

# Calculus II Notes

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# 1 Integration Techniques

## 1.1 Fundamental Theorem of Calculus

**Theorem 1.1.1 (Fundamental Theorem of Calculus):** Let  $f$  be a function defined on an open interval  $I$  that contains  $a$ . If  $f$  is continuous on  $I$ , then the function  $F$  defined by

$$F(x) = \int_a^x f(t) \, dt$$

is uniformly continuous on  $I$ , differentiable on the open interval, and

$$F'(x) = f(x)$$

for all  $x$  in the open interval.

## 1.2 Common Differentiation and Integration Formulae

Derivative	Integral
$\frac{d}{dx}x = 1$ , $\frac{d}{dx}c = 0$	$\int c \, dx = cx + c$
$\frac{d}{dx}x^n$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln x  + c$
$\frac{d}{dx}e^{mx} = me^{mx}$	$\int e^{mx} \, dx = \frac{1}{m}e^{mx} + c$
$\frac{d}{dx}a^x = a^x \ln(a)$	$\int a^x \, dx = \frac{1}{\ln(a)}a^x + c$
$\frac{d}{dx}\sin(mx) = m \cos(mx)$	$\int \cos(mx) \, dx = \frac{1}{m}\sin(mx) + c$
$\frac{d}{dx}\cos(mx) = -m \sin(mx)$	$\int \sin(mx) \, dx = -\frac{1}{m}\cos(mx) + c$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\int \sec^2(x) \, dx = \tan(x) + c$
$\frac{d}{dx}\cot(x) = -\csc^2(x)$	$\int \csc^2(x) \, dx = -\cot(x) + c$
$\frac{d}{dx}\sec(x) = \sec(x) \tan(x)$	$\int \sec(x) \tan(x) \, dx = \sec(x) + c$
$\frac{d}{dx}\csc(x) = -\csc(x) \cot(x)$	$\int \csc(x) \cot(x) \, dx = -\csc(x) + c$
$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + c$
$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = -\cos^{-1}(x) + c$
$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \tan^{-1}(x) + c$
$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = -\cot^{-1}(x) + c$
$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1}(x) + c$
$\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = -\csc^{-1}(x) + c$
$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x} + c$

Table 1: Common Differentiation and Integration Formulae

### 1.3 More Formulae

1.  $\int \tan(x) \, dx = \ln|\sec(x)| + c$
2.  $\int \csc(x) \, dx = \ln|\tan \frac{x}{2}| + c$
3.  $\int \sec(x) \, dx = \ln|\sec(x) + \tan(x)| + c$
4.  $\int \sec(x) \, dx = \ln|\tan(\frac{\pi}{4} + \frac{x}{2})|$
5.  $\int \csc(x) \, dx = -\ln|\csc(x) + \cot(x)| + c$
6.  $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + c$
7.  $\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \ln|\frac{a+x}{a-x}| + c$
8.  $\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \ln|\frac{a-x}{a+x}| + c$
9.  $\int \frac{1}{\sqrt{x^2+a^2}} \, dx = \ln|x + \sqrt{x^2+a^2}| + c$
10.  $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}(\frac{x}{a}) + c$
11.  $\int \sqrt{a^2-x^2} \, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$
12.  $\int e^x [f(x) + f'(x)] \, dx = e^x f(x) + c$

### 1.4 Integration by Parts

**Theorem 1.4.1 (Integration by Parts):** Let  $u$  and  $v$  be differentiable functions of  $x$ . Then,

$$\int uv \, dx = u \int v \, dx - \int \left[ \frac{du}{dx} \int v \, dx \right] dx$$

Or,

$$\int u \, dv = uv - \int v \, du$$

#### Proof:

Let  $u = u(x)$  and  $w = w(x)$ . Then,

$$\frac{d(uw)}{dx} = u \frac{dw}{dx} + w \frac{du}{dx}$$

Integrating both sides, we get

$$\int \frac{d(uw)}{dx} \, dx = \int u \frac{dw}{dx} \, dx + \int w \frac{du}{dx} \, dx$$

Or,

$$uw = \int u \frac{dw}{dx} \, dx + \int w \frac{du}{dx} \, dx$$

Rearranging, we get

$$\int u \frac{dw}{dx} \, dx = uw + c - \int w \frac{du}{dx} \, dx$$

Let  $v = \frac{dw}{dx}$ , then  $w = \int v \, dx$ . Hence,

$$\int uv \, dx = u \int v \, dx - \int \left[ \frac{du}{dx} \int v \, dx \right] dx$$

□

## 1.5 Method of Substitution

- A.  $\int \frac{1}{(ax+b)\sqrt{cx+d}} dx$  Let  $cx + d = z^2$
- B.  $\int \frac{1}{\sin^m x \cos^m x} dx$  If  $m + n = p$  is even, multiply and divide by  $\sec^p x$  and let  $\tan x = z$ .
- C.  $\int \frac{1}{\sin^m x + \cos^m x} dx$  If  $m$  is even, multiply and divide by  $\sec^m x$ .
- D.  $\int \frac{\cos x}{a \cos x + b \sin x} dx$  Write  $nom = l \times (denom) + m \times (denom)'$ , then determine  $l$  and  $m$ .
- E.  $\int \frac{\cos x}{a \cos x + b \sin x} dx + c$  Write  $\sin x$  and  $\cos x$  as  $\tan \frac{x}{2}$ .
- F.  $\int \frac{1}{\sqrt{x^2+a^2}} dx$  Let  $x = a \tan \theta$
- G.  $\int \frac{1}{\sqrt{x^2-a^2}} dx$  Let  $x = a \sec \theta$
- H.  $\int \sqrt{a^2 - x^2} dx$  Let  $x = a \sin \theta$

## 1.6 Trigonometric Integrals

Form	Looks like	Substitution	Limit Assumption
$\sqrt{b^2x^2 - a^2}$	$\sec^2 \theta - 1 = \tan^2 \theta$	$x = \frac{a}{b} \sec \theta$	$0 \leq \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \leq \pi$
$\sqrt{a^2 - b^2x^2}$	$1 - \sin^2 \theta = \cos^2 \theta$	$x = \frac{a}{b} \sin \theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2 + b^2x^2}$	$\tan^2 \theta + 1 = \sec^2 \theta$	$x = \frac{a}{b} \tan \theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Table 2: Trigonometric Integral Substitution

## 1.7 Partial Fractions

Factor in denominator	Term in partial fraction decomposition
$ax + b$	$\frac{A}{ax+b}$
$(ax + b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+b}{ax^2+bx+c}$
$(ax^2 + bx + c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Table 3: Partial Fraction Decomposition

## 1.8 Integrals Involving Roots

For integrals involving  $\sqrt{a^2 - x^2}$ ,  $\sqrt{a^2 + x^2}$ , or  $\sqrt{x^2 - a^2}$ , use trigonometric substitution:

**Example 1.1: Evaluate**  $\int \frac{1}{\sqrt{9-x^2}} dx$

Expression	Substitution	Identity Used
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$

Table 4: Trigonometric Substitutions for Roots

Here  $a = 3$ , so let  $x = 3 \sin \theta$ , thus  $dx = 3 \cos \theta d\theta$ .

$$\begin{aligned}
 \int \frac{1}{\sqrt{9 - x^2}} dx &= \int \frac{3 \cos \theta}{\sqrt{9 - 9 \sin^2 \theta}} d\theta \\
 &= \int \frac{3 \cos \theta}{3 \cos \theta} d\theta \\
 &= \int 1 d\theta \\
 &= \theta + c \\
 &= \sin^{-1} \left( \frac{x}{3} \right) + c
 \end{aligned}$$

## 1.9 Integrals Involving Quadratics

For integrals involving  $ax^2 + bx + c$ , complete the square first:

$$ax^2 + bx + c = a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right]$$

Then use substitution  $u = x + \frac{b}{2a}$  to convert to standard forms.

**Example 1.2: Evaluate**  $\int \frac{1}{x^2 + 4x + 13} dx$

Complete the square:

$$x^2 + 4x + 13 = (x + 2)^2 + 9 = (x + 2)^2 + 3^2$$

Let  $u = x + 2$ , so  $du = dx$ :

$$\begin{aligned}
 \int \frac{1}{x^2 + 4x + 13} dx &= \int \frac{1}{u^2 + 9} du \\
 &= \frac{1}{3} \tan^{-1} \left( \frac{u}{3} \right) + c \\
 &= \frac{1}{3} \tan^{-1} \left( \frac{x + 2}{3} \right) + c
 \end{aligned}$$

## 1.10 Integration Strategy

When faced with an integral, use the following strategy:

1. **Simplify the integrand:** Expand, factor, or use algebraic manipulation

2. **Look for obvious substitutions:** If  $u' = g'(x)$  appears, try  $u = g(x)$
3. **Classify by type:**
  - Trigonometric integrals  $\rightarrow$  use trig identities
  - Rational functions  $\rightarrow$  use partial fractions
  - Products  $\rightarrow$  try integration by parts
  - Roots of quadratics  $\rightarrow$  complete the square or trig substitution
4. **Try different techniques:** If one method fails, try another
5. **Use tables or computer algebra systems:** For complex integrals

## 1.11 Approximating Definite Integrals

When an antiderivative cannot be found in closed form, use numerical approximation methods.

