Calculus II Notes

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Contents

1 Integration Techniques

1.1 Fundamental Theorem of Calculus

Theorem 1.1.1* (Fundamental Theorem of Calculus)*: *Let f be a function defined on an open interval I that contains a. If f is continuous on I, then the function F defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is uniformly continuous on I, differentiable on the open interval, and

$$F'(x) = f(x)$$

for all x in the open interval.

1.2 Common Differentiation and Integration Formulae

Derivative	**Integral**
$\frac{\mathrm{d}}{\mathrm{d}x}x = 1$, $\quad \frac{\mathrm{d}}{\mathrm{d}x}c = 0$	$\int c\,\mathrm{d}x = cx + c$
$\frac{\mathrm{d}}{\mathrm{d}x}x^n$	$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}$	$\int \frac{1}{x}\,\mathrm{d}x = \ln(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}e^{mx} = me^{mx}$	$\int e^{mx}\,\mathrm{d}x = \frac{1}{m}e^{mx} + c$
$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x\ln(a)$	$\int a^x\,\mathrm{d}x = \frac{1}{\ln(a)}a^x + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sin(mx) = m\cos(mx)$	$\int \cos(mx)\,\mathrm{d}x = \frac{1}{m}\sin(mx) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cos(mx) = -m\sin(mx)$	$\int \sin(mx)\,\mathrm{d}x = -\frac{1}{m}\cos(mx) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$	$\int \sec^2(x)\,\mathrm{d}x = \tan(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot(x) = -\csc^2(x)$	$\int \csc^2(x)\,\mathrm{d}x = -\cot(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec(x) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,\mathrm{d}x = \sec(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc(x) = -\csc(x)\cot(x)$	$\int \csc(x)\cot(x)\,\mathrm{d}x = -\csc(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = \sec^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\cos^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}(x) = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,\mathrm{d}x = -\cot^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = -\csc^{-1}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x} + c$

Table 1: Common Differentiation and Integration Formulae

1.3 More Formulae

1. $\int \tan(x)\,\mathrm{d}x = \ln|\sec(x)| + c$
2. $\int \csc(x)\,\mathrm{d}x = \ln|\tan\frac{x}{2}| + c$
3. $\int \sec(x)\,\mathrm{d}x = \ln|\sec(x) + \tan(x)| + c$
4. $\int \sec(x)\,\mathrm{d}x = \ln|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)|$
5. $\int \csc(x)\,\mathrm{d}x = -\ln|\csc(x) + \cot(x)| + c$
6. $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$
7. $\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln|\frac{a+x}{a-x}| + c$
8. $\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln|\frac{a-x}{a+x}| + c$
9. $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln|x + \sqrt{x^2+a^2}| + c$
10. $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}(\frac{x}{a}) + c$
11. $\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$
12. $\int e^x\left[f(x) + f'(x)\right]\mathrm{d}x = e^x f(x) + c$

1.4 Integration by Parts

***Theorem 1.4.1* (Integration by Parts):** *Let u and v be differentiable functions of x. Then,*

$$\int uv\,\mathrm{d}x = u\int v\,\mathrm{d}x - \int\left[\frac{\mathrm{d}u}{\mathrm{d}x}\int v\,\mathrm{d}x\right]\mathrm{d}x$$

Or,

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

Proof:
Let $u = u(x)$ and $w = w(x)$. Then,

$$\frac{\mathrm{d}(uw)}{\mathrm{d}x} = u\frac{\mathrm{d}w}{\mathrm{d}x} + w\frac{\mathrm{d}u}{\mathrm{d}x}$$

Integrating both sides, we get

$$\int \frac{\mathrm{d}(uw)}{\mathrm{d}x}\,\mathrm{d}x = \int u\frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x + \int w\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Or,

$$uw = \int u\frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x + \int w\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Rearranging, we get

$$\int u\frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x = uw + c - \int w\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Let $v = \frac{\mathrm{d}w}{\mathrm{d}x}$, then $w = \int v\,\mathrm{d}x$. Hence,

$$\int uv\,\mathrm{d}x = u\int v\,\mathrm{d}x - \int\left[\frac{\mathrm{d}u}{\mathrm{d}x}\int v\,\mathrm{d}x\right]\mathrm{d}x$$

□

1.5 Method of Substitution

A. $\int \frac{1}{(ax+b)\sqrt{cx+d}}\,\mathrm{d}x$ Let $cx+d=z^2$

B. $\int \frac{1}{\sin^m x\cos^m x}\,\mathrm{d}x$ If $m+n=p$ is even, multiply and divide by $\sec^p x$ and let $\tan x = z$.

C. $\int \frac{1}{\sin^m x+\cos^m x}\,\mathrm{d}x$ If m is even, mulpliply and divide by $\sec^m x$.

D. $\int \frac{\cos x}{a\cos x+b\sin x}\,\mathrm{d}x$ Write $nom = l\times(denom)+m\times(denom)'$, then determine l and m.

E. $\int \frac{\cos x}{a\cos x+b\sin x}\,\mathrm{d}x + c$ Write $\sin x$ and $\cos x$ as $\tan\frac{x}{2}$.

F. $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x$ Let $x=a\tan\theta$

G. $\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x$ Let $x=a\sec\theta$

H. $\int \sqrt{a^2-x^2}\,\mathrm{d}x$ Let $x=a\sin\theta$

1.6 Trigonometric Integrals

Form	**Looks like**	**Substitution**	**Limit Assumption**
$\sqrt{b^2x^2-a^2}$	$\sec^2\theta-1=\tan^2\theta$	$x=\frac{a}{b}\sec\theta$	$0\le\theta<\frac{\pi}{2}, \frac{\pi}{2}<\theta\le\pi$
$\sqrt{a^2-b^2x^2}$	$1-\sin^2\theta=\cos^2\theta$	$x=\frac{a}{b}\sin\theta$	$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$
$\sqrt{a^2+b^2x^2}$	$\tan^2\theta+1=\sec^2\theta$	$x=\frac{a}{b}\tan\theta$	$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

Table 2: Trigonometric Integral Substitution

1.7 Partial Fractions

Factor in denominator	**Term in partial fraction decomposition**
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+b}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Table 3: Partial Fraction Decomposition

1.8 Improper Integrals

Definition 1.8.1: Improper Integrals

An integral is said to be **improper** if one of the following conditions is met:

1. The interval of integration is infinite.
2. The integrand is discontinuous at one or more points in the interval of integration.

The integral is said to **converge** if the limit of the integral exists, and **diverge** otherwise.

