, which is a parabola opening upwards. So, the point $(0, 0)$ looks like a minimum along this path. In the y-axis, the function becomes $f(0, y) = -y^2$, which is a parabola opening downwards. So, the point $(0, 0)$ looks like a maximum along this path. Since the point $(0, 0)$ is a minimum in one direction and a maximum in another, it's neither a relative minimum nor a relative maximum. This type of critical point is also called a **saddle point**.

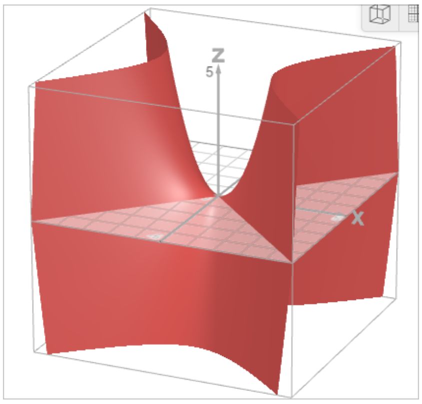

Figure 3.3.1: Saddle point

Definition 3.3.3: Saddle Point

A point (a, b) is a **saddle point** of the function $f(x, y)$ if it is a critical point but neither a relative minimum nor a relative maximum.

Definition 3.3.4: Hessian Matrix

The Hessian matrix of a function $f(x, y)$ is given by

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

where f_{xx}, f_{yy}, and f_{xy} are the second order partial derivatives of $f(x, y)$.
The Hessian matrix is used in the second derivative test to classify critical points of the function $f(x, y)$.

Relative Extrema

Suppose that (a, b) is a critical point of $f(x, y)$ and that the second order partial derivatives are continuous in some region that contains (a, b). Next, let's use the determinant of the Hessian matrix, denoted by D, to classify the critical point (a, b).

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - \left[f_{xy}(a, b)\right]^2$$

We then have the following classifications of the critical point:

1. If $D > 0$ and $f_{xx}(a, b) > 0$ then there is a relative minimum at (a, b).
2. If $D > 0$ and $f_{xx}(a, b) < 0$ then there is a relative maximum at (a, b).
3. If $D < 0$ then there is a saddle point at (a, b).
4. If $D = 0$ then the test is inconclusive. In this case, we need to use other methods to determine the nature of the critical point.

Note that if $D > 0$ then both $f_{xx}(a, b)$ and $f_{yy}(a, b)$ will have the same sign and so in the first two cases, we can conclude that both $f_{xx}(a, b)$ and $f_{yy}(a, b)$ are either both positive or both negative.

Example 3.4: Find and classify all the critical points for $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

First, we need to find the first and second order partial derivatives of the function:

$$f_x = 6xy - 6x, \qquad f_y = 3x^2 + 3y^2 - 6y$$

$$f_{xx} = 6y - 6, \qquad f_{yy} = 6y - 6, \qquad f_{xy} = 6x$$

Now, we need to find the critical points by solving the equations $f_x = 0$ and $f_y = 0$ simultaneously:

$$6xy - 6x = 0$$
$$3x^2 + 3y^2 - 6y = 0$$

From the first equation,

$$6x(y - 1) = 0 \implies x = 0, y = 1$$

For $x = 0$:

$$3y^2 - 6y = 3y(y-2) = 0 \implies y = 0, 2$$

For $y = 1$:

$$3x^2 + 3 - 6 = 0 \implies x^2 = 1 \implies x = 1, -1$$

Hence, the critical points are $(0,0)$, $(0,2)$, $(1,1)$, and $(-1,1)$. Now, we need to classify these critical points using the second derivative test:

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2 = (6y-6)^2 - 36x^2$$

For $(0,0)$:

$$D(0,0) = 36 > 0 \qquad f_{xx}(0,0) = -6 < 0$$

For $(0,2)$:

$$D(0,2) = 36 > 0 \qquad f_{xx}(0,2) = 6 > 0$$

For $(1,1)$:

$$D(1,1) = -36 < 0$$

For $(-1,1)$:

$$D(-1,1) = -36 < 0$$

So, we have the following classifications:

- $(0,0)$ is a relative maximum.
- $(0,2)$ is a relative minimum.
- $(1,1)$ and $(-1,1)$ are saddle points.