### 1.11.1 Midpoint Rule

Divide  $[a, b]$  into  $n$  subintervals of width  $\Delta x = \frac{b-a}{n}$ . Let  $\bar{x}_i$  be the midpoint of the  $i$ -th subinterval.

$$\int_a^b f(x) dx \approx M_n = \Delta x \sum_{i=1}^n f(\bar{x}_i)$$

### 1.11.2 Trapezoidal Rule

$$\int_a^b f(x) dx \approx T_n = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

### 1.11.3 Simpson's Rule

Requires  $n$  to be even. Approximates the function with parabolas instead of lines:

$$\int_a^b f(x) dx \approx S_n = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

#### Note:-

Simpson's Rule is generally more accurate than the Trapezoidal Rule, which is more accurate than the Midpoint Rule for the same number of subintervals.

## 1.12 Improper Integrals

### Definition 1.12.1: Improper Integrals

An integral is said to be **improper** if one of the following conditions is met:

1. The interval of integration is infinite.
2. The integrand is discontinuous at one or more points in the interval of integration.

The integral is said to **converge** if the limit of the integral exists, and **diverge** otherwise.

**Type-1:** If  $\int_a^t f(x) \, dx$  exists for all  $t > a$ , then

$$\int_a^\infty f(x) \, dx = \lim_{t \rightarrow \infty} \int_a^t f(x) \, dx$$

provided the limit exists and is finite.

**Type-2:** If  $\int_t^b f(x) \, dx$  exists for all  $t < b$ , then

$$\int_{-\infty}^b f(x) \, dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) \, dx$$

provided the limit exists and is finite.

**Type-3:** If  $\int_{-\infty}^c f(x) \, dx$  and  $\int_c^\infty f(x) \, dx$  are both convergent, then

$$\int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^c f(x) \, dx + \int_c^\infty f(x) \, dx$$

**Type-4:**  $\int_a^b f(x) \, dx$  If  $f(x)$  is discontinuous at  $x = c$ , then

$$\int_a^b f(x) \, dx = \lim_{\epsilon \rightarrow 0} \left[ \int_a^{c-\epsilon} f(x) \, dx + \int_{c+\epsilon}^b f(x) \, dx \right]$$

**Note:-**

If  $a > 0$  then

$$\int_a^\infty \frac{1}{x^p} \, dx$$

converges if  $p > 1$  and diverges if  $p \leq 1$ .

### 1.12.1 Comparison Test

**Theorem 1.12.2 (Comparison Theorem):** If  $f(x) \geq g(x) \geq 0$  on the interval  $[a, \infty)$ , then

1. If  $\int_a^\infty f(x) \, dx$  converges, then so does  $\int_a^\infty g(x) \, dx$ .
2. If  $\int_a^\infty g(x) \, dx$  diverges, then so does  $\int_a^\infty f(x) \, dx$ .

**Example 1.3:** Determine if the following integral is convergent or divergent:

$$\int_2^\infty \frac{\cos^2 x}{x^2} \, dx$$

Notice that the numerator is bounded since

$$0 \leq \cos^2 x \leq 1$$

Hence, it's likely that the denominator will determine the convergence of the integral. Since  $p = 2 > 1$ ,

$$\int_2^\infty \frac{1}{x^2} \, dx$$



is convergent. Since

$$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

and  $\int_2^\infty \frac{1}{x^2} dx$  is convergent, by the comparison test,

$$\int_2^\infty \frac{\cos^2 x}{x^2} dx$$

is convergent.

**Example 1.4:** Determine if the following integral is convergent or divergent:

$$\int_3^\infty \frac{1}{x + e^x} dx$$

In this case, the denominator determines the convergence of the integral. If we can find a larger function that converges, then the integral will converge. Notice that

$$\frac{1}{x + e^x} < \frac{1}{e^x} = e^{-x}$$

Also,

$$\begin{aligned} \int_3^\infty e^{-x} dx &= \lim_{t \rightarrow \infty} \int_3^t e^{-x} dx \\ &= \lim_{t \rightarrow \infty} (-e^{-t} + e^{-3}) \\ &= e^{-3} \end{aligned}$$

So,  $\int_3^\infty e^{-x} dx$  is convergent. Therefore, by the Comparison test,

$$\int_3^\infty \frac{1}{x + e^x} dx$$

is also convergent.

## 2 Applications of Integrals

### 2.1 Arc Length

Consider a curve  $y = f(x)$ . We want to find the length of the curve from  $x = a$  to  $x = b$ . We can approximate the curve by a series of line segments. The length of each line segment is given by the Pythagorean theorem:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

The total length of the curve is given by the sum of the lengths of the line segments:

$$L = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Similarly, for a curve  $x = h(y)$ , the length of the curve from  $y = c$  to  $y = d$  is given by:

$$L = \int ds = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

#### Arc Length Formula

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{for } y = f(x), a \leq x \leq b$$
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \quad \text{for } x = h(y), c \leq y \leq d$$

### 2.2 Surface Area

Consider a curve  $y = f(x)$  rotated about the  $x$ -axis. We want to find the surface area of the resulting surface. We can approximate the surface by a series of frustums. The surface area of each frustum is given by:

$$dS = 2\pi y ds = 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

The total surface area of the surface is given by the sum of the surface areas of the frustums:

$$S = \int dS = \int 2\pi y ds = \int 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Similarly, for a curve  $x = h(y)$  rotated about the  $y$ -axis, the surface area of the resulting surface is given by:

$$A = \int 2\pi x ds = \int 2\pi h(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

### Surface Area Formula

$$\begin{aligned} S &= \int dS \\ &= \int 2\pi y \, ds \quad \text{rotation about } x\text{-axis} \\ &= \int 2\pi x \, ds \quad \text{rotation about } y\text{-axis} \end{aligned}$$

where,

$$\begin{aligned} dS &= 2\pi y \, ds = 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{for } y = f(x), a \leq x \leq b \\ dS &= 2\pi x \, ds = 2\pi h(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \quad \text{for } x = h(y), c \leq y \leq d \end{aligned}$$

## 2.3 Center of Mass

Suppose we want to find the center of mass of a region bounded by two curves  $f(x)$  and  $g(x)$  on the interval  $[a, b]$ .

The mass is

$$M = \rho \int_a^b (f(x) - g(x)) \, dx$$

Next, we need the **moments** of the region. There are two moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2} [f(x)^2 - g(x)^2] \, dx \\ M_y &= \rho \int_a^b x [f(x) - g(x)] \, dx \end{aligned}$$

The coordinates of the center of mass,  $(\bar{x}, \bar{y})$ , are given by:

### Center of Mass Formula

$$\begin{aligned} \bar{x} &= \frac{M_y}{M} = \frac{1}{A} \int_a^b x [f(x) - g(x)] \, dx \\ \bar{y} &= \frac{M_x}{M} = \frac{1}{A} \int_a^b \frac{1}{2} [f(x)^2 - g(x)^2] \, dx \end{aligned}$$

where,

$$A = \int_a^b [f(x) - g(x)] \, dx$$

## 2.4 Probability

Every continuous random variable  $X$ , has a probability density function  $f(x)$ . Probability density functions satisfy the following conditions:

1.  $f(x) \geq 0$  for all  $x$ .
2.  $\int_{-\infty}^{\infty} f(x) \, dx = 1$

Probability density functions can be used to determine the probability that a continuous random variable lies between two values, say  $a$  and  $b$ . This probability is denoted by  $P(a \leq X \leq b)$ .

**Note:-**

$$P(a \leq X \leq b) = \int_a^b f(x) \, dx$$

Probability density functions can also be used to determine the mean of a continuous random variable. The mean is given by:

**Mean of a Continuous Random Variable**

$$\mu = \int_{-\infty}^{\infty} x f(x) \, dx$$

### 3 Parametric Equations and Polar Coordinates

There are a great many curves out there that cannot be expressed in a single equation in terms of only  $x$  and  $y$ . To deal with such problems, we introduce **parametric equations**. Instead of defining  $y$  in terms of  $x$  ( $y = f(x)$ ) or  $x$  in terms of  $y$  ( $x = h(y)$ ), we define both  $x$  and  $y$  in terms of a third variable called a parameter as follows:

$$x = f(t) \quad y = g(t)$$

This third variable is usually denoted by  $t$ . Each value of  $t$  defines a point  $(x, y) = (f(t), g(t))$  that we can plot. The collection of points that we get by letting  $t$  be all possible values is the graph of the parametric equations and is called a **parametric curve**.

#### 3.1 Parametric Equations and Curves

Unlike graphs of functions  $y = f(x)$ , parametric curves can trace out shapes that fail the vertical line test. For instance, a circle cannot be written as a single function  $y = f(x)$ , but can be easily expressed parametrically.

**Example 3.1: Sketch the parametric curve  $x = t^2$ ,  $y = 2t - 1$  for  $-2 \leq t \leq 2$ .**

We can create a table of values:

$t$	$x = t^2$	$y = 2t - 1$
-2	4	-5
-1	1	-3
0	0	-1
1	1	1
2	4	3

Plotting these points  $(x, y)$  and connecting them traces out a parabola opening to the right.

To eliminate the parameter, note that  $x = t^2$ , so  $t = \pm\sqrt{x}$ . Then:

$$y = 2t - 1 = \pm 2\sqrt{x} - 1$$

Since the parameter  $t$  ranges from  $-2$  to  $2$ , we get both branches. However, note that  $x \geq 0$  always (since  $x = t^2$ ).

