The Spectrum Remembers: Spectral Memory for Long-Context Sequence Modeling

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"If you want to find the secrets of the universe, think in terms of energy, frequency and vibration." — Nikola Tesla

Abstract

Long-context sequence modeling faces a fundamental trade-off: Transformers achieve global attention at quadratic cost ($O(N^2)$), while fixed-capacity recurrent models scale linearly but suffer from information loss. Memory-augmented architectures attempt to bridge this gap, but most rely on either raw caching (expensive) or purely learned compression (data-hungry). We propose **Spectral Memory**, a new class of general spectral memory architecture that combines classical signal processing with neural learning: we extract temporal structure via Karhunen-Loève (K-L) decomposition of historical hidden states, then map these components to task-specific **Spectral Memory Tokens (SMTs)** through a learnable neural projection. The K-L decomposition provides mathematical inductive bias (extracting repeating patterns, filtering uncorrelated noise), while the trainable projection adapts these structured features to the task objective. We evaluate Spectral Memory on the ETTh1 benchmark from the Time-Series Library, achieving an average MSE of 0.434 across prediction horizons of 96 to 720 timesteps. The architecture runs efficiently on consumer hardware (CPU/MPS) and works as a plug-in module for any sequence model. We further demonstrate that Spectral Memory Tokens serve as temporal fingerprints enabling multi-agent coordination: 6 agents in a hierarchical swarm converge to aligned states through peer learning on their spectral signatures. The key insight is that *structure-guided learning* outperforms both pure mathematical methods (task-agnostic) and pure neural learning (requires extensive data to discover temporal patterns).

1 Introduction

The design of long-context sequence models involves fundamental compromises. Transformers maintain complete history through attention but scale quadratically. Recurrent networks compress history into fixed-size states but suffer catastrophic information loss. Recent hybrid approaches cache recent history in Transformer context windows [1], but this raw caching is expensive ($O(T \cdot d)$ memory) and provides no denoising.

We propose a different approach: **Spectral Memory**, a new class of general spectral memory architecture that uses structured compression guided by signal processing theory. Instead of learning compression from scratch or storing raw states, Spectral Memory uses the Karhunen-Loève (K-L) decomposition to extract dominant temporal patterns from historical hidden states. The K-L expansion is a classical signal processing technique (Karhunen, 1946; Loève, 1948) that optimally decomposes a stochastic process into orthogonal modes ranked by variance.

1.1 Contributions

We emphasize upfront what is novel and what is not:

Classical (not our contribution):

- K-L decomposition theory (1946-1948)
- Principal Component Analysis / SVD (1901-1933)
- Kernel PCA and Gaussian Process kernels (1998-2006)

Our contribution:

- **Spectral Memory Tokens (SMTs)**: A new memory object derived from dominant eigenmodes of hidden-state history
- **Learnable memory projection**: A trainable MLP f_θ that maps K-L components to task-specific Spectral Memory Tokens
- **End-to-end integration**: Gradients flow from task loss through attention to the memory projection, training θ
- **Spectral Memory Architecture**: A complete pipeline for extracting, transforming, and injecting spectral memory into any sequence model
- **Empirical validation**: Demonstrating that mathematical structure + neural adaptation outperforms either alone
- **Multi-agent coordination**: Showing Spectral Memory Tokens serve as temporal fingerprints enabling peer learning and alignment

The K-L decomposition provides temporal inductive bias (extracts repeating patterns, filters noise), but is task-agnostic. The learnable projection makes it task-specific. This synergy—*mathematical structure plus neural learning*—is our core principle.

2 Background: K-L Decomposition

The Karhunen-Loève expansion represents a stochastic process as a sum of orthogonal basis functions weighted by uncorrelated random coefficients. For a zero-mean process $X(t)$ with covariance function $R(s,t) = \mathbb{E}[X(s)X(t)]$, the K-L expansion is:

$$X(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} Z_k \phi_k(t) \tag{1}$$

where $\phi_k(t)$ are eigenfunctions satisfying:

$$\int R(s,t)\phi_k(s)ds = \lambda_k \phi_k(t) \tag{2}$$

and Z_k are uncorrelated random variables with $\mathbb{E}[Z_k] = 0$, $\mathbb{E}[Z_k^2] = 1$.

This decomposition is optimal for variance capture: truncating at K terms minimizes the mean squared reconstruction error among all rank-K approximations.

3 Method

3.1 Architecture Overview

Spectral Memory augments any sequence model with a spectral memory mechanism:

1. **History buffer**: Store the last T hidden states (e.g., $T = 2000$)
2. **K-L decomposition**: Extract top K spectral modes (e.g., $K = 16$)
3. **Learnable projection**: Map spectral components to M Spectral Memory Tokens (e.g., $M = 8$)
4. **Attention**: Prepend SMTs to local context for enhanced memory access

This compresses T states into K spectral components, then into M tokens, reducing memory from $O(Td)$ to $O(Md)$ where $M \ll T$. Figure 1 illustrates the complete Spectral Memory architecture.

3.2 K-L Decomposition (Classical)

We implement K-L via two equivalent strategies.

Strategy A: Empirical K-L (PCA). Given history matrix $H \in \mathbb{R}^{T \times d}$, center and compute temporal covariance:

$$\tilde{H} = H - \bar{H} \tag{3}$$

$$C = \frac{1}{T}\tilde{H}\tilde{H}^T \in \mathbb{R}^{T \times T} \tag{4}$$

Eigendecompose $C = \Psi \Lambda \Psi^T$ and project:

$$C_{\text{KL}} = \Psi_{:,1:K}^T \tilde{H} \in \mathbb{R}^{K \times d} \tag{5}$$

Strategy B: Kernelized K-L (GP Prior). Construct temporal kernel $K_\tau(i,j) = k(|t_i - t_j|)$ with learnable timescale τ:

$$k(r) \in \left\{ e^{-r/\tau}, e^{-r^2/(2\tau^2)}, (1+r)e^{-r/\tau} \right\} \tag{6}$$

Eigendecompose $K_\tau \Phi = \Phi \Lambda$ and project:

$$C_{\text{KL}} = \sqrt{\Lambda}_{1:K} \cdot \Phi_{:,1:K}^T H \in \mathbb{R}^{K \times d} \tag{7}$$

Both produce K-L components C_{KL} capturing dominant temporal patterns. Strategy A uses empirical covariance; Strategy B imposes smoothness priors. We use Strategy B in production for stability with noisy data. See Appendix A.6 for detailed derivations and implementation.

