Multirobot Cooperative Localization

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For state s, $\hat{s}$ stands for the estimate of s, and $\tilde{s}$ represents the error between s and $\tilde{s}$, with $\tilde{s} = \hat{s} - s$. The boldface is preserved for random variables, and capital notation is for matrix. The time index of all variables is shown in superscript.

$C(\theta)$	rotation matrix with angle θ
d_{ij}	relative distance measurement of robot j by robot i
H	observation matrix of the system
I_n	$n \times n$ identity matrix
N	number of robots
s	state of all the parameters in the system
s_r	state of all the parameters in the robots
$s_{r,i}$	state of robot i
u_i^t	control input of robot i at time t
v_i^t	speed of robot i at time t
$\mathbf{w}_{z,ij}$	relative observation noise in z_{ij}
z_{ij}	relative observation taken by robot i to robot j
ϕ_{ij}	relative bearing measurement of robot j by robot i
ω_i^t	angular velocity of robot i at time t

1 General System Model

We consider a system with N robots and M landmarks. The state of all robots is denoted by

$$s_r = \begin{bmatrix} s_{r,1} \\ s_{r,2} \\ \vdots \\ s_{r,N} \end{bmatrix}, \quad s_{r,i} = \begin{bmatrix} x_{r,i} \\ y_{r,i} \\ \theta_{r,i} \end{bmatrix}. \tag{1}$$

The states of landmarks are simply the locations, which can be expressed as

$$s_l = \begin{bmatrix} s_{l,1} \\ s_{l,2} \\ \vdots \\ s_{l,M} \end{bmatrix}, \quad s_{l,i} = \begin{bmatrix} x_{l,i} \\ y_{l,i} \end{bmatrix}. \tag{2}$$

1.1 Propagation Model

The motion model captures the state variation over a time interval. For simplicity, we use discrete index t for time, and the duration between consecutive time index is δt. In reality, the time interval may not be a constant, but we leave this issue for later discussion. The propagation model for robots can be modeled as

$$s_{r,i}^{t+1} = f(s_{r,i}^t, u_{r,i}^t + \mathbf{w}_u^t), \tag{3}$$

where $u_{r,i}^t$ is the control input and $\mathbf{w}_{r,i}^t$ is the propagation disturbance. The propagation model f is assumed to be invariant with time and identical for all robots. The landmarks are assumed stationary, with $s_l^{t+1} = s_l^t$ for all t.

In common scenario, the control input can model the odometry input, such as translational velocity v and angular velocity ω.

$$u_{r,i}^t = \begin{bmatrix} v_{r,i}^t \\ \omega_{r,i}^t \end{bmatrix}. \tag{4}$$

The disturbance of control input is captured by $\mathbf{w}_u^t$.[1] In this model, we have the explicit propagation model for robots as

$$s_{r,i}^{t+1} = \begin{bmatrix} x_{r,i}^{t+1} \\ y_{r,i}^{t+1} \\ \theta_{r,i}^{t+1} \end{bmatrix} = f(s_{r,i}^t, u_{r,i}^t + \mathbf{w}_u^t) = \begin{bmatrix} x_{r,i}^t + \cos(\theta_{r,i}^t)\,(v_{r,i}^t + \mathbf{w}_v^t)\,\delta t \\ y_{r,i}^t + \sin(\theta_{r,i}^t)\,(v_{r,i}^t + \mathbf{w}_v^t)\,\delta t \\ \theta_{r,i}^t + (\omega_{r,i}^t + \mathbf{w}_\omega^t)\,\delta t \end{bmatrix}. \tag{5}$$

1.2 Observation Model

Robot i can observe either landmark or other robots to make the localization more accurate. The observation result is denoted by $z_{i,j}^t$ where i is the observer and j is the observed robot or landmark. We can relate those terms by an observation model as

$$z_{i,j} = h(s_i, s_j) + \mathbf{w}_{o,i}. \tag{6}$$

Since the observation does not involve time elapse, we omit the time index. The random term stands for the observation noise, and it is determined by the reliability of the sensor in the observer robot i. Therefore, a subscript i is added to emphasize the concern.