**Example 3.2: Sketch the parametric curve  $x = \cos(t)$ ,  $y = \sin(t)$  for  $0 \leq t \leq 2\pi$ .**

This is the unit circle! At  $t = 0$ , we're at  $(1, 0)$ . At  $t = \pi/2$ , we're at  $(0, 1)$ . At  $t = \pi$ , we're at  $(-1, 0)$ . At  $t = 3\pi/2$ , we're at  $(0, -1)$ , and at  $t = 2\pi$  we return to  $(1, 0)$ . To eliminate the parameter, use the Pythagorean identity:

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$$

So the curve traces the circle  $x^2 + y^2 = 1$ , starting at  $(1, 0)$  and moving counterclockwise.

## 3.2 Tangents with Parametric Equations

### Derivative for Parametric Equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{provided } \frac{dx}{dt} \neq 0$$

### Tangents for Parametric Equations

Horizontal Tangent:

$$\frac{dy}{dt} = 0, \quad \text{provided } \frac{dx}{dt} \neq 0$$

Vertical Tangent:

$$\frac{dx}{dt} = 0, \quad \text{provided } \frac{dy}{dt} \neq 0$$

### Second Derivative for Parametric Equations

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}}$$

## 3.3 Area with Parametric Equations

### Area with Parametric Equations

For the area between the parametric curve and the  $x$ -axis:

$$A = \int_{t_1}^{t_2} y(t) \frac{dx}{dt} dt$$

Use this formula when the curve is traced vertically (from bottom to top or top to bottom).  
For the area between the parametric curve and the  $y$ -axis:

$$A = \int_{t_1}^{t_2} x(t) \frac{dy}{dt} dt$$

Use this formula when the curve is traced horizontally (from left to right or right to left).

## 3.4 Arc Length with Parametric Equations

The arc length of a curve is given by

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx \quad \text{if } y = f(x), a \leq x \leq b$$

$$ds = \sqrt{1 + \left( \frac{dx}{dy} \right)^2} dy \quad \text{if } x = h(y), c \leq y \leq d$$

Using the first  $ds$ , we can write

$$dx = \frac{dx}{dt} dt$$

Then the arc length formula becomes,

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \frac{dx}{dt} dt \\ &= \int_{\alpha}^{\beta} \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \frac{dx}{dt} dt \end{aligned}$$

#### • Arc Length with Parametric Equations •

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### 3.5 Surface Area with Parametric Equations

#### • Surface Area with Parametric Equations •

$$\begin{aligned} S &= \int 2\pi y \, ds \quad \text{rotation about } x\text{-axis} \\ S &= \int 2\pi x \, ds \quad \text{rotation about } y\text{-axis} \end{aligned}$$

where,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### 3.6 Polar Coordinates

In polar coordinates, each point is determined by a distance  $r$  from the origin and an angle  $\theta$  measured counterclockwise from the positive  $x$ -axis. This provides an alternative way to describe points in the plane.

#### • Polar to Cartesian Conversion •

To convert from polar coordinates  $(r, \theta)$  to Cartesian coordinates  $(x, y)$ :

$$x = r \cos \theta \quad y = r \sin \theta$$

#### • Cartesian to Polar Conversion •

To convert from Cartesian coordinates  $(x, y)$  to polar coordinates  $(r, \theta)$ :

$$\begin{aligned} r^2 &= x^2 + y^2 & \tan \theta &= \frac{y}{x} \\ r &= \sqrt{x^2 + y^2} & \theta &= \tan^{-1} \left( \frac{y}{x} \right) \end{aligned}$$

Note: The formula for  $\theta$  requires care with quadrants. The functions  $r^2 = x^2 + y^2$  and  $\tan \theta = y/x$  are the fundamental relationships.

### 3.7 Tangents with Polar Coordinates

To find  $\frac{dy}{dx}$  for a polar curve  $r = f(\theta)$ , we use the parametric derivatives. Since  $x = r \cos \theta$  and  $y = r \sin \theta$ , applying the product rule gives:

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta$$

#### Tangents with Polar Coordinates

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$$

where  $r' = \frac{dr}{d\theta}$ .

### 3.8 Area with Polar Coordinates

#### Area with Polar Coordinates

For the area between two polar curves  $r = r_o(\theta)$  (outer radius) and  $r = r_i(\theta)$  (inner radius):

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_o^2 - r_i^2) d\theta$$

For the area enclosed by a single polar curve  $r = f(\theta)$ , set  $r_i = 0$ :

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

### 3.9 Arc Length with Polar Coordinates

Let

$$r = f(\theta) \quad \alpha \leq \theta \leq \beta$$

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

$$x = f(\theta) \cos \theta$$

$$y = f(\theta) \sin \theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

Now,

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta} \cos \theta - r \sin \theta\right)^2 + \left(\frac{dr}{d\theta} \sin \theta + r \cos \theta\right)^2 \\ &= \left(\frac{dr}{d\theta}\right)^2 \cos^2 \theta - 2r \frac{dr}{d\theta} \cos \theta \sin \theta + r^2 \sin^2 \theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2 \theta + 2r \frac{dr}{d\theta} \sin \theta \cos \theta + r^2 \cos^2 \theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2 \end{aligned}$$



### • Arc Length with Polar Coordinates •

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

### 3.10 Surface Area with Polar Coordinates

Let

$$r = f(\theta) \quad \alpha \leq \theta \leq \beta$$

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

$$x = f(\theta) \cos \theta$$

$$y = f(\theta) \sin \theta$$

### • Surface Area with Polar Coordinates •

$$S = \int 2\pi y \, ds \quad \text{rotation about } x\text{-axis}$$

$$S = \int 2\pi x \, ds \quad \text{rotation about } y\text{-axis}$$

where,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad r = f(\theta), \alpha \leq \theta \leq \beta$$

## 4 Sequences

### 4.1 Definition

#### Definition 4.1.1: Sequence

A **sequence** is a function whose domain is the set of natural numbers  $\mathbb{N}$ . The sequence is denoted by  $\{a_n\}$  and the value of the function at  $n$  (the  $n$ -th term) is denoted by  $a_n$ . Various ways of representing a sequence are:

$$\{a_1, a_2, \dots, a_n, a_{n+1}, \dots\} \quad \{a_n\} \quad \{a_n\}_{n=1}^{\infty}$$

### 4.2 Precise Definition of Limit of a Sequence

#### Precise Definition of Limit

1. We say that

$$\lim_{n \rightarrow \infty} a_n = L$$

if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - L| < \epsilon$$

2. We say that

$$\lim_{n \rightarrow \infty} a_n = \infty$$

if

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n > M$$

3. We say that

$$\lim_{n \rightarrow \infty} a_n = -\infty$$

if

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n < M$$

### 4.3 Convergence of Sequences

**Theorem 4.3.1 (Convergence of Sequences):** A sequence  $\{a_n\}$  is said to be **convergent** if there exists a real number  $L$  such that

$$\lim_{n \rightarrow \infty} a_n = L$$

**Theorem 4.3.2 (Uniqueness of Limits):** If a sequence  $\{a_n\}$  converges, then its limit is unique.

**Theorem 4.3.3:** Given the sequence  $\{a_n\}$  if we have a function  $f(x)$  such that  $f(n) = a_n$  and  $\lim_{x \rightarrow \infty} f(x) = L$  then  $\lim_{n \rightarrow \infty} a_n = L$ .

#### Properties of Convergent Sequences

If  $\{a_n\}$  and  $\{b_n\}$  are both convergent sequences, then:

1.

$$\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$$

2.

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n$$

3.

$$\lim_{n \rightarrow \infty} (a_n b_n) = \left( \lim_{n \rightarrow \infty} a_n \right) \left( \lim_{n \rightarrow \infty} b_n \right)$$

4.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}, \quad \text{provided } \lim_{n \rightarrow \infty} b_n \neq 0$$

5.

$$\lim_{n \rightarrow \infty} a_n^p = \left[ \lim_{n \rightarrow \infty} a_n \right]^p, \quad \text{provided } a_n \geq 0$$

**Theorem 4.3.4 (Squeeze Theorem for Sequences):** If  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  are sequences such that  $a_n \leq b_n \leq c_n$  for all  $n \geq N$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$  then  $\lim_{n \rightarrow \infty} b_n = L$ .

**Theorem 4.3.5:**

$$\text{If } \lim_{n \rightarrow \infty} |a_n| = 0, \text{ then } \lim_{n \rightarrow \infty} a_n = 0$$

**Proof:**

The main thing to this proof is to note that,

$$-|a_n| \leq a_n \leq |a_n|$$

Then note that,

$$\lim_{n \rightarrow \infty} -|a_n| = - \lim_{n \rightarrow \infty} |a_n| = 0$$

We then have that,

$$0 \leq \lim_{n \rightarrow \infty} a_n \leq 0$$

and so by the Squeeze Theorem,

$$\lim_{n \rightarrow \infty} a_n = 0$$

**Theorem 4.3.6 (Convergent Sequences are Bounded):** If a sequence  $\{a_n\}$  is convergent, then it is bounded.

**Example 4.1: Determining Convergence**

Determine if the following sequences converge or diverge:

1.  $\left\{ \frac{n^2}{2n^2 + 1} \right\}$

2.  $\left\{ \frac{(-1)^n}{n} \right\}$

$$3. \left\{ \frac{n!}{n^n} \right\}$$

**Solution:**

1. We can use the theorem about converting sequences to functions. Let  $f(x) = \frac{x^2}{2x^2 + 1}$ .

Then

$$\lim_{x \rightarrow \infty} \frac{x^2}{2x^2 + 1} = \lim_{x \rightarrow \infty} \frac{1}{2 + 1/x^2} = \frac{1}{2}$$

Therefore,  $\lim_{n \rightarrow \infty} \frac{n^2}{2n^2 + 1} = \frac{1}{2}$  and the sequence converges.