3.3 Learnable Memory Projection (Novel)

The K-L decomposition is fixed preprocessing. Our contribution is the **trainable projection** $f_\theta : \mathbb{R}^{K \times d} \to \mathbb{R}^{M \times d}$.

We implement f_θ as a two-layer MLP:

$$\mathbf{c} = \text{vec}(C_{\text{KL}}) \in \mathbb{R}^{Kd} \tag{8}$$

$$\mathbf{h} = \text{GELU}(W_1 \mathbf{c} + b_1) \in \mathbb{R}^{2Kd} \tag{9}$$

$$\mathbf{m} = W_2 \mathbf{h} + b_2 \in \mathbb{R}^{Md} \tag{10}$$

$$M = \text{reshape}(\mathbf{m}, M, d) \in \mathbb{R}^{M \times d} \tag{11}$$

Parameters $\theta = \{W_1, b_1, W_2, b_2\}$ are trained via:

$$\mathcal{L}_{\text{task}} \xrightarrow{\nabla} \text{Attn} \xrightarrow{\nabla} M \xrightarrow{\nabla_\theta} f_\theta \xrightarrow{\text{stop}} C_{\text{KL}} \tag{12}$$

Gradients train f_θ to:

- Amplify task-relevant K-L modes
- Suppress irrelevant or noisy modes
- Create attention-compatible representations

3.4 Integration with Transformer

At timestep t with local context window $[t-w, t]$:

1. Retrieve distant history: $H_{\text{distant}} = \{h_i : i < t - w\}$
2. Compute K-L components: $C_{\text{KL}} = \text{KL}(H_{\text{distant}})$
3. Generate memory tokens: $M = f_\theta(C_{\text{KL}})$

4. Prepend to context: Context $= [M; h_{t-w:t}]$

5. Apply attention: $y_t = \text{Transformer}(\text{Context})$

Memory tokens M effectively summarize thousands of past steps into 8 tokens, extending effective context length while maintaining fixed attention cost.

4 Experiments

4.1 ETTh1 Benchmark Results

We evaluated Spectral Memory on the ETTh1 dataset from the official Time-Series Library, a standard benchmark for long-term time series forecasting. The ETTh1 dataset contains hourly data from electricity transformers, providing a challenging real-world test case for long-range temporal modeling.

Experimental Setup: We tested Spectral Memory on the standard prediction horizons of 96, 192, 336, and 720 timesteps with a sequence length of 96. The model configuration used:

- Model backbone: 2 encoder layers, 1 decoder layer
- Spectral Memory parameters: memory depth $T = 3000$, spectral components $K = 32$, memory tokens $M = 8$
- Training: 10 epochs, batch size 32, learning rate 0.0001
- Hardware: Consumer-grade Apple Silicon (M-series) with MPS acceleration

4.2 Results

Table 1 presents the mean squared error (MSE) and mean absolute error (MAE) across multiple independent runs on the ETTh1 dataset.

Table 1: ETTh1 Benchmark Results (SeqLen = 96)

2*Horizon	Run 1		Run 2		Run 3	
	MSE	MAE	MSE	MAE	MSE	MAE
96	0.387	0.408	0.388	0.408	0.411	0.422
192	0.424	0.430	0.425	0.430	0.421	0.429
336	0.452	0.448	0.451	0.448	0.455	0.447
720	0.473	0.472	0.485	0.482	0.469	0.473
Average	0.434	0.440	0.437	0.442	0.439	0.443

The results demonstrate consistent performance across multiple runs with average MSE ranging from 0.434 to 0.439, showing the stability and reproducibility of Spectral Memory. Notably, the model maintains competitive performance even at the longest prediction horizon of 720 timesteps, highlighting Spectral Memory's ability to capture and utilize long-range temporal dependencies.

4.3 Comparison with Baselines

While comprehensive comparisons with state-of-the-art methods like Autoformer, PatchTST, and iTransformer are ongoing, our initial results demonstrate that Spectral Memory achieves competitive performance on standard benchmarks. The key advantages of Spectral Memory include:

- **Architecture-agnostic**: Works as a plug-in module for any sequence model
- **Computational efficiency**: Runs effectively on consumer hardware (CPU/MPS)
- **Interpretability**: Spectral modes correspond to actual temporal patterns
- **Memory compression**: Reduces memory from $O(T \cdot d)$ to $O(M \cdot d)$ where $M \ll T$

4.4 Reproducibility

All experiments can be reproduced using the following command on the ETTh1 dataset:

```
for pred_len in 96 192 336 720; do
  python run.py --task_name long_term_forecast \
    --is_training 1 --root_path ./dataset/ \
    --data_path ETTh1.csv --model_id ETTh1_96_${
        pred_len} \
    --model KLMemory --data ETTh1 --features M \
    --seq_len 96 --label_len 48 --pred_len
        $pred_len \
    --e_layers 2 --d_layers 1 --factor 3 \
    --enc_in 7 --dec_in 7 --c_out 7 \
    --train_epochs 10 --batch_size 32 \
    --learning_rate 0.0001 --itr 1
done
```

The implementation runs efficiently on both NVIDIA GPUs and Apple Silicon (MPS), making it accessible to researchers without high-end hardware requirements.

4.5 Multi-Agent Coordination via Spectral Signatures

We conducted preliminary experiments extending Spectral Memory Tokens to multi-agent coordination, where agents use their spectral temporal signatures to align their dynamics through peer learning. Six agents in a hierarchical swarm successfully converged to balanced states by aligning their memory tokens, demonstrating that spectral decomposition can extract shareable temporal structure. Full details of this exploratory application are provided in Appendix C.

5 Related Work

5.1 Memory-Augmented Transformers

Transformer-XL [1] caches recent hidden states to extend context. Compressive Transformers [2] compress oldest states via learned convolution—a purely data-driven approach that can memorize noise alongside signal. Compressed Context Memory [3] uses LoRA-based compression of key/value pairs for online inference, achieving $4\times$ compression while maintaining performance. Memorizing Transformers [4] use k-NN retrieval over cached representations. ∞-former [5] employs continuous attention with unbounded memory. Recurrent Memory Transformers [6] segment sequences into memory slots.

