Split Covariance Intersection Filter: Theory and Its Application to Vehicle Localization

Hao Li, Fawzi Nashashibi, and Ming Yang

***Abstract*—Data fusion is an important process in a variety of tasks in the intelligent transportation systems field. Most existing data fusion methods rely on the assumption of conditional independence or known statistics of data correlation. In contrast, the split covariance intersection filter (split CIF) was heuristically presented in literature, which aims at providing a mechanism to reasonably handle both known independent information and unknown correlated information in source data. In this paper, we provide a theoretical foundation for the split CIF. First, we clearly specify the *consistency* definition (coined as *split consistency*) for estimates in split form. Second, we provide a theoretical proof for the fusion consistency of the split CIF. Finally, we provide a theoretical derivation of the split CIF for the partial observation case. We also present a general architecture of decentralized vehicle localization, which serves as a concrete application example of the split CIF to demonstrate the advantages of the split CIF and how it can potentially benefit vehicle localization (noncooperative and cooperative). In general, this paper aims at providing a baseline for researchers who might intend to incorporate the split CIF (a useful tool for general data fusion) into their prospective research works.**

***Index Terms*—Cooperative intelligent systems, data fusion, intelligent transportation systems (ITS), Kalman filter, split covariance intersection filter (split CIF), vehicle localization.**

I. Introduction

In THE intelligent transportation system (ITS) field, many tasks require real-time estimation of a certain state, such as the vehicle state [1], the object state [2], [3], the environment state [4], etc. A state might be observed (partially) from multiple sources, i.e., data of multiple instants or of multiple sensors. Data fusion, which aims at directly or indirectly fusing the data of multiple sources, is usually an important process to achieve a better state estimate.

For real-time state estimation (normally in a recursive way, see [5]), a commonly used data fusion method is the Kalman filter (including its variants such as the extended Kalman filter and the unscented Kalman filter) [6]–[8]. It has been applied in a variety of tasks, such as vehicle localization [9]–[12], lane tracking [13]–[15], moving object tracking [16]–[19], etc. Several factors account for the popularity of the Kalman filter: Its formalism is simple, it is easy to implement, it is a computationally efficient way to maintain a full-state distribution estimate, and it is generally effective for unimodal estimation problems.

Another well-known data fusion method is the particle filter (or a sequential Monte Carlo method) [20], [21], which enjoys a natural mechanism to handle multimodal ambiguities [22]–[24]. On the other hand, compared with the Kalman filter, the particle filter is not computationally efficient due to the comparatively large number of particles needed to approximate full-state distribution. Applications of the maximum-likelihood method [25], [26] and the maximum *a posteriori* method [27] have also been reported.

Most existing data fusion methods rely on the assumption of conditional independence or known statistics of data correlation; they cannot guarantee the fusion consistency when fusing data of unknown correlation. The authors in [28] propose a new data fusion method named covariance intersection filter (CIF), and its fusion consistency is theoretically guaranteed. On the other hand, the CIF has a drawback of yielding pessimistic estimates because it treats the source data as fully correlated and neglects possible independent information in them.

In [29], the split CIF is introduced, which provides the ability to incorporate and maintain known independent information in the estimates. The split CIF can be regarded as a generalization of the Kalman filter. It also enjoys formalism simplicity and implementation convenience, whereas it overcomes the drawback of the Kalman filter when fusing correlated data. Recently, it has been applied to multivehicle cooperative localization [30].

In [29], the split CIF is only briefly presented as a heuristic extension of the CIF, and no theoretical analysis has been given for the split CIF. In contrast, as a contribution of this paper, we provide a theoretical foundation for the split CIF.

1) We clearly specify the *consistency* definition (coined as *split consistency* in this paper) for estimates in split form.
2) We provide a theoretical proof for the fusion consistency of the split CIF.
3) We provide a theoretical derivation of the split CIF for the partial observation case.

The motivation of providing a theoretical foundation for the split CIF is to provide a theoretical guide for researchers who might intend to incorporate the split CIF into their prospective research works.

In this paper, in addition to the description of the split CIF theory, we examine the application case of vehicle localization (ground vehicles), which is an important task for intelligent vehicle systems. Traditionally, a vehicle performs this task

Manuscript received March 17, 2013; revised June 7, 2013; accepted June 9, 2013. Date of publication July 3, 2013; date of current version November 26, 2013. The Associate Editor for this paper was M. Á. Sotelo Vázquez.

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Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TITS.2013.2267800

based only on its own sensor data. In recent years, multivehicle cooperative localization that takes advantage of data sharing has been arousing more research interest [30].

The real traffic environment is open for thousands of vehicles. Compared with a *centralized architecture*, a *decentralized architecture*, where each vehicle only handles local information, would be a desirable solution for cooperative localization because of a comparatively low computational burden and the flexibility for dealing with dynamic vehicle networks.