2. Note that  $\left| \frac{(-1)^n}{n} \right| = \frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ . By the theorem about absolute values,  
 $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$  and the sequence converges.

3. For large  $n$ ,  $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$  grows much slower than  $n^n = n \cdot n \cdot n \cdots n \cdot n$ . We have

$$0 < \frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n} \cdot \frac{1}{n} \leq \frac{n}{n} \cdot \frac{n}{n} \cdot \frac{1}{n} = \frac{1}{n}$$

Since  $\lim_{n \rightarrow \infty} 0 = 0$  and  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , by the Squeeze Theorem,  $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$  and the sequence converges.

**Theorem 4.3.7:** For the sequence  $\{a_n\}$  if both  $\lim_{n \rightarrow \infty} a_{2n} = L$  and  $\lim_{n \rightarrow \infty} a_{2n+1} = L$ , then  $\{a_n\}$  is convergent and  $\lim_{n \rightarrow \infty} a_n = L$ .

**Theorem 4.3.8:** The sequence  $\{r^n\}_{n=0}^{\infty}$  converges if  $-1 < r \leq 1$  and diverges for all other values of  $r$ . Also,

$$\lim_{n \rightarrow \infty} r^n \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \end{cases}$$

**Theorem 4.3.9:** For the sequence  $\{a_n\}$  if both  $\lim_{n \rightarrow \infty} a_{2n} = L$  and  $\lim_{n \rightarrow \infty} a_{2n+1} = L$ , then  $\{a_n\}$  is convergent and  $\lim_{n \rightarrow \infty} a_n = L$ .

**Proof:**

Let  $\epsilon > 0$ . Then since  $\lim_{n \rightarrow \infty} a_{2n} = L$ ,

$$\exists N_1 \in \mathbb{N} : \forall n \geq N_1, |a_{2n} - L| < \epsilon$$

Similarly, because  $\lim_{n \rightarrow \infty} a_{2n+1} = L$ ,

$$\exists N_2 \in \mathbb{N} : \forall n \geq N_2, |a_{2n+1} - L| < \epsilon$$

Now, let  $N = \max\{2N_1, 2N_2 + 1\}$  and let  $n > N$ . Then either  $a_n = a_{2k}$  for some  $k > N_1$  or  $a_n = a_{2k+1}$  for some  $k > N_2$ . And so in either case we have that

$$|a_n - L| < \epsilon$$

Therefore,  $\lim_{n \rightarrow \infty} a_n = L$  and so  $\{a_n\}$  is convergent.

## 4.4 Bounded and Monotonic Sequences

### Definition 4.4.1: Bounded Sequence

A sequence  $\{a_n\}$  is **bounded** if

$$\exists M \in \mathbb{R} : \forall n \in \mathbb{N}, |a_n| \leq M$$

### Upper and Lower Bounds

If

$$\exists m \in \mathbb{R} : \forall n \in \mathbb{N}, m \leq a_n$$

the sequence  $\{a_n\}$  is said to be **bounded below** and  $m$  is a **lower bound** of the sequence. Similarly, if

$$\exists M \in \mathbb{R} : \forall n \in \mathbb{N}, M \geq a_n$$

the sequence  $\{a_n\}$  is said to be **bounded above** and  $M$  is an **upper bound** of the sequence.

**Theorem 4.4.2 (Bounded Sequence):** If a sequence  $\{a_n\}$  is both bounded above and below, then it is bounded. That is, if

$$\exists m, M \in \mathbb{R} : \forall n \in \mathbb{N}, m \leq a_n \leq M$$

then  $\{a_n\}$  is bounded.

### Example 4.2: Bounded Sequences

Determine if the following sequences are bounded:

1.  $\left\{ \frac{\sin(n)}{n} \right\}$
2.  $\{(-1)^n n\}$
3.  $\left\{ \frac{2n}{n+1} \right\}$

**Solution:**

1. Since  $-1 \leq \sin(n) \leq 1$  for all  $n$ , and  $n > 0$  for all  $n \in \mathbb{N}$ , we have

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

For  $n \geq 1$ , this gives us  $-1 \leq \frac{\sin(n)}{n} \leq 1$ . Therefore, the sequence is bounded with  $m = -1$  and  $M = 1$ .

2. The terms of this sequence alternate:  $-1, 2, -3, 4, -5, 6, \dots$ . As  $n$  increases,  $|(-1)^n n| = n$  grows without bound. Therefore, the sequence is unbounded.

3. We can rewrite  $\frac{2n}{n+1} = \frac{2n+2-2}{n+1} = \frac{2(n+1)-2}{n+1} = 2 - \frac{2}{n+1}$ . Since  $n \geq 1$ , we have  $\frac{2}{n+1} \leq \frac{2}{2} = 1$ , so  $a_n \geq 2 - 1 = 1$ . Also, since  $\frac{2}{n+1} > 0$ , we have  $a_n < 2$ . Therefore,  $1 \leq a_n < 2$  for all  $n$ , and the sequence is bounded.

### Definition 4.4.3: Monotonic Sequence

A sequence  $\{a_n\}$  is **monotonic** if for all  $n \in \mathbb{N}$

$$a_{n+1} \geq a_n \quad \text{or} \quad a_{n+1} \leq a_n$$

**Theorem 4.4.4:** If a sequence  $\{a_n\}$  is both bounded and monotonic, then it is convergent.

### Example 4.3: Monotonic Sequences

Determine if the following sequences are monotonic:

1.  $\left\{ \frac{n}{n+1} \right\}$
2.  $\left\{ \frac{n+3}{n^2} \right\}$

**Solution:**

1. Consider  $a_n = \frac{n}{n+1}$ . To check if the sequence is monotonic, we can check if  $a_{n+1} \geq a_n$  or  $a_{n+1} \leq a_n$ . We have

$$a_{n+1} = \frac{n+1}{n+2} \quad \text{and} \quad a_n = \frac{n}{n+1}$$

Comparing:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0$$

Since  $a_{n+1} - a_n > 0$ , we have  $a_{n+1} > a_n$  for all  $n$ , so the sequence is monotonically increasing.

2. Consider  $a_n = \frac{n+3}{n^2}$ . Let's check the first few terms:  $a_1 = 4$ ,  $a_2 = \frac{5}{4}$ ,  $a_3 = \frac{6}{9} = \frac{2}{3}$ ,  $a_4 = \frac{7}{16}$ . The sequence appears to be decreasing, but let's verify. We have

$$a_{n+1} - a_n = \frac{n+4}{(n+1)^2} - \frac{n+3}{n^2} = \frac{n^2(n+4) - (n+3)(n+1)^2}{n^2(n+1)^2}$$

Expanding the numerator:

$$n^3 + 4n^2 - (n+3)(n^2 + 2n + 1) = n^3 + 4n^2 - n^3 - 2n^2 - n - 3n^2 - 6n - 3 = -n^2 - 7n - 3 < 0$$

Since  $a_{n+1} - a_n < 0$ , we have  $a_{n+1} < a_n$  for all  $n$ , so the sequence is monotonically decreasing.

### Looking Ahead

Sequences form the foundation for understanding **series**. While a sequence is simply a list of numbers, a series is the sum of the terms of a sequence. In the next section, we will explore infinite series and develop tests to determine whether such sums converge to a finite value.

## 5 Series

### Properties of Convergent Series

If  $\sum a_n$  and  $\sum b_n$  are both convergent series, then:

1.  $\sum ca_n$ , where  $c$  is a constant, is also convergent and

$$\sum ca_n = c \sum a_n$$

2.  $\sum(a_n \pm b_n)$  is also convergent and

$$\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$$

### 5.1 Convergence of Series

**Theorem 5.1.1 (Convergence of Series):** The series  $\sum a_n$  converges if and only if the sequence of partial sums  $\{s_n\}$  is convergent. That is,

$$\sum a_n \text{ converges} \iff \lim_{n \rightarrow \infty} s_n \text{ exists}$$

If the series  $\sum a_n$  converges, then

$$\lim_{n \rightarrow \infty} s_n = s$$

where  $s$  is the sum of the series.

#### Example 5.1:

$\lim_{n \rightarrow \infty} n = \infty$	(diverges)
$\lim_{n \rightarrow \infty} \frac{1}{n^2 - 1} = 0$	(converges)
$\lim_{n \rightarrow \infty} (-1)^n$ does not exist	(diverges)
$\lim_{n \rightarrow \infty} \frac{1}{3^{n-1}} = 0$	(converges)

**Theorem 5.1.2:** If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

### 5.2 Divergence Test

**Theorem 5.2.1 (Divergence Test):** If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then the series  $\sum a_n$  diverges.

#### Note:-

The Divergence Test is the contrapositive of the previous theorem. This is often the first test to try when testing a series for convergence, as it is quick to apply. However, if  $\lim_{n \rightarrow \infty} a_n = 0$ , the test is **inconclusive** and we must use other tests.