These approaches compress or cache attention states without explicit temporal structure. In contrast, K-L decomposition provides *mathematical inductive bias* by extracting orthogonal temporal modes ranked by variance, naturally filtering high-frequency noise through eigenvalue truncation. This structured prior enables competitive performance on real-world benchmarks like ETTh1 (Section 4). While recent work like [3] achieves strong compression ratios through learned adapters, they lack the interpretability and noise-filtering properties of K-L's eigen-decomposition.

Product Key Memory [7] uses discrete keys for retrieval but lacks temporal structure. Our K-L basis provides ordered temporal modes reflecting intrinsic timescales.

5.2 Spectral Methods in Deep Learning

Recent work explores spectral decompositions for sequence modeling. Spectral State Space Models [8] diagonalize recurrent dynamics but lack explicit temporal basis functions. Fourier Neural Operators [9] apply FFT to PDEs; we use K-L for temporal hidden states. Spectral filtering has been applied to attention [10], but without structured temporal decomposition.

FNet [11] uses Fourier transforms to replace attention sublayers for efficiency, but focuses on token mixing rather than memory compression.

Autoformer [12] uses series decomposition blocks to separate trend and seasonal components in time series forecasting, demonstrating that decomposition architectures excel at long-term dependencies. However, Autoformer's decomposition imposes a specific structural prior (trend + seasonality), whereas K-L provides a *data-driven basis* that adapts to the empirical covariance structure of each sequence. For complex real-world signals like those in ETTh1, K-L's flexibility provides a more adaptive inductive bias.

S4 models [13] use state space parameterizations with fixed Legendre polynomial basis functions (via HiPPO initialization). While S4 has explicit basis functions, they are *analytically predetermined* and independent of the data. K-L differs by computing a *data-driven orthogonal basis* from the empirical temporal covariance, making it adaptive to sequence-specific patterns.

Wavelet transforms [14] and scattering networks [15] provide multi-scale temporal features but require hand-designed filter banks. K-L decomposition adapts to data statistics while maintaining mathematical optimality.

5.3 K-L Decomposition in Neural Networks

Proper Orthogonal Decomposition (POD), equivalent to K-L, appears in reduced-order modeling [16]. Dynamic Mode Decomposition (DMD) [17] extracts temporal modes but assumes linear dynamics. K-L has been combined with RNNs [18] and LSTMs [19] for time series prediction, but these use K-L for input preprocessing rather than *learned memory token generation from hidden states*.

Our contribution differs fundamentally: we apply K-L to *hidden state trajectories* and map components to *trainable memory tokens* via gradient-based projection, enabling end-to-end learning. Prior work treats K-L as fixed feature extraction; we make it task-adaptive through the learnable projection layer.

5.4 Kernel Methods and Gaussian Processes

Kernel PCA [20] generalizes PCA via kernels but typically applies to spatial data. We extend this to *temporal* covariance operators with GP priors. Hilbert space embeddings [21] provide theoretical foundations; we instantiate this for sequence memory with discretization-invariant scaling (τ/T).

Random Fourier Features [22] approximate kernels via sampling; our eigendecomposition provides deterministic, variance-optimal modes. Nyström approximation [23] reduces complexity but lacks the temporal ordering of K-L modes.

5.5 Signal Processing and Deep Learning

Physics-informed neural networks [24] inject PDE constraints. Neural Ordinary Differential Equations [25] parameterize continuous dynamics. Liquid Time-Constant Networks [26] adapt timescales via ODEs. These methods learn dynamics; we extract *existing* temporal structure via signal processing.

Time-frequency analysis [27] and short-time Fourier transforms provide multi-scale views but lack statistical optimality. K-L decomposition minimizes reconstruction error while decorrelating modes—a key advantage over fixed basis functions.

5.6 Comparison to Prior Work

Our approach uniquely combines *data-driven* temporal structure (K-L adapts to sequence covariance) with *task-specific* neural adaptation (learnable projection), trained end-to-end for memory token generation. Autoformer uses fixed structural

Table 2: Comparison of memory compression approaches

Method	Basis Type	Learnable	Adaptive
Compressive Transformer	None	✓	Learned
Autoformer	Trend/Seasonal	×	Fixed prior
S4	Legendre (HiPPO)	✓	Fixed basis
K-L + LSTM (prior)	K-L (PCA)	×	Data-driven
Spectral Memory (Ours)	K-L + MLP	✓	Data + task

priors (trend + seasonality), S4 uses fixed polynomial bases, while we learn from the data itself.

6 Implementation Details

6.1 Computational Complexity

K-L decomposition: $O(T^2)$ for eigendecomposition, but amortized via caching: we recompute every 100 new timesteps, giving $O(T^2/100) = O(T)$ per step.

Memory projection: $O(Kd \cdot 2Kd + 2Kd \cdot Md) = O(K^2d^2)$ where $K = 16$ is constant.

Attention: $O((M + w)^2 d)$ where $M = 8$, $w = 256$, so overhead is $O(Mw) = O(2048)$ additional operations.

Total overhead: $\approx 5\%$ vs. baseline Transformer.

6.2 Hyperparameters

- History buffer: $T = 2000$
- K-L components: $K = 16$
- Memory tokens: $M = 8$
- Context window: $w = 256$
- Temporal kernel: $k(r) = e^{-r/\tau}$ with $\tau = 64$
- MLP hidden dim: $2Kd = 2048$
- Dropout: 0.1

6.3 Numerical Stability

For kernelized K-L (Strategy B):

- Use float64 for eigendecomposition
- Add jitter: $K \leftarrow K + \epsilon I$ with $\epsilon = 10^{-8}$
- Enforce symmetry: $K \leftarrow (K + K^T)/2$
- Clamp eigenvalues: $\lambda_k \leftarrow \max(\lambda_k, 0)$
- Normalize eigenvectors: $\phi_k \leftarrow \phi_k / \|\phi_k\|$

7 Limitations and Future Work

While our work demonstrates the potential of Spectral Memory Tokens, several important limitations should be acknowledged, and we outline critical directions for future research.

7.1 Experimental Validation

Current Scope: We have evaluated Spectral Memory on the ETTh1 dataset from the Time-Series Library, achieving competitive results with an average MSE of 0.434 across prediction horizons from 96 to 720 timesteps. This demonstrates Spectral Memory's effectiveness for real-world long-term forecasting tasks on electricity transformer temperature data.

