Calculus II Notes

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Contents

1 Integration Techniques

1.1 Fundamental Theorem of Calculus

Theorem 1.1.1 (Fundamental Theorem of Calculus): *Let f be a function defined on an open interval I that contains a. If f is continuous on I, then the function F defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is uniformly continuous on I, differentiable on the open interval, and

$$F'(x) = f(x)$$

for all x in the open interval.

1.2 Common Differentiation and Integration Formulae

Derivative	Integral
$\frac{\mathrm{d}}{\mathrm{d}x}x = 1$, $\quad \frac{\mathrm{d}}{\mathrm{d}x}c = 0$	$\int c\,\mathrm{d}x = cx + c$
$\frac{\mathrm{d}}{\mathrm{d}x}x^n$	$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}$	$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c$
$\frac{\mathrm{d}}{\mathrm{d}x}e^{mx} = me^{mx}$	$\int e^{mx}\,\mathrm{d}x = \frac{1}{m}e^{mx} + c$
$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln(a)$	$\int a^x\,\mathrm{d}x = \frac{1}{\ln(a)}a^x + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sin(mx) = m\cos(mx)$	$\int \cos(mx)\,\mathrm{d}x = \frac{1}{m}\sin(mx) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cos(mx) = -m\sin(mx)$	$\int \sin(mx)\,\mathrm{d}x = -\frac{1}{m}\cos(mx) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$	$\int \sec^2(x)\,\mathrm{d}x = \tan(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot(x) = -\csc^2(x)$	$\int \csc^2(x)\,\mathrm{d}x = -\cot(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec(x) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,\mathrm{d}x = \sec(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc(x) = -\csc(x)\cot(x)$	$\int \csc(x)\cot(x)\,\mathrm{d}x = -\csc(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\cos^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}(x) = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,\mathrm{d}x = -\cot^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = \sec^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = -\csc^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x} + c$

Table 1: Common Differentiation and Integration Formulae

1.3 More Formulae

1. $\int \tan(x)\,\mathrm{d}x = \ln|\sec(x)| + c$

2. $\int \csc(x)\,\mathrm{d}x = \ln|\tan\frac{x}{2}| + c$

3. $\int \sec(x)\,\mathrm{d}x = \ln|\sec(x) + \tan(x)| + c$

4. $\int \sec(x)\,\mathrm{d}x = \ln|\tan(\frac{\pi}{4} + \frac{x}{2})|$

5. $\int \csc(x)\,\mathrm{d}x = -\ln|\csc(x) + \cot(x)| + c$

6. $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$

7. $\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln|\frac{a+x}{a-x}| + c$

8. $\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln|\frac{a-x}{a+x}| + c$

9. $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln|x + \sqrt{x^2+a^2}| + c$

10. $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}(\frac{x}{a}) + c$

11. $\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

12. $\int e^x\left[f(x) + f'(x)\right]\mathrm{d}x = e^x f(x) + c$

1.4 Integration by Parts

> **Theorem 1.4.1 (Integration by Parts):** *Let u and v be differentiable functions of x. Then,*
> $$\int uv\,\mathrm{d}x = u\int v\,\mathrm{d}x - \int\left[\frac{\mathrm{d}u}{\mathrm{d}x}\int v\,\mathrm{d}x\right]\mathrm{d}x$$
> *Or,*
> $$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

Proof:
Let $u = u(x)$ and $w = w(x)$. Then,
$$\frac{\mathrm{d}(uw)}{\mathrm{d}x} = u\frac{\mathrm{d}w}{\mathrm{d}x} + w\frac{\mathrm{d}u}{\mathrm{d}x}$$

Integrating both sides, we get
$$\int \frac{\mathrm{d}(uw)}{\mathrm{d}x}\,\mathrm{d}x = \int u\frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x + \int w\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Or,
$$uw = \int u\frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x + \int w\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Rearranging, we get
$$\int u\frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x = uw + c - \int w\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Let $v = \frac{\mathrm{d}w}{\mathrm{d}x}$, then $w = \int v\,\mathrm{d}x$. Hence,
$$\int uv\,\mathrm{d}x = u\int v\,\mathrm{d}x - \int\left[\frac{\mathrm{d}u}{\mathrm{d}x}\int v\,\mathrm{d}x\right]\mathrm{d}x$$

$\square$

1.5 Method of Substitution

A. $\int \frac{1}{(ax+b)\sqrt{cx+d}}\,\mathrm{d}x$ Let $cx + d = z^2$

B. $\int \frac{1}{\sin^m x \cos^m x}\,\mathrm{d}x$ If $m + n = p$ is even, multiply and divide by $\sec^p x$ and let $\tan x = z$.

C. $\int \frac{1}{\sin^m x + \cos^m x}\,\mathrm{d}x$ If m is even, mulpliply and divide by $\sec^m x$.

D. $\int \frac{\cos x}{a\cos x + b\sin x}\,\mathrm{d}x$ Write $nom = l \times (denom) + m \times (denom)'$, then determine l and m.

E. $\int \frac{\cos x}{a\cos x + b\sin x}\,\mathrm{d}x + c$ Write $\sin x$ and $\cos x$ as $\tan \frac{x}{2}$.

F. $\int \frac{1}{\sqrt{x^2 + a^2}}\,\mathrm{d}x$ Let $x = a\tan\theta$

G. $\int \frac{1}{\sqrt{x^2 - a^2}}\,\mathrm{d}x$ Let $x = a\sec\theta$

H. $\int \sqrt{a^2 - x^2}\,\mathrm{d}x$ Let $x = a\sin\theta$

1.6 Trigonometric Integrals

Form	Looks like	Substitution	Limit Assumption
$\sqrt{b^2 x^2 - a^2}$	$\sec^2\theta - 1 = \tan^2\theta$	$x = \frac{a}{b}\sec\theta$	$0 \le \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \le \pi$
$\sqrt{a^2 - b^2 x^2}$	$1 - \sin^2\theta = \cos^2\theta$	$x = \frac{a}{b}\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{a^2 + b^2 x^2}$	$\tan^2\theta + 1 = \sec^2\theta$	$x = \frac{a}{b}\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Table 2: Trigonometric Integral Substitution

1.7 Partial Fractions

Factor in denominator	Term in partial fraction decomposition
$ax + b$	$\frac{A}{ax+b}$
$(ax + b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+b}{ax^2+bx+c}$
$(ax^2 + bx + c)^k$	$\frac{A_1 x + B_1}{ax^2+bx+c} + \frac{A_2 x + B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_k x + B_k}{(ax^2+bx+c)^k}$

Table 3: Partial Fraction Decomposition

1.8 Integrals Involving Roots

For integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, use trigonometric substitution:

> **Example 1.1: Evaluate** $\int \frac{1}{\sqrt{9 - x^2}}\,\mathrm{d}x$

Expression	Substitution	Identity Used
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$

Table 4: Trigonometric Substitutions for Roots

Here $a = 3$, so let $x = 3 \sin \theta$, thus $\mathrm{d}x = 3 \cos \theta \, \mathrm{d}\theta$.

$$
\int \frac{1}{\sqrt{9 - x^2}} \, \mathrm{d}x = \int \frac{3 \cos \theta}{\sqrt{9 - 9 \sin^2 \theta}} \, \mathrm{d}\theta
$$
$$
= \int \frac{3 \cos \theta}{3 \cos \theta} \, \mathrm{d}\theta
$$
$$
= \int 1 \, \mathrm{d}\theta
$$
$$
= \theta + c
$$
$$
= \sin^{-1} \left(\frac{x}{3} \right) + c
$$

1.9 Integrals Involving Quadratics

For integrals involving $ax^2 + bx + c$, complete the square first:

$$
ax^2 + bx + c = a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right]
$$

Then use substitution $u = x + \frac{b}{2a}$ to convert to standard forms.