An important issue for decentralized architectures is how to handle interestimate correlation. A naïve implementation of decentralized cooperative localization, which neglects this correlation, will lead to the overconvergence problem [30], [31]. A common idea to handle interestimate correlation is to avoid fusing correlated data, which is realized by controlling the data flow within vehicle networks. For example, the authors in [31] and [32] propose some simple heuristic rules to avoid circular reasoning of correlated data. However, these methods do not have complete control of the data flow, and the risk of circular reasoning may still exist. The authors in [33] propose a complicated data transfer scheme to thoroughly remove circular reasoning. However, in this method, the estimates are output with uncertain length of delay; moreover, large data pedigrees have to be relayed within the networks. The authors in [34] and [35] propose the state-exchange-based method, which only allows independent estimates to be shared within vehicle networks. The method is efficient and the risk of fusing correlated data can be thoroughly removed. The major shortcoming of this method is that a vehicle cannot benefit from the whole connective vehicle networks.

In the given methods, the used data fusion methods, such as the Kalman filter, cannot guarantee the fusion consistency when fusing data of unknown correlation. Due to this limitation, the given methods require controlling the data flow in order to avoid circular reasoning of unknown correlated information. On the other hand, if we use a data fusion method that can always guarantee the fusion consistency, then the practice of controlling the data flow to avoid circular reasoning will become unnecessary. Motivated by this idea, the authors in [30] proposes a cooperative localization method using the split CIF. This method possesses several merits. First, its implementation is rather simple, exempt from complicated techniques to control the data flow. Second, it enables good estimates to be naturally spread within connective vehicle networks, which is particularly apparent for vehicles with heterogeneous self-positioning ability.

Absolute positioning measurements (APMs) are essential for vehicle localization. They can be provided by commonly used GPSs [9], [10], by certain *ad hoc* techniques [11], [36], or by hybrid techniques [37]–[39]. Usually, APMs are modeled as independent variables; however, this modeling might not be suitable. Take GPS measurements as an example. Some research works [40] and our own experiences have shown that GPS errors can be modeled by Gaussian white noise combined with a slowly changing bias vector. Because of the existence of this bias vector, there is a certain (unknown) temporal correlation among consecutive GPS measurements in a long enough time interval. Similarly, APMs obtained by other techniques might also contain certain systematic bias error, which can cause a temporal correlation among the measurements. Treating APMs simply as independent variables and neglecting their temporal correlation might result in overconfident estimates, as will be shown in this paper.

In our previous works [30], the split CIF has been used to handle the spatial correlation among multivehicle estimates; however, the issue of temporal data correlation has not been taken into account. As an extension of the previous works [30], a general architecture of decentralized vehicle localization, which incorporates a new strategy of using the split CIF to handle unknown temporal correlation among APMs, is presented in this paper. The presented architecture is intended to serve as a concrete application example of the split CIF, which aims at demonstrating the advantages of the split CIF and how it can be useful in concrete applications in which unknown data correlation exists.

II. Split CIF Theory

A. Estimate Consistency and Fusion Consistency

Given estimate $\{\mathbf{X}, \mathbf{P}\}$, where $\mathbf{X}$ denotes the estimated state vector and $\mathbf{P}$ denotes the estimated covariance matrix, let $\overline{\mathbf{X}}$ denote the ground truth of $\mathbf{X}$, $\widetilde{\mathbf{X}}$ denote the error of $\mathbf{X}$, and $\mathbf{P}^*$ denote the true covariance of $\mathbf{X}$, i.e.,

$$\mathbf{P}^* = E[\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^T] = E\left[(\mathbf{X}-\overline{\mathbf{X}})(\mathbf{X}-\overline{\mathbf{X}})^T\right].$$

Then, *consistency* is defined as a property of an estimate that the estimated covariance matrix is no smaller than the true covariance of the estimated state vector (in simple words, an estimate is *consistent* if it is not overconfident), i.e.,

$$\mathbf{P} - \mathbf{P}^* \geq \mathbf{0}. \tag{1}$$

Given two consistent source estimates $\{\mathbf{X}_i, \mathbf{P}_i\}$ $(i = 1, 2)$ to be fused, if the fusion estimate is also consistent, then the fusion is regarded as consistent. We hope that the fusion consistency can be always guaranteed because we do not want to establish any extra confidence in the fusion estimate than what the source estimates can convey. Consider the Kalman filter, which can be equivalently given as

$$\begin{aligned}\mathbf{P}^{-1} &= \mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\\ \mathbf{X} &= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{X}_1 + \mathbf{P}_2^{-1}\mathbf{X}_2\right).\end{aligned}$$

The fusion consistency of the Kalman filter cannot be guaranteed if source estimates are correlated. For example, let $\{\mathbf{X}_1, \mathbf{P}_1\}$ and $\{\mathbf{X}_2, \mathbf{P}_2\}$ be the two copies of an estimate. Their proper fusion will yield the same estimate, satisfying $\mathbf{P} = \mathbf{P}_1 = \mathbf{P}_2$. However, via the Kalman filter, $\mathbf{P}$ will become a half of $\mathbf{P}_1$, which is obviously inconsistent; in other words, the fusion estimate is overconfident.