Type-1: If $\int_a^t f(x)\,\mathrm{d}x$ exists for all $t > a$, then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x$$

provided the limit exists and is finite.

Type-2: If $\int_t^b f(x)\,\mathrm{d}x$ exists for all $t < b$, then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\,\mathrm{d}x$$

provided the limit exists and is finite.

Type-3: If $\int_{-\infty}^c f(x)\,\mathrm{d}x$ and $\int_c^\infty f(x)\,\mathrm{d}x$ are both convergent, then

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^c f(x)\,\mathrm{d}x + \int_c^\infty f(x)\,\mathrm{d}x$$

Type-4: $\int_a^b f(x)\,\mathrm{d}x$ If $f(x)$ is discontinuous at $x = c$, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0}\left[\int_a^{c-\epsilon} f(x)\,\mathrm{d}x + \int_{c+\epsilon}^b f(x)\,\mathrm{d}x\right]$$

Note:-

If $a > 0$ then

$$\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$$

converges if $p > 1$ and diverges if $p \leq 1$.

1.8.1 Comparison Test

Theorem 1.8.2* (Comparison Theorem)*: *If $f(x) \geq g(x) \geq 0$ on the interval $[a,\infty)$, then*

1. *If $\int_a^\infty f(x)\,\mathrm{d}x$ converges, then so does $\int_a^\infty g(x)\,\mathrm{d}x$.*
2. *If $\int_a^\infty g(x)\,\mathrm{d}x$ diverges, then so does $\int_a^\infty f(x)\,\mathrm{d}x$.*

Example 1.1: Determine if the following integral is convergent or divergent:

$$\int_2^\infty \frac{\cos^2 x}{x^2}\,\mathrm{d}x$$

Notice that the numerator is bounded since

$$0 \leq \cos^2 x \leq 1$$

Hence, it's likely that the denominator will determine the convergence of the integral. Since $p = 2 > 1$,

$$\int_2^\infty \frac{1}{x^2}\,\mathrm{d}x$$

is convergent. Since

$$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

and $\int_2^\infty \frac{1}{x^2}\,\mathrm{d}x$ is convergent, by the comparison test,

$$\int_2^\infty \frac{\cos^2 x}{x^2}\,\mathrm{d}x$$

is convergent.

Example 1.2: Determine if the following integral is convergent or divergent:

$$\int_3^\infty \frac{1}{x + e^x}\,\mathrm{d}x$$

In this case, the denominator determines the convergence of the integral. If we can find a larger function that converges, then the integral will converge. Notice that

$$\frac{1}{x + e^x} < \frac{1}{e^x} = e^{-x}$$

Also,

$$\begin{aligned}\int_3^\infty e^{-x}\,\mathrm{d}x &= \lim_{t\to\infty} \int_3^t e^{-x}\,\mathrm{d}x \\ &= \lim_{t\to\infty} \left(-e^{-t} + e^{-3}\right) \\ &= e^{-3}\end{aligned}$$

So, $\int_3^\infty e^{-x}\,\mathrm{d}x$ is convergent. Therefore, by the Comparison test,

$$\int_3^\infty \frac{1}{x + e^x}\,\mathrm{d}x$$

is also convergent.

2 Applications of Integrals

2.1 Arc Length

Consider a curve $y = f(x)$. We want to find the length of the curve from $x = a$ to $x = b$. We can approximate the curve by a series of line segments. The length of each line segment is given by the Pythagorean theorem:

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

The total length of the curve is given by the sum of the lengths of the line segments:

$$L = \int \mathrm{d}s = \int \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Similarly, for a curve $x = h(y)$, the length of the curve from $y = c$ to $y = d$ is given by:

$$L = \int \mathrm{d}s = \int \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

Arc Length Formula

$$L = \int \mathrm{d}s$$

where,

$$ds = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \quad \text{for } y = f(x), a \leq x \leq b$$

$$ds = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y \quad \text{for } x = h(y), c \leq y \leq d$$

2.2 Surface Area

Consider a curve $y = f(x)$ rotated about the x-axis. We want to find the surface area of the resulting surface. We can approximate the surface by a series of frustums. The surface area of each frustum is given by:

$$\mathrm{d}S = 2\pi y\,\mathrm{d}s = 2\pi f(x)\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

The total surface area of the surface is given by the sum of the surface areas of the frustums:

$$S = \int \mathrm{d}S = \int 2\pi y\,\mathrm{d}s = \int 2\pi f(x)\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Similarly, for a curve $x = h(y)$ rotated about the y-axis, the surface area of the resulting surface is given by:

$$A = \int 2\pi x\,\mathrm{d}s = \int 2\pi h(y)\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

Surface Area Formula

$$\begin{aligned} S &= \int \mathrm{d}S \\ &= \int 2\pi y \, \mathrm{d}s \quad \text{rotation about } x\text{–axis} \\ &= \int 2\pi x \, \mathrm{d}s \quad \text{rotation about } y\text{–axis} \end{aligned}$$

where,

$$dS = 2\pi y \, \mathrm{d}s = 2\pi f(x) \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x \quad \text{for } y = f(x), a \leq x \leq b$$

$$dS = 2\pi x \, \mathrm{d}s = 2\pi h(y) \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \, \mathrm{d}y \quad \text{for } x = h(y), c \leq y \leq d$$

2.3 Center of Mass

Suppose we want to find the center of mass of a region bounded by two curves $f(x)$ and $g(x)$ on the interval $[a, b]$.
The mass is

$$M = \rho \int_a^b \big(f(x) - g(x)\big) \, \mathrm{d}x$$

Next, we need the **moments** of the region. There are two moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2} \left[f(x)^2 - g(x)^2\right] \mathrm{d}x \\ M_y &= \rho \int_a^b x \left[f(x) - g(x)\right] \mathrm{d}x \end{aligned}$$

The coordinates of the center of mass, $(\bar{x}, \bar{y})$, are given by:

Center of Mass Formula

$$\bar{x} = \frac{M_y}{M} = \frac{1}{A} \int_a^b x \left[f(x) - g(x)\right] \mathrm{d}x$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{A} \int_a^b \frac{1}{2} \left[f(x)^2 - g(x)^2\right] \mathrm{d}x$$

where,

$$A = \int_a^b \left[f(x) - g(x)\right] \mathrm{d}x$$

2.4 Probability

Every continuous random variable X, has a probability density function $f(x)$. Probability density functions satisfy the following conditions:

1. $f(x) \geq 0$ for all x.
2. $\displaystyle\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = 1$

Probability density finctions can be used to determine the probability that a continuous random variable lies between two values, say a and b. This probability is denoted by $P(a \leq X \leq b)$.