1.3 Discussion

SLAM and localization The model is general enough to comprise two main categories in multi-robot state estimation problem: SLAM and cooperative localization. The main difference is whether the state of landmark s_l is known in advance. In SLAM, s_l is the part of the parameter to estimate, while s_l serves a source of absolute state information in localization problem.

centralized or distributive

2 Extended Kalman Filter Estimation

The Extended Kalman filter (EKF) estimation resides on the assumption that the noise term and the initial state term is Gaussian distribution. Therefore, the covariance matrix Σ of all estimates should be maintained and updated with each action.

Due to the inconsistency issue in linearization, we consider only position term with

$$s_{r,i} = \begin{bmatrix} x_{r,i} \\ y_{r,i} \end{bmatrix}. \tag{7}$$

In addition, the variance of orientation estimate ${\sigma_\theta}^2$ is bounded.

There are three steps in analysis:

1. estimation update
2. error propagation
3. covariance update

[1] The disturbance $\mathbf{w}_u^t$ can be modeled either as that of control input or that of state.

2.1 Position Propagation

We assume that the time elapse between every step is δt. The position propagation model gives

$$\hat{s}_i^{t+1} = \begin{bmatrix} \hat{x}_i^{t+1} \\ \hat{y}_i^{t+1} \end{bmatrix} = f(\hat{s}_i^t, u_i^t) = \begin{bmatrix} \hat{x}_i^t + \cos(\hat{\theta}_i^t) v_i^t \delta t \\ \hat{y}_i^t + \sin(\hat{\theta}_i^t) v_i^t \delta t \end{bmatrix}. \tag{8}$$

By linearizing (8), the error propagation is derived as

$$\begin{bmatrix} \tilde{x}_i^{t+1} \\ \tilde{y}_i^{t+1} \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_i^t \\ \tilde{y}_i^t \end{bmatrix} + \delta t \underbrace{\begin{bmatrix} \cos(\hat{\theta}_i^t) & -\sin(\hat{\theta}_i^t) v_i^t \\ \sin(\hat{\theta}_i^t) & \cos(\hat{\theta}_i^t) v_i^t \end{bmatrix}}_{G_i} \begin{bmatrix} \mathbf{w}_v \\ \tilde{\theta}_i^t \end{bmatrix}. \tag{9}$$

In position propagation, the additional covariance term induced by control disturbance is

$$\begin{aligned} \Sigma_{u_i} &= (\delta t)^2 \,\mathsf{E}\left[G_i \begin{bmatrix} \mathbf{w}_v \\ \tilde{\theta}_i^t \end{bmatrix} \begin{bmatrix} \mathbf{w}_v \\ \tilde{\theta}_i^t \end{bmatrix}^\mathsf{T} G_i^\mathsf{T} \right] \\ &= (\delta t)^2 C(\hat{\theta}_i) \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & v_i^2 \sigma_{\tilde{\theta}_i}^2 \end{bmatrix} C^\mathsf{T}(\hat{\theta}_i), \end{aligned}$$

where $C(\theta)$ is the 2×2 rotation matrix associated with θ. Since the position propagation of one robot is independent of that of the rest, the covariance update is given by

$$\Sigma_s^{t+1} = \Sigma_s^t + \begin{bmatrix} \Sigma_{u_1} & \cdots & 0_{2\times 2} \\ \vdots & \ddots & \vdots \\ 0_{2\times 2} & \cdots & \Sigma_{u_N} \end{bmatrix}. \tag{10}$$

2.2 Landmark Observation

The positions of landmarks are known for robots in advance. Consequently, the landmark observation can be regarded as a source of absolute positioning.

The relative position measurement z_{il} between robot i and landmark l is

$$z_{il} = C^\mathsf{T}(\theta_i)(s_l - s_i) + \mathbf{w}_{z,il}.$$

The position of landmark s_l is determined, so it does not contribute any error in the measurement, nor the estimate. The error in the measurement can be approximated as

$$\begin{aligned} \tilde{z}_{il} &= \hat{z}_{il} - z_{il} \\ &\simeq H_i \tilde{s} + \Gamma_{il} \begin{bmatrix} \mathbf{w}_{z,il} \\ \tilde{\theta}_i \end{bmatrix}, \end{aligned} \tag{11}$$

where

$$H_i = C^\mathsf{T}(\hat{\theta}_i) H_{o_i}, \tag{12}$$

$$H_{o_i} = \begin{bmatrix} 0_{2\times 2} & \cdots & \underbrace{-I_2}_{i} & \cdots & 0_{2\times 2} \end{bmatrix}_{2\times 2N},$$

$$\Gamma_{il} = \begin{bmatrix} I_2 & -C(\hat{\theta}_i) J \Delta \hat{s}_{il} \end{bmatrix}_{2\times 3}, \tag{13}$$