## 5.3 Special Series

### 5.3.1 Geometric Series

#### Definition 5.3.1: Geometric Series

A series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

is called a **geometric series**. The sum of the series is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r| < 1$$

### 5.3.2 Telescoping Series

#### Definition 5.3.2: Telescoping Series

A series of the form

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - a_2 + a_2 - a_3 + \cdots$$

is called a **telescoping series**. The sum of the series is

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n \rightarrow \infty} a_{n+1}$$

**Example 5.2: Find the sum of the series**  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

**Solution:** First, we use partial fraction decomposition:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

Multiplying both sides by  $n(n+1)$  gives  $1 = A(n+1) + Bn$ . Setting  $n = 0$  gives  $A = 1$ , and setting  $n = -1$  gives  $B = -1$ . Thus:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Now we can write the partial sum:

$$\begin{aligned} s_N &= \sum_{n=1}^N \frac{1}{n(n+1)} = \sum_{n=1}^N \left( \frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{N} - \frac{1}{N+1} \right) \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

Taking the limit as  $N \rightarrow \infty$ :

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N \rightarrow \infty} s_N = \lim_{N \rightarrow \infty} \left( 1 - \frac{1}{N+1} \right) = 1$$



### 5.3.3 Harmonic Series

#### Definition 5.3.3: Harmonic Series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is called the **harmonic series**. The harmonic series diverges.

## 5.4 Integral Test

**Theorem 5.4.1 (Integral Test):** Let  $f(x)$  be a continuous, positive, and decreasing function for  $x \geq 1$ . Then the series

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x) dx$$

converges. That is,

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x) dx \text{ converges}$$

#### Proof:

Let  $s_n = f(1) + f(2) + \cdots + f(n)$  and  $s_{n+1} = f(1) + f(2) + \cdots + f(n) + f(n+1)$ . Then

$$s_{n+1} - s_n = f(n+1) \geq 0$$

and

$$\int_n^{n+1} f(x) dx \leq f(n) \leq \int_{n-1}^n f(x) dx$$

Summing from 1 to  $n$  gives

$$s_{n+1} - s_1 \leq \int_1^n f(x) dx \leq s_n$$

Taking the limit as  $n \rightarrow \infty$  gives

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} s_{n+1} = s$$

which implies that the series converges if and only if the integral converges.  $\square$

### 5.4.1 The $p$ -series Test

**Theorem 5.4.2 ( $p$ -series Test):** The series

$$\sum_{n=k}^{\infty} \frac{1}{n^p} \quad \text{where } k \in \mathbb{N}$$

converges if  $p > 1$  and diverges if  $p \leq 1$ .

**Proof:**

If  $p > 1$ , then the integral

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \left[ \frac{x^{1-p}}{1-p} \right]_1^b = \frac{1}{p-1}$$

converges. If  $p \leq 1$ , then the integral diverges. □

## 5.5 Comparison Test/Limit Comparison Test

**Theorem 5.5.1 (Comparison Test):** Let  $\sum a_n$  and  $\sum b_n$  be two series with  $a_n, b_n \geq 0$  for all  $n$  and  $a_n \leq b_n$  for all  $n$ . Then:

1. If  $\sum b_n$  converges, then  $\sum a_n$  converges.
2. If  $\sum a_n$  diverges, then  $\sum b_n$  diverges.

**Proof:**

Let the partial sums be

$$s_n = \sum_{i=1}^n a_i \quad \text{and} \quad t_n = \sum_{i=1}^n b_i$$

Since  $a_n, b_n \geq 0$ , we know that

$$\begin{aligned} s_n \leq s_n + a_{n+1} &= \sum_{i=1}^n a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = s_{n+1} && \implies s_n \leq s_{n+1} \\ t_n \leq t_n + b_{n+1} &= \sum_{i=1}^n b_i + b_{n+1} = \sum_{i=1}^{n+1} b_i = t_{n+1} && \implies t_n \leq t_{n+1} \end{aligned}$$

Also, since  $a_n \leq b_n$ , we have

$$s_n = \sum_{i=1}^n a_i \leq \sum_{i=1}^n b_i = t_n$$

If  $\sum b_n$  converges, then  $\{t_n\}$  is bounded and increasing, so it converges. Since  $s_n \leq t_n$ ,  $\{s_n\}$  is also bounded and increasing, so it converges. □

**Theorem 5.5.2 (Limit Comparison Test):** Let  $\sum a_n$  and  $\sum b_n$  be two series with  $a_n, b_n > 0$  for all  $n$ . If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where  $c \in \mathbb{R}_+$  and  $c < \infty$ , then  $\sum a_n$  and  $\sum b_n$  either both converge or both diverge.

**Proof:**

Since  $0 < c < \infty$ , there exists two positive finite numbers  $m$  and  $M$  such that  $m < c < M$ . Now we have

$$\begin{aligned} m &< \frac{a_n}{b_n} < M \\ mb_n &< a_n < Mb_n \end{aligned}$$

Now, if  $\sum b_n$  converges, then so does  $\sum Mb_n$ , and since  $a_n < Mb_n$ , for all  $n$  by the Comparison Test,  $\sum a_n$  also converges.

Similarly, if  $\sum b_n$  diverges, then so does  $\sum mb_n$ , and since  $mb_n < a_n$ , for all  $n$  by the Comparison Test,  $\sum a_n$  also diverges. □

## 5.6 Alternating Series Test

**Theorem 5.6.1 (Alternating Series Test):** Let  $\sum a_n$  be an alternating series, that is,  $a_n = (-1)^n b_n$  or  $a_n = (-1)^{n+1} b_n$  where  $b_n > 0$  for all  $n$ . Then if:

1.  $\lim_{n \rightarrow \infty} b_n = 0$

2.  $\forall n, b_{n+1} \leq b_n$

the series  $\sum a_n$  is convergent.

**Proof:**

Let  $s_n = \sum_{i=1}^n a_i$ . Since  $b_n$  is decreasing and  $\lim_{n \rightarrow \infty} b_n = 0$ , we can say

$$\forall n, b_n - b_{n+1} \geq 0$$

Now, we have

$$\begin{aligned} s_{2n} &= b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots + b_{2n-1} - b_{2n} \\ &= b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \end{aligned}$$

Since  $b_n$  is decreasing,  $s_{2n}$  is increasing and bounded above by  $b_1$ .

Let's assume that its limit is  $s$ , that is

$$\lim_{n \rightarrow \infty} s_{2n} = s$$

Then

$$\lim_{n \rightarrow \infty} s_{2n+1} = \lim_{n \rightarrow \infty} (s_{2n} + b_{2n+1}) = \lim_{n \rightarrow \infty} s_{2n} + \lim_{n \rightarrow \infty} b_{2n+1} = s + 0 = s$$

So, we know that both  $\{s_{2n}\}$  and  $\{s_{2n+1}\}$  are convergent sequences and they both have the same limit.

We also know that  $\{s_n\}$  is a convergent sequence with a limit of  $s$ . This in turn implies that the series  $\sum a_n$  is convergent □

## 5.7 Absolute and Conditional Convergence

### Definition 5.7.1: Absolute and Conditional Convergence

A series  $\sum a_n$  is said to be **absolutely convergent** if  $\sum |a_n|$  converges. A series that converges but not absolutely is said to be **conditionally convergent**.

**Theorem 5.7.2:** If  $\sum a_n$  is absolutely convergent, then  $\sum a_n$  is convergent.

**Note:-**

The converse is not true: a series can be convergent without being absolutely convergent. Such a series is called conditionally convergent.

**Theorem 5.7.3 (Riemann Rearrangement Theorem):** Given the series  $\sum a_n$ :

1. If  $\sum a_n$  is absolutely convergent with sum  $s$ , then any rearrangement of  $\sum a_n$  also converges to  $s$ .
2. If  $\sum a_n$  is conditionally convergent, then for any real number  $r$  (including  $\pm\infty$ ), there exists a rearrangement of  $\sum a_n$  that converges to  $r$  or diverges to  $\pm\infty$ .

**Note:-**

This remarkable theorem shows that conditionally convergent series are very sensitive to the order of their terms. Rearranging the terms can produce any desired sum or even cause divergence!

## 5.8 Ratio Test

**Theorem 5.8.1 (Ratio Test):** Let  $\sum a_n$  be a series and let

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then:

1. If  $L < 1$ , then  $\sum a_n$  is absolutely convergent.
2. If  $L > 1$  or  $L = \infty$ , then  $\sum a_n$  diverges.
3. If  $L = 1$ , then the test is inconclusive.

**Proof:**

Let  $L < 1$  and  $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ . Then there exists a number  $r$  such that  $L < r < 1$ . Since  $L < r$ , there exists a number  $N$  such that for all  $n > N$ ,

$$\left| \frac{a_{n+1}}{a_n} \right| < r$$

$$|a_{n+1}| < r |a_n|$$

$$|a_{n+1}| < r |a_n| < r^2 |a_{n-1}| < \cdots < r^{n-N} |a_N|$$

$$|a_{n+1}| < r^{n-N} |a_N|$$

Since  $r < 1$ , the series  $\sum r^{n-N} |a_N|$  converges by the geometric series test. By the Comparison Test,  $\sum a_n$  also converges.  $\square$

## 5.9 Root Test

**Theorem 5.9.1 (Root Test):** Let  $\sum a_n$  be a series and let

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$

Then:

1. If  $L < 1$ , then  $\sum a_n$  is absolutely convergent.
2. If  $L > 1$  or  $L = \infty$ , then  $\sum a_n$  diverges.
3. If  $L = 1$ , then the test is inconclusive.