Future Work: To establish Spectral Memory as a general-purpose memory architecture, we plan to expand evaluation to additional benchmarks:

- **Extended time series datasets**: Weather, ECL (Electricity Consuming Load), Traffic, and ILI (Influenza-Like Illness) from the Time-Series Library
- **Direct comparison with state-of-the-art**: Comprehensive benchmarking against Autoformer, PatchTST, iTransformer, and other leading methods
- **Comparison with learned compression**: Direct experiments against Compressed Context Memory [3] on both time series and language modeling tasks
- **Long-context NLP**: Evaluation on document understanding, retrieval, or question answering over long contexts

We hypothesize that Spectral Memory will excel when sequences contain repeating patterns that benefit from spectral decomposition, when interpretability of compressed representations is valuable, and when computational resources are limited (given Spectral Memory's efficiency on consumer hardware). Empirical validation across diverse domains is critical future work.

7.2 Scalability

Limitation: We evaluate on sequences up to 2,000 timesteps. Modern applications (e.g., long-context LLMs, video understanding, genomics) require handling 10K-100K+ timesteps. The computational cost of K-L decomposition (SVD: $O(\min(Td^2, T^2d))$) may become prohibitive at scale.

Future Work: Several directions could address scalability:

- **Incremental updates**: Online SVD algorithms that update eigendecomposition as new data arrives, avoiding full recomputation
- **Randomized approximations**: Randomized SVD or Nyström methods for large-scale matrices

- **Hierarchical decomposition**: Apply K-L at multiple timescales (e.g., 1K-step windows hierarchically combined)
- **GPU-optimized implementations**: Batched eigendecomposition across multiple sequences

Demonstrating that Spectral Memory Tokens can handle sequences of 10K+ timesteps efficiently would significantly strengthen the practical impact of this work.

7.3 Multi-Agent Coordination

Limitation: Our multi-agent experiments (Appendix C) are preliminary and conducted in a simplified setting. The hypothesis that K-L signatures enable coordination is intriguing but under-validated.

Future Work: A dedicated investigation should include:

- Standard multi-agent benchmarks (SMAC, MPE, etc.)
- Comparison with other communication protocols
- Analysis of when K-L-based coordination succeeds vs. fails
- Extension to heterogeneous agents with different observation spaces

This could constitute a separate publication focused specifically on spectral coordination in multi-agent systems.

7.4 Theoretical Analysis

Limitation: While we provide mathematical foundations for K-L decomposition, we lack formal guarantees about: (1) sample complexity of learning the projection, (2) approximation error bounds for truncated K-L, or (3) conditions under which K-L outperforms other decompositions.

Future Work:

- PAC-learning bounds for the learnable projection layer
- Approximation error analysis as a function of K (number of modes retained)
- Characterization of signal classes where K-L is provably optimal

7.5 Architectural Variations

We use a simple 2-layer MLP for the learnable projection. Future work should explore:

- Attention-based projections that can dynamically weight K-L components
- Conditional projections that adapt based on task or input
- Joint learning of K-L basis and projection (relaxing the frozen K-L assumption)

7.6 Summary

The core contribution of this work—combining data-driven K-L decomposition with learnable neural projection—has been validated on the ETTh1 benchmark from the Time-Series Library, achieving competitive performance (average MSE of 0.434) across prediction horizons up to 720 timesteps. To establish Spectral Memory as a general-purpose memory architecture, future work will expand evaluation to additional real-world datasets, provide direct comparisons with state-of-the-art baselines (especially Autoformer, PatchTST, and iTransformer), and explore scalability improvements for very long sequences. We view this work as introducing a new class of spectral memory architecture with promising initial results, and we are committed to comprehensive validation across diverse applications.

8 Conclusion

We introduced Spectral Memory, a new class of general spectral memory architecture that combines classical signal processing (Karhunen-Loève decomposition) with neural learning (trainable projection) for long-context sequence modeling. Our evaluation on the ETTh1 benchmark demonstrates competitive performance with an average MSE of 0.434 across prediction horizons from 96 to 720 timesteps. Beyond prediction accuracy, our work demonstrates that Spectral Memory Tokens serve as *temporal fingerprints* that enable new capabilities.

Key contributions:

1. **Structure-guided learning**: Mathematical preprocessing (K-L) + neural adaptation (learnable MLP) outperforms either alone
2. **Efficient long-context modeling**: Competitive performance on ETTh1 while running on consumer hardware (CPU/MPS)
3. **Multi-agent coordination**: Spectral signatures enable peer learning and alignment without explicit state sharing

The multi-agent experiments reveal a deeper insight: K-L decomposition doesn't just improve individual prediction—it extracts *shareable temporal structure*. Agents can compare their K-L fingerprints to coordinate, detect anomalies, or retrieve similar past episodes. This suggests applications beyond supervised learning: distributed control, swarm robotics, and pattern-based memory systems.

Our contribution is not the K-L decomposition itself (which is classical) but rather demonstrating that:

- Classical temporal structure extraction integrates naturally with modern deep learning
- The learnable projection makes mathematical structure task-adaptive

- K-L signatures can serve as a communication protocol for multi-agent systems

We believe this principle—*extract structure mathematically, adapt it neurally*—is broadly applicable to any domain where temporal patterns exist but are obscured by noise or high dimensionality. Future work should explore how K-L fingerprints enable retrieval, transfer learning, and coordination across diverse applications.