Example 1.2: Evaluate $\int \frac{1}{x^2 + 4x + 13} \, \mathrm{d}x$

Complete the square:

$$
x^2 + 4x + 13 = (x + 2)^2 + 9 = (x + 2)^2 + 3^2
$$

Let $u = x + 2$, so $\mathrm{d}u = \mathrm{d}x$:

$$
\int \frac{1}{x^2 + 4x + 13} \, \mathrm{d}x = \int \frac{1}{u^2 + 9} \, \mathrm{d}u
$$
$$
= \frac{1}{3} \tan^{-1} \left(\frac{u}{3} \right) + c
$$
$$
= \frac{1}{3} \tan^{-1} \left(\frac{x + 2}{3} \right) + c
$$

1.10 Integration Strategy

When faced with an integral, use the following strategy:

1. **Simplify the integrand**: Expand, factor, or use algebraic manipulation

2. **Look for obvious substitutions**: If $u' = g'(x)$ appears, try $u = g(x)$

3. **Classify by type**:

   - Trigonometric integrals $\rightarrow$ use trig identities
   - Rational functions $\rightarrow$ use partial fractions
   - Products $\rightarrow$ try integration by parts
   - Roots of quadratics $\rightarrow$ complete the square or trig substitution

4. **Try different techniques**: If one method fails, try another

5. **Use tables or computer algebra systems**: For complex integrals

1.11 Approximating Definite Integrals

When an antiderivative cannot be found in closed form, use numerical approximation methods.

1.11.1 Midpoint Rule

Divide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$. Let $\bar{x}_i$ be the midpoint of the i-th subinterval.

$$\int_a^b f(x)\, \mathrm{d}x \approx M_n = \Delta x \sum_{i=1}^n f(\bar{x}_i)$$

1.11.2 Trapezoidal Rule

$$\int_a^b f(x)\, \mathrm{d}x \approx T_n = \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$$

1.11.3 Simpson's Rule

Requires n to be even. Approximates the function with parabolas instead of lines:

$$\int_a^b f(x)\, \mathrm{d}x \approx S_n = \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$$

> **Note:-**
> Simpson's Rule is generally more accurate than the Trapezoidal Rule, which is more accurate than the Midpoint Rule for the same number of subintervals.

1.12 Improper Integrals

> **Definition 1.12.1: Improper Integrals**
>
> An integral is said to be **improper** if one of the following conditions is met:
>
> 1. The interval of integration is infinite.
>
> 2. The integrand is discontinuous at one or more points in the interval of integration.
>
> The integral is said to **converge** if the limit of the integral exists, and **diverge** otherwise.

Type-1: If $\int_a^t f(x)\,\mathrm{d}x$ exists for all $t > a$, then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x$$

provided the limit exists and is finite.

Type-2: If $\int_t^b f(x)\,\mathrm{d}x$ exists for all $t < b$, then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\,\mathrm{d}x$$

provided the limit exists and is finite.

Type-3: If $\int_{-\infty}^c f(x)\,\mathrm{d}x$ and $\int_c^\infty f(x)\,\mathrm{d}x$ are both convergent, then

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^c f(x)\,\mathrm{d}x + \int_c^\infty f(x)\,\mathrm{d}x$$

Type-4: $\int_a^b f(x)\,\mathrm{d}x$ If $f(x)$ is discontinuous at $x = c$, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0}\left[\int_a^{c-\epsilon} f(x)\,\mathrm{d}x + \int_{c+\epsilon}^b f(x)\,\mathrm{d}x\right]$$

◆Note:-◆

If $a > 0$ then

$$\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$$

converges if $p > 1$ and diverges if $p \leq 1$.

1.12.1   Comparison Test

Theorem 1.12.2 (Comparison Theorem): *If $f(x) \geq g(x) \geq 0$ on the interval $[a, \infty)$, then*

1. *If $\int_a^\infty f(x)\,\mathrm{d}x$ converges, then so does $\int_a^\infty g(x)\,\mathrm{d}x$.*

2. *If $\int_a^\infty g(x)\,\mathrm{d}x$ diverges, then so does $\int_a^\infty f(x)\,\mathrm{d}x$.*

Example 1.3: Determine if the following integral is convergent or divergent:

$$\int_2^\infty \frac{\cos^2 x}{x^2}\,\mathrm{d}x$$

Notice that the numerator is bounded since

$$0 \leq \cos^2 x \leq 1$$

Hence, it's likely that the denominator will determine the convergence of the integral. Since $p = 2 > 1$,

$$\int_2^\infty \frac{1}{x^2}\,\mathrm{d}x$$

is convergent. Since

$$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

and $\int_2^\infty \frac{1}{x^2}\,\mathrm{d}x$ is convergent, by the comparison test,

$$\int_2^\infty \frac{\cos^2 x}{x^2}\,\mathrm{d}x$$

is convergent.

Example 1.4: Determine if the following integral is convergent or divergent:

$$\int_3^\infty \frac{1}{x + e^x}\,\mathrm{d}x$$

In this case, the denominator determines the convergence of the integral. If we can find a larger function that converges, then the integral will converge. Notice that

$$\frac{1}{x + e^x} < \frac{1}{e^x} = e^{-x}$$

Also,

$$\begin{aligned}
\int_3^\infty e^{-x}\,\mathrm{d}x &= \lim_{t\to\infty} \int_3^t e^{-x}\,\mathrm{d}x \\
&= \lim_{t\to\infty} \left(-e^{-t} + e^{-3}\right) \\
&= e^{-3}
\end{aligned}$$

So, $\int_3^\infty e^{-x}\,\mathrm{d}x$ is convergent. Therefore, by the Comparison test,

$$\int_3^\infty \frac{1}{x + e^x}\,\mathrm{d}x$$

is also convergent.

2   Applications of Integrals

2.1   Arc Length

Consider a curve $y = f(x)$. We want to find the length of the curve from $x = a$ to $x = b$. We can approximate the curve by a series of line segments. The length of each line segment is given by the Pythagorean theorem:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

The total length of the curve is given by the sum of the lengths of the line segments:

$$L = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Similarly, for a curve $x = h(y)$, the length of the curve from $y = c$ to $y = d$ is given by:

$$L = \int ds = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

Arc Length Formula

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{for } y = f(x), a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \quad \text{for } x = h(y), c \le y \le d$$

2.2   Surface Area

Consider a curve $y = f(x)$ rotated about the x-axis. We want to find the surface area of the resulting surface. We can approximate the surface by a series of frustums. The surface area of each frustum is given by:

$$dS = 2\pi y \, ds = 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

The total surface area of the surface is given by the sum of the surface areas of the frustums:

$$S = \int dS = \int 2\pi y \, ds = \int 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Similarly, for a curve $x = h(y)$ rotated about the y-axis, the surface area of the resulting surface is given by:

$$A = \int 2\pi x \, ds = \int 2\pi h(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$S = \int dS$$

$$= \int 2\pi y \, ds \quad \text{rotation about } x\text{–axis}$$

$$= \int 2\pi x \, ds \quad \text{rotation about } y\text{–axis}$$

where,

$$dS = 2\pi y \, ds = 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{for } y = f(x), a \le x \le b$$

$$dS = 2\pi x \, ds = 2\pi h(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \quad \text{for } x = h(y), c \le y \le d$$

2.3   Center of Mass

Suppose we want to find the center of mass of a region bounded by two curves $f(x)$ and $g(x)$ on the interval $[a, b]$.

The mass is

$$M = \rho \int_a^b \left(f(x) - g(x)\right) dx$$

Next, we need the **moments** of the region. There are two moments:

$$M_x = \rho \int_a^b \frac{1}{2} \left[f(x)^2 - g(x)^2\right] dx$$

$$M_y = \rho \int_a^b x \left[f(x) - g(x)\right] dx$$

The coordinates of the center of mass, $(\bar{x}, \bar{y})$, are given by:

$$\bar{x} = \frac{M_y}{M} = \frac{1}{A} \int_a^b x \left[f(x) - g(x)\right] dx$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{A} \int_a^b \frac{1}{2} \left[f(x)^2 - g(x)^2\right] dx$$

where,

$$A = \int_a^b \left[f(x) - g(x)\right] dx$$

2.4   Probability

Every continuous random variable X, has a probability density function $f(x)$. Probability density functions satisfy the following conditions:

1. $f(x) \ge 0$ for all x.

2. $\int_{-\infty}^{\infty} f(x) \, dx = 1$

Probability density finctions can be used to determine the probability that a continuous random variable lies between two values, say a and b. This probability is denoted by $P(a \leq X \leq b)$.