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{14}$$

The covariance update can be expressed as

$$
\begin{aligned}
\Sigma_s^{t+} &= \Sigma_s^{t-} - \Sigma_s^{t-} H_i^\mathsf{T} S^{-1} H_i \Sigma_s^{t-} \\
&= \Sigma_s^{t-} - \Sigma_s^{t-} H_{o_i}^\mathsf{T} S_{o_i}^{-1} H_{o_i} \Sigma_s^{t-} \\
&= \Sigma_s^{t-} - \begin{bmatrix} \Sigma_{1i} S_{o_i}^{-1} \Sigma_{i1} & \cdots & \Sigma_{1i} S_{o_i}^{-1} \sigma_{iN} \\ \Sigma_{2i} S_{o_i}^{-1} \Sigma_{i1} & \cdots & \Sigma_{2i} S_{o_i}^{-1} \sigma_{iN} \\ \vdots & \ddots & \vdots \\ \Sigma_{Ni} S_{o_i}^{-1} \Sigma_{i1} & \cdots & \Sigma_{Ni} S_{o_i}^{-1} \sigma_{iN} \end{bmatrix}, \\
S_{o_i} &= H_{o_i} \Sigma_s^t H_{o_i}^\mathsf{T} + \left(\sigma_{d_i}^2 I_2 + \sigma_{\phi_i}^2 - \frac{\sigma_{d_i}^2}{d_{il}^2} \right) J \Delta\hat{s}_{il} \Delta\hat{s}_{il}{}^\mathsf{T} J^\mathsf{T}.
\end{aligned}
$$

2.3 Robot Observation

The relative position measurement z_{ij} between robot i and j is

$$
\begin{aligned}
z_{ij} &= C^\mathsf{T}(\theta_i)(s_j - s_i) + \mathbf{w}_{z,ij} \\
&= C^\mathsf{T}(\theta_i) \Delta s_{ij} + \mathbf{w}_{z,ij}.
\end{aligned} \tag{15}
$$

By linearization to separate the source of noise, the measurement error is obtained

$$
\begin{aligned}
\tilde{z}_{ij} &= \hat{z}_{ij} - z_{ij} \\
&\simeq H_{ij} \tilde{s} + \Gamma_{ij} \begin{bmatrix} \mathbf{w}_{z,ij} \\ \tilde{\theta}_i \end{bmatrix},
\end{aligned} \tag{16}
$$

where

$$
H_{ij} = C^\mathsf{T}(\hat{\theta}_i) H_{o_{ij}}, \tag{17}
$$

$$
H_{o_{ij}} = \begin{bmatrix} 0_{2\times 2} & \cdots & \underbrace{-I_2}_{i} & \cdots & \underbrace{I_2}_{j} & \cdots & 0_{2\times 2} \end{bmatrix}_{2\times 2N},
$$

$$
\Gamma_{ij} = \begin{bmatrix} I_2 & -C(\hat{\theta}_i) J \Delta\hat{s}_{ij} \end{bmatrix}_{2\times 3}. \tag{18}
$$

From (16), two sources of measurement error are specified: state error $\tilde{s}$ and observation error, including distance, bearing, and angle. The measurement covariance $\Sigma_{o,ij}$ is given by

$$
\begin{aligned}
{}_i\Sigma_{o,ij} &= \mathsf{E}\left[\Gamma_{ij} \begin{bmatrix} \mathbf{w}_{z,ij} \\ \tilde{\theta}_i \end{bmatrix} \begin{bmatrix} \mathbf{w}_{z,ij} \\ \tilde{\theta}_i \end{bmatrix}^\mathsf{T} \Gamma_{ij}^\mathsf{T} \right] \\
&= \mathsf{E}[\mathbf{w}_{z,ij} \mathbf{w}_{z,ij}^\mathsf{T}] + \sigma_{\theta_i}^2 C^\mathsf{T}(\hat{\theta}_i) J \Delta\hat{s}_{ij} \Delta\hat{s}_{ij}^\mathsf{T} J^\mathsf{T} C(\hat{\theta}_i).
\end{aligned}
$$