B. Split CIF

The authors in [28] propose a data fusion method named CIF, which forms the fusion estimate by taking a convex combination of the source estimates. The formula of the CIF can be written as

$$\mathbf{P}^{-1} = (\mathbf{P}_1/w)^{-1} + (\mathbf{P}_2/(1-w))^{-1}$$
$$\mathbf{X} = \mathbf{P}\left[(\mathbf{P}_1/w)^{-1}\mathbf{X}_1 + (\mathbf{P}_2/(1-w))^{-1}\mathbf{X}_2\right].$$

For any choice of w in the interval [0, 1], the CIF is guaranteed to yield consistent fusion estimates even when facing source estimates of unknown correlation. However, the CIF has a drawback of yielding a pessimistic estimate because it treats the source estimates as being totally correlated and neglects possible independent information in them. In [29], the split CIF is introduced, which provides the ability to incorporate and maintain known independent information in the estimates.

For the split CIF, an estimate is always represented in split form $\{\mathbf{X}, \mathbf{P}_d + \mathbf{P}_i\}$, where covariance component $\mathbf{P}_d$ represents the maximum degree to which the estimate is possibly correlated with others, and covariance component $\mathbf{P}_i$ represents the degree of its absolute independence. To facilitate understanding of the split form, we can imagine that $\mathbf{X}$ consists of correlated component $\mathbf{X}_d$ and independent component $\mathbf{X}_i$, i.e., $\mathbf{X} = \mathbf{X}_d + \mathbf{X}_i$; then, $\mathbf{P}_d$ and $\mathbf{P}_i$ can be regarded as the estimated covariance matrices, respectively, for $\mathbf{X}_d$ and $\mathbf{X}_i$. Given two source estimates $\{\mathbf{X}_1, \mathbf{P}_{1d} + \mathbf{P}_{1i}\}$ and $\{\mathbf{X}_2, \mathbf{P}_{2d} + \mathbf{P}_{2i}\}$, let the fusion estimate be denoted as $\{\mathbf{X}, \mathbf{P}_d + \mathbf{P}_i\}$. The formula of the split CIF is given as

$$\begin{aligned}
\mathbf{P}_1 &= \mathbf{P}_{1d}/w + \mathbf{P}_{1i}\\
\mathbf{P}_2 &= \mathbf{P}_{2d}/(1-w) + \mathbf{P}_{2i}\\
\mathbf{P}^{-1} &= \mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\\
\mathbf{X} &= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{X}_1 + \mathbf{P}_2^{-1}\mathbf{X}_2\right)\\
\mathbf{P}_i &= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{P}_{1i}\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\mathbf{P}_{2i}\mathbf{P}_2^{-1}\right)\mathbf{P}\\
\mathbf{P}_d &= \mathbf{P} - \mathbf{P}_i.
\end{aligned} \tag{2}$$

w belongs to the interval [0, 1]. In practice, w can be determined by optimizing an objective function in terms of w, such as the determinant of the new covariance [29]. Notice that the split CIF in (2) can be regarded as a generalization of the Kalman filter. Let $\mathbf{P}_{1d}$ and $\mathbf{P}_{2d}$ be zero, and (2) will become similar to the Kalman filter. In other words, the Kalman filter can be treated as a special case of the split CIF, where the source estimates are known to be independent.

C. Theoretical Analysis for the Split CIF

The consistency definition (1) is not enough to characterize the consistency property of an estimate in split form $\{\mathbf{X}, \mathbf{P}_d + \mathbf{P}_i\}$. For example, consider two estimates $\{\mathbf{X}_1, \mathbf{P}_{1d} + \mathbf{P}_{1i}\}$ and $\{\mathbf{X}_2, \mathbf{P}_{2d} + \mathbf{P}_{2i}\}$, where $\mathbf{X}_1 = \mathbf{X}_2$, $\mathbf{P}_{1d} = \mathbf{P}_{2i}$, and $\mathbf{P}_{1i} = \mathbf{P}_{2d} = 0$. They are different because $\mathbf{X}_1$ conveys correlated information, whereas $\mathbf{X}_2$ conveys independent information. However, their consistency properties defined by (1) are exactly the same, which is not reasonable. In addition, the consistency defined by (1) is not sufficient to guarantee yielding a consistent fusion estimate via the split CIF. These limitations of the consistency definition (1) for the split CIF have not been pointed out in [29]. To address these limitations, we propose the augmented version of the consistency definition (1), which is coined as the *split consistency* (either the *A-split consistency* or the *B-split consistency*).