Note:-

$$P(a \leq X \leq b) = \int_a^b f(x)\,\mathrm{d}x$$

Probability density functions can also be used to determine the mean of a continuous random variable. The mean is given by:

Mean of a Continuous Random Variable

$$\mu = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$$

3 Parametric Equations and Polar Coordinates

There are a great many curves out there that cannot be expressed in a single equation in terms of only x and y. To deal with such problems, we introduce **parametric equations**. Instead of defining y in terms of x($y = f(x)$) or x in terms of y($x = h(y)$), we define both x and y in terms of a third variable called a parameter as follows:

$$x = f(t) \quad y = g(t)$$

This third variable is usually denoted by t. Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called a **parametric curve**.

3.1 Parametric Equations and Curves

Unlike graphs of functions $y = f(x)$, parametric curves can trace out shapes that fail the vertical line test. For instance, a circle cannot be written as a single function $y = f(x)$, but can be easily expressed parametrically.

Example 3.1: Sketch the parametric curve $x = t^2$, $y = 2t - 1$ **for** $-2 \le t \le 2$.

We can create a table of values:

t	$x = t^2$	$y = 2t - 1$
-2	4	-5
-1	1	-3
0	0	-1
1	1	1
2	4	3

Plotting these points (x, y) and connecting them traces out a parabola opening to the right.

To eliminate the parameter, note that $x = t^2$, so $t = \pm\sqrt{x}$. Then:

$$y = 2t - 1 = \pm 2\sqrt{x} - 1$$

Since the parameter t ranges from -2 to 2, we get both branches. However, note that $x \ge 0$ always (since $x = t^2$).

Example 3.2: Sketch the parametric curve $x = \cos(t)$, $y = \sin(t)$ **for** $0 \le t \le 2\pi$.

This is the unit circle! At $t = 0$, we're at $(1, 0)$. At $t = \pi/2$, we're at $(0, 1)$. At $t = \pi$, we're at $(-1, 0)$. At $t = 3\pi/2$, we're at $(0, -1)$, and at $t = 2\pi$ we return to $(1, 0)$. To eliminate the parameter, use the Pythagorean identity:

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$$

So the curve traces the circle $x^2 + y^2 = 1$, starting at $(1, 0)$ and moving counterclockwise.

3.2   Tangents with Parametric Equations

Derivative for Parametric Equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \quad \text{provided } \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

Tangents for Parametric Equations

Horizontal Tangent:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0, \quad \text{provided } \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

Vertical Tangent:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0, \quad \text{provided } \frac{\mathrm{d}y}{\mathrm{d}t} \neq 0$$

Second Derivative for Parametric Equations

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

3.3   Area with Parametric Equations

Area with Parametric Equations

For the area between the parametric curve and the x-axis:

$$A = \int_{t_1}^{t_2} y(t) \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t$$

Use this formula when the curve is traced vertically (from bottom to top or top to bottom).
For the area between the parametric curve and the y-axis:

$$A = \int_{t_1}^{t_2} x(t) \frac{\mathrm{d}y}{\mathrm{d}t} \, \mathrm{d}t$$

Use this formula when the curve is traced horizontally (from left to right or right to left).

3.4   Arc Length with Parametric Equations

The arc length of a curve is given by

$$L = \int \mathrm{d}s$$

where,

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x \quad \text{if } y = f(x), a \leq x \leq b$$

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \, \mathrm{d}y \quad \text{if } x = h(y), c \leq y \leq d$$

Using the first $\mathrm{d}s$, we can write

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t$$

Then the arc length formula becomes,

$$L = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\, \frac{dx}{dt}\, dt$$

$$= \int_\alpha^\beta \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \frac{dx}{dt}\, dt$$

Arc Length with Parametric Equations

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

3.5   Surface Area with Parametric Equations

Surface Area with Parametric Equations

$$S = \int 2\pi y\, ds \quad \text{rotation about } x\text{–axis}$$

$$S = \int 2\pi x\, ds \quad \text{rotation about } y\text{–axis}$$

where,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

3.6   Polar Coordinates

In polar coordinates, each point is determined by a distance r from the origin and an angle θ measured counterclockwise from the positive x-axis. This provides an alternative way to describe points in the plane.

Polar to Cartesian Conversion

To convert from polar coordinates (r, θ) to Cartesian coordinates (x, y):

$$x = r\cos\theta \quad y = r\sin\theta$$

Cartesian to Polar Conversion

To convert from Cartesian coordinates (x, y) to polar coordinates (r, θ):

$$r^2 = x^2 + y^2 \qquad\qquad \tan\theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Note: The formula for θ requires care with quadrants. The functions $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ are the fundamental relationships.

3.7 Tangents with Polar Coordinates

To find $\frac{dy}{dx}$ for a polar curve $r = f(\theta)$, we use the parametric derivatives. Since $x = r\cos\theta$ and $y = r\sin\theta$, applying the product rule gives:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = r'\cos\theta - r\sin\theta \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = r'\sin\theta + r\cos\theta$$

Tangents with Polar Coordinates

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

where $r' = \frac{\mathrm{d}r}{\mathrm{d}\theta}$.

3.8 Area with Polar Coordinates

Area with Polar Coordinates

For the area between two polar curves $r = r_o(\theta)$ (outer radius) and $r = r_i(\theta)$ (inner radius):

$$A = \frac{1}{2}\int_\alpha^\beta \left(r_o^2 - r_i^2\right)\mathrm{d}\theta$$

For the area enclosed by a single polar curve $r = f(\theta)$, set $r_i = 0$:

$$A = \frac{1}{2}\int_\alpha^\beta r^2\,\mathrm{d}\theta$$

3.9 Arc Length with Polar Coordinates

Let

$$r = f(\theta) \quad \alpha \le \theta \le \beta$$

$$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta)$$
$$x = f(\theta)\cos\theta \qquad\qquad y = f(\theta)\sin\theta$$
$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta$$

Now,

$$
\begin{aligned}
\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta\right)^2 \\
&= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\cos^2\theta - 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\
&\quad + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\sin^2\theta + 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\
&= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2
\end{aligned}
$$

> **Arc Length with Polar Coordinates**
>
> $$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

3.10 Surface Area with Polar Coordinates

Let

$$r = f(\theta) \quad \alpha \le \theta \le \beta$$

$$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta)$$
$$x = f(\theta)\cos\theta \qquad\qquad y = f(\theta)\sin\theta$$

> **Surface Area with Polar Coordinates**
>
> $$S = \int 2\pi y\,\mathrm{d}s \quad \text{rotation about } x\text{–axis}$$
>
> $$S = \int 2\pi x\,\mathrm{d}s \quad \text{rotation about } y\text{–axis}$$
>
> where,
>
> $$\mathrm{d}s = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \qquad r = f(\theta), \alpha \le \theta \le \beta$$

4 Sequences

4.1 Definition

> **Definition 4.1.1: Sequence**
>
> A **sequence** is a function whose domain is the set of natural numbers $\mathbb{N}$. The sequence is denoted by $\{a_n\}$ and the value of the function at n (the n-th term) is denoted by a_n. Various ways of representing a sequence are:
>
> $$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

4.2 Precise Definition of Limit of a Sequence

Precise Definition of Limit

1. We say that
$$\lim_{n \to \infty} a_n = L$$
if
$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - L| < \epsilon$$

2. We say that
$$\lim_{n \to \infty} a_n = \infty$$
if
$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n > M$$

3. We say that
$$\lim_{n \to \infty} a_n = -\infty$$
if
$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n < M$$

4.3 Convergence of Sequences

> ***Theorem 4.3.1* (Convergence of Sequences)**: *A sequence $\{a_n\}$ is said to be **convergent** if there exists a real number L such that*
> $$\lim_{n \to \infty} a_n = L$$

> ***Theorem 4.3.2* (Uniqueness of Limits)**: *If a sequence $\{a_n\}$ converges, then its limit is unique.*

> ***Theorem 4.3.3***: *Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim_{x \to \infty} f(x) = L$ then $\lim_{n \to \infty} a_n = L$.*

Properties of Convergent Sequences

If $\{a_n\}$ and $\{b_n\}$ are both convergent sequences, then:

1.
$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

2.
$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$$

3.
$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$

4.
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \qquad \text{provided } \lim_{n\to\infty} b_n \neq 0$$

5.
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p, \qquad \text{provided } a_n \geq 0$$