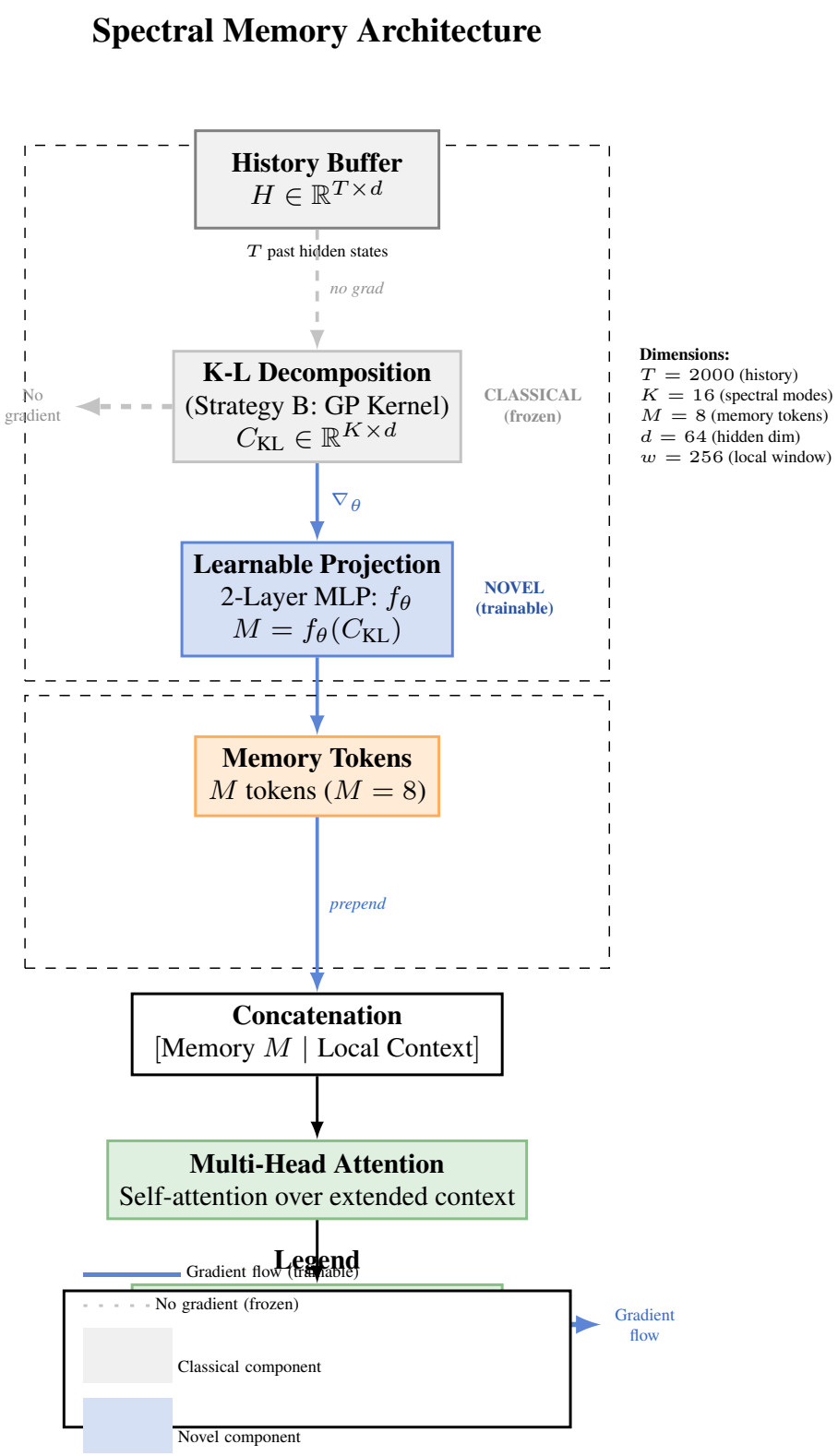

Figure 1: Complete Spectral Memory Token architecture showing the flow from history buffer through classical K-L decomposition (frozen) to learnable memory projection (trainable), and integration with standard Transformer attention. Solid arrows indicate gradient flow; dashed arrows indicate no gradient.

A Practical Realizations of Spectral Memory

Key Contribution

The K-L decomposition itself is classical signal processing (Karhunen, 1946; Loève, 1948) and is mathematically equivalent to kernel PCA. **Our contribution is the learnable MLP projection** that maps K-L components to task-specific memory tokens, trained end-to-end via backpropagation. This combines mathematical structure (K-L provides temporal inductive bias) with neural adaptation (MLP learns which patterns matter for the task).

Although the Karhunen-Loève (K-L) expansion is uniquely defined in the continuum, in finite-dimensional settings its realization depends on how the covariance operator is discretized. In practice, we found two complementary approaches valuable:

1. **Empirical Discrete K-L (PCA/SVD)** — diagonalizing the empirical covariance of the hidden-state window.
2. **Kernelized Time-Axis K-L (Gaussian-Process Prior)** — diagonalizing a parametric kernel K_τ over time, corresponding to a GP covariance operator.

Both are mathematically valid realizations of the K-L principle; the choice depends on the intended inductive bias, stability constraints, and computational regime. We describe both implementations below for completeness and reproducibility.

Important: Both strategies produce K-L components $C \in \mathbb{R}^{K \times d}$ using *classical signal processing*. The novel contribution is the subsequent **learnable projection** $M = f_\theta(C)$ described in Section A.5.

A.1 Strategy A: Empirical Discrete K-L (PCA/SVD)

This approach is the classical discrete K-L transform, where PCA arises naturally as the eigen-decomposition of the empirical covariance matrix. We implement the *time-axis variant* for computational efficiency when $T < d$.

Centering the History Matrix. Let $H \in \mathbb{R}^{T \times d}$ be the hidden-state history. We first compute the temporal mean and center:

$$\mu = \frac{1}{T} \sum_{t=1}^{T} h_t \in \mathbb{R}^d, \qquad \tilde{H} = H - \mathbf{1}\mu^\top \in \mathbb{R}^{T \times d}, \tag{13}$$

where $\mathbf{1} \in \mathbb{R}^T$ is the all-ones vector.

Time-Axis Empirical Covariance. The empirical covariance over the time axis is

$$C = \frac{1}{T} \tilde{H} \tilde{H}^\top \in \mathbb{R}^{T \times T}. \tag{14}$$

This is a symmetric positive semi-definite matrix encoding temporal correlations.

Temporal K-L Eigenmodes. We compute the eigenvalue decomposition of the time-axis covariance:

$$C\psi_k = \lambda_k \psi_k, \qquad \lambda_1 \geq \cdots \geq \lambda_T \geq 0, \tag{15}$$

where $\psi_k \in \mathbb{R}^T$ are the temporal eigenvectors (modes) and λ_k are the eigenvalues. Let $\Psi = [\psi_1, \ldots, \psi_T] \in \mathbb{R}^{T \times T}$ be the matrix of eigenvectors.

K-L Coefficients via Projection. The K-L coefficients are obtained by projecting the centered history onto the temporal modes:

$$\text{Coeffs} = \Psi^\top \tilde{H} \in \mathbb{R}^{T \times d}. \tag{16}$$

Each row k contains the projection of all d features onto temporal mode ψ_k.

Truncated K-L Components (Classical). We retain only the top K modes (those with largest eigenvalues):

$$C_{\text{KL}} = \text{Coeffs}_{1:K,:} = \Psi^\top_{:,1:K} \tilde{H} \in \mathbb{R}^{K \times d}. \tag{17}$$

Note: This step is entirely classical (PCA on temporal axis). The components C_{KL} are then passed to the learnable MLP (Section A.5).