A-split consistency: Given an estimate in split form $\{\mathbf{X}, \mathbf{P}_d + \mathbf{P}_i\}$ (or, imaginarily, as $\{\mathbf{X}_d + \mathbf{X}_i, \mathbf{P}_d + \mathbf{P}_i\}$), it is *A-split consistent* if it satisfies

$$\mathbf{P}_d \geq \mathbf{P}_d^* = E\left[\widetilde{\mathbf{X}}_d\widetilde{\mathbf{X}}_d^T\right] \qquad \mathbf{P}_i \geq \mathbf{P}_i^* = E\left[\widetilde{\mathbf{X}}_i\widetilde{\mathbf{X}}_i^T\right]. \tag{3}$$

B-split consistency: Given an estimate in split form $\{\mathbf{X}, \mathbf{P}_d + \mathbf{P}_i\}$ (or, imaginarily, as $\{\mathbf{X}_d + \mathbf{X}_i, \mathbf{P}_d + \mathbf{P}_i\}$), it is *B-split consistent* if it satisfies

$$\mathbf{P}_d \geq \mathbf{P}_d^* = E\left[\widetilde{\mathbf{X}}_d\widetilde{\mathbf{X}}_d^T\right] \quad \mathbf{P}_i + \mathbf{P}_d \geq \mathbf{P}_i^* + \mathbf{P}_d^* = E\left[\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^T\right]. \tag{4}$$

Theorem 1: If the two source estimates are both A-split consistent, then for any choice of w in the interval [0, 1] the fusion estimate of the split CIF in (2) is guaranteed to be A-split consistent.

Proof: Let the two source estimates be denoted as $\{\mathbf{X}_1, \mathbf{P}_{1d} + \mathbf{P}_{1i}\}$ and $\{\mathbf{X}_2, \mathbf{P}_{2d} + \mathbf{P}_{2i}\}$. Let the fusion estimate of the split CIF in (2) be denoted as $\{\mathbf{X}, \mathbf{P}_d + \mathbf{P}_i\}$. Since the source estimates are both A-split consistent, we have

$$\begin{aligned}
\mathbf{P}_{1d} &\geq E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right] \qquad & \mathbf{P}_{1i} &\geq E\left[\widetilde{\mathbf{X}}_{1i}\widetilde{\mathbf{X}}_{1i}^T\right]\\
\mathbf{P}_{2d} &\geq E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right] \qquad & \mathbf{P}_{2i} &\geq E\left[\widetilde{\mathbf{X}}_{2i}\widetilde{\mathbf{X}}_{2i}^T\right].
\end{aligned}$$

We examine the independent component of $\mathbf{X}$, i.e.,

$$\begin{aligned}
\mathbf{P}_i &= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{P}_{1i}\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\mathbf{P}_{2i}\mathbf{P}_2^{-1}\right)\mathbf{P}\\
\widetilde{\mathbf{X}}_i &= \mathbf{P}\left(\mathbf{P}_1^{-1}\widetilde{\mathbf{X}}_{1i} + \mathbf{P}_2^{-1}\widetilde{\mathbf{X}}_{2i}\right)\\
\mathbf{P}_i &- E\left[\widetilde{\mathbf{X}}_i\widetilde{\mathbf{X}}_i^T\right]\\
&= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{P}_{1i}\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\mathbf{P}_{2i}\mathbf{P}_2^{-1}\right)\mathbf{P}\\
&\quad - \mathbf{P}\left\{\mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1i}\widetilde{\mathbf{X}}_{1i}^T\right]\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2i}\widetilde{\mathbf{X}}_{2i}^T\right]\mathbf{P}_2^{-1}\right\}\mathbf{P}\\
&= \mathbf{P}\left\{\mathbf{P}_1^{-1}\left(\mathbf{P}_{1i} - E\left[\widetilde{\mathbf{X}}_{1i}\widetilde{\mathbf{X}}_{1i}^T\right]\right)\mathbf{P}_1^{-1}\right.\\
&\qquad \left. + \mathbf{P}_2^{-1}\left(\mathbf{P}_{2i} - E\left[\widetilde{\mathbf{X}}_{2i}\widetilde{\mathbf{X}}_{2i}^T\right]\right)\mathbf{P}_2^{-1}\right\}\mathbf{P} \geq \mathbf{0}.
\end{aligned}$$

We examine the correlated component of $\mathbf{X}$ (for $0 < w < 1$). Denote $\overline{w} = 1 - w$ as that shown at the bottom of the next page. In summary, the fusion estimate $\{\mathbf{X}, \mathbf{P}_d + \mathbf{P}_i\}$ is also A-split consistent. □

Theorem 2: If the two source estimates are both B-split consistent, then for any choice of w in the interval [0, 1], the fusion estimate of the split CIF in (2) is guaranteed to be B-split consistent.

Proof: Let the two source estimates be denoted as $\{\mathbf{X}_1, \mathbf{P}_{1d} + \mathbf{P}_{1i}\}$ and $\{\mathbf{X}_2, \mathbf{P}_{2d} + \mathbf{P}_{2i}\}$. Let the fusion estimate of the split CIF in (2) be denoted as $\{\mathbf{X}, \mathbf{P}_d + \mathbf{P}_i\}$. Since the source estimates are both B-split consistent, we have

$$\begin{aligned}
\mathbf{P}_{1d} &\geq E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right] \qquad & \mathbf{P}_{1d} + \mathbf{P}_{1i} &\geq E\left[\widetilde{\mathbf{X}}_1\widetilde{\mathbf{X}}_1^T\right]\\
\mathbf{P}_{2d} &\geq E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right] \qquad & \mathbf{P}_{2d} + \mathbf{P}_{2i} &\geq E\left[\widetilde{\mathbf{X}}_2\widetilde{\mathbf{X}}_2^T\right].
\end{aligned}$$