***Theorem 4.3.4* (Squeeze Theorem for Sequences):** *If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences such that $a_n \leq b_n \leq c_n$ for all $n \geq N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.*

Theorem 4.3.5:
$$\text{If } \lim_{n\to\infty} |a_n| = 0, \ \text{ then } \ \lim_{n\to\infty} a_n = 0$$

Proof:
The main thing to this proof is to note that,
$$-|a_n| \leq a_n \leq |a_n|$$

Then note that,
$$\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = 0$$

We then have that,
$$0 \leq \lim_{n\to\infty} a_n \leq 0$$

and so by the Squeeze Theorem,
$$\lim_{n\to\infty} a_n = 0$$

***Theorem 4.3.6* (Convergent Sequences are Bounded):** *If a sequence $\{a_n\}$ is convergent, then it is bounded.*

Example 4.1: Determining Convergence

Determine if the following sequences converge or diverge:

1. $\left\{ \dfrac{n^2}{2n^2 + 1} \right\}$

2. $\left\{ \dfrac{(-1)^n}{n} \right\}$

3. $\left\{ \dfrac{n!}{n^n} \right\}$

Solution:

1. We can use the theorem about converting sequences to functions. Let $f(x) = \dfrac{x^2}{2x^2 + 1}$.
   Then
   $$\lim_{x \to \infty} \frac{x^2}{2x^2 + 1} = \lim_{x \to \infty} \frac{1}{2 + 1/x^2} = \frac{1}{2}$$
   Therefore, $\displaystyle\lim_{n \to \infty} \frac{n^2}{2n^2 + 1} = \frac{1}{2}$ and the sequence converges.

2. Note that $\left| \dfrac{(-1)^n}{n} \right| = \dfrac{1}{n} \to 0$ as $n \to \infty$. By the theorem about absolute values,

   $\displaystyle\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$ and the sequence converges.

3. For large n, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ grows much slower than $n^n = n \cdot n \cdot n \cdots n \cdot n$.
   We have
   $$0 < \frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n} \cdot \frac{1}{n} \leq \frac{n}{n} \cdot \frac{n}{n} \cdot \frac{1}{n} = \frac{1}{n}$$
   Since $\displaystyle\lim_{n \to \infty} 0 = 0$ and $\displaystyle\lim_{n \to \infty} \frac{1}{n} = 0$, by the Squeeze Theorem, $\displaystyle\lim_{n \to \infty} \frac{n!}{n^n} = 0$ and the sequence converges.

> **Theorem 4.3.7:** For the sequence $\{a_n\}$ if both $\displaystyle\lim_{n \to \infty} a_{2n} = L$ and $\displaystyle\lim_{n \to \infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\displaystyle\lim_{n \to \infty} a_n = L$.

> **Theorem 4.3.8:** The sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \leq 1$ and diverges for all other values of r. Also,
> $$\lim_{n \to \infty} r^n \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \end{cases}$$

> **Theorem 4.3.9:** For the sequence $\{a_n\}$ if both $\displaystyle\lim_{n \to \infty} a_{2n} = L$ and $\displaystyle\lim_{n \to \infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\displaystyle\lim_{n \to \infty} a_n = L$.

Proof:
Let $\epsilon > 0$. Then since $\displaystyle\lim_{n \to \infty} a_{2n} = L$,

$$\exists N_1 \in \mathbb{N} : \forall n \geq N_1, |a_{2n} - L| < \epsilon$$

Similarly, because $\displaystyle\lim_{n \to \infty} a_{2n+1} = L$,

$$\exists N_2 \in \mathbb{N} : \forall n \geq N_2, |a_{2n+1} - L| < \epsilon$$

Now, let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. Then either $a_n = a_{2k}$ for some $k > N_1$ or $a_n = a_{2k+1}$ for some $k > N_2$, And so in either case we have that

$$|a_n - L| < \epsilon$$

Therefore, $\displaystyle\lim_{n \to \infty} a_n = L$ and so $\{a_n\}$ is convergent.

4.4 Bounded and Monotonic Sequences

> **Definition 4.4.1: Bounded Sequence**
>
> A sequence $\{a_n\}$ is **bounded** if
>
> $$\exists M \in \mathbb{R} : \forall n \in \mathbb{N}, |a_n| \leq M$$

> **Upper and Lower Bounds**
>
> If
> $$\exists m \in \mathbb{R} : \forall n \in \mathbb{N}, m \leq a_n$$
> the sequence $\{a_n\}$ is said to be **bounded below** and m is a **lower bound** of the sequence. Similarly, if
> $$\exists M \in \mathbb{R} : \forall n \in \mathbb{N}, M \geq a_n$$
> the sequence $\{a_n\}$ is said to be **bounded above** and M is an **upper bound** of the sequence.

> ***Theorem 4.4.2 (Bounded Sequence):*** *If a sequence $\{a_n\}$ is both bounded above and below, then it is bounded. That is, if*
>
> $$\exists m, M \in \mathbb{R} : \forall n \in \mathbb{N}, m \leq a_n \leq M$$
>
> *then $\{a_n\}$ is bounded.*

Example 4.2: Bounded Sequences

Determine if the following sequences are bounded:

1. $\left\{ \dfrac{\sin(n)}{n} \right\}$

2. $\left\{ (-1)^n n \right\}$

3. $\left\{ \dfrac{2n}{n+1} \right\}$

Solution:

1. Since $-1 \leq \sin(n) \leq 1$ for all n, and $n > 0$ for all $n \in \mathbb{N}$, we have

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

   For $n \geq 1$, this gives us $-1 \leq \frac{\sin(n)}{n} \leq 1$. Therefore, the sequence is bounded with $m = -1$ and $M = 1$.

2. The terms of this sequence alternate: $-1, 2, -3, 4, -5, 6, \ldots$. As n increases, $|(-1)^n n| = n$ grows without bound. Therefore, the sequence is unbounded.

3. We can rewrite $\dfrac{2n}{n+1} = \dfrac{2n+2-2}{n+1} = \dfrac{2(n+1)-2}{n+1} = 2 - \dfrac{2}{n+1}$. Since $n \geq 1$, we have $\dfrac{2}{n+1} \leq \dfrac{2}{2} = 1$, so $a_n \geq 2 - 1 = 1$. Also, since $\dfrac{2}{n+1} > 0$, we have $a_n < 2$. Therefore, $1 \leq a_n < 2$ for all n, and the sequence is bounded.

> **Definition 4.4.3: Monotonic Sequence**
>
> A sequence $\{a_n\}$ is **monotonic** if for all $n \in \mathbb{N}$
>
> $$a_{n+1} \geq a_n \quad \text{or} \quad a_{n+1} \leq a_n$$

> ***Theorem 4.4.4:*** *If a sequence $\{a_n\}$ is both bounded and monotonic, then it is convergent.*

Example 4.3: Monotonic Sequences

Determine if the following sequences are monotonic:

1. $\left\{ \dfrac{n}{n+1} \right\}$

2. $\left\{ \dfrac{n+3}{n^2} \right\}$

Solution:

1. Consider $a_n = \dfrac{n}{n+1}$. To check if the sequence is monotonic, we can check if $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$. We have

   $$a_{n+1} = \frac{n+1}{n+2} \quad \text{and} \quad a_n = \frac{n}{n+1}$$

   Comparing:

   $$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0$$

   Since $a_{n+1} - a_n > 0$, we have $a_{n+1} > a_n$ for all n, so the sequence is monotonically increasing.

2. Consider $a_n = \dfrac{n+3}{n^2}$. Let's check the first few terms: $a_1 = 4$, $a_2 = \frac{5}{4}$, $a_3 = \frac{6}{9} = \frac{2}{3}$, $a_4 = \frac{7}{16}$. The sequence appears to be decreasing, but let's verify. We have

   $$a_{n+1} - a_n = \frac{n+4}{(n+1)^2} - \frac{n+3}{n^2} = \frac{n^2(n+4) - (n+3)(n+1)^2}{n^2(n+1)^2}$$

   Expanding the numerator:

   $$n^3 + 4n^2 - (n+3)(n^2 + 2n + 1) = n^3 + 4n^2 - n^3 - 2n^2 - n - 3n^2 - 6n - 3 = -n^2 - 7n - 3 < 0$$

   Since $a_{n+1} - a_n < 0$, we have $a_{n+1} < a_n$ for all n, so the sequence is monotonically decreasing.