Example 3.5: Determine the point on the plane $4x-2y+z=1$ that is closest to the point $(-2,-1,5)$

We can rewrite the equation of the plane as

$$z = 1 - 4x + 2y$$

The distance between a point (x,y,z) on the plane and the point $(-2,-1,5)$ is given by

$$d = \sqrt{(x+2)^2 + (y+1)^2 + (2y-4x-4)^2}$$

Now, since the finding the mininmum value of d is equivalent to finding the minimum value of d^2, let

$$\begin{aligned} f(x,y) &= d^2 = (x+2)^2 + (y+1)^2 + (2y-4x-4)^2 \\ &= (x^2+4x+4) + (y^2+2y+1) + (16x^2+4y^2-16xy+32x-16y+16) \\ f(x,y) &= 17x^2 + 5y^2 - 16xy + 36x - 14y + 21 \end{aligned}$$

We can now find the derivatives:

$$f_x(x,y) = 34x - 16y + 36, \qquad f_y(x,y) = 10y - 16x - 14$$

$$f_{xx}(x,y) = 34, \qquad f_{yy}(x,y) = 10, \qquad f_{xy} = -16$$

Before finding the critical points, notice that

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ &= 34 \times 10 - (-16)^2 \\ &= 340 - 256 = 84 > 0 \end{aligned}$$

Since $D > 0$ and $f_{xx} > 0$, all the critical points will be relative minimums. Now, to find the critical points, solve the equations

$$\begin{aligned} f_x &= 34x - 16y + 36 = 0 \\ f_y &= 10y - 16x - 14 = 0 \end{aligned}$$

From the first equation, we get

$$x = \frac{1}{34}(16y - 36) = \frac{1}{17}(8y - 18)$$

Substituting this in the second equation, we get

$$\begin{aligned} 10y - \frac{16}{17}(8y - 18) - 14 &= 0 \\ 170y - 16(8y - 18) - 238 &= 0 \\ 42y + 50 &= 0 \\ \therefore y &= -\frac{25}{21} \end{aligned}$$

And substituting this value in the equation for x, we get

$$\begin{aligned} x &= \frac{1}{17}\left(\frac{-200}{21} - 18\right) \\ &= \frac{-200 - 378}{17 \times 21} \\ \therefore x &= -\frac{34}{21} \end{aligned}$$

Finally, substituting these values in the equation of the plane, we get

$$z = 1 + 4 \times \frac{34}{21} - 2 \times \frac{25}{21} = \frac{107}{21}$$

So, the point on the plane that is closest to the point $(-2, -1, 5)$ is $\left(-\frac{34}{21}, -\frac{25}{21}, \frac{107}{21}\right)$

3.4 Absolute Extrema

Definition 3.4.1: Closed, Open, and Bounded Region

1. A region $\mathbb{R}^2$ is called **closed** if it contains all its boundary points.
2. A region $\mathbb{R}^2$ is called **open** if it does not contain any of its boundary points.
3. A region $\mathbb{R}^2$ is called **bounded** if it can be contained in a circle of finite radius.

***Theorem 3.4.2* (Extreme Value Theorem):** *If $f(x,y)$ is continuous on a closed and bounded region D in $\mathbb{R}^2$, then $f(x,y)$ there are two points (x_1,y_1) and (x_2,y_2) in region D such that $f(x_1,y_1)$ is the absolute maximum and $f(x_2,y_2)$ is the absolute minimum of $f(x,y)$ in D.*

Finding Absolute Extrema

1. Find all the critical points of the function $f(x,y)$ that lie in the region D and determine the function value at each of these points.

2. Find all extrema of the function $f(x,y)$ on the boundary of the region D and determine the function value at each of these points.

3. The largest and the smallest values found in the first two steps are the absolute minimum and the absolute maximum of the function $f(x,y)$ in the region D.