Concerning the correlated component of $\mathbf{X}$ (for $0 < w < 1$), we can also have (the proof for this part is the same to that in the proof for Theorem 1)

$$\mathbf{P}_d - E\left[\widetilde{\mathbf{X}}_d\widetilde{\mathbf{X}}_d^T\right] \geq \mathbf{0}.$$

Therefore, we only have to examine the entire $\mathbf{X}$ (for $0<w<1$). Denote $\overline{w} = 1 - w$ as that shown at the bottom of the next page. In summary, the fusion estimate $\{\mathbf{X}, \mathbf{P}_\mathrm{d} + \mathbf{P}_\mathrm{i}\}$ is also B-split consistent. □

We can see that the consistency defined by (1) is naturally satisfied if either the A-split consistency or the B-split consistency is satisfied. No matter which of the two split-consistency definitions is used, the split CIF itself is the same, and the consistency defined by (1) can be always satisfied. This is why we use the general term split consistency to indicate either the A-split consistency or the B-split consistency, without always explicitly distinguishing between them.

In addition, we can also see from the given proofs that, if the source estimates are assumed split consistent (no matter whether they are split consistent in reality), then the fusion estimate of the split CIF will be also split consistent from this assumption perspective. In other words, the split CIF itself will not add any extra confidence on the fusion estimate. Therefore, in this sense, the fusion consistency of the split CIF is always guaranteed, even without the specification of the split consistency definition.

D. Split CIF for Partial Observation Case

Corollary 1: Given two source estimates $\{\mathbf{X}_1, \mathbf{P}_{1d} + \mathbf{P}_{1i}\}$ and $\{\mathbf{X}_2, \mathbf{P}_{2d} + \mathbf{P}_{2i}\}$, suppose that $\mathbf{X}_1$ is a complete observation, i.e., $\mathbf{X}_1 = \mathbf{X}_\mathrm{true}$, whereas $\mathbf{X}_2$ is a partial observation, i.e., $\mathbf{X}_2 = \mathbf{H}\mathbf{X}_\mathrm{true}$ ($\mathbf{H}$ is not of full rank). Let the fusion estimate be denoted as $\{\mathbf{X}, \mathbf{P}_d + \mathbf{P}_i\}$. Then, the split CIF can be given as

$$\begin{aligned}
\mathbf{P}_1 &= \mathbf{P}_{1d}/w + \mathbf{P}_{1i}\\
\mathbf{P}_2 &= \mathbf{P}_{2d}/(1-w) + \mathbf{P}_{2i}\\
\mathbf{P}^{-1} &= \mathbf{P}_1^{-1} + \mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{H}\\
\mathbf{X} &= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{X}_1 + \mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{X}_2\right)\\
\mathbf{P}_i &= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{P}_{1i}\mathbf{P}_1^{-1} + \mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{P}_{2i}\mathbf{P}_2^{-1}\mathbf{H}\right)\mathbf{P}\\
\mathbf{P}_d &= \mathbf{P} - \mathbf{P}_i.
\end{aligned} \tag{5}$$

Proof: Complement $\mathbf{H}$ with $\mathbf{H}_0$ to make an invertible matrix $\mathbf{H}_A$, i.e.,