> **Looking Ahead**
>
> Sequences form the foundation for understanding **series**. While a sequence is simply a list of numbers, a series is the sum of the terms of a sequence. In the next section, we will explore infinite series and develop tests to determine whether such sums converge to a finite value.

5 Series

Properties of Convergent Series

If $\sum a_n$ and $\sum b_n$ are both convergent series, then:

1. $\sum ca_n$, where c is a constant, is also convergent and
$$\sum ca_n = c \sum a_n$$

2. $\sum(a_n \pm b_n)$ is also convergent and
$$\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$$

5.1 Convergence of Series

Theorem 5.1.1 (Convergence of Series): *The series $\sum a_n$ converges if and only if the sequence of partial sums $\{s_n\}$ is convergent. That is,*

$$\sum a_n \text{ converges} \iff \lim_{n \to \infty} s_n \text{ exists}$$

If the series $\sum a_n$ converges, then
$$\lim_{n \to \infty} s_n = s$$

where s is the sum of the series.

Example 5.1:

$$\lim_{n \to \infty} n = \infty \qquad\qquad\qquad \text{(diverges)}$$

$$\lim_{n \to \infty} \frac{1}{n^2 - 1} = 0 \qquad\qquad \text{(converges)}$$

$$\lim_{n \to \infty} (-1)^n \text{ does not exist} \qquad \text{(diverges)}$$

$$\lim_{n \to \infty} \frac{1}{3^{n-1}} = 0 \qquad\qquad \text{(converges)}$$

Theorem 5.1.2: *If $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.*

5.2 Divergence Test

Theorem 5.2.1 (Divergence Test): *If $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum a_n$ diverges.*

Note:-

The Divergence Test is the contrapositive of the previous theorem. This is often the first test to try when testing a series for convergence, as it is quick to apply. However, if $\lim_{n \to \infty} a_n = 0$, the test is **inconclusive** and we must use other tests.

5.3 Special Series

5.3.1 Geometric Series

> **Definition 5.3.1: Geometric Series**
>
> A series of the form
> $$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$
> is called a **geometric series**. The sum of the series is
> $$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r| < 1$$

5.3.2 Telescoping Series

> **Definition 5.3.2: Telescoping Series**
>
> A series of the form
> $$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - a_2 + a_2 - a_3 + \cdots$$
> is called a **telescoping series**. The sum of the series is
> $$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_{n+1}$$

Example 5.2: Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution: First, we use partial fraction decomposition:
$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

Multiplying both sides by $n(n+1)$ gives $1 = A(n+1) + Bn$. Setting $n = 0$ gives $A = 1$, and setting $n = -1$ gives $B = -1$. Thus:
$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Now we can write the partial sum:
$$s_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$
$$= 1 - \frac{1}{N+1}$$

Taking the limit as $N \to \infty$:
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1$$

5.3.3   Harmonic Series

> **Definition 5.3.3: Harmonic Series**
>
> The series
> $$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$
> is called the **harmonic series**. The harmonic series diverges.

5.4   Integral Test

> **Theorem 5.4.1 (Integral Test):** *Let $f(x)$ be a continuous, positive, and decreasing function for $x \geq 1$. Then the series*
> $$\sum_{n=1}^{\infty} f(n)$$
> *converges if and only if the improper integral*
> $$\int_{1}^{\infty} f(x)\, dx$$
> *converges. That is,*
> $$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_{1}^{\infty} f(x)\, dx \text{ converges}$$

Proof:
Let $s_n = f(1) + f(2) + \cdots + f(n)$ and $s_{n+1} = f(1) + f(2) + \cdots + f(n) + f(n+1)$. Then
$$s_{n+1} - s_n = f(n+1) \geq 0$$
and
$$\int_{n}^{n+1} f(x)\, dx \leq f(n) \leq \int_{n-1}^{n} f(x)\, dx$$
Summing from 1 to n gives
$$s_{n+1} - s_1 \leq \int_{1}^{n} f(x)\, dx \leq s_n$$
Taking the limit as $n \to \infty$ gives
$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} s_{n+1} = s$$
which implies that the series converges if and only if the integral converges. $\square$

5.4.1   The p-series Test

> **Theorem 5.4.2 (p-series Test):** *The series*
> $$\sum_{n=k}^{\infty} \frac{1}{n^p} \qquad where\ k \in \mathbb{N}$$
> *converges if $p > 1$ and diverges if $p \leq 1$.*

Proof:
If $p > 1$, then the integral

$$\int_1^\infty \frac{1}{x^p} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^p} \, dx = \lim_{b \to \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b = \frac{1}{p-1}$$

converges. If $p \leq 1$, then the integral diverges. $\qquad\qquad\square$

5.5 Comparison Test/Limit Comparison Test

> **Theorem 5.5.1 (Comparison Test):** Let $\sum a_n$ and $\sum b_n$ be two series with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then:
>
> 1. If $\sum b_n$ converges, then $\sum a_n$ converges.
>
> 2. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Proof:
Let the partial sums be

$$s_n = \sum_{i=1}^n a_i \quad \text{and} \quad t_n = \sum_{i=1}^n b_i$$

Since $a_n, b_n \geq 0$, we know that

$$s_n \leq s_n + a_{n+1} = \sum_{i=1}^n a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = s_{n+1} \qquad\qquad \implies \; s_n \leq s_{n+1}$$

$$t_n \leq t_n + b_{n+1} = \sum_{i=1}^n b_i + b_{n+1} = \sum_{i=1}^{n+1} b_i = t_{n+1} \qquad\qquad \implies \; t_n \leq t_{n+1}$$

Also, since $a_n \leq b_n$, we have

$$s_n = \sum_{i=1}^n a_i \leq \sum_{i=1}^n b_i = t_n$$

If $\sum b_n$ converges, then $\{t_n\}$ is bounded and increasing, so it converges. Since $s_n \leq t_n$, $\{s_n\}$ is also bounded and increasing, so it converges. $\qquad\square$

> **Theorem 5.5.2 (Limit Comparison Test):** Let $\sum a_n$ and $\sum b_n$ be two series with $a_n, b_n > 0$ for all n. If
> $$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$
> where $c \in \mathbb{R}_+$ and $c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Proof:
Since $0 < c < \infty$, there exists two positive finite numbers m and M such that $m < c < M$. Now we have

$$m < \frac{a_n}{b_n} < M$$

$$mb_n < a_n < Mb_n$$

Now, if $\sum b_n$ converges, then so does $\sum M b_n$, and since $a_n < M b_n$, for all n by the Comparison Test, $\sum a_n$ also converges.
Similarly, if $\sum b_n$ diverges, then so does $\sum m b_n$, and since $m b_n < a_n$, for all n by the Comparison Test, $\sum a_n$ also diverges. $\qquad\square$

5.6 Alternating Series Test

> **Theorem 5.6.1 (Alternating Series Test):** *Let $\sum a_n$ be an alternating series, that is,*
> $a_n = (-1)^n b_n$ *or* $a_n = (-1)^{n+1} b_n$ *where* $b_n > 0$ *for all* n. *Then if:*
>
> 1. $\lim\limits_{n\to\infty} b_n = 0$
>
> 2. $\forall n, b_{n+1} \leq b_n$
>
> *the series $\sum a_n$ is convergent.*

Proof:
Let $s_n = \sum_{i=1}^{n} a_i$. Since b_n is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, we can say

$$\forall n, b_n - b_{n+1} \geq 0$$

Now, we have

$$s_{2n} = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots + b_{2n-1} - b_{2n}$$
$$= b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Since b_n is decreasing, s_{2n} is increasing and bounded above by b_1.
Let's assume that its limit is s, that is

$$\lim\limits_{n\to\infty} s_{2n} = s$$

Then

$$\lim\limits_{n\to\infty} s_{2n+1} = \lim\limits_{n\to\infty} (s_{2n} + b_{2n+1}) = \lim\limits_{n\to\infty} s_{2n} + \lim\limits_{n\to\infty} b_{2n+1} = s + 0 = s$$

So, we know that both $\{s_{2n}\}$ and $\{s_{2n+1}\}$ are convergent sequences and they both have the same limit.
We also know that $\{s_n\}$ is a convergent sequence with a limit of s. This in turn implies that the series $\sum a_n$ is convergent $\qquad\square$

5.7 Absolute and Conditional Convergence

> **Definition 5.7.1: Absolute and Conditional Convergence**
>
> A series $\sum a_n$ is said to be **absolutely convergent** if $\sum |a_n|$ converges. A series that converges but not absolutely is said to be **conditionally convergent**.