Note:-

$$P(a \leq X \leq b) = \int_a^b f(x)\,\mathrm{d}x$$

Probability density functions can also be used to determine the mean of a continuous random variable. The mean is given by:

Mean of a Continuous Random Variable

$$\mu = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$$

3 Parametric Equations and Polar Coordinates

There are great many curves out there that cannot be expressed in a single equation in terms of only x and y. To deal with such problems, we introduce **parametric equations**. Instead of defining y in terms of $x(y = f(x))$ or x in terms of $y(x = h(y))$, we define both x and y in terms of a third variable called a parameter as follows:

$$x = f(t) \quad y = g(t)$$

This third variable is usually denoted byu t. Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letteing t be all possible values if the graph of the parametric equations and is called a **parametric curve**.

3.1 Tangents with Parametric Equations

Derivative for Parametric Equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \quad \text{provided } \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

Tangents for Parametric Equations

Horizontal Tangent:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0, \quad \text{provided } \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

Vertical Tangent:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0, \quad \text{provided } \frac{\mathrm{d}y}{\mathrm{d}t} \neq 0$$

Second Derivative for Parametric Equations

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

3.2 Area with Parametric Equations

Area with Parametric Equations

$$A = \int_{t_1}^{t_2} y(t)\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

or

$$A = \int_{t_1}^{t_2} x(t)\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t$$

3.3 Arc Length with Parametric Equations

The arc length of a curve is given by

$$L = \int \mathrm{d}s$$

where,

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \qquad \text{if } y = f(x), a \leq x \leq b$$
$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y \qquad \text{if } x = h(y), c \leq y \leq d$$

Using the first $\mathrm{d}s$, we can write

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

Then the arc length formula becomes,

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}\right)^2}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_\alpha^\beta \frac{1}{\left|\frac{\mathrm{d}x}{\mathrm{d}t}\right|}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \end{aligned}$$

Arc Length with Parametric Equations

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

3.4 Surface Area with Parametric Equations

Surface Area with Parametric Equations

$$S = \int 2\pi y\,\mathrm{d}s \qquad \text{rotation about } x\text{–axis}$$
$$S = \int 2\pi x\,\mathrm{d}s \qquad \text{rotation about } y\text{–axis}$$

where,

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

3.5 Polar Coordinates

Polar to Cartesian Conversion

$$x = r\cos\theta \quad y = r\sin\theta$$

Cartesian to Polar Conversion

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$
$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

3.6 Tangents with Polar Coordinates

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = r' \cos\theta - r\sin\theta \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = r' \sin\theta + r\cos\theta$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\theta} + \frac{\mathrm{d}r}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}\theta}$$

Tangents with Polar Coordinates

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{r' \sin\theta + r\cos\theta}{r' \cos\theta - r\sin\theta}$$

3.7 Area with Polar Coordinates

Area with Polar Coordinates

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_o^2 - r_i^2\right) \mathrm{d}\theta$$

3.8 Arc Length with Polar Coordinates

Let

$$r = f(\theta) \quad \alpha \leq \theta \leq \beta$$

$$\begin{aligned} x &= r\cos(\theta) & y &= r\sin(\theta) \\ x &= f(\theta)\cos\theta & y &= f(\theta)\sin\theta \\ \frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta & \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta \end{aligned}$$

Now,

$$\begin{aligned} \left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta\right)^2 \\ &= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \cos^2\theta - 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &+ \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \sin^2\theta + 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2 \end{aligned}$$

Arc Length with Polar Coordinates

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta$$

3.9 Surface Area with Polar Coordinates

Let

$$r = f(\theta) \quad \alpha \leq \theta \leq \beta$$

$$\begin{aligned} x &= r\cos(\theta) & y &= r\sin(\theta) \\ x &= f(\theta)\cos\theta & y &= f(\theta)\sin\theta \end{aligned}$$

Surface Area with Polar Coordinates

$$S = \int 2\pi y \, \mathrm{d}s \qquad \text{rotation about } x\text{–axis}$$

$$S = \int 2\pi x \, \mathrm{d}s \qquad \text{rotation about } y\text{–axis}$$

where,

$$\mathrm{d}s = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} \, \mathrm{d}\theta \qquad r = f(\theta), \alpha \leq \theta \leq \beta$$

4 Sequences

4.1 Definition

Definition 4.1.1: Sequence

A **sequence** is a function whose domain is the set of natural numbers $\mathbb{N}$. The function is denoted by a_n and the value of the function at n is denoted by a_n.
Various ways of representing a sequence are:

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

4.2 Precise Definition of Limit of a Sequence

Precise Definition of Limit

1. We say that
$$\lim_{n\to\infty} a_n = L$$
if
$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - L| < \epsilon$$
2. We say that
$$\lim_{n\to\infty} a_n = \infty$$
if
$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n > M$$
3. We say that
$$\lim_{n\to\infty} a_n = -\infty$$
if
$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n < M$$

4.3 Convergence of Sequences

Theorem 4.3.1* (Convergence of Sequences)*: *A sequence $\{a_n\}$ is said to be* ***convergent*** *if there exists a real number L such that*

$$\lim_{n\to\infty} a_n = L$$

Theorem 4.3.2: *Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.*

Properties of Convergent Sequences

If $\{a_n\}$ and $\{b_n\}$ are both convergent sequences, then:

1.
$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$
2.
$$\lim_{n\to\infty} ca_n = c \lim_{n\to oo} a_n$$

3.
$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$

4.
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \qquad \text{provided } \lim_{n\to\infty} b_n \neq 0$$

5.
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p, \qquad \text{provided } a_n \geq 0$$

Theorem 4.3.3* (Squeeze Theorem for Sequences)*: *If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences such that $a_n \leq b_n \leq c_n$ for all $n \geq N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.*

Theorem 4.3.4:

$$\textit{If } \lim_{n\to\infty}|a_n| = 0, \textit{ then } \lim_{n\to\infty} a_n = 0$$

Proof:

The main thing to this proof is to note that,

$$-|a_n| \leq a_n \leq |a_n|$$

Then note that,

$$\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty}|a_n| = 0$$

We then have that,

$$0 \leq \lim_{n\to\infty} a_n \leq 0$$

and so by the Squeeze Theorem,

$$\lim_{n\to\infty} a_n = 0$$

Theorem 4.3.5: *The sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \leq 1$ and diverges for all other values of r. Also,*

$$\lim_{n\to\infty} r^n \begin{cases} 0, & \textit{if } -1 < r < 1 \\ 1, & \textit{if } r = 1 \end{cases}$$