$$\mathbf{H}_A = \begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}.$$

$$\begin{aligned}
\mathbf{P}_d &= \mathbf{P} - \mathbf{P}_i = \mathbf{P} - \mathbf{P}(\mathbf{P}_1^{-1}\mathbf{P}_{1i}\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\mathbf{P}_{2i}\mathbf{P}_2^{-1})\mathbf{P}\\
&= \mathbf{P}\left(\mathbf{P}^{-1} - \mathbf{P}_1^{-1}\mathbf{P}_{1i}\mathbf{P}_1^{-1} - \mathbf{P}_2^{-1}\mathbf{P}_{2i}\mathbf{P}_2^{-1}\right)\mathbf{P}\\
&= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{P}_1\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\mathbf{P}_2\mathbf{P}_2^{-1} - \mathbf{P}_1^{-1}\mathbf{P}_{1i}\mathbf{P}_1^{-1} - \mathbf{P}_2^{-1}\mathbf{P}_{2i}\mathbf{P}_2^{-1}\right)\mathbf{P}\\
&= \mathbf{P}\left(\mathbf{P}_1^{-1}\frac{\mathbf{P}_{1d}}{w}\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\frac{\mathbf{P}_{2d}}{\overline{w}}\mathbf{P}_2^{-1}\right)\mathbf{P}\\
&\geq \mathbf{P}\left(\mathbf{P}_1^{-1}\frac{E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]}{w}\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\frac{E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]}{\overline{w}}\mathbf{P}_2^{-1}\right)\mathbf{P}\\
\widetilde{\mathbf{X}}_d &= \mathbf{P}\left(\mathbf{P}_1^{-1}\widetilde{\mathbf{X}}_{1d} + \mathbf{P}_2^{-1}\widetilde{\mathbf{X}}_{2d}\right)\\
\mathbf{P}_d - E\left[\widetilde{\mathbf{X}}_d\widetilde{\mathbf{X}}_d^T\right] &\geq \mathbf{P}\left\{\mathbf{P}_1^{-1}\frac{E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]}{w}\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\frac{E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]}{\overline{w}}\mathbf{P}_2^{-1}\right\}\mathbf{P}\\
&\quad - \mathbf{P}\left\{\mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1} + \mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1}\right.\\
&\quad\quad \left. + \mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1}\right\}\mathbf{P}\\
&= \mathbf{P}\left\{\mathbf{P}_1^{-1}\frac{\overline{w}}{w}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\frac{w}{\overline{w}}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1} - \mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1}\right.\\
&\quad\quad \left. - \mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1}\right\}\mathbf{P}\\
&= \frac{1}{w\overline{w}}\mathbf{P}\left\{\overline{w}^2\mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1} + w^2\mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1} - w\overline{w}\mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1}\right.\\
&\quad\quad \left. - w\overline{w}\mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1}\right\}\mathbf{P}\\
&= \frac{\mathbf{P}}{w\overline{w}}E\left\{\left[\overline{w}\mathbf{P}_1^{-1}\widetilde{\mathbf{X}}_{1d} - w\mathbf{P}_2^{-1}\widetilde{\mathbf{X}}_{2d}\right]\left[\overline{w}\mathbf{P}_1^{-1}\widetilde{\mathbf{X}}_{1d} - w\mathbf{P}_2^{-1}\widetilde{\mathbf{X}}_{2d}\right]^T\right\}\mathbf{P}\\
&\geq \mathbf{0}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{P}_d + \mathbf{P}_i &= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{P}_1\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\mathbf{P}_2\mathbf{P}_2^{-1}\right)\mathbf{P} \\
&= \mathbf{P}\left\{\mathbf{P}_1^{-1}\left(\frac{\mathbf{P}_{1d}}{w} + \mathbf{P}_{1i}\right)\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\left(\frac{\mathbf{P}_{2d}}{\overline{w}} + \mathbf{P}_{2i}\right)\mathbf{P}_2^{-1}\right\}\mathbf{P} \\
&= \mathbf{P}\left\{\mathbf{P}_1^{-1}\left(\mathbf{P}_{1i} + \frac{\mathbf{P}_{1d} - E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]}{w} + \frac{E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]}{w}\right)\mathbf{P}_1^{-1}\right. \\
&\qquad\left. + \mathbf{P}_2^{-1}\left(\mathbf{P}_{2i} + \frac{\mathbf{P}_{2d} - E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]}{\overline{w}} + \frac{E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]}{\overline{w}}\right)\mathbf{P}_2^{-1}\right\}\mathbf{P} \\
&\geq \mathbf{P}\left\{\mathbf{P}_1^{-1}\left(\mathbf{P}_{1i} + \mathbf{P}_{1d} - E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right] + \frac{E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]}{w}\right)\mathbf{P}_1^{-1}\right. \\
&\qquad\left. + \mathbf{P}_2^{-1}\left(\mathbf{P}_{2i} + \mathbf{P}_{2d} - E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right] + \frac{E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]}{\overline{w}}\right)\mathbf{P}_2^{-1}\right\}\mathbf{P} \\
&\geq \mathbf{P}\left\{\mathbf{P}_1^{-1}\left(E\left[\widetilde{\mathbf{X}}_{1i}\widetilde{\mathbf{X}}_{1i}^T\right] + \frac{E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]}{w}\right)\mathbf{P}_1^{-1}\right. \\
&\qquad\left. + \mathbf{P}_2^{-1}\left(E\left[\widetilde{\mathbf{X}}_{2i}\widetilde{\mathbf{X}}_{2i}^T\right] + \frac{E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]}{\overline{w}}\right)\mathbf{P}_2^{-1}\right\}\mathbf{P} \\
\widetilde{\mathbf{X}}_i &= \mathbf{P}\left(\mathbf{P}_1^{-1}\widetilde{\mathbf{X}}_{1i} + \mathbf{P}_2^{-1}\widetilde{\mathbf{X}}_{2i}\right) \\
\widetilde{\mathbf{X}}_d &= \mathbf{P}\left(\mathbf{P}_1^{-1}\widetilde{\mathbf{X}}_{1d} + \mathbf{P}_2^{-1}\widetilde{\mathbf{X}}_{2d}\right) \\
E[\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^T] &= E\left[\widetilde{\mathbf{X}}_i\widetilde{\mathbf{X}}_i^T\right] + E\left[\widetilde{\mathbf{X}}_d\widetilde{\mathbf{X}}_d^T\right] \\
\mathbf{P}_d + \mathbf{P}_i - E[\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^T] &\geq \mathbf{P}\left\{\mathbf{P}_1^{-1}\left(E\left[\widetilde{\mathbf{X}}_{1i}\widetilde{\mathbf{X}}_{1i}^T\right] + \frac{E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]}{w}\right)\mathbf{P}_1^{-1}\right. \\
&\qquad\left. + \mathbf{P}_2^{-1}\left(E\left[\widetilde{\mathbf{X}}_{2i}\widetilde{\mathbf{X}}_{2i}^T\right] + \frac{E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]}{\overline{w}}\right)\mathbf{P}_2^{-1}\right\}\mathbf{P} \\
&\quad - \mathbf{P}\left\{\mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1i}\widetilde{\mathbf{X}}_{1i}^T\right]\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2i}\widetilde{\mathbf{X}}_{2i}^T\right]\mathbf{P}_2^{-1}\right\}\mathbf{P} \\
&\quad - \mathbf{P}\left\{\mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1}\right. \\
&\qquad\left. + \mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1} + \mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1}\right\}\mathbf{P} \\
&= \mathbf{P}\left\{\mathbf{P}_1^{-1}\frac{E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]}{w}\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\frac{E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]}{\overline{w}}\mathbf{P}_2^{-1}\right\}\mathbf{P} \\
&\quad - \mathbf{P}\left\{\mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1}\right. \\
&\qquad\left. + \mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1} + \mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1}\right\}\mathbf{P} \\
&= \mathbf{P}\left\{\mathbf{P}_1^{-1}\frac{\overline{w}}{w}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1}\frac{w}{\overline{w}}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1}\right. \\
&\qquad\left. - \mathbf{P}_1^{-1}E\left[\widetilde{\mathbf{X}}_{1d}\widetilde{\mathbf{X}}_{2d}^T\right]\mathbf{P}_2^{-1} - \mathbf{P}_2^{-1}E\left[\widetilde{\mathbf{X}}_{2d}\widetilde{\mathbf{X}}_{1d}^T\right]\mathbf{P}_1^{-1}\right\}\mathbf{P} \\
&= \frac{\mathbf{P}}{w\overline{w}}E\left\{\left[\overline{w}\mathbf{P}_1^{-1}\widetilde{\mathbf{X}}_{1d} - w\mathbf{P}_2^{-1}\widetilde{\mathbf{X}}_{2d}\right]\right. \\
&\qquad\left.\left[\overline{w}\mathbf{P}_1^{-1}\widetilde{\mathbf{X}}_{1d} - w\mathbf{P}_2^{-1}\widetilde{\mathbf{X}}_{2d}\right]^T\right\}\mathbf{P} \\
&\geq \mathbf{0}.
\end{aligned}
$$