**Note:-**

$$\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

**Proof:**

Let  $L < 1$  and  $L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ . Then there exists a number  $r$  such that  $L < r < 1$ . Since  $L < r$ , there exists a number  $N$  such that for all  $n > N$ ,

$$\sqrt[n]{|a_n|} < r$$

$$|a_n| < r^n$$

Since  $r < 1$ , the series  $\sum r^n$  converges by the geometric series test. By the Comparison Test,  $\sum a_n$  also converges.  $\square$

## 5.10 Strategies for Series Test

**Divergence Test** If  $\lim_{n \rightarrow \infty} a_n \neq 0$ .

**Geometric Series Test** If  $a_n = ar^n$  or  $a_n = ar^{n-1}$

**Integral Test** If  $a_n = f(n)$  and  $f(x)$  is continuous, positive, and decreasing

**p-series Test** If  $a_n = \frac{1}{n^p}$

**Comparison Test** If  $a_n$  is hard to work with, but  $b_n$  is easy to work with

**Limit Comparison Test** If  $a_n$  is a rational expression involving only polynomials.

**Alternating Series Test** If  $a_n = (-1)^n b_n$  or  $a_n = (-1)^{n+1} b_n$

**Ratio Test** If  $a_n$  is a product of terms

**Root Test** If  $a_n$  is a power of terms

## 5.11 Estimating the Value of a Series

### 5.11.1 Integral Test

#### Integral Test

$$s_n + \int_{n+1}^{\infty} f(x) \, dx \leq s \leq s_n + \int_n^{\infty} f(x) \, dx$$

### 5.11.2 Comparison Test

Given a series  $\sum a_n$ , let's assume that we've used the comparison test to show that it's convergent. Therefore we found a second series  $\sum b_n$  that converged and  $a_n \leq b_n$  for all  $n$ . Now, let

$$R_n = \sum_{k=n+1}^{\infty} a_k \quad \text{and} \quad T_n = \sum_{k=n+1}^{\infty} b_k$$

Since,  $a_n \leq b_n$  we also know that  $R_n \leq T_n$ .

#### Comparison Test

$$R_n \leq T_n \leq \int_n^{\infty} g(x) \, dx, \quad \text{where } g(n) = b_n$$

### 5.11.3 Alternating Series Test

#### Alternating Series Test

$$|R_n| = |s - s_n| \leq b_{n+1}$$

### 5.11.4 Ratio Test

#### Ratio Test

To get an estimate of the remainder, let's first define the following sequence,

$$r_n = \frac{a_{n+1}}{a_n}$$

We now have two possible cases:

1. If  $\{r_n\}$  is a decreasing sequence and  $r_{n+1} < 1$ , then

$$R_n \leq \frac{a_{n+1}}{1 - r_{n+1}}$$

2. If  $\{r_n\}$  is an increasing sequence, then

$$R_n \leq \frac{a_{n+1}}{1 - L}, \quad \text{where } L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

## 5.12 Power Series

### Definition 5.12.1: Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n(x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots$$

where  $c_n$  are constants and called coefficients and  $a$  is a fixed number. The number  $a$  is called the **center** of the power series.

**Theorem 5.12.2 (Convergence of Power Series):** Given the power series  $\sum c_n(x - a)^n$ , there exists a number  $R$  called the **radius of convergence** such that:

1. The series converges absolutely for all  $x$  with  $|x - a| < R$
2. The series diverges for all  $x$  with  $|x - a| > R$

To find  $R$ , apply the **Ratio Test** or **Root Test**:

- **Ratio Test:** Compute  $L = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}(x - a)^{n+1}}{c_n(x - a)^n} \right|$ . The series converges when  $L < 1$ , which gives  $|x - a| < R$ .
- **Root Test:** Compute  $L = \lim_{n \rightarrow \infty} \sqrt[n]{|c_n(x - a)^n|}$ . The series converges when  $L < 1$ , which gives  $|x - a| < R$ .

**N.B:** the series always converges at  $x = a$ .

### Interval vs Radius of Convergence

The **radius of convergence**  $R$  is a non-negative number (or  $\infty$ ) that tells us how far from the center  $a$  the series converges. The **interval of convergence** is the actual set of  $x$ -values for which the series converges.

Given radius  $R$  and center  $a$ :

- The series converges absolutely on the open interval  $(a - R, a + R)$
- The series diverges outside the closed interval  $[a - R, a + R]$
- At the endpoints  $x = a - R$  and  $x = a + R$ , convergence must be tested separately using other tests (p-series, alternating series, etc.)
- The interval of convergence is one of:  $(a - R, a + R)$ ,  $[a - R, a + R)$ ,  $(a - R, a + R]$ , or  $[a - R, a + R]$

## 5.13 Power Series and Functions

**Theorem 5.13.1 (Power Series and Functions):** Let  $f(x) = \sum c_n(x - a)^n$  be a power series with radius of convergence  $R$ . Then:

1.  $f(x)$  is continuous and differentiable on the interval  $(a - R, a + R)$

2. 
$$f'(x) = \sum_{n=1}^{\infty} n c_n (x - a)^{n-1}$$

3. 
$$\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{c_n (x - a)^{n+1}}{n + 1}$$

4. The radius of convergence of  $f'(x)$  and  $\int f(x) dx$  are also  $R$ .

## 5.14 Properties of Power Series

**Theorem 5.14.1:** Suppose  $f(x) = \lim_{n=k \rightarrow \infty} a_n x^n$  and  $g(x) = \lim_{n=k \rightarrow \infty} b_n x^n$  on an interval  $I$  and fix some  $c \neq 0$ . Then:

1. **Addition/Subtraction:**

$$f(x) \pm g(x) = \lim_{n=k \rightarrow \infty} (a_n \pm b_n) x^n \text{ for } x \in I$$

2. **Multiplication:** If  $k + m \geq 0$  then

$$x^m f(x) = \lim_{n=k \rightarrow \infty} a_n x^{n+m} \quad \forall x \in I, x \neq 0$$

with

$$\lim_{n=k \rightarrow \infty} a_n x^{n+m} = \lim_{x \rightarrow 0} x^m f(x) \text{ when } x = 0.$$

3. **Composition:**

$$f(cx^m) = \lim_{n=k \rightarrow \infty} a_n (cx^m)^n \text{ for } cx^m \in I$$

**Theorem 5.14.2 (Term-by-Term Differentiation and Integration):** Let  $f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$  be a power series with radius of convergence  $R > 0$ . Then  $F$  is differentiable on  $(c - R, c + R)$  and:

1. **Differentiation:**

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x - a)^{n-1} \text{ for } |x - a| < R$$

2. **Integration:**

$$\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x - a)^{n+1} \text{ for } |x - a| < R$$

**Theorem 5.14.3 (Abel's Theorem):** If a power series  $F(x) = \lim_{n=0 \rightarrow \infty} a_n (x - c)^n$  has a radius of convergence  $R \in (0, \infty)$  and  $F$  converges at an endpoint, then  $F$  is continuous at that endpoint.



## 5.15 Taylor Series

### Definition 5.15.1: Taylor Series

The **Taylor series** of a function  $f(x)$  about  $x = a$  is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

where  $f^{(n)}(a)$  is the  $n$ th derivative of  $f(x)$  evaluated at  $x = a$ .

To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the **n-th degree Taylor polynomial** of  $f(x)$  as

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x - a)^i$$

Notice that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

the  $n$ -th degree Taylor polynomial is just the partial sum for the series. The **remainder** is defined to be

$$R_n(x) = f(x) - T_n(x)$$

So, the remainder is just the *error* between the function  $f(x)$  and the  $n$ -th degree Taylor polynomial  $T_n(x)$  for a given  $n$ .

With this definition, we can then write the function as

$$f(x) = T_n(x) + R_n(x)$$

**Theorem 5.15.2:** Suppose that  $f(x) = T_n(x) + R_n(x)$ . Then if

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

for  $|x - a| < R$ , then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

on  $|x - a| < R$ .

**Theorem 5.15.3 (Taylor's Remainder Theorem):** If  $f(x)$  has  $n+1$  continuous derivatives on the interval containing  $a$  and  $x$ , then there exists a number  $c$  between  $a$  and  $x$  such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}$$

This is called the **Lagrange form** of the remainder.

**Theorem 5.15.4 (Lagrange Error Bound):** If  $|f^{(n+1)}(x)| \leq M$  for  $|x - a| \leq d$ , then for  $|x - a| \leq d$ ,

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \leq \frac{M}{(n+1)!} d^{n+1}$$

### 5.15.1 Maclaurin Series

#### Definition 5.15.5: Maclaurin Series

The **Maclaurin series** of a function  $f(x)$  is the Taylor series of  $f(x)$  about  $x = 0$ . That is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

#### Common Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all } x$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } -1 < x \leq 1$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } -1 \leq x \leq 1$$

### 5.16 Binomial Series

**Theorem 5.16.1 (Binomial Theorem):** If  $n$  is a positive integer, then

$$\begin{aligned} (a+b)^n &= \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \cdots + nab^{n-1} + b^n \end{aligned}$$

where,

$$\begin{aligned} \binom{n}{i} &= \frac{n!}{i!(n-i)!} = \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!} \\ \binom{n}{0} &= 1 \end{aligned}$$

### Binomial Series

If  $k$  is any real number and  $|x| < 1$ , then

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \binom{k}{n}x^n + \cdots\end{aligned}$$

where,

$$\begin{aligned}\binom{k}{n} &= \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \\ \binom{k}{0} &= 1\end{aligned}$$

## 6 3-Dimensional Space

The 3-D coordinate system is often denoted by  $\mathbb{R}^3$ . Likewise, the 2-D coordinate system is denoted by  $\mathbb{R}^2$ , and the 1-D coordinate system is denoted by  $\mathbb{R}$ .