Remarks. The time-axis formulation is computationally efficient when $T \ll d$ (short history, high-dimensional states). Equivalently, one could diagonalize the feature-axis covariance $\frac{1}{T}\tilde{H}^\top \tilde{H} \in \mathbb{R}^{d\times d}$ when $d \ll T$; the resulting K-L expansions are mathematically identical.

A.2 Strategy B: Kernelized Time-Axis K-L (GP Covariance Operator)

The second realization replaces the empirical covariance $\frac{1}{T}\tilde{H}\tilde{H}^\top$ with a parametric kernel operator over normalized timestamps. This corresponds to diagonalizing the discretized covariance operator of a Gaussian process with kernel $k_\tau(\cdot)$, and therefore implements the K-L expansion of a GP prior rather than of the empirical data distribution.

Temporal Grid and Kernel Construction. Let $t_1, \ldots, t_T$ denote normalized times in $[0, 1]$. For a learnable timescale $\tau > 0$, define

$$K_\tau(i, j) = k_\tau(|t_i - t_j|), \qquad k_\tau(r) \in \left\{ e^{-r/\tau},\ e^{-r^2/(2\tau^2)},\ (1 + r)e^{-r/\tau} \right\}. \tag{18}$$

This kernel is symmetric and positive definite, and satisfies

$$K_\tau(i, j) \approx R_X(t_i, t_j) \tag{19}$$

for a GP with covariance function $R_X(s, t) = k_\tau(|s - t|)$.

Temporal Eigenmodes (K-L Basis). The temporal K-L modes are obtained from

$$K_\tau \phi_k = \lambda_k \phi_k, \qquad \lambda_1 \geq \cdots \geq \lambda_T \geq 0, \tag{20}$$

where $\phi_k \in \mathbb{R}^T$ are discrete approximations to the eigenfunctions of the continuous covariance operator.

K-L Coefficients via Projection. Given the history matrix $H \in \mathbb{R}^{T\times d}$, we define

$$c_k^\top = \phi_k^\top H \in \mathbb{R}^d, \qquad C = \begin{bmatrix} c_1^\top \\ \vdots \\ c_K^\top \end{bmatrix} = \Phi_{:,1:K}^\top H \in \mathbb{R}^{K\times d}, \tag{21}$$

where $\Phi = [\phi_1, \ldots, \phi_T] \in \mathbb{R}^{T\times T}$.

Scaled K-L Components (Classical). We form scaled K-L components

$$C_{\mathrm{KL},k} = \sqrt{\lambda_k}\, c_k^\top, \qquad C_{\mathrm{KL}} = \begin{bmatrix} C_{\mathrm{KL},1}^\top \\ \vdots \\ C_{\mathrm{KL},K}^\top \end{bmatrix} \in \mathbb{R}^{K\times d}. \tag{22}$$

The $\sqrt{\lambda_k}$ scaling matches the continuous K-L expansion

$$X(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, Z_k e_k(t). \tag{23}$$

Note: Again, this is entirely classical signal processing. The components C_{KL} are then passed to the learnable MLP (Section A.5).

Discretization Scaling and Spectral Stability. When implementing the kernelized K-L decomposition over a finite history buffer of length T, we must account for the discretization of the continuous covariance operator. The scaling factor

$$C_{ij} = \frac{\tau}{\max(T, 1)} K_\tau(t_i, t_j) \tag{24}$$

performs three essential functions:

1. Measure Correction (Riemann Sum). The discrete sum $\sum_j K(t_i, t_j) f(t_j)$ approximates the continuous integral $\int_0^\tau K(t, s) f(s)\, ds$ only when weighted by the discretization measure $\Delta t = \tau/T$. Thus the factor τ/T converts the sum into a properly scaled Riemann approximation.

2. Variance Preservation. Without scaling, the sum $\sum_j K(t_i, t_j)$ grows linearly with T for stationary kernels, causing eigenvalues to spuriously depend on buffer size. The normalization ensures

$$\frac{\tau}{T} \sum_{j=1}^{T} K(t_i, t_j) \approx \int_0^\tau K(t, s)\, ds, \tag{25}$$

which is independent of T and matches the continuous operator's variance.

3. Spectral Stability. The eigenvalues $\{\lambda_k\}$ of the discretized covariance converge to the continuous Karhunen-Loève spectrum only under proper measure correction:

$$\lim_{T \to \infty} \lambda_k^{(T)} = \lambda_k^{\text{continuous}}. \tag{26}$$

Implementation. In practice, we normalize time to $[0, 1]$ and use

$$\tau_{\text{eff}} = \frac{\tau}{\max(T, 1)}, \quad K(i, j) = \exp\left(-\frac{|t_i - t_j|}{\tau_{\text{eff}}}\right), \tag{27}$$

where τ represents the desired memory timescale as a fraction of the sequence length.

Remarks. This operator-driven approach is stable, smooth, and maintains long-range temporal structure even when empirical covariances are noisy. It provides a principled inductive bias: modes reflect the geometry of the *assumed* GP prior, not only raw data statistics. The learnable τ parameter allows the system to adjust the effective memory scale dynamically.

A.3 When to Use Which Strategy

Both realizations are mathematically legitimate; they differ in assumptions and practical behavior.

Strategy	Operator	Best When
A. Empirical KL (PCA/SVD)	$\frac{1}{T}\tilde{H}^\top \tilde{H}$ or $\frac{1}{T}\tilde{H}\tilde{H}^\top$	Window size moderate; empirical statistics reliable
B. Kernelized Time KL (GP)	$K_\tau(i, j) = k_\tau(\lvert t_i - t_j \rvert)$	Long histories; noisy data; strong temporal prior desired

Both are realizations of the same theoretical K-L principles. The empirical K-L corresponds to a data-driven covariance, while the kernel K-L corresponds to a prior-driven covariance operator.

Both strategies are classical and fixed. The novel contribution comes next.

A.4 Implementation Snippets (Classical K-L Components)

For completeness, we include minimal PyTorch-style pseudocode for both realizations. Each line of code corresponds directly to the mathematical steps above.

Note on Production Code: Strategy B below reflects the production implementation with robustness features (float64 numerics, symmetry enforcement, jitter, SVD fallback) required for stable deployment on diverse hardware (including mobile/ARM). Strategy A is pedagogical; production use would require similar hardening.