Augment $\mathbf{X}_2$ to a complete observation $\mathbf{X}_{2A}$ with the covariance $\mathbf{P}_{2Ad}+\mathbf{P}_{2Ai}$, which satisfies

$$\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}\mathbf{X}_{2A}\sim N\left(\begin{bmatrix}\mathbf{X}_2\\ \mathbf{0}\end{bmatrix},\begin{bmatrix}\mathbf{P}_{2d}+\mathbf{P}_{2i} & \mathbf{0}\\ \mathbf{0} & \infty\end{bmatrix}\right).$$

Then, we have

$$\begin{aligned}
\mathbf{X}_{2A} &= \begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{X}_2\\ \mathbf{0}\end{bmatrix}\\
\mathbf{P}_{2Ad}+\mathbf{P}_{2Ai} &= \begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{P}_{2d}+\mathbf{P}_{2i} & \mathbf{0}\\ \mathbf{0} & \infty\end{bmatrix}\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-T}\\
&= \underbrace{\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{P}_{2d} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-T}}_{\mathbf{P}_{2Ad}}\\
&\quad+\underbrace{\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{P}_{2i} & \mathbf{0}\\ \mathbf{0} & \infty\end{bmatrix}\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-T}}_{\mathbf{P}_{2Ai}}.
\end{aligned}$$

We apply the split CIF in (2) on $\{\mathbf{X}_1,\mathbf{P}_{1d}+\mathbf{P}_{1i}\}$ and $\{\mathbf{X}_{2A},\mathbf{P}_{2Ad}+\mathbf{P}_{2Ai}\}$ as