Theorem 4.3.6: *For the sequence $\{a_n\}$ if both $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.*

Proof:

Let $\epsilon > 0$. Then since $\lim_{n\to\infty} a_{2n} = L$,

$$\exists N_1 \in \mathbb{N} : \forall n \geq N_1, |a_{2n} - L| < \epsilon$$

Similarly, because $\lim_{n\to\infty} a_{2n+1} = L$,

$$\exists N_2 \in \mathbb{N} : \forall n \geq N_2, |a_{2n+1} - L| < \epsilon$$

Now, let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. Then either $a_n = a_{2k}$ for some $k > N_1$ or $a_n = a_{2k+1}$ for some $k > N_2$, And so in either case we have that

$$|a_n - L| < \epsilon$$

Therefore, $\lim\limits_{n\to\infty} a_n = L$ and so $\{a_n\}$ is convergent.

4.4 Bounded and Monotonic Sequences

Definition 4.4.1: Bounded Sequence

A sequence $\{a_n\}$ is **bounded** if

$$\exists M \in \mathbb{R} : \forall n \in \mathbb{N}, |a_n| \leq M$$

Upper and Lower Bounds

If

$$\forall n, \exists m : m \leq a_n$$

the sequence $\{a_n\}$ is said to be **bounded below** and m is a **lower bound** of the sequence. Similarly, if

$$\forall n, \exists M : M \geq a_n$$

the sequence $\{a_n\}$ is said to be **bounded above** and M is an **upper bound** of the sequence.

***Theorem 4.4.2* (Bounded Sequence):** *If a sequence $\{a_n\}$ is both bounded above and below, then it is bounded. That is, if*

$$\exists m, M \in \mathbb{R} : \forall n \in \mathbb{N}, m \leq a_n \leq M$$

then $\{a_n\}$ is bounded.

Definition 4.4.3: Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if for all $n \in \mathbb{N}$

$$a_{n+1} \geq a_n \quad \text{or} \quad a_{n+1} \leq a_n$$

Theorem 4.4.4: *If a sequence $\{a_n\}$ is both bounded and monotonic, then it is convergent.*

5 Series

Properties of Convergent Series

If $\sum a_n$ and $\sum b_n$ are both convergent series, then:

1. $\sum ca_n$, where c is a constant, is also convergent and
$$\sum ca_n = c\sum a_n$$
2. $\sum(a_n \pm b_n)$ is also convergent and
$$\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$$

5.1 Convergence of Series

Theorem 5.1.1* (Convergence of Series)*: *The series $\sum a_n$ converges if and only if the sequence of partial sums $\{s_n\}$ is convergent. That is,*

$$\sum a_n \textit{ converges} \iff \lim_{n\to\infty} s_n \textit{ exists}$$

If the series $\sum a_n$ converges, then
$$\lim_{n\to\infty} s_n = s$$
where s is the sum of the series.

Example 5.1:

$$\begin{aligned}
&\lim_{n\to\infty} n = \infty && \text{(diverges)}\\
&\lim_{n\to\infty} \frac{1}{n^2-1} = 0 && \text{(converges)}\\
&\lim_{n\to\infty} (-1)^n \text{ does not exist} && \text{(diverges)}\\
&\lim_{n\to\infty} \frac{1}{3^{n-1}} = 0 && \text{(converges)}
\end{aligned}$$

Theorem 5.1.2: *If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

5.2 Absolute and Conditional Convergence

Definition 5.2.1: Absolute and Conditional Convergence

A series $\sum a_n$ is said to be **absolutely convergent** if $\sum |a_n|$ converges. A series that converges but not absolutely is said to be **conditionally convergent**.

Note:-

If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is also convergent. But if $\sum a_n$ is conditionally convergent, then $\sum a_n$ is not necessarily absolutely convergent.

Note:-

Given the series $\sum a_n$:

1. If $\sum a_n$ is absolutely convergent and its value is s then any rearrangement of $\sum a_n$ will also have a value of s.

2. If $\sum a_n$ is conditionally convergent and r is any real number then there is a rearrangement of $\sum a_n$, say $\sum b_n$, such that $\sum b_n = r$.

5.3 Special Series

5.3.1 Geometric Series

Definition 5.3.1: Geometric Series

A series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

is called a **geometric series**. The sum of the series is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad \text{if } |r| < 1$$

5.3.2 Telescoping Series

Definition 5.3.2: Telescoping Series

A series of the form

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - a_2 + a_2 - a_3 + \cdots$$

is called a **telescoping series**. The sum of the series is

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_{n+1}$$

5.3.3 Harmonic Series

Definition 5.3.3: Harmonic Series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is called the **harmonic series**. The harmonic series diverges.

5.4 Divergence Test

Theorem 5.4.1 **(Divergence Test)***: If* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum a_n$ *diverges.*

5.5 Integral Test

Theorem 5.5.1* (Integral Test)*: *Let $f(x)$ be a continuous, positive, and decreasing function for $x \geq 1$. Then the series*

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x)\,dx$$

converges. That is,

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

Proof:
Let $s_n = f(1) + f(2) + \cdots + f(n)$ and $s_{n+1} = f(1) + f(2) + \cdots + f(n) + f(n+1)$. Then

$$s_{n+1} - s_n = f(n+1) \geq 0$$

and

$$\int_n^{n+1} f(x)\,dx \leq f(n) \leq \int_{n-1}^{n} f(x)\,dx$$

Summing from 1 to n gives

$$s_{n+1} - s_1 \leq \int_0^n f(x)\,dx \leq s_n$$

Taking the limit as $n \to \infty$ gives

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1} = s$$

which implies that the series converges if and only if the integral converges. □

5.5.1 The p-series Test

Theorem 5.5.2* (p-series Test)*: *The series*

$$\sum_{n=k}^{\infty} \frac{1}{n^p} \qquad \text{where } k > 0$$

converges if $p > 1$ and diverges if $p \leq 1$.