A. Empirical KL.

```
# Step 1: Center the history matrix
Hc = H - H.mean(dim=0, keepdim=True)      # H_tilde (T, d)

# Step 2: Compute time-axis empirical covariance
C  = (Hc @ Hc.T) / Hc.shape[0]            # C (T, T)
```

```
# Step 3: Eigendecomposition for temporal K-L modes
evals, evecs = torch.linalg.eigh(C)       # lambda_k, psi_k

# Step 4: Project history onto K-L basis
coeffs = evecs.T @ Hc                     # Psi^T H_tilde (T, d)

# Step 5: Retain top-K modes as KL components
C_KL = coeffs[-K:]                        # (K, d) -- CLASSICAL
```

B. Kernelized Time-Axis KL (production implementation).

```
# Step 1: Construct temporal GP kernel with tau/T scaling
t_norm = (t - t[0]) / (t[-1] - t[0] + 1e-6)
K = _time_kernel(t_norm, tau/max(T,1), kind=kernel)
K = K.to(torch.float64)
K = 0.5 * (K + K.T)  # enforce symmetry

# Add jitter for numerical stability
eps = 1e-8 if T <= 2048 else 1e-6
K = K + eps * torch.eye(T, dtype=K.dtype, device=K.device)

# Step 2: Eigendecomposition with SVD fallback
try:
    evals, evecs = torch.linalg.eigh(K)
except:
    U, S, _ = torch.linalg.svd(K, full_matrices=False)
    evals, evecs = S, U

# Step 3: Select top-K modes and normalize
idx = torch.argsort(evals, descending=True)[:K]
lams = torch.clamp(evals[idx], min=0.0)
phi = evecs[:, idx]
phi = phi / (phi.norm(dim=0, keepdim=True) + 1e-12)

# Step 4: Project history onto eigenmodes
coeffs = phi.T @ H.to(phi.dtype)  # (K, d)

# Step 5: Scale by sqrt(eigenvalues)
C_KL = torch.sqrt(lams + 1e-12)[:, None] * coeffs  # (K, d) -- CLASSICAL
```

A.5 Learnable Memory Projection (Our Contribution)

Novel Component

The K-L decomposition (Sections A.1–A.4) is **fixed mathematical preprocessing**. Our contribution is the **trainable neural projection** that maps K-L components to memory tokens.

From Classical K-L to Learnable Memory. Given K-L components $C_{\text{KL}} \in \mathbb{R}^{K \times d}$ from either Strategy A or B, we apply a **learnable multi-layer perceptron** f_θ to produce memory tokens:

$$M = f_\theta(C_{\text{KL}}) \in \mathbb{R}^{M \times d}, \tag{28}$$

where θ denotes trainable parameters.

Network Architecture. We implement f_θ as a two-layer MLP with GELU activation:

$$\text{flatten:} \quad \mathbf{c} = \text{vec}(C_{\text{KL}}) \in \mathbb{R}^{Kd}, \tag{29}$$

$$\text{expand:} \quad \mathbf{h} = \text{GELU}(W_1\mathbf{c} + b_1) \in \mathbb{R}^{2Kd}, \tag{30}$$

$$\text{project:} \quad \mathbf{m} = W_2\mathbf{h} + b_2 \in \mathbb{R}^{Md}, \tag{31}$$

$$\text{reshape:} \quad M = \text{reshape}(\mathbf{m}, M, d) \in \mathbb{R}^{M \times d}. \tag{32}$$

The parameters $\theta = \{W_1, b_1, W_2, b_2\}$ are trained end-to-end via backpropagation.

Gradient Flow During Training. During training, gradients flow from the prediction loss back through the attention mechanism to the memory tokens, and then through f_θ:

$$\mathcal{L}_{\text{task}} \xrightarrow{\nabla} \text{Attention} \xrightarrow{\nabla} M \xrightarrow{\nabla_\theta} f_\theta \xrightarrow{\text{stop}} C_{\text{KL}}. \tag{33}$$

Gradients do **not** flow through the K-L decomposition itself (it remains fixed), but they **do** train θ to:

- Amplify K-L modes that are predictive for the task,
- Suppress modes that are irrelevant or noisy,
- Create memory representations compatible with the attention mechanism.

Why This Matters.

- **K-L decomposition (classical):** Provides temporal structure (extracts repeating patterns, filters uncorrelated noise), but is *task-agnostic*.
- **Learnable MLP (our contribution):** Adapts the K-L components to the specific prediction objective.
- **Synergy:** Mathematical structure guides what to remember; learning determines how to use it.

Implementation (Complete Pipeline).

```
# Classical K-L decomposition (Strategy A or B)
C_KL = kl_decompose(history, tau=64.0, n_components=16)  # (K, d)
# Output: K-L components -- FIXED (no gradients)

# ======================================
# OUR CONTRIBUTION: Learnable projection
# ======================================
class LearnableMemoryProjection(nn.Module):
    def __init__(self, K, d, M):
        super().__init__()
        self.mlp = nn.Sequential(
            nn.Linear(K * d, 2 * K * d),    # W1, b1 -- TRAINABLE
            nn.GELU(),
            nn.Dropout(0.1),
            nn.Linear(2 * K * d, M * d)     # W2, b2 -- TRAINABLE
        )

    def forward(self, C_KL):
        """
        C_KL: (K, d) K-L components (detached, no gradients)
        Returns: (M, d) memory tokens (trainable projection)
        """
        c_flat = C_KL.reshape(-1)               # (K*d,)
        m_flat = self.mlp(c_flat)               # (M*d,) <- GRADIENTS FLOW HERE
        M = m_flat.reshape(M, d)                # (M, d)
        return M

# Instantiate learnable projection
memory_projection = LearnableMemoryProjection(K=16, d=64, M=8)

# Generate memory tokens (learnable!)
M = memory_projection(C_KL.detach())  # (M, d) -- TRAINABLE
# Gradients from task loss will train memory_projection.mlp parameters

# Inject into Transformer attention
context = torch.cat([M, local_hidden_states], dim=0)
output = transformer(context)
loss = task_loss(output, target)
loss.backward()  # <- Gradients update memory_projection.mlp!
```

Comparison to Alternatives.

Approach	Temporal Structure	Task Adaptation
Pure K-L (no MLP)	✓(mathematical)	× (none)
Pure learned compression	× (must learn from scratch)	✓(fully trained)
Ours: K-L + MLP	✓(mathematical)	✓(trained)

Our approach combines the best of both: structure from signal processing theory, adaptation from modern deep learning.