Example 3.6: Find the absolute minimum and absolute maximum of $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ on the rectangle given by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

First, we need to find the critical points of the function $f(x,y)$ in the region D. The first order partial derivatives are given by

$$f_x = 2x - 4xy$$
$$f_y = 8y - 2x^2$$

Setting these equal to zero, we get the following equations:

$$2x - 4xy = 0$$
$$8y - 2x^2 = 0$$

Solving the equations, we get

$$(x,y) = (0,0), \left(\sqrt{2}, \frac{1}{2}\right), \left(-\sqrt{2}, \frac{1}{2}\right)$$

Since $-1 \leq x \leq 1$, the only critical point in the region D is $(0,0)$. Value of the function at this point is

$$f(0,0) = 0^2 + 4(0)^2 - 2(0)^2(0) + 4 = 4$$

Now, we need to find the extrema on the boundary of the region. The boundary consists of four line segments:

1. $(-1,-1)$ to $(1,-1)$ where $y = -1$ and $-1 \leq x \leq 1$

2. $(1,-1)$ to $(1,1)$ where $x = 1$ and $-1 \leq y \leq 1$

3. $(1,1)$ to $(-1,1)$ where $y = 1$ and $-1 \leq x \leq 1$

4. $(-1,1)$ to $(-1,-1)$ where $x = -1$ and $-1 \leq y \leq 1$

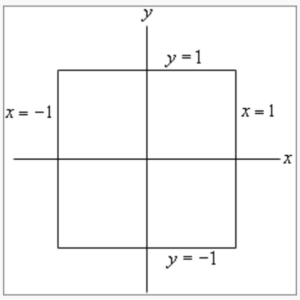

Figure 3.4.1: Boundary of the region

For segment 1:
Let $g(x) = f(x,-1) = 3x^2 + 8$

For critical point on the edge, $g'(x) = 0$ or $6x = 0 \implies x = 0$.

$$\begin{aligned} g(0) &= f(0,-1) = 8 \\ g(-1) &= f(-1,-1) = 11 \\ g(1) &= f(1,-1) = 11 \end{aligned}$$

For segment 2:
Let $h(y) = f(1, y) = 1 + 4y^2 - 2(1)y + 4 = 5 + 4y^2 - 2y$
For critical point on the edge, $h'(y) = 0$ or $8y - 2 = 0 \implies y = \frac{1}{4}$.

$$\begin{aligned} h\left(\frac{1}{4}\right) &= f(1, \frac{1}{4}) = \frac{19}{4} = 4.75 \\ h(-1) &= f(1,-1) = 11 \\ h(1) &= f(1,1) = 7 \end{aligned}$$

For segment 3:
Let $k(x) = f(x, 1) = x^2 + 4(1)^2 - 2x^2(1) + 4 = 5 - 2x^2$
For critical point on the edge, $k'(x) = 0$ or $-4x = 0 \implies x = 0$.

$$\begin{aligned} k(0) &= f(0,1) = 8 \\ k(-1) &= f(-1,1) = 7 \\ k(1) &= f(1,1) = 7 \end{aligned}$$

For segment 4:
Let $l(y) = f(-1, y) = (-1)^2 + 4y^2 - 2(-1)y + 4 = 5 + 4y^2 + 2y$
For critical point on the edge, $l'(y) = 0$ or $8y + 2 = 0 \implies y = -\frac{1}{4}$.

$$\begin{aligned} l\left(-\frac{1}{4}\right) &= f(-1, -\frac{1}{4}) = \frac{19}{4} = 4.75 \\ l(-1) &= f(-1,-1) = 11 \\ l(1) &= f(-1,1) = 7 \end{aligned}$$

Now, we can summarize the function values at the critical point and the boundary points:

$$\begin{aligned} &f(0,0) = 4, \\ &f(0,-1) = 8, && f(-1,-1) = 11, && f(1,-1) = 11, \\ &f\left(1, \frac{1}{4}\right) = 4.75, && f(1,-1) = 11, && f(1,1) = 7, \\ &f(0,1) = 8, && f(-1,1) = 7, && f(1,1) = 7, \\ &f\left(-1, -\frac{1}{4}\right) = 4.75, && f(-1,-1) = 11, && f(-1,1) = 7 \end{aligned}$$