Proof:
If $p > 1$, then the integral

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^b = \frac{1}{p-1}$$

converges. If $p \leq 1$, then the integral diverges. □

5.6 Comparison Test/Limit Comparison Test

Theorem 5.6.1* (Comparison Test)*: *Let $\sum a_n$ and $\sum b_n$ are two series with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then:*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*
2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Proof:
Let the partial sums be

$$s_n = \sum_{i=1}^{n} a_i \quad \text{and} \quad t_n = \sum_{i=1}^{n} b_i$$

Since $a_n, b_n \geq 0$, we know that

$$s_n \leq s_n + a_{n+1} = \sum_{i=1}^{n} a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = s_{n+1} \qquad \implies s_n \leq s_{n+1}$$

$$t_n \leq t_n + b_{n+1} = \sum_{i=1}^{n} b_i + b_{n+1} = \sum_{i=1}^{n+1} b_i = t_{n+1} \qquad \implies t_n \leq t_{n+1}$$

Also, since $a_n \leq b_n$, we have

$$s_n = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i = t_n$$

If $\sum b_n$ converges, then $\{t_n\}$ is bounded and increasing, so it converges. Since $s_n \leq t_n$, $\{s_n\}$ is also bounded and increasing, so it converges. □

Theorem 5.6.2* (Limit Comparison Test)*: *Let $\sum a_n$ and $\sum b_n$ be two series with $a_n, b_n > 0$ for all n. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $c \in \mathbb{R}_+$ and $c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Proof:
Since $0 < c < \infty$, there exists two positive finite numbers m and M such that $m < c < M$. Now we have

$$m < \frac{a_n}{b_n} < M$$

$$mb_n < a_n < Mb_n$$

Now, if $\sum b_n$ converges, then so does $\sum Mb_n$, and since $a_n < Mb_n$, for all n by the Comparison Test, $\sum a_n$ also converges.
Similarly, if $\sum b_n$ diverges, then so does $\sum mb_n$, and since $mb_n < a_n$, for all n by the Comparison Test, $\sum a_n$ also diverges. □

5.7 Alternating Series Test

***Theorem 5.7.1* (Alternating Series Test):** *Let $\sum a_n$ be an alternating series, that is, $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n > 0$ for all n. Then if:*

1. $\lim\limits_{n\to\infty} b_n = 0$
2. $\forall n, b_{n+1} \leq b_n$

the series $\sum a_n$ is convergent.

Proof:
Let $s_n = \sum_{i=1}^{n} a_i$. Since b_n is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, we can say

$$\forall n, b_n - b_{n+1} \geq 0$$

Now, we have

$$\begin{aligned} s_{2n} &= b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots + b_{2n-1} - b_n \\ &= b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \end{aligned}$$

Since b_n is decreasing, s_{2n} is increasing and bounded above by b_1.
Let's assume that its limit is s, that is

$$\lim_{n\to\infty} s_{2n} = s$$

Then

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} (s_{2n} + b_{2n+1}) = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} = s + 0 = s$$

So, we know that both $\{s_{2n}\}$ and $\{s_{2n+1}\}$ are convergent sequences and they both have the same limit.
We also know that $\{s_n\}$ is a convergent sequence with a limit of s. This in turn implies that the series $\sum a_n$ is convergent □

5.8 Ratio Test

***Theorem 5.8.1* (Ratio Test):** *Let $\sum a_n$ be a series and let*

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then:

1. *If $L < 1$, then $\sum a_n$ is absolutely convergent.*
2. *If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.*
3. *If $L = 1$, then the test is inconclusive.*

Proof:
Let $L < 1$ and $L = \lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then there exists a number r such that $L < r < 1$. Since $L < r$, there exists a number N such that for all $n > N$,

$$\left| \frac{a_{n+1}}{a_n} \right| < r$$

$$|a_{n+1}| < r\,|a_n|$$
$$|a_{n+1}| < r\,|a_n| < r^2\,|a_{n-1}| < \cdots < r^{n-N}\,|a_N|$$
$$|a_{n+1}| < r^{n-N}\,|a_N|$$

Since $r < 1$, the series $\sum r^{n-N}\,|a_N|$ converges by the p-series test. By the Comparison Test, $\sum a_n$ also converges. □

5.9 Root Test

***Theorem 5.9.1* (Root Test):** *Let $\sum a_n$ be a series and let*

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then:

1. *If $L < 1$, then $\sum a_n$ is absolutely convergent.*
2. *If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.*
3. *If $L = 1$, then the test is inconclusive.*

Note:-

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Proof:

Let $L < 1$ and $L = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$. Then there exists a number r such that $L < r < 1$. Since $L < r$, there exists a number N such that for all $n > N$,

$$\sqrt[n]{|a_n|} < r$$

$$|a_n| < r^n$$

Since $r < 1$, the series $\sum r^n$ converges by the geometric series test. By the Comparison Test, $\sum a_n$ also converges. □

5.10 Strategies for Series Test

Divergence Test If $\lim\limits_{n\to\infty} a_n \neq 0$.

Geometric Series Test If $a_n = ar^n$ or $a_n = ar^{n-1}$

Integral Test If $a_n = f(n)$ and $f(x)$ is continuous, positive, and decreasing

p-series Test If $a_n = \frac{1}{n^p}$

Comparison Test If a_n is hard to work with, but b_n is easy to work with

Limit Comparison Test If a_n is a rational expression involving only polynomials.

Alternating Series Test If $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$

Ratio Test If a_n is a product of terms

Root Test If a_n is a power of terms

5.11 Estimating the Value of a Series

5.11.1 Integral Test

Integral Test

$$s_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \le s \le s_n + \int_{n}^{\infty} f(x)\,\mathrm{d}x$$

5.11.2 Comparison Test

Given a series $\sum a_n$, let's assume that we've used the comparison test to show that it's convergent. Therefore we found a second series $\sum b_n$ that converged and $a_n \le b_n$ for all n.
Now, let

$$R_n = \sum_{k=n+1}^{\infty} a_k \quad \text{and} \quad T_n = \sum_{k=n+1}^{\infty} b_k$$

Since, $a_n \le b_n$ we also know that $R_n \le T_n$.

Comparison Test

$$R_n \le T_n \le \int_{n}^{\infty} g(x)\,\mathrm{d}x\,, \qquad \text{where } g(n) = b_n$$

5.11.3 Alternating Series Test

Alternating Series Test

$$|R_n| = |s - s_n| \le b_{n+1}$$

5.11.4 Ratio Test

Ratio Test

To get an estimate of the remainder, let's first define the following sequence,

$$r_n = \frac{a_{n+1}}{a_n}$$

We now have two possible cases:

1. If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, then

$$R_n \le \frac{a_{n+1}}{1 - r_{n+1}}$$

2. If $\{r_n\}$ is an increasing sequence, then

$$R_n \le \frac{a_{n+1}}{1 - L}, \qquad \text{where } L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

5.12 Power Series

Definition 5.12.1: Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where c_n are constants and called coefficients and a is a fixed number. The number a is called the **center** of the power series.