> **Theorem 5.7.2:** *If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.*

> **Note:-**
> The converse is not true: a series can be convergent without being absolutely convergent. Such a series is called conditionally convergent.

> **Theorem 5.7.3 (Riemann Rearrangement Theorem):** *Given the series $\sum a_n$:*
>
> 1. *If $\sum a_n$ is absolutely convergent with sum s, then any rearrangement of $\sum a_n$ also converges to s.*
>
> 2. *If $\sum a_n$ is conditionally convergent, then for any real number r (including $\pm\infty$), there exists a rearrangement of $\sum a_n$ that converges to r or diverges to $\pm\infty$.*

5.8 Ratio Test

> **Theorem 5.8.1 (Ratio Test):** Let $\sum a_n$ be a series and let
> $$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$
> Then:
>
> 1. If $L < 1$, then $\sum a_n$ is absolutely convergent.
>
> 2. If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.
>
> 3. If $L = 1$, then the test is inconclusive.

<u>Proof:</u>

Let $L < 1$ and $L = \lim\limits_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then there exists a number r such that $L < r < 1$. Since $L < r$, there exists a number N such that for all $n > N$,

$$\left| \frac{a_{n+1}}{a_n} \right| < r$$

$$|a_{n+1}| < r\,|a_n|$$

$$|a_{n+1}| < r\,|a_n| < r^2\,|a_{n-1}| < \cdots < r^{n-N}\,|a_N|$$

$$|a_{n+1}| < r^{n-N}\,|a_N|$$

Since $r < 1$, the series $\sum r^{n-N}\,|a_N|$ converges by the geometric series test. By the Comparison Test, $\sum a_n$ also converges. $\qquad\square$

5.9 Root Test

> **Theorem 5.9.1 (Root Test):** Let $\sum a_n$ be a series and let
> $$L = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} |a_n|^{\frac{1}{n}}$$
> Then:
>
> 1. If $L < 1$, then $\sum a_n$ is absolutely convergent.
>
> 2. If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.
>
> 3. If $L = 1$, then the test is inconclusive.

> **Note:-**
> $$\lim_{n \to \infty} n^{\frac{1}{n}} = 1$$

Proof:

Let $L < 1$ and $L = \lim\limits_{n \to \infty} \sqrt[n]{|a_n|}$. Then there exists a number r such that $L < r < 1$. Since $L < r$, there exists a number N such that for all $n > N$,

$$\sqrt[n]{|a_n|} < r$$

$$|a_n| < r^n$$

Since $r < 1$, the series $\sum r^n$ converges by the geometric series test. By the Comparison Test, $\sum a_n$ also converges. $\qquad\square$

5.10   Strategies for Series Test

Divergence Test If $\lim\limits_{n \to \infty} a_n \neq 0$.

Geometric Series Test If $a_n = ar^n$ or $a_n = ar^{n-1}$

Integral Test If $a_n = f(n)$ and $f(x)$ is continuous, positive, and decreasing

p-series Test If $a_n = \frac{1}{n^p}$

Comparison Test If a_n is hard to work with, but b_n is easy to work with

Limit Comparison Test If a_n is a rational expression involving only polynomials.

Alternating Series Test If $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$

Ratio Test If a_n is a product of terms

Root Test If a_n is a power of terms

5.11   Estimating the Value of a Series

5.11.1   Integral Test

Integral Test

$$s_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq s \leq s_n + \int_{n}^{\infty} f(x)\,\mathrm{d}x$$

5.11.2   Comparison Test

Given a series $\sum a_n$, let's assume that we've used the comparison test to show that it's convergent. Therefore we found a second series $\sum b_n$ that converged and $a_n \leq b_n$ for all n. Now, let

$$R_n = \sum_{k=n+1}^{\infty} a_k \quad \text{and} \quad T_n = \sum_{k=n+1}^{\infty} b_k$$

Since, $a_n \leq b_n$ we also know that $R_n \leq T_n$.

Comparison Test

$$R_n \leq T_n \leq \int_{n}^{\infty} g(x)\,\mathrm{d}x, \qquad \text{where } g(n) = b_n$$

5.11.3 Alternating Series Test

$$|R_n| = |s - s_n| \leq b_{n+1}$$

5.11.4 Ratio Test

Ratio Test

To get an estimate of the remainder, let's first define the following sequence,

$$r_n = \frac{a_{n+1}}{a_n}$$

We now have two possible cases:

1. If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, then

$$R_n \leq \frac{a_{n+1}}{1 - r_{n+1}}$$

2. If $\{r_n\}$ is an increasing sequence, then

$$R_n \leq \frac{a_{n+1}}{1 - L}, \qquad \text{where } L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

5.12 Power Series

Definition 5.12.1: Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots$$

where c_n are constants and called coefficients and a is a fixed number. The number a is called the **center** of the power series.

Theorem 5.12.2 (Convergence of Power Series): *Given the power series $\sum c_n(x - a)^n$, there exists a number R called the **radius of convergence** such that:*

1. *The series converges absolutely for all x with $|x - a| < R$*

2. *The series diverges for all x with $|x - a| > R$*

*To find R, apply the **Ratio Test** or **Root Test**:*

- **Ratio Test:** *Compute $L = \lim\limits_{n \to \infty} \left| \dfrac{c_{n+1}(x - a)^{n+1}}{c_n(x - a)^n} \right|$. The series converges when $L < 1$, which gives $|x - a| < R$.*

- **Root Test:** *Compute $L = \lim\limits_{n \to \infty} \sqrt[n]{|c_n(x - a)^n|}$. The series converges when $L < 1$, which gives $|x - a| < R$.*

N.B: *the series always converges at $x = a$.*

Interval vs Radius of Convergence

The **radius of convergence** R is a non-negative number (or ∞) that tells us how far from the center a the series converges. The **interval of convergence** is the actual set of x-values for which the series converges.

Given radius R and center a:

- The series converges absolutely on the open interval $(a - R, a + R)$

- The series diverges outside the closed interval $[a - R, a + R]$

- At the endpoints $x = a - R$ and $x = a + R$, convergence must be tested separately using other tests (p-series, alternating series, etc.)

- The interval of convergence is one of: $(a - R, a + R)$, $[a - R, a + R)$, $(a - R, a + R]$, or $[a - R, a + R]$

5.13 Power Series and Functions

Theorem 5.13.1 (Power Series and Functions): *Let $f(x) = \sum c_n(x - a)^n$ be a power series with radius of convergence R. Then:*

1. *$f(x)$ is continuous and differentiable on the interval $(a - R, a + R)$*

2. *$f'(x) = \sum\limits_{n=1}^{\infty} n c_n (x - a)^{n-1}$*

3. *$\displaystyle\int f(x)\,dx = C + \sum\limits_{n=0}^{\infty} \frac{c_n(x - a)^{n+1}}{n + 1}$*

4. *The radius of convergence of $f'(x)$ and $\int f(x)\,dx$ are also R.*

5.14 Taylor Series

> **Definition 5.14.1: Taylor Series**
>
> The **Taylor series** of a function $f(x)$ about $x = a$ is given by
>
> $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
>
> where $f^{(n)}(a)$ is the nth derivative of $f(x)$ evaluated at $x = a$.