### 6.1 Equations of Lines

#### Vector form

If  $\vec{a}$  and  $\vec{v}$  are parallel vectors, then  $\vec{a} = t\vec{v}$  for some scalar  $t$ .  
Now if we have a vector  $\vec{r}$  as follows

$$\vec{r} = \vec{r}_0 + \vec{a}$$

Then we can write

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

This is called the **vector form of the equation of a line**.

#### Parametric form

We can rewrite the vector form as

$$\begin{aligned}\vec{r} &= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \\ \langle x, y, z \rangle &= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle\end{aligned}$$

In other words

$$\begin{aligned}x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc\end{aligned}$$

This set of equations is called the **parametric form of the equation of a line**.

#### Symmetric Equations of a Line

If we assume that  $a, b$ , and  $c$  are non-zero numbers, then we can solve each of the parametric equations for  $t$ . This gives us

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

#### Example 6.1: Find the Equations of lines:

1. Through the points  $(7, -3, 1)$  and  $(-2, 1, 4)$
2. Through the point  $(1, -5, 0)$  and parallel to the line given by  $\vec{r}(t) = \langle 8 - 3t, -10 + 9t, -1 - t \rangle$
3. Through the point  $(-7, 2, 4)$  and orthogonal to both  $\vec{v} = \langle 0, -9, 1 \rangle$  and  $\vec{w} = 3\hat{i} + \hat{j} - 4\hat{k}$

1.

Direction vector  $\vec{d} = \langle -2 - 7, 1 + 3, 4 - 1 \rangle = \langle -9, 4, 3 \rangle$

Now, the vector form of the line is

$$\vec{r} = \langle 7, -3, 1 \rangle + t\langle -9, 4, 3 \rangle$$

The parametric form is

$$x = 7 - 9t, \quad y = -3 + 4t, \quad z = 1 + 3t$$

The symmetric form is

$$\frac{x-7}{-9} = \frac{y+3}{4} = \frac{z-1}{3}$$

**2.**

The direction vector is  $\vec{d} = \langle -3, 9, -1 \rangle$

Hence, the vector form of the line is

$$\vec{r} = \langle 1, -5, 0 \rangle + t\langle -3, 9, -1 \rangle$$

The parametric form is

$$x = 1 + 3t, \quad y = -5 + 9t, \quad z = -t$$

And the symmetric form is

$$\frac{x-1}{3} = \frac{y+5}{9} = -z$$

**3.**

Direction vector

$$\vec{d} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -9 & 1 \\ 3 & 1 & -4 \end{vmatrix} = \langle 35, 3, 27 \rangle$$

Hence, the vector form of the line is

$$\vec{r} = \langle -7, 2, 4 \rangle + t\langle 35, 3, 27 \rangle$$

The parametric form is

$$x = -7 + 35t, \quad y = 2 + 3t, \quad z = 4 + 27t$$

The symmetric form is

$$\frac{x+7}{35} = \frac{y-2}{3} = \frac{z-4}{27}$$

**Example 6.2: Determine if the two lines are parallel, orthogonal, or neither:**

**1. The line given by  $\vec{r}(t) = \langle 4 - 7t, -10 + 5t, 21 - 4t \rangle$  and the line given by  $\vec{r}(t) = \langle -2 + 3t, 7 + 5t, 5 + t \rangle$**

**2. The line given by  $x = 29, y = -3 - 6t, z = 12 - t$  and the line given by  $\vec{r}(t) = \langle 12 - 14t, 2 + 7t, -10 + 3t \rangle$**

**1.**

The direction vectors are

$$\vec{d}_1 = \langle -7, 5, -4 \rangle, \quad \vec{d}_2 = \langle 3, 5, 1 \rangle$$

To check if they are parallel, we can check:

$$\frac{-7}{3} \neq \frac{5}{5} \neq \frac{-4}{1}$$

which means they are not parallel.

To check if they are orthogonal, we can check:

$$\vec{d}_1 \cdot \vec{d}_2 = -7(3) + 5(5) + (-4)(1) = -21 + 25 - 4 = 0$$

Hence, they are orthogonal.

**2.**

The direction vectors are

$$\vec{d}_1 = \langle 0, -6, -1 \rangle, \quad \vec{d}_2 = \langle -14, 7, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{0}{-14} \neq \frac{-6}{7} \neq \frac{-1}{3}$$

which means they are not parallel.

To check if they are orthogonal, we can check:

$$\vec{d}_1 \cdot \vec{d}_2 = 0(-14) + (-6)(7) + (-1)(3) = -42 - 3 = -45 \neq 0$$

Hence, they are neither parallel nor orthogonal.

**Example 6.3: Determine the intersection point of the two lines or show that they don't intersect:**

**1. The line passing through the point  $(0, -9, -1)$  and  $(1, 6, -3)$  and the line given by  $\vec{r}(t) = \langle -9 - 4t, 10 + 6t, 1 - 2t \rangle$**

**2. The line given by  $x = 1 + 6t, y = -1 - 3t, z = 4 + 12t$  and the line given by  $x = 4 + t, y = -10 - 8t, z = 3 - 5t$**

**1.**

The direction vector of the first line is

$$\vec{d}_1 = \langle 1 - 0, 6 + 9, -3 + 1 \rangle = \langle 1, 15, -2 \rangle$$

We can write the parametric equations of the first line as:

$$x = s, y = -9 + 15s, z = -1 - 2s$$

And the parametric equations of the second line as:

$$x = -9 - 4t, y = 10 + 6t, z = 1 - 2t$$

Setting them equal to each other we get,

$$\begin{aligned} 0 + t &= -9 - 4s \\ -9 + 15t &= 10 + 6s \\ -1 - 2t &= 1 - 2s \end{aligned}$$

Solving the first two equations, we get

$$t = -\frac{7}{3}, \quad s = \frac{1}{3}$$

Now, verifying the third equation, we get

$$\begin{aligned} -1 - 2\left(-\frac{7}{3}\right) &= 1 - 2\left(\frac{1}{3}\right) \\ -1 + \frac{14}{3} &= 1 - \frac{2}{3} \\ \frac{11}{3} &\neq \frac{1}{3} \end{aligned}$$

Since the third equation is not satisfied, the two lines do not intersect.

**2.**

The lines are given in parametric form.

Setting them equal to each other we get,

$$\begin{aligned}1 + 6s &= 4 + t \\ -1 - 3s &= -10 - 8t \\ 4 + 12s &= 3 - 5t\end{aligned}$$

Solving the first two equations, we get

$$s = \frac{1}{3}, \quad t = -1$$

Now, verifying the third equation, we get

$$\begin{aligned}4 + 12\left(\frac{1}{3}\right) &= 3 - 5(-1) \\ 8 &= 8\end{aligned}$$

That means, the lines intersect. Substituting the values in the parametric equation, we get

$$\begin{aligned}x &= 1 + 6\left(\frac{1}{3}\right) = 3 \\ y &= -1 - 3\left(\frac{1}{3}\right) = -2 \\ z &= 4 + 12\left(\frac{1}{3}\right) = 8\end{aligned}$$

Hence, the intersection point is  $(3, -2, 8)$ .

**Example 6.4: Which of the three coordinate planes does the line given by  $x = 16t, y = -4 - 9t, z = 34$  intersect?**

To intersect the  $xy$ -plane, we need  $z = 0$ . But here  $z = 34$  is constant. Hence, the line does not intersect the  $xy$ -plane.

To intersect the  $yz$ -plane, we need  $x = 0$ . Hence,

$$16t = 0 \implies t = 0$$

And the intersection point is  $(0, -4 - 9 \times 0, 34)$  or  $(0, -4, 34)$ .

To intersect the  $xz$ -plane, we need  $y = 0$ . Hence,

$$-4 - 9t = 0 \implies t = -\frac{4}{9}$$

And the intersection point is  $\left(16\left(-\frac{4}{9}\right), 0, 34\right)$  or  $\left(-\frac{64}{9}, 0, 34\right)$ .

## 6.2 Equations of Planes

### Vector form

Let's assume  $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$  and  $\vec{r} = \langle x, y, z \rangle$  are two position vectors and  $\vec{r} - \vec{r}_0$  is a vector in the plane.

If  $\vec{n} = \langle a, b, c \rangle$  is a normal to the plane (which means it's orthogonal to the vector  $\vec{r} - \vec{r}_0$ ), then we can write

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \implies \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

This is called the **vector form of the equation of a plane**.

### Scalar form

If we expand the vector equation in the following way,

$$\begin{aligned} \vec{n} \cdot (\vec{r} - \vec{r}_0) &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \end{aligned}$$

Computing the dot product, we get

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the **scalar form of the equation of a plane**.