***Theorem 5.12.2* (Convergence of Power Series):** *Given the power series $\sum c_n(x-a)^n$, there exists a number R called the **radius of convergence** such that:*

1. *The series converges absolutely for all x with $|x-a| < R$*
2. *The series diverges for all x with $|x-a| > R$*

*To find R, apply the **Ratio Test** or **Root Test**:*

- ***Ratio Test:** Compute $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ where $a_n = c_n(x-a)^n$. The series converges when $L < 1$, which gives $|x-a| < R$.*
- ***Root Test:** Compute $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ where $a_n = c_n(x-a)^n$. The series converges when $L < 1$, which gives $|x-a| < R$.*

***N.B:** the series always converges at $x = a$.*

5.13 Power Series and Functions

***Theorem 5.13.1* (Power Series and Functions):** *Let $f(x) = \sum c_n(x-a)^n$ be a power series with radius of convergence R. Then:*

1. *$f(x)$ is continuous and differentiable on the interval $(-R, R)$*
2. $f'(x) = \sum nc_n(x-a)^{n-1}$
3. $\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$
4. *The radius of convergence of $f'(x)$ and $\int f(x)\,\mathrm{d}x$ are also R.*

5.14 Taylor Series

Definition 5.14.1: Taylor Series

The **Taylor series** of a function $f(x)$ about $x = a$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $f^{(n)}(a)$ is the nth derivative of $f(x)$ evaluated at $x = a$.

To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the **n-th degree Taylor polynomial** of $f(x)$ as

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Notice that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the n-th degree Taylor polynomial is just the partial sum for the series.
The **remainder** is defined to be

$$R_n(x) = f(x) - T_n(x)$$

So, the remainder is just the *error* between the function $f(x)$ and the n-th degree Taylor polynomial $T_n(x)$ for a given n.
With this definition, we can then write the function as

$$f(x) = T_n(x) + R_n(x)$$

Theorem 5.14.2: *Suppose that $f(x) = T_n(x) + R_n(x)$. Then if*

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

on $|x-a| < R$.

5.14.1 Maclaurin Series

Definition 5.14.3: Maclaurin Series

The **Maclaurin series** of a function $f(x)$ is the Taylor series of $f(x)$ about $x = 0$. That is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Note:-

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

5.15 Binomial Series

Theorem 5.15.1* (Binomial Theorem)*: *If n is a positive integer, then*

$$\begin{aligned}(a+b)^n &= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n\end{aligned}$$

where,

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} = \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!}$$

$$\binom{n}{0} = 1$$

Binomial Series

If k is a number and $|x| < 1$, then

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \binom{k}{n} x^n\end{aligned}$$

where,

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

$$\binom{k}{0} = 1$$

6 3-Dimensional Space

The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise, the 2-D coordinate system is denoted by $\mathbb{R}^2$, and the 1-D coordinate system is denoted by $\mathbb{R}$.

6.1 Equations of Lines

Vector form

If $\vec{a}$ and $\vec{v}$ are parallel vectors, then $\vec{a} = t\vec{v}$ for some scalar t.
Now if we have a vector $\vec{r}$ as follows

$$\vec{r} = \vec{r_0} + \vec{a}$$

Then we can write

$$\vec{r} = \vec{r_0} + t\vec{v} = \langle x_0, y_0, z_0\rangle + t\langle a, b, b\rangle$$

This is called the **vector form of the equation of a line**.

Parametric form

We can rewrite the vector form as

$$\vec{r} = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$
$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

In other words

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

This set of equations is called the **parametric form of the equation of a line**.

Symmetric Equations of a Line

If we assume that a, b, and c are non-zero numbers, then we can solve each of the parametric equations for t. This gives us

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 6.1: Find the Equations of lines:
1. Through the points $(7, -3, 1)$ **and** $(-2, 1, 4)$
2. Through the point $(1, -5, 0)$ **and parallel to the line given by** $\vec{r}(t) = \langle 8 - 3t, -10 + 9t, -1 - t\rangle$
3. Through the point $(-7, 2, 4)$ **and orthogonal to both** $\vec{v} = \langle 0, -9, 1\rangle$ **and** $\vec{w} = 3\hat{i} + \hat{j} - 4\hat{k}$

1.
Direction vector $\vec{d} = \langle -2 - 7, 1 + 3, 4 - 1\rangle = \langle -9, 4, 3\rangle$
Now, the vector form of the line is

$$\vec{r} = \langle 7, -3, 1\rangle + t\langle -9, 4, 3\rangle$$

The parametric form is

$$x = 7 - 9t, \quad y = -3 + 4t, \quad z = 1 + 3t$$

The symmetric form is

$$\frac{x-7}{-9} = \frac{y+3}{4} = \frac{z-1}{3}$$

2.
The direction vector is $\vec{d} = \langle 3, 9, -1 \rangle$
Hence, the vector form of the line is

$$\vec{r} = \langle 1, -5, 0 \rangle + t\langle 3, 9, -1 \rangle$$

The parametric form is

$$x = 1 + 3t, \quad y = -5 + 9t, \quad z = -t$$

And the symmetric form is

$$\frac{x-1}{3} = \frac{y+5}{9} = -z$$

3.
Direction vector

$$\vec{d} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -9 & 1 \\ 3 & 1 & -4 \end{vmatrix} = \langle 35, 3, 27 \rangle$$

Hence, the vector form of the line is

$$\vec{r} = \langle -7, 2, 4 \rangle + t\langle 35, 3, 27 \rangle$$

The parametric form is

$$x = -7 + 35t, \quad y = 2 + 3t, \quad z = 4 + 27t$$

The symmetric form is

$$\frac{x+7}{35} = \frac{y-2}{3} = \frac{z-4}{27}$$

Example 6.2: Determine if the two lines are parallel, orthogonal, or neither:
1. The line given by $\vec{r}(t) = \langle 4 - 7t, -10 + 5t, 21 - 4t \rangle$ **and the line given by** $\vec{r}(t) = \langle -2 + 3t, 7 + 5t, 5 + t \rangle$
2. The line given by $x = 29, y = -3 - 6t, z = 12 - t$ **and the line given by** $\vec{r}(t) = \langle 12 - 14t, 2 + 7t, -10 + 3t \rangle$
1.
The direction vectors are

$$\vec{d_1} = \langle -7, 5, -4 \rangle, \quad \vec{d_2} = \langle 3, 5, 1 \rangle$$