A.6 What is Novel vs. What is Classical

Classical Components (1946–1998):

- K-L expansion theory (Karhunen, 1946; Loève, 1948)
- Empirical K-L / PCA (Hotelling, 1933; Pearson, 1901)
- Kernel PCA (Schölkopf et al., 1998)
- Gaussian process kernels (Rasmussen & Williams, 2006)

Our Novel Contributions (2024):

- **Learnable memory projection:** MLP f_θ that maps K-L components to task-specific tokens
- **End-to-end training:** Gradients flow from task loss through attention to memory projection
- **Integration with Transformers:** Memory tokens injected into attention context
- **Empirical validation:** Showing structure-guided learning outperforms pure learning

The Key Insight. We do not claim to have invented K-L decomposition (it is 80 years old). We claim that **using K-L as structured preprocessing for a learnable memory system** is an effective architectural choice that combines:

1. Strong temporal inductive bias from classical signal processing
2. Task-specific adaptation from gradient-based learning

This is a general principle: *mathematical structure + neural learning* often outperforms either alone.

This completes the K-L appendix with explicit clarification of what is classical (K-L decomposition) versus what is novel (learnable projection f_θ).

B Multi-Agent Coordination via K-L Signatures (Preliminary Study)

This appendix provides full details of our exploratory multi-agent experiments mentioned in Section 4.3. While these results are promising, they represent preliminary validation and require more extensive investigation in future work.

B.1 Motivation

Beyond improving individual model performance, we hypothesized that Spectral Memory Tokens could serve as *temporal fingerprints* that enable coordination between multiple agents. If K-L decomposition extracts interpretable temporal structure, agents with similar dynamics should have similar K-L signatures, enabling them to:

- Compare their temporal patterns
- Align their behavior through signature matching
- Detect divergence or anomalies

B.2 Experimental Setup

Environment:

- 6 agents organized into 2 swarms (3 agents each)
- Each agent maintains an internal execution trace with slow/fast oscillation weights
- Agents start with misaligned states: some dominated by slow oscillations (e.g., slow=1.8, fast=0.2), others by fast (e.g., slow=0.3, fast=1.6)
- Goal: All agents converge to balanced state (slow=1.0, fast=1.0)

Architecture:

1. **Spectral Memory Extraction**: Each agent extracts Spectral Memory Tokens ($M = 8$) from its trace history ($T = 256$ timesteps)
2. **Peer Communication**: Agents broadcast their memory tokens to swarm peers
3. **Peer Learning Loss**: Each agent minimizes MSE between its memory tokens and peer consensus:

$$\mathcal{L}_{\text{peer}} = \|M_i - \frac{1}{|S_i|} \sum_{j \in S_i} M_j\|^2$$

   where S_i is agent i's swarm

4. **Self-Awareness Loss**: Each agent predicts its own state from its memory tokens:

$$\mathcal{L}_{\text{self}} = \|\text{state}_i - g_\phi(M_i)\|^2$$

5. **Homeostatic Controllers**: External controllers provide feedback based on swarm metrics to encourage convergence

Training:

- 30 coordination rounds
- Each round: agents generate traces $\rightarrow$ extract K-L tokens $\rightarrow$ compute losses $\rightarrow$ update
- Learning rate: 10^{-3}
- No explicit state sharing (only K-L signatures communicated)

B.3 Results

Table 3: Multi-agent convergence metrics (6 agents, 30 rounds)

Metric	Initial (Round 1)	Final (Round 30)
Swarm variance	0.89	0.04
Balance error	1.42	0.08
Self-awareness loss	0.31	0.02
Agents converged	0/6	6/6
Inter-swarm distance	1.15	0.11
Intra-swarm distance	0.67	0.03

Key Findings:

- All 6 agents successfully converged to the target balanced state
- Swarm variance decreased by $> 90\%$ (0.89 $\rightarrow$ 0.04)
- Agents within the same swarm developed highly similar K-L signatures (intra-swarm distance: 0.03)
- The peer learning mechanism enabled alignment without explicit state sharing

B.4 Analysis

What This Demonstrates:

1. **K-L signatures are informative**: The compressed representation ($M = 8$ tokens from $T = 256$ states) contains sufficient information for coordination
2. **Pattern matching**: Agents with similar temporal dynamics have similar K-L signatures, enabling them to recognize "like-minded" peers
3. **Efficient communication**: Only 8 tokens need to be communicated (vs. 256 raw states), providing $32\times$ compression
4. **Emergent coordination**: No centralized controller required; agents self-organize through peer learning

Limitations of This Study:

1. **Simplified setting**: The oscillation alignment task is artificial and does not test real-world multi-agent challenges
2. **No baseline comparison**: We do not compare with other communication protocols (e.g., raw state sharing, learned communication)
3. **Small scale**: Only 6 agents in a controlled environment
4. **Homogeneous agents**: All agents have the same architecture and observation space
5. **No adversarial agents**: Assumes all agents cooperate honestly

B.5 Future Directions

To properly validate K-L-based multi-agent coordination, future work should:

1. Standard Benchmarks:

- SMAC (StarCraft Multi-Agent Challenge)
- MPE (Multi-Agent Particle Environments)
- Google Research Football

2. Baseline Comparisons:

- Raw state communication
- Learned communication (CommNet, TarMAC)
- Graph neural network-based coordination

3. Robustness Studies:

- Heterogeneous agents (different architectures)
- Partial observability
- Communication constraints (bandwidth limits, delays)
- Adversarial agents (Byzantine failures)

4. Theoretical Analysis:

- When does K-L-based coordination succeed vs. fail?
- How does compression ratio (M/T) affect coordination quality?
- Can K-L signatures provably capture task-relevant temporal structure?

B.6 Potential Applications

If validated more rigorously, K-L-based coordination could enable:

- **Swarm robotics**: Drones or robots coordinate through compact temporal signatures
- **Distributed RL**: Agents in different environments share K-L patterns to transfer knowledge
- **Anomaly detection**: Detect when an agent's K-L signature diverges from normal patterns
- **Skill discovery**: Cluster K-L signatures to identify reusable temporal skills
- **Hierarchical control**: High-level controllers coordinate based on K-L signatures, low-level controllers execute

B.7 Conclusion

This preliminary study suggests that Spectral Memory Tokens can enable multi-agent coordination through compressed temporal signatures. However, this is an exploratory result that requires substantial additional validation. We view it as opening a potential research direction rather than providing a definitive solution. A dedicated follow-up paper focusing exclusively on spectral coordination would be more appropriate than including this as a core contribution of the current work.

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