The relative position z_{ij} is retrieved from a range measurement d_{ij} and a bearing measurement ϕ_{ij}. In this case, we have

$$
\mathsf{E}[\mathbf{w}_{z,ij}\, \mathbf{w}_{z,ij}^\mathsf{T}] = C(\phi_{ij}) \begin{bmatrix} \sigma_{d_i}^2 & 0 \\ 0 & d_{ij}^2 \sigma_{\phi_i}^2 \end{bmatrix} C^\mathsf{T}(\phi_{ij}). \tag{19}
$$

The next step is to distinguish the role of the angle $\hat{\theta}_i$ from the rest terms. We rewrite the second term in (16) as

$$
\Gamma_{ij} \begin{bmatrix} \mathbf{w}_{z,ij} \\ \tilde{\theta}_i \end{bmatrix} = \begin{bmatrix} \cos(\phi_{ij}) & -d_{ij}\sin(\phi_{ij}) \\ \sin(\phi_{ij}) & d_{ij}\cos(\phi_{ij}) \end{bmatrix} \begin{bmatrix} \tilde{d_{ij}} \\ \tilde{\phi}_{ij} \end{bmatrix} \tag{20}
$$

$$
\begin{aligned}
C(\hat{\theta}_i)\,\Gamma_{ij}\begin{bmatrix}\mathbf{w}_{z,ij}\\ \tilde{\theta}_i\end{bmatrix} &= \begin{bmatrix}\cos(\hat{\theta}_i) & -\sin(\hat{\theta}_i)\\ \sin(\hat{\theta}_i) & \cos(\hat{\theta}_i)\end{bmatrix}\begin{bmatrix}\cos(\phi_{ij}) & -d_{ij}\sin(\phi_{ij})\\ \sin(\phi_{ij}) & d_{ij}\cos(\phi_{ij})\end{bmatrix}\begin{bmatrix}\tilde{d_{ij}}\\ \tilde{\phi}_{ij}\end{bmatrix}\\
&= \begin{bmatrix}\cos(\hat{\theta}_i+\phi_{ij}) & -d_{ij}\sin(\hat{\theta}_i+\phi_{ij})\\ \sin(\hat{\theta}_i+\phi_{ij}) & d_{ij}\cos(\hat{\theta}_i+\phi_{ij})\end{bmatrix}\begin{bmatrix}\tilde{d_{ij}}\\ \tilde{\phi}_{ij}\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{d_{ij}}I_2 & J\end{bmatrix}(\hat{s}_j-\hat{s}_i)\begin{bmatrix}\tilde{d_{ij}}\\ \tilde{\phi}_{ij}\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}_i\Sigma_{o,ij} &= \mathsf{E}\left[\Gamma_{ij}\begin{bmatrix}\mathbf{w}_{z,ij}\\ \tilde{\theta}_i\end{bmatrix}\begin{bmatrix}\mathbf{w}_{z,ij}\\ \tilde{\theta}_i\end{bmatrix}^{\mathsf{T}}\Gamma_{ij}{}^{\mathsf{T}}\right]\\
&= C^{\mathsf{T}}(\hat{\theta}_i)\begin{bmatrix}\frac{1}{d_{ij}}\Delta\hat{s}_{ij} & J\Delta\hat{s}_{ij}\end{bmatrix}\mathsf{E}\left[\begin{bmatrix}\tilde{d_{ij}}\\ \tilde{\phi}_{ij}\end{bmatrix}\begin{bmatrix}\tilde{d_{ij}}\\ \tilde{\phi}_{ij}\end{bmatrix}^{\mathsf{T}}\right]\begin{bmatrix}\frac{1}{d_{ij}}\Delta\hat{s}_{ij} & J\Delta\hat{s}_{ij}\end{bmatrix}^{\mathsf{T}}C(\hat{\theta}_i)\\
&= C^{\mathsf{T}}(\hat{\theta}_i)\begin{bmatrix}\frac{1}{d_{ij}}\Delta\hat{s}_{ij} & J\Delta\hat{s}_{ij}\end{bmatrix}\begin{bmatrix}\sigma_{d_i}^2 & 0\\ 0 & \sigma_{\phi_i}^2\end{bmatrix}\begin{bmatrix}\frac{1}{d_{ij}}\Delta\hat{s}_{ij} & J\Delta\hat{s}_{ij}\end{bmatrix}^{\mathsf{T}}C(\hat{\theta}_i)\\
&= C^{\mathsf{T}}(\hat{\theta}_i)\left(\frac{\sigma_{d_i}^2}{d_{ij}^2}\Delta\hat{s}_{ij}\Delta\hat{s}_{ij}{}^{\mathsf{T}}+\sigma_{\phi_i}^2 J\Delta\hat{s}_{ij}\Delta\hat{s}_{ij}{}^{\mathsf{T}}J^{\mathsf{T}}\right)C(\hat{\theta}_i)\\
&= C^{\mathsf{T}}(\hat{\theta}_i)\left(\frac{\sigma_{d_i}^2}{d_{ij}^2}\left(d_{ij}^2 I_2 - J\Delta\hat{s}_{ij}\Delta\hat{s}_{ij}{}^{\mathsf{T}}J^{\mathsf{T}}\right)+\sigma_{\phi_i}^2 J\Delta\hat{s}_{ij}\Delta\hat{s}_{ij}{}^{\mathsf{T}}J^{\mathsf{T}}\right)C(\hat{\theta}_i)\\
&= C^{\mathsf{T}}(\hat{\theta}_i)\left(\sigma_{d_i}^2 I_2+\left(\sigma_{\phi_i}^2-\frac{\sigma_{d_i}^2}{d_{ij}^2}\right)J\Delta\hat{s}_{ij}\Delta\hat{s}_{ij}{}^{\mathsf{T}}J^{\mathsf{T}}\right)C(\hat{\theta}_i)
\end{aligned}
$$