To check if they are parallel, we can check:

$$\frac{-7}{3} \neq \frac{5}{5} \neq \frac{-4}{1}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{d_1} \cdot \vec{d_2} = -7(3) + 5(5) + (-4)(1) = -21 + 25 - 4 = 0$$

Hence, they are orthogonal.
2.
The direction vectors are

$$\vec{d_1} = \langle 0, -6, -1 \rangle, \quad \vec{d_2} = \langle -14, 7, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{0}{-14} \neq \frac{-6}{7} \neq \frac{-1}{3}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{d_1} \cdot \vec{d_2} = 0(-14) + (-6)(7) + (-1)(3) = -42 - 3 = -45 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 6.3: Determine the intersection point of the two lines or show that they don't not intersect:
1. The line passing through the point $(0, -9, -1)$ **and** $(1, 6, -3)$ **and the line given by** $\vec{r}(t) = \langle -9 - 4t, 10 + 6t, 1 - 2t \rangle$
2. The line given by $x = 1 + 6t, t = -1 - 3t, z = 4 + 12t$ **and the line given by** $x = 4 + t, y = -10 - 8t, z = 3 - 5t$
1.
The direction vector of the first line is

$$\vec{d_1} = \langle 1 - 0, 6 + 9, -3 + 1 \rangle = \langle 1, 15, -2 \rangle$$

We can write the parametric equations of the first line as:

$$x = s, y = -9 + 15s, z = -1 - 2s$$

And the parametric equations of the second line as:

$$x = -9 - 4t, y = 10 + 6t, z = 1 - 2t$$

Setting them equal to each other we get,

$$\begin{aligned} 0 + t &= -9 - 4s \\ -9 + 15t &= 10 + 6s \\ -1 - 2t &= 1 - 2s \end{aligned}$$

Solving the first two equations, we get

$$t = -\frac{7}{3}, \quad s = \frac{1}{3}$$

Now, verifying the third equation, we get

$$\begin{aligned} -1 - 2(-\frac{7}{3}) &= 1 - 2(\frac{1}{3}) \\ -1 + \frac{14}{3} &= 1 - \frac{2}{3} \\ \frac{11}{3} &\neq \frac{1}{3} \end{aligned}$$

Since the third equation is not satisfied, the two lines do not intersect.
2.
The lines are given in parametric form.
Setting them equal to each other we get,

$$\begin{aligned} 1+6s &= 4+t \\ -1-3s &= -10-8t \\ 4+12s &= 3-5t \end{aligned}$$

Solving the first two equations, we get

$$s=\frac{1}{3}, \quad t=-1$$

Now, verifying the third equation, we get

$$\begin{aligned} 4+12\left(\frac{1}{3}\right) &= 3-5\langle -1\rangle \\ 8 &= 8 \end{aligned}$$

That means, the lines intersect. Substituting the values in the parametric equation, we get

$$\begin{aligned} x &= 1+6\left(\frac{1}{3}\right)=3 \\ y &= -1-3\left(\frac{1}{3}\right)=-2 \\ z &= 4+12\left(\frac{1}{3}\right)=8 \end{aligned}$$

Hence, the intersection point is $(3,-2,8)$.

Example 6.4: Which of the three coordinate planes does the line given by $x = 16t, y=-4-9t, z=34$ intersect?

To intersect the xy-plane, we need $z=0$. But here $z=34$ is constant. Hence, the line does not intersect the xy-plane.
To intersect the yz-plane, we need $x=0$. Hence,

$$16t=0 \implies t=0$$

And the intersection point is $(0,-4-9\times 0,34)$ or $(0,-4,34)$.
To intersect the xz-plane, we need $y=0$. Hence,

$$-4-9t=0 \implies t=-\frac{4}{9}$$

And the intersection point is $\left(16\left(-\frac{4}{9}\right),0,43\right)$ or $\left(-\frac{64}{9},0,34\right)$.

6.2 Equations of Planes

Vector form

Let's assume $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ and $\vec{r} = \langle x, y, z \rangle$ are two position vectors and $\vec{r} - \vec{r_0}$ is a vector in the plane.
If $\vec{n} = \langle a, b, c \rangle$ is a normal to the plane (which means it's orthogonal to the vector $\vec{r} - \vec{r_0}$), then we can write

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \Longrightarrow \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

This is called the **vector form of the equation of a plane**.

Scalar form

If we expand the vector equation in the following way,

$$\begin{aligned} \vec{n} \cdot (\vec{r} - \vec{r_0}) &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \end{aligned}$$

Computing the dot product, we get

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the **scalar form of the equation of a plane**.
This equation can also be written as

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

Example 6.5: Find the equation of the plane:
1. Through the point $(6, -3, 1)$**,** $(5, -4, 1)$**, and** $(3, -4, 0)$
2. The plane containing the point $(1, -5, 8)$ **and orthogonal to the line given by** $x = -3 + 15t, y = 14 - t, z = 9 - 3t$
3. The plane containing the point $(-8, 3, 7)$ **and parallel to the plane given by** $4x + 8y - 2z = 45$
4. The plane containing the two lines given by $\vec{r}(t) = \langle 7 + 5t, 2 + t, 6t \rangle$ **and** $\vec{r}(t) = \langle 7 - 6t, 2 - 2t, 10t \rangle$

1.
The given points are

$$A(6, -3, 1), B(5, -4, 1), C(3, -4, 0)$$

Two vectors in the plane are

$$\begin{aligned} \vec{AB} &= \langle 5 - 6, -4 + 3, 1 - 1 \rangle = \langle -1, -1, 0 \rangle \\ \vec{BC} &= \langle 3 - 5, -4 + 4, 0 - 1 \rangle = \langle -2, 0, -1 \rangle \end{aligned}$$

Normal vector on the place:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 0 \\ -2 & 0 & -1 \end{vmatrix} = \hat{i} - \hat{j} - 2\hat{k}$$

Now, using the point A, we can write the equation of the plane as

$$\begin{aligned}(x-6)-(y+3)-2(z-1)&=0\\ x-y-2z&=7\end{aligned}$$

2.
The normal vector is

$$\vec{n}=\langle 15,-1,-3\rangle$$

Using the point $(1,-5,8)$, the equation of the plane is

$$\begin{aligned}15(x-1)-(y+5)-3(z-8)&=0\\ 15x-y-3z&=15+5-24\\ 15x-y-3z+4&=0\end{aligned}$$