Hence, the absolute extrema are given by

$$\begin{aligned} &\text{Absolute Maximum: } (-1,-1) \text{ and } (1,-1) \text{ with value } 11, \\ &\text{Absolute Minimum: } (0,0) \text{ with value } 4 \end{aligned}$$

Example 3.7: Find the absolute minimum and absolute maximum of $f(x,y) = 2x^2 - y^2 + 6y$ on the disk of radius 4, $x^2 + y^2 \leq 16$

First, we need to find the critical points of the function $f(x, y)$ in the region D. The first order partial derivatives are given by

$$\begin{aligned} f_x &= 4x \\ f_y &= -2y + 6 \end{aligned}$$

Setting these equal to zero, we get the following equations:

$$\begin{aligned} 4x = 0 &\implies x = 0 \\ -2y + 6 = 0 &\implies y = 3 \end{aligned}$$

So, the only critical point in the region D is $(0, 3)$. The value of the function at this point is

$$f(0, 3) = 2(0)^2 - (3)^2 + 6(3) = 15$$

Now, we need to find the extrema on the boundary of the region. The boundary consists of the circle $x^2 + y^2 = 16$. We can rewrite it as

$$x^2 = 16 - y^2$$

Let

$$\begin{aligned} g(y) &= f\left(\sqrt{16 - y^2}, y\right) \\ &= 2(16 - y^2) - y^2 + 6y \\ &= 32 - 3y^2 + 6y \end{aligned}$$

The critical points on the boundary are given by

$$g'(y) = -6y + 6 = 0 \implies y = 1$$

$$x = \pm\sqrt{y^2 - 1} = \pm\sqrt{15}$$

Now, we can find the value of the function at this point and at the endpoints of the boundary:

$$\begin{aligned} g(1) &= f(\pm\sqrt{15}, 1) = 35 \\ g(-4) &= f(0, -4) = -40 \\ g(4) &= f(0, 4) = 8 \end{aligned}$$

Now, we can summarize the function values at the critical point and the boundary points:

$$\begin{aligned} &f(0, 3) = 15, \\ &f(\sqrt{15}, 1) = 35, &\quad f(-\sqrt{15}, 1) = 35, \\ &f(0, -4) = -40, &\quad f(0, 4) = 8 \end{aligned}$$

Hence, the absolute extrema are given by

$$\begin{aligned} &\text{Absolute Maximum: } (\sqrt{15}, 1) \text{ and } (-\sqrt{15}, 1) \text{ with value } 35, \\ &\text{Absolute Minimum: } (0, -4) \text{ with value } -40 \end{aligned}$$

Example 3.8: Find the absolute minimum and absolute maximum of $f(x,y) = 18x^2 + 4y^2 - y^2x - 2$ on the triangle with vertices $(-1,-1)$, $(5,-1)$, and $(5,17)$.

First, we need to find the critical points of the function $f(x,y)$ in the region D. The first order partial derivatives are given by

$$f_x = 3x - y^2$$
$$f_y = 8y - 2xy$$

Setting these equal to zero, we get the following equations:

$$3x - y^2 = 0$$
$$8y - 2xy = 0$$

Solving the equations, we get

$$(x,y) = (0,0), (4,-12), (4,12)$$

Notice that the triangle is bounded by

1. $y = -1$
2. $x = 5$
3. $y + 1 = 3(x+1) \implies y = 3x + 2$

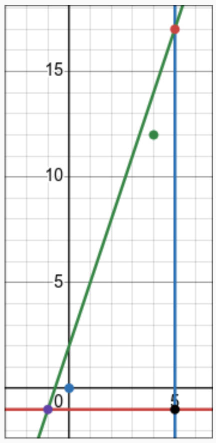

Figure 3.4.2: Triangle region

Among the critical points,

1. $(0,0)$ lies inside the triangle.
2. $(4,12)$ lies inside the triangle. $[3 \times 4 + 2 = 14 > 12]$
3. $(4,-12)$ lies outside the triangle. $[-1 \leq y \leq 17]$