The covariance update can be expressed as

$$
\begin{aligned}
\Sigma_s^{t+} &= \Sigma_s^{t-} - \Sigma_s^{t-}H_{o_{ij}}^{\mathsf{T}}S_o^{-1}H_{o_{ij}}\Sigma_s^{t-}\\
&= \Sigma_s^{t-} - \begin{bmatrix}(\Sigma_{1j}-\Sigma_{1i})S_{o_i}^{-1}(\Sigma_{j1}-\Sigma_{i1}) & \cdots & (\Sigma_{Nj}-\Sigma_{Ni})S_{o_i}^{-1}(\Sigma_{jN}-\Sigma_{iN})\\ (\Sigma_{2j}-\Sigma_{2i})S_{o_i}^{-1}(\Sigma_{j2}-\Sigma_{i2}) & \cdots & (\Sigma_{Nj}-\Sigma_{Ni})S_{o_i}^{-1}(\Sigma_{jN}-\Sigma_{iN})\\ \vdots & \ddots & \vdots\\ (\Sigma_{Nj}-\Sigma_{Ni})S_{o_i}^{-1}(\Sigma_{jN}-\Sigma_{iN}) & \cdots & (\Sigma_{Nj}-\Sigma_{Ni})S_{o_i}^{-1}(\Sigma_{jN}-\Sigma_{iN})\end{bmatrix},\\
S_o &= H_{o_{ij}}\Sigma_s^{t-}H_{o_{ij}}^{\mathsf{T}}+\sigma_{d_i}^2 I_2+\left(\sigma_{\phi_i}^2-\frac{\sigma_{d_i}^2}{d_{ij}^2}\right)J\Delta\hat{s}_{ij}\Delta\hat{s}_{ij}{}^{\mathsf{T}}J^{\mathsf{T}}.
\end{aligned}
$$

3 Theoretical Analysis

4 Cooperative Localization Algorithms

4.1 Distributive Algorithm

4.2 Partially Distributive Algorithm