3.
The normal vector is

$$\vec{n}=\langle 4,8,-2\rangle$$

Using the point $(-8,3,7)$, the equation of the plane is

$$\begin{aligned}4(x+8)+8(y-3)-2(z-7)&=0\\ 4x+8y-2z&=-32+24+14\\ 4x+8y-2z+6&=0\end{aligned}$$

4.
The direction vectors of the two lines are

$$\vec{d_1}=\langle 5,1,6\rangle,\quad \vec{d_2}=\langle -6,-2,10\rangle$$

The normal vector is

$$\vec{n}=\vec{d_1}\times\vec{d_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & 1 & 6\\ -6 & -2 & 10\end{vmatrix}=\langle 22,-86,-4\rangle$$

Using the point $A(7,2,0)$, the equation of the plane is

$$\begin{aligned}22(x-7)-86(y-2)-4(z-0)&=0\\ 22x-86y-4z-154+172&=0\\ 22x-86y-4z+18&=0\end{aligned}$$

Example 6.6: Determine if the two planes are parallel, orthogonal, or neither: The plane given by $3x+9y+7z=-1$ and the plane containing the points $(1,-1,9),(4,-1,2),(-2,3,4)$

The normal vector of the first plane is

$$\vec{n_1}=\langle 3,9,7\rangle$$

Let the points be

$$A(1,-1,9), B(4,-1,2), C(-2,3,4)$$

Two vectors in the plane are

$$\begin{aligned}\vec{AB} &= \langle 4-1, -1+1, 2-9\rangle = \langle 3, 0, -7\rangle \\ \vec{AC} &= \langle -2-1, 3+1, 4-9\rangle = \langle -3, 4, -5\rangle\end{aligned}$$

The normal vector of the second plane is

$$\vec{n_2} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -7 \\ -3 & 4 & -5 \end{vmatrix} = \langle 28, 36, 12\rangle = \langle 7, 9, 3\rangle$$

To check if they are parallel, we can check:

$$\frac{3}{7} \neq \frac{9}{9} \neq \frac{7}{3}$$

which means they are not parallel.
To check if they are orthogonal, we can check:

$$\vec{n_1} \cdot \vec{n_2} = 3(7) + 9(9) + 7(3) = 21 + 81 + 21 = 123 \neq 0$$

Hence, they are neither parallel nor orthogonal.

Example 6.7: Find the intersection of the plane given by $4x + y + 10z = -2$ and the plane given by $-8x + 2y + 3z = -8$
The two planes are

$$\begin{aligned}4x + y + 10z &= -2 \\ -8x + 2y + 3z &= -8\end{aligned}$$

Multiplying the first equation by 2 and adding it to the second equation, we get

$$4y + 23z = -12 \quad \Longrightarrow \quad y = -3 - \frac{23}{4}z$$

Substituting the value of y in the first equation, we get

$$16x - 3 - \frac{23}{4}z + 10z = -2 \quad \Longrightarrow \quad x = \frac{1}{4} - \frac{17}{16}z$$

Let $z = t$ (a parameter). Then we get

$$\begin{aligned}x &= \frac{1}{4} - \frac{17}{16}t \\ y &= -3 - \frac{23}{4}t \\ z &= t\end{aligned}$$

This is the parametric form of the line of intersection.
We can also write it in vector form as

$$\vec{r} = \langle \frac{1}{4}, -3, 0\rangle + t\langle -\frac{17}{16}, -\frac{23}{4}, 1\rangle$$

6.3 Quadratic Surfaces

General form

The general form of a quadratic surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where $A, B, C, D, E, F, G, H, I, J$ are constants.

Ellipsoid

The general equation of an ellipsoid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the ellipsoid and a, b, c are the semi-axis lengths.
If $a = b = c$, we get a sphere.

Cone

The general equation of a cone that opens along the z-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{(z-l)^2}{c^2}$$

where (h, k, l) is the center of the cone and a, b, c are the semi-axis lengths.

Cylinder

The general equation of a cylinder that opens along the z-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where (h, k) is the center of the cylinder and a, b are the semi-axis lengths.
If $a = b$, we get a circular cylinder.

Hyperboloid of One Sheet

The general equation of a hyperboloid of one sheet is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

Hyperboloid of Two Sheets

The general equation of a hyperboloid of two sheets is

$$-\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where (h, k, l) is the center of the hyperboloid and a, b, c are the semi-axis lengths.

Elliptic Paraboloid

The general equation of an elliptic paraboloid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

Hyperbolic Paraboloid

The general equation of a hyperbolic paraboloid is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where (h, k, l) is the center of the paraboloid and a, b are the semi-axis lengths.

6.4 Calculus with Vector Functions

Let

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Note:-

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

Note:-

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{u} + \vec{v}\right) = \vec{u}' + \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c\vec{u}\right) = c\vec{u}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{u} \cdot \vec{v}\right) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{u} \times \vec{v}\right) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\vec{u}f(t)\big) = f'(t)\vec{u}'(f(t))$$

Note:-

$$\int \vec{r}(t)\,\mathrm{d}t = \langle \int f(t)\,\mathrm{d}t\,, \int g(t)\,\mathrm{d}t\,, \int h(t)\,\mathrm{d}t \rangle$$

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \left\langle \int_a^b f(t)\,\mathrm{d}t + \int_a^b g(t)\,\mathrm{d}t + \int_a^b h(t)\,\mathrm{d}t \right\rangle$$

6.5 Tangent, Normal, and Binormal Vectors

Unit Tangent vector

Given the vector function $\vec{r}(t)$, we call $\vec{r}'(t)$ the **tangent vector** The unit tangent vector is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Unit Normal vector

If $\vec{T}(t)$ is the unit tangent vector, then the **unit normal vector** is given by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Note:-

If $\vec{r}(t)$ is a vector such that $\|\vec{r}(t)\| = c$ for all t, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$

Binormal vector

The **binormal vector** is given by

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The binormal vector is orthogonal to both the tangent and normal vectors.

6.6 Arc Length with Vector Functions

Note:-

The arc length of a vector function $\vec{r}(t)$ from $t = a$ to $t = b$ is given by

$$L = \int_a^b \|\vec{r}'(t)\| \,\mathrm{d}t$$

Or,

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2} \,\mathrm{d}t$$

6.7 Curvature

Curvature of a curve in 3-D space

The curvature of a curve in 3-D space is given by

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

where $\vec{T}(t)$ is the unit tangent vector and $\vec{r}(t)$ is the position vector.
This can also be written as

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$