The values of the function at the critical points are:

$$f(0,0) = -2 \qquad \text{and} \qquad f(4,12) = 286$$

Now, we need to find the extrema on the boundary of the region. The boundary consists of three line segments:

1. $(-1,-1)$ to $(5,-1)$ where $y = -1$ and $-1 \leq x \leq 5$
2. $(5,-1)$ to $(5,17)$ where $x = 5$ and $-1 \leq y \leq 17$
3. $(-1,-1)$ to $(5,17)$ where $y = 3x + 2$ and $-1 \leq x \leq 5$

For segment 1:
Let $g(x) = f(x,-1) = 18x^2 + 4 - x - 2 = 18x^2 - x + 2$
For critical point on the edge, $g'(x) = 0$ or $36x - 1 = 0 \implies x = \frac{1}{36}$.

$$g\left(\frac{1}{36}\right) = f\left(\frac{1}{36}, -1\right) \approx 1.986$$
$$g(-1) = f(-1,-1) = 21$$
$$g(5) = f(5,-1) = 447$$

For segment 2:
Let $h(y) = f(5, y) = 450 + 4y^2 - 5y^2 - 2 = 448 - 9y^2$

For critical point on the edge, $h'(y) = 0$ or $448 - 9y^2 = 0 \implies y = \pm\dfrac{8\sqrt{7}}{3}$

$$h\left(\frac{8\sqrt{7}}{3}\right) = f\left(5, \frac{8\sqrt{7}}{3}\right) \approx 398.22$$
$$h\left(-\frac{8\sqrt{7}}{3}\right) = f\left(5, -\frac{8\sqrt{7}}{3}\right) \approx 398.22$$
$$h(-1) = f(5, -1) = 447$$
$$h(17) = f(5, 17) = 159$$

For segment 3:
Let $k(x) = f(x, 3x + 2) = 18x^2 + 4(3x + 2)^2 - (3x + 2)^2 x - 2$

$$\begin{aligned}
k(x) &= 18x^2 + 4(3x + 2)^2 - (3x + 2)^2 x - 2 \\
&= 18x^2 + 4(9x^2 + 12x + 4) - (9x^3 + 12x^2 + 4x) - 2 \\
&= 18x^2 + 36x^2 + 48x + 16 - 9x^3 - 12x^2 - 4x - 2 \\
&= -9x^3 + 42x^2 + 44x + 14
\end{aligned}$$

For critical point on the edge, $k'(x) = 0$ or $-27x^2 + 84x + 44 = 0$. This gives us the points $(x, y) \approx (3.568, 12.704), (-0.457, 0.629)$

$$k(3.568) = f(3.568, 12.704) \approx 296.872$$
$$k(-0.457) = f(-0.457, 0.629) \approx 3.523$$

Hence, the absolute extrema are given by

$$\begin{aligned}
&\text{Absolute Maximum: } (5, \pm 1) \text{ with value } 447, \\
&\text{Absolute Minimum: } (0, 0) \text{ with value } -2
\end{aligned}$$

3.5 Lagrange Multipliers

Method of Lagrange Multipliers

Given a function $f(x, y, z)$ and a constraint $g(x, y, z) = k$. To find the absolute extrema:

1. Solve the following system of equations:

$$\begin{aligned}
\nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\
g(x, y, z) &= k
\end{aligned}$$

2. Plug in all solutions, (x, y, z), from the first step into the function $f(x, y, z)$ and identify the minimum and maximum values, provided they exist and $\nabla g \neq \vec{0}$ at the point.

The constant λ is called the **Lagrange Multiplier**.

Lagrange Multipliers for Multiple Constraints

If there are multiple constraints, say $g_1(x,y,z) = k_1$ and $g_2(x,y,z) = k_2$, then the system of equations becomes

$$\begin{aligned}\nabla f(x,y,z) &= \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z)\\ g_1(x,y,z) &= k_1\\ g_2(x,y,z) &= k_2\end{aligned}$$

where λ_1 and λ_2 are the Lagrange Multipliers for the constraints g_1 and g_2, respectively. The method can be extended to any number of constraints by adding more terms to the right-hand side of the first equation.