This equation can also be written as

$$ax + by + cz = d$$

where  $d = ax_0 + by_0 + cz_0$ .

**Example 6.5: Find the equation of the plane:**

1. Through the point  $(6, -3, 1)$ ,  $(5, -4, 1)$ , and  $(3, -4, 0)$
2. The plane containing the point  $(1, -5, 8)$  and orthogonal to the line given by  $x = -3 + 15t$ ,  $y = 14 - t$ ,  $z = 9 - 3t$
3. The plane containing the point  $(-8, 3, 7)$  and parallel to the plane given by  $4x + 8y - 2z = 45$
4. The plane containing the two lines given by  $\vec{r}(t) = \langle 7 + 5t, 2 + t, 6t \rangle$  and  $\vec{r}(t) = \langle 7 - 6t, 2 - 2t, 10t \rangle$

1.

The given points are

$$A(6, -3, 1), B(5, -4, 1), C(3, -4, 0)$$

Two vectors in the plane are

$$\begin{aligned} \vec{AB} &= \langle 5 - 6, -4 + 3, 1 - 1 \rangle = \langle -1, -1, 0 \rangle \\ \vec{BC} &= \langle 3 - 5, -4 + 4, 0 - 1 \rangle = \langle -2, 0, -1 \rangle \end{aligned}$$

Normal vector on the plane:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 0 \\ -2 & 0 & -1 \end{vmatrix} = \hat{i} - \hat{j} - 2\hat{k}$$



Now, using the point  $A$ , we can write the equation of the plane as

$$\begin{aligned}(x - 6) - (y + 3) - 2(z - 1) &= 0 \\ x - y - 2z &= 7\end{aligned}$$

**2.**

The normal vector is

$$\vec{n} = \langle 15, -1, -3 \rangle$$

Using the point  $(1, -5, 8)$ , the equation of the plane is

$$\begin{aligned}15(x - 1) - (y + 5) - 3(z - 8) &= 0 \\ 15x - y - 3z &= 15 + 5 - 24 \\ 15x - y - 3z + 4 &= 0\end{aligned}$$

**3.**

The normal vector is

$$\vec{n} = \langle 4, 8, -2 \rangle$$

Using the point  $(-8, 3, 7)$ , the equation of the plane is

$$\begin{aligned}4(x + 8) + 8(y - 3) - 2(z - 7) &= 0 \\ 4x + 8y - 2z &= -32 + 24 + 14 \\ 4x + 8y - 2z + 6 &= 0\end{aligned}$$

**4.**

The direction vectors of the two lines are

$$\vec{d}_1 = \langle 5, 1, 6 \rangle, \quad \vec{d}_2 = \langle -6, -2, 10 \rangle$$

The normal vector is

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 1 & 6 \\ -6 & -2 & 10 \end{vmatrix} = \langle 22, -86, -4 \rangle$$

Using the point  $A(7, 2, 0)$ , the equation of the plane is

$$\begin{aligned}22(x - 7) - 86(y - 2) - 4(z - 0) &= 0 \\ 22x - 86y - 4z - 154 + 172 &= 0 \\ 22x - 86y - 4z + 18 &= 0\end{aligned}$$

**Example 6.6: Determine if the two planes are parallel, orthogonal, or neither:**  
**The plane given by  $3x + 9y + 7z = -1$  and the plane containing the points  $(1, -1, 9), (4, -1, 2), (-2, 3, 4)$**

The normal vector of the first plane is

$$\vec{n}_1 = \langle 3, 9, 7 \rangle$$

Let the points be

$$A(1, -1, 9), B(4, -1, 2), C(-2, 3, 4)$$

Two vectors in the plane are

$$\begin{aligned}\vec{AB} &= \langle 4 - 1, -1 + 1, 2 - 9 \rangle = \langle 3, 0, -7 \rangle \\ \vec{AC} &= \langle -2 - 1, 3 + 1, 4 - 9 \rangle = \langle -3, 4, -5 \rangle\end{aligned}$$

The normal vector of the second plane is

$$\vec{n}_2 = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -7 \\ -3 & 4 & -5 \end{vmatrix} = \langle 28, 36, 12 \rangle = \langle 7, 9, 3 \rangle$$

To check if they are parallel, we can check:

$$\frac{3}{7} \neq \frac{9}{9} \neq \frac{7}{3}$$

which means they are not parallel.

To check if they are orthogonal, we can check:

$$\vec{n}_1 \cdot \vec{n}_2 = 3(7) + 9(9) + 7(3) = 21 + 81 + 21 = 123 \neq 0$$

Hence, they are neither parallel nor orthogonal.

**Example 6.7:** Find the intersection of the plane given by  $4x + y + 10z = -2$  and the plane given by  $-8x + 2y + 3z = -8$

The two planes are

$$\begin{aligned}4x + y + 10z &= -2 \\ -8x + 2y + 3z &= -8\end{aligned}$$

Multiplying the first equation by 2 and adding it to the second equation, we get

$$4y + 23z = -12 \quad \implies \quad y = -3 - \frac{23}{4}z$$

Substituting the value of  $y$  in the first equation, we get

$$16x - 3 - \frac{23}{4}z + 10z = -2 \quad \implies \quad x = \frac{1}{4} - \frac{17}{16}z$$

Let  $z = t$  (a parameter). Then we get

$$\begin{aligned}x &= \frac{1}{4} - \frac{17}{16}t \\ y &= -3 - \frac{23}{4}t \\ z &= t\end{aligned}$$

This is the parametric form of the line of intersection.

We can also write it in vector form as

$$\vec{r} = \langle \frac{1}{4}, -3, 0 \rangle + t \langle -\frac{17}{16}, -\frac{23}{4}, 1 \rangle$$

## 6.3 Quadratic Surfaces

### General form

The general form of a quadratic surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where  $A, B, C, D, E, F, G, H, I, J$  are constants.

### Ellipsoid

The general equation of an ellipsoid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where  $(h, k, l)$  is the center of the ellipsoid and  $a, b, c$  are the semi-axis lengths. If  $a = b = c$ , we get a sphere.

### Cone

The general equation of a cone that opens along the  $z$ -axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{(z-l)^2}{c^2}$$

where  $(h, k, l)$  is the center of the cone and  $a, b, c$  are the semi-axis lengths.

### Cylinder

The general equation of a cylinder that opens along the  $z$ -axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where  $(h, k)$  is the center of the cylinder and  $a, b$  are the semi-axis lengths. If  $a = b$ , we get a circular cylinder.

### Hyperboloid of One Sheet

The general equation of a hyperboloid of one sheet is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1$$

where  $(h, k, l)$  is the center of the hyperboloid and  $a, b, c$  are the semi-axis lengths.

### Hyperboloid of Two Sheets

The general equation of a hyperboloid of two sheets is

$$-\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

where  $(h, k, l)$  is the center of the hyperboloid and  $a, b, c$  are the semi-axis lengths.

### •Elliptic Paraboloid•

The general equation of an elliptic paraboloid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where  $(h, k, l)$  is the center of the paraboloid and  $a, b$  are the semi-axis lengths.

### •Hyperbolic Paraboloid•

The general equation of a hyperbolic paraboloid is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \frac{z-l}{c}$$

where  $(h, k, l)$  is the center of the paraboloid and  $a, b$  are the semi-axis lengths.

## 6.4 Calculus with Vector Functions

Let

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

### •Note:-•

$$\lim_{t \rightarrow a} \vec{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle$$

### •Note:-•

$$\frac{d}{dt} (\vec{u} + \vec{v}) = \vec{u}' + \vec{v}'$$

$$\frac{d}{dt} (c\vec{u}) = c\vec{u}'$$

$$\frac{d}{dt} (\vec{u} \cdot \vec{v}) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

$$\frac{d}{dt} (\vec{u} \times \vec{v}) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$$

$$\frac{d}{dt} (\vec{u} f(t)) = f'(t) \vec{u}'(f(t))$$

### •Note:-•

$$\int \vec{r}(t) dt = \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle$$

$$\int_a^b \vec{r}(t) dt = \left\langle \int_a^b f(t) dt + \int_a^b g(t) dt + \int_a^b h(t) dt \right\rangle$$

## 6.5 Tangent, Normal, and Binormal Vectors

### •Unit Tangent vector•

Given the vector function  $\vec{r}(t)$ , we call  $\vec{r}'(t)$  the **tangent vector** The unit tangent vector is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

### Unit Normal vector

If  $\vec{T}(t)$  is the unit tangent vector, then the **unit normal vector** is given by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

### Note:-

If  $\vec{r}(t)$  is a vector such that  $\|\vec{r}(t)\| = c$  for all  $t$ , then  $\vec{r}'(t)$  is orthogonal to  $\vec{r}(t)$

### Binormal vector

The **binormal vector** is given by

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The binormal vector is orthogonal to both the tangent and normal vectors.

## 6.6 Arc Length with Vector Functions

### Note:-

The arc length of a vector function  $\vec{r}(t)$  from  $t = a$  to  $t = b$  is given by

$$L = \int_a^b \|\vec{r}'(t)\| dt$$

Or,

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2} dt$$

## 6.7 Curvature

### Curvature of a curve in 3-D space

The curvature of a curve in 3-D space is given by

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

where  $\vec{T}(t)$  is the unit tangent vector and  $\vec{r}(t)$  is the position vector.

This can also be written as

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$