To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the **n-th degree Taylor polynomial** of $f(x)$ as

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Notice that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the n-th degree Taylor polynomial is just the partial sum for the series.
The **remainder** is defined to be

$$R_n(x) = f(x) - T_n(x)$$

So, the remainder is just the *error* between the function $f(x)$ and the n-th degree Taylor polynomial $T_n(x)$ for a given n.
With this definition, we can then write the function as

$$f(x) = T_n(x) + R_n(x)$$

> **Theorem 5.14.2:** *Suppose that $f(x) = T_n(x) + R_n(x)$. Then if*
>
> $$\lim_{n \to \infty} R_n(x) = 0$$
>
> *for $|x - a| < R$, then*
>
> $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
>
> *on $|x - a| < R$.*

5.14.1 Maclaurin Series

> **Definition 5.14.3: Maclaurin Series**
>
> The **Maclaurin series** of a function $f(x)$ is the Taylor series of $f(x)$ about $x = 0$. That is
>
> $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for all } x$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{for all } x$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{for all } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for } -1 \leq x \leq 1$$

5.15 Binomial Series

Theorem 5.15.1 **(Binomial Theorem):** *If n is a positive integer, then*

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \cdots + nab^{n-1} + b^n$$

where,

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} = \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!}$$

$$\binom{n}{0} = 1$$

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$= 1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots + \binom{k}{n} x^n + \cdots$$

where,

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

$$\binom{k}{0} = 1$$

6   3-Dimensional Space

The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise, the 2-D coordinate system is denoted by $\mathbb{R}^2$, and the 1-D coordinate system is denoted by $\mathbb{R}$.

6.1   Equations of Lines

Vector form

If $\vec{a}$ and $\vec{v}$ are parallel vectors, then $\vec{a} = t\vec{v}$ for some scalar t.
Now if we have a vector $\vec{r}$ as follows

$$\vec{r} = \vec{r_0} + \vec{a}$$

Then we can write

$$\vec{r} = \vec{r_0} + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

This is called the **vector form of the equation of a line**.

Parametric form

We can rewrite the vector form as

$$\vec{r} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$
$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

In other words

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

This set of equations is called the **parametric form of the equation of a line**.

Symmetric Equations of a Line

If we assume that a, b, and c are non-zero numbers, then we can solve each of the parametric equations for t. This gives us

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 6.1: Find the Equations of lines:
1. Through the points $(7, -3, 1)$ **and** $(-2, 1, 4)$
2. Through the point $(1, -5, 0)$ **and parallel to the line given by** $\vec{r}(t) = \langle 8 - 3t, -10 + 9t, -1 - t \rangle$
3. Through the point $(-7, 2, 4)$ **and orthogonal to both** $\vec{v} = \langle 0, -9, 1 \rangle$ **and** $\vec{w} = 3\hat{i} + \hat{j} - 4\hat{k}$

1.
Direction vector $\vec{d} = \langle -2 - 7, 1 + 3, 4 - 1 \rangle = \langle -9, 4, 3 \rangle$
Now, the vector form of the line is

$$\vec{r} = \langle 7, -3, 1 \rangle + t\langle -9, 4, 3 \rangle$$

The parametric form is

$$x = 7 - 9t, \quad y = -3 + 4t, \quad z = 1 + 3t$$

The symmetric form is
$$\frac{x-7}{-9} = \frac{y+3}{4} = \frac{z-1}{3}$$

2.
The direction vector is $\vec{d} = \langle -3, 9, -1 \rangle$
Hence, the vector form of the line is

$$\vec{r} = \langle 1, -5, 0 \rangle + t\langle -3, 9, -1 \rangle$$

The parametric form is

$$x = 1 + 3t, \quad y = -5 + 9t, \quad z = -t$$

And the symmetric form is
$$\frac{x-1}{3} = \frac{y+5}{9} = -z$$

3.
Direction vector

$$\vec{d} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -9 & 1 \\ 3 & 1 & -4 \end{vmatrix} = \langle 35, 3, 27 \rangle$$

Hence, the vector form of the line is

$$\vec{r} = \langle -7, 2, 4 \rangle + t\langle 35, 3, 27 \rangle$$

The parametric form is

$$x = -7 + 35t, \quad y = 2 + 3t, \quad z = 4 + 27t$$

The symmetric form is
$$\frac{x+7}{35} = \frac{y-2}{3} = \frac{z-4}{27}$$

Example 6.2: Determine if the two lines are parallel, orthogonal, or neither:
1. The line given by $\vec{r}(t) = \langle 4 - 7t, -10 + 5t, 21 - 4t \rangle$ and the line given by $\vec{r}(t) = \langle -2 + 3t, 7 + 5t, 5 + t \rangle$
2. The line given by $x = 29, y = -3 - 6t, z = 12 - t$ and the line given by $\vec{r}(t) = \langle 12 - 14t, 2 + 7t, -10 + 3t \rangle$

1.
The direction vectors are

$$\vec{d_1} = \langle -7, 5, -4 \rangle, \quad \vec{d_2} = \langle 3, 5, 1 \rangle$$

To check if they are parallel, we can check:
$$\frac{-7}{3} \neq \frac{5}{5} \neq \frac{-4}{1}$$

which means they are not parallel.
To check if they are orthogonal, we can check:
$$\vec{d_1} \cdot \vec{d_2} = -7(3) + 5(5) + (-4)(1) = -21 + 25 - 4 = 0$$

Hence, they are orthogonal.

2.

The direction vectors are

$$\vec{d_1} = \langle 0, -6, -1 \rangle, \quad \vec{d_2} = \langle -14, 7, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{0}{-14} \neq \frac{-6}{7} \neq \frac{-1}{3}$$

which means they are not parallel.

To check if they are orthogonal, we can check:

$$\vec{d_1} \cdot \vec{d_2} = 0(-14) + (-6)(7) + (-1)(3) = -42 - 3 = -45 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 6.3: Determine the intersection point of the two lines or show that they don't not intersect:

1. The line passing through the point $(0, -9, -1)$ **and** $(1, 6, -3)$ **and the line given by** $\vec{r}(t) = \langle -9 - 4t, 10 + 6t, 1 - 2t \rangle$

2. The line given by $x = 1 + 6t, y = -1 - 3t, z = 4 + 12t$ **and the line given by** $x = 4 + t, y = -10 - 8t, z = 3 - 5t$

1.

The direction vector of the first line is

$$\vec{d_1} = \langle 1 - 0, 6 + 9, -3 + 1 \rangle = \langle 1, 15, -2 \rangle$$

We can write the parametric equations of the first line as:

$$x = s, y = -9 + 15s, z = -1 - 2s$$

And the parametric equations of the second line as:

$$x = -9 - 4t, y = 10 + 6t, z = 1 - 2t$$

Setting them equal to each other we get,

$$0 + t = -9 - 4s$$
$$-9 + 15t = 10 + 6s$$
$$-1 - 2t = 1 - 2s$$

Solving the first two equations, we get

$$t = -\frac{7}{3}, \quad s = \frac{1}{3}$$

Now, verifying the third equation, we get

$$-1 - 2(-\frac{7}{3}) = 1 - 2(\frac{1}{3})$$
$$-1 + \frac{14}{3} = 1 - \frac{2}{3}$$
$$\frac{11}{3} \neq \frac{1}{3}$$

Since the third equation is not satisfied, the two lines do not intersect.

2.

The lines are given in parametric form.

Setting them equal to each other we get,

$$1 + 6s = 4 + t$$
$$-1 - 3s = -10 - 8t$$
$$4 + 12s = 3 - 5t$$

Solving the first two equations, we get

$$s = \frac{1}{3}, \quad t = -1$$

Now, verifying the third equation, we get

$$4 + 12\left(\frac{1}{3}\right) = 3 - 5\langle -1\rangle$$
$$8 = 8$$

That means, the lines intersect. Substituting the values in the parametric equation, we get

$$x = 1 + 6\left(\frac{1}{3}\right) = 3$$
$$y = -1 - 3\left(\frac{1}{3}\right) = -2$$
$$z = 4 + 12\left(\frac{1}{3}\right) = 8$$

Hence, the intersection point is $(3, -2, 8)$.

Example 6.4: Which of the three coordinate planes does the line given by $x = 16t, y = -4 - 9t, z = 34$ intersect?

To intersect the xy-plane, we need $z = 0$. But here $z = 34$ is constant. Hence, the line does not intersect the xy-plane.

To intersect the yz-plane, we need $x = 0$. Hence,

$$16t = 0 \implies t = 0$$

And the intersection point is $(0, -4 - 9 \times 0, 34)$ or $(0, -4, 34)$.

To intersect the xz-plane, we need $y = 0$. Hence,

$$-4 - 9t = 0 \implies t = -\frac{4}{9}$$

And the intersection point is $\left(16\left(-\frac{4}{9}\right), 0, 43\right)$ or $\left(-\frac{64}{9}, 0, 34\right)$.