Example 3.9: Find the maximum and minimum values of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$.

Using Lagrange multipliers, we need to solve the following system of equations:

$$\begin{aligned}\nabla f(x,y) &= \lambda \nabla g(x,y)\\ g(x,y) &= x^2 + y^2 = 4\end{aligned}$$

That is, we need to solve the following equations:

$$\begin{aligned}8x &= 2\lambda x\\ 20y &= 2\lambda y\\ x^2 + y^2 &= 4\end{aligned}$$

From the first equation, we get:

$$2x(4-\lambda) = 0 \implies x = 0 \text{ or } \lambda = 4$$

If we have $x = 0$ then the constraint gives us

$$y = \sqrt{x^2 - 4} \implies y = \pm 2$$

If we have $\lambda = 4$ the second equation gives us

$$20y = 8y \implies y = 0$$

Then the constraint gives us

$$x^2 + 0^2 = 4 \implies x = \pm 2$$

Now, we can find the values of the function at these points:

$$\begin{aligned}&f(0,0) = 0 & &\\ &f(0,2) = 40 & &f(0,-2) = 40\\ &f(2,0) = 16 & &f(-2,0) = 16\end{aligned}$$

Hence, the maximum value is 40 at the points $(0,2)$ and $(0,-2)$, and the minimum value is 0 at the point $(0,0)$.

Example 3.10: Find the maximum and minimum values of $f(x, y, z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2 = 1$.

Using Lagrange multipliers method, we need to solve the following system of equations:

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \\ g(x, y, z) &= 2x - y - z = 2 \\ h(x, y, z) &= x^2 + y^2 = 1 \end{aligned}$$

That is, we need to solve the following equations:

$$\begin{aligned} 0 &= 2\lambda + 2\mu x \\ 4 &= -\lambda + 2\mu y \\ -2 &= -\lambda \\ 2x - y - z &= 2 \\ x^2 + y^2 &= 1 \end{aligned}$$

From the third equation, we get $\lambda = 2$. Plugging this into the first and second equations, we get respectively

$$\begin{aligned} -4 = 2\mu x &\implies x = -\frac{2}{\mu} \\ 6 = 2\mu y &\implies y = \frac{3}{\mu} \end{aligned}$$

Now, substituting these values in the fifth equation, we get

$$\frac{4}{\mu^2} + \frac{9}{\mu^2} = 1 \implies \mu = \pm\sqrt{13}$$

Now, for $\mu = \sqrt{13}$, we have

$$\begin{aligned} x &= -\frac{2}{\sqrt{13}} \\ y &= \frac{3}{\sqrt{13}} \\ z &= 2x - y - 2 = -\frac{4}{\sqrt{13}} - \frac{3}{\sqrt{13}} - 2 = -2 - \frac{7}{\sqrt{13}} \end{aligned}$$

And for $\mu = -\sqrt{13}$, we have

$$\begin{aligned} x &= \frac{2}{\sqrt{13}} \\ y &= -\frac{3}{\sqrt{13}} \\ z &= 2x - y - 2 = \frac{4}{\sqrt{13}} + \frac{3}{\sqrt{13}} - 2 = -2 + \frac{7}{\sqrt{13}} \end{aligned}$$

Now, we can find the values of the function at these points:

$$\begin{aligned} f\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2 - \frac{7}{\sqrt{13}}\right) &\approx 11.211 \\ f\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2 + \frac{7}{\sqrt{13}}\right) &\approx -3.211 \end{aligned}$$

Hence, the maximum value is approximately 11.211 at the point $\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2 - \frac{7}{\sqrt{13}}\right)$, and the minimum value is approximately -3.211 at the point $\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2 + \frac{7}{\sqrt{13}}\right)$.

4 Multiple Integrals

4.1 Double Integrals