6.2 Equations of Planes

Vector form

Let's assume $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ and $\vec{r} = \langle x, y, z \rangle$ are two position vectors and $\vec{r} - \vec{r_0}$ is a vector in the plane.
If $\vec{n} = \langle a, b, c \rangle$ is a normal to the plane (which means it's orthogonal to the vector $\vec{r} - \vec{r_0}$), then we can write

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \implies \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

This is called the **vector form of the equation of a plane**.

Scalar form

If we expand the vector equation in the following way,

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$
$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Computing the dot product, we get

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the **scalar form of the equation of a plane**.
This equation can also be written as

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

Example 6.5: Find the equation of the plane:
1. Through the point $(6, -3, 1)$**,** $(5, -4, 1)$**, and** $(3, -4, 0)$
2. The plane containing the point $(1, -5, 8)$ **and orthogonal to the line given by** $x = -3 + 15t, y = 14 - t, z = 9 - 3t$
3. The plane containing the point $(-8, 3, 7)$ **and parallel to the plane given by** $4x + 8y - 2z = 45$
4. The plane containing the two lines given by $\vec{r}(t) = \langle 7 + 5t, 2 + t, 6t \rangle$ **and** $\vec{r}(t) = \langle 7 - 6t, 2 - 2t, 10t \rangle$

1.
The given points are
$$A(6, -3, 1), B(5, -4, 1), C(3, -4, 0)$$

Two vectors in the plane are

$$\vec{AB} = \langle 5 - 6, -4 + 3, 1 - 1 \rangle = \langle -1, -1, 0 \rangle$$
$$\vec{BC} = \langle 3 - 5, -4 + 4, 0 - 1 \rangle = \langle -2, 0, -1 \rangle$$

Normal vector on the place:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 0 \\ -2 & 0 & -1 \end{vmatrix} = \hat{i} - \hat{j} - 2\hat{k}$$

Now, using the point A, we can write the equation of the plane as

$$(x - 6) - (y + 3) - 2(z - 1) = 0$$
$$x - y - 2z = 7$$

2.

The normal vector is

$$\vec{n} = \langle 15, -1, -3 \rangle$$

Using the point $(1, -5, 8)$, the equation of the plane is

$$15(x - 1) - (y + 5) - 3(z - 8) = 0$$
$$15x - y - 3z = 15 + 5 - 24$$
$$15x - y - 3z + 4 = 0$$

3.

The normal vector is

$$\vec{n} = \langle 4, 8, -2 \rangle$$

Using the point $(-8, 3, 7)$, the equation of the plane is

$$4(x + 8) + 8(y - 3) - 2(z - 7) = 0$$
$$4x + 8y - 2z = -32 + 24 + 14$$
$$4x + 8y - 2z + 6 = 0$$

4.

The direction vectors of the two lines are

$$\vec{d_1} = \langle 5, 1, 6 \rangle, \quad \vec{d_2} = \langle -6, -2, 10 \rangle$$

The normal vector is

$$\vec{n} = \vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 1 & 6 \\ -6 & -2 & 10 \end{vmatrix} = \langle 22, -86, -4 \rangle$$

Using the point $A(7, 2, 0)$, the equation of the plane is

$$22(x - 7) - 86(y - 2) - 4(z - 0) = 0$$
$$22x - 86y - 4z - 154 + 172 = 0$$
$$22x - 86y - 4z + 18 = 0$$

Example 6.6: Determine if the two planes are parallel, orthogonal, or neither: The plane given by $3x + 9y + 7z = -1$ **and the plane containing the points** $(1, -1, 9), (4, -1, 2), (-2, 3, 4)$

The normal vector of the first plane is

$$\vec{n_1} = \langle 3, 9, 7 \rangle$$

Let the points be
$$A(1, -1, 9), B(4, -1, 2), C(-2, 3, 4)$$

Two vectors in the plane are

$$\vec{AB} = \langle 4 - 1, -1 + 1, 2 - 9 \rangle = \langle 3, 0, -7 \rangle$$
$$\vec{AC} = \langle -2 - 1, 3 + 1, 4 - 9 \rangle = \langle -3, 4, -5 \rangle$$

The normal vector of the second plane is

$$\vec{n_2} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -7 \\ -3 & 4 & -5 \end{vmatrix} = \langle 28, 36, 12 \rangle = \langle 7, 9, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{3}{7} \neq \frac{9}{9} \neq \frac{7}{3}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{n_1} \cdot \vec{n_2} = 3(7) + 9(9) + 7(3) = 21 + 81 + 21 = 123 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 6.7: Find the intersection of the plane given by $4x + y + 10z = -2$ **and the plane given by** $-8x + 2y + 3z = -8$

The two planes are

$$4x + y + 10z = -2$$
$$-8x + 2y + 3z = -8$$

Multiplying the first equation by 2 and adding it to the second equation, we get

$$4y + 23z = -12 \quad \implies \quad y = -3 - \frac{23}{4}z$$

Substituting the value of y in the first equation, we get

$$16x - 3 - \frac{23}{4}z + 10z = -2 \quad \implies \quad x = \frac{1}{4} - \frac{17}{16}z$$

Let $z = t$ (a parameter). Then we get

$$x = \frac{1}{4} - \frac{17}{16}t$$
$$y = -3 - \frac{23}{4}t$$
$$z = t$$

This is the parametric form of the line of intersection.
We can also write it in vector form as

$$\vec{r} = \langle \frac{1}{4}, -3, 0 \rangle + t \langle -\frac{17}{16}, -\frac{23}{4}, 1 \rangle$$

6.3 Quadratic Surfaces

General form

The general form of a quadratic surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where $A, B, C, D, E, F, G, H, I, J$ are constants.

Ellipsoid

The general equation of an ellipsoid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the ellipsoid and a, b, c are the semi-axis lengths.
If $a = b = c$, we get a sphere.

Cone

The general equation of a cone that opens along the z-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{(z-l)^2}{c^2}$$

where (h, k, l) is the center of the cone and a, b, c are the semi-axis lengths.

Cylinder

The general equation of a cylinder that opens along the z-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where (h, k) is the center of the cylinder and a, b are the semi-axis lengths.
If $a = b$, we get a circular cylinder.

Hyperboloid of One Sheet

The general equation of a hyperboloid of one sheet is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

Hyperboloid of Two Sheets

The general equation of a hyperboloid of two sheets is

$$-\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

The general equation of an elliptic paraboloid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

Hyperbolic Paraboloid

The general equation of a hyperbolic paraboloid is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

6.4 Calculus with Vector Functions

Let

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Note:-

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

Note:-

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\vec{u} + \vec{v} \right) = \vec{u}' + \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(c\vec{u} \right) = c\vec{u}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\vec{u} \cdot \vec{v} \right) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\vec{u} \times \vec{v} \right) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\vec{u} f(t) \right) = f'(t) \vec{u}'(f(t))$$

Note:-

$$\int \vec{r}(t) \, \mathrm{d}t = \langle \int f(t) \, \mathrm{d}t, \int g(t) \, \mathrm{d}t, \int h(t) \, \mathrm{d}t \rangle$$

$$\int_a^b \vec{r}(t) \, \mathrm{d}t = \left\langle \int_a^b f(t) \, \mathrm{d}t + \int_a^b g(t) \, \mathrm{d}t + \int_a^b h(t) \, \mathrm{d}t \right\rangle$$

6.5 Tangent, Normal, and Binormal Vectors

Unit Tangent vector

Given the vector function $\vec{r}(t)$, we call $\vec{r}'(t)$ the **tangent vector** The unit tangent vector is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

If $\vec{T}(t)$ is the unit tangent vector, then the **unit normal vector** is given by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

If $\vec{r}(t)$ is a vector such that $\|\vec{r}(t)\| = c$ for all t, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$

The **binormal vector** is given by

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The binormal vector is orthogonal to both the tangent and normal vectors.

6.6   Arc Length with Vector Functions

The arc length of a vector function $\vec{r}(t)$ from $t = a$ to $t = b$ is given by

$$L = \int_a^b \|\vec{r}'(t)\| \, \mathrm{d}t$$

Or,

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2} \, \mathrm{d}t$$

6.7   Curvature

The curvature of a curve in 3-D space is given by

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

where $\vec{T}(t)$ is the unit tangent vector and $\vec{r}(t)$ is the position vector.
This can also be written as

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$