$$\begin{aligned}
\mathbf{P}_1 &= \mathbf{P}_{1d}/w+\mathbf{P}_{1i}\\
(\mathbf{P}_2 &= \mathbf{P}_{2d}/(1-w)+\mathbf{P}_{2i})\\
\mathbf{P}_{2A}^{-1} &= \left(\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{P}_2 & \mathbf{0}\\ \mathbf{0} & \infty\end{bmatrix}\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-T}\right)^{-1}\\
&= \mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{H}\\
\mathbf{P}^{-1} &= \mathbf{P}_1^{-1}+\mathbf{P}_{2A}^{-1} = \mathbf{P}_1^{-1}+\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{H}\\
\mathbf{X} &= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{X}_1+\mathbf{P}_{2A}^{-1}\mathbf{X}_{2A}\right)\\
&= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{X}_1+\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{H}\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{X}_2\\ \mathbf{0}\end{bmatrix}\right)\\
&= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{X}_1+\mathbf{H}^T\mathbf{P}_2^{-1}[\mathbf{I}\quad\mathbf{0}]\begin{bmatrix}\mathbf{X}_2\\ \mathbf{0}\end{bmatrix}\right)\\
&= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{X}_1+\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{X}_2\right)\\
\mathbf{P}_{2A}^{-1}\mathbf{P}_{2Ai}\mathbf{P}_{2A}^{-1} &= \mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{H}\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-1}\\
&\quad\begin{bmatrix}\mathbf{P}_{2i} & \mathbf{0}\\ \mathbf{0} & \infty\end{bmatrix}\begin{bmatrix}\mathbf{H}\\ \mathbf{H}_0\end{bmatrix}^{-T}\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{H}\\
&= \mathbf{H}^T\mathbf{P}_2^{-1}[\mathbf{I}\quad\mathbf{0}]\begin{bmatrix}\mathbf{P}_{2i} & \mathbf{0}\\ \mathbf{0} & \infty\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix}\mathbf{P}_2^{-1}\mathbf{H}\\
&= \mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{P}_{2i}\mathbf{P}_2^{-1}\mathbf{H}\\
\mathbf{P}_i &= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{P}_{1i}\mathbf{P}_1^{-1}+\mathbf{P}_{2A}^{-1}\mathbf{P}_{2Ai}\mathbf{P}_{2A}^{-1}\right)\mathbf{P}\\
&= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{P}_{1i}\mathbf{P}_1^{-1}+\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{P}_{2i}\mathbf{P}_2^{-1}\mathbf{H}\right)\mathbf{P}\\
\mathbf{P}_d &= \mathbf{P}-\mathbf{P}_i.
\end{aligned}$$

Therefore, we have (5). □

Corollary 2: Corollary 2 has the same conditions with those in Corollary 1. the split CIF can be given as

$$\begin{aligned}
\mathbf{P}_1 &= \mathbf{P}_{1d}/w+\mathbf{P}_{1i}\\
\mathbf{P}_2 &= \mathbf{P}_{2d}/(1-w)+\mathbf{P}_{2i}\\
\mathbf{K} &= \mathbf{P}_1\mathbf{H}^T\left(\mathbf{H}\mathbf{P}_1\mathbf{H}^T+\mathbf{P}_2\right)^{-1}\\
\mathbf{X} &= \mathbf{X}_1+\mathbf{K}(\mathbf{X}_2-\mathbf{H}\mathbf{X}_1)\\
\mathbf{P} &= (\mathbf{I}-\mathbf{K}\mathbf{H})\mathbf{P}_1\\
\mathbf{P}_i &= (\mathbf{I}-\mathbf{K}\mathbf{H})\mathbf{P}_{1i}(\mathbf{I}-\mathbf{K}\mathbf{H})^T+\mathbf{K}\mathbf{P}_{2i}\mathbf{K}^T\\
\mathbf{P}_d &= \mathbf{P}-\mathbf{P}_i.
\end{aligned}\tag{6}$$

Proof: From Corollary 1, we have

$$\begin{aligned}
\mathbf{P} &= \left(\mathbf{P}_1^{-1}+\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{H}\right)^{-1}\\
&= \left[\mathbf{P}_1^{-1}\left(\mathbf{I}+\mathbf{P}_1\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{H}\right)\right]^{-1}\\
&= (\mathbf{I}+\mathbf{P}_1\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{H})^{-1}\mathbf{P}_1\\
&= \left\{\sum_{i=0}^{\infty}\left(-\mathbf{P}_1\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{H}\right)^i\right\}\mathbf{P}_1\\
&= \left\{\mathbf{I}-\mathbf{P}_1\mathbf{H}^T\left[\sum_{i=0}^{\infty}\left(-\mathbf{P}_2^{-1}\mathbf{H}\mathbf{P}_1\mathbf{H}^T\right)^i\right]\mathbf{P}_2^{-1}\mathbf{H}\right\}\mathbf{P}_1\\
&= \left\{\mathbf{I}-\mathbf{P}_1\mathbf{H}^T\left(\mathbf{I}+\mathbf{P}_2^{-1}\mathbf{H}\mathbf{P}_1\mathbf{H}^T\right)^{-1}\mathbf{P}_2^{-1}\mathbf{H}\right\}\mathbf{P}_1\\
&= \left\{\mathbf{I}-\mathbf{P}_1\mathbf{H}^T(\mathbf{P}_2+\mathbf{H}\mathbf{P}_1\mathbf{H}^T)^{-1}\mathbf{H}\right\}\mathbf{P}_1\\
&= (\mathbf{I}-\mathbf{K}\mathbf{H})\mathbf{P}_1\\
\mathbf{P}\mathbf{H}^T\mathbf{P}_2^{-1} &= \left\{\mathbf{P}_1-\mathbf{P}_1\mathbf{H}^T(\mathbf{P}_2+\mathbf{H}\mathbf{P}_1\mathbf{H}^T)^{-1}\mathbf{H}\mathbf{P}_1\right\}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{H}^T\mathbf{P}_2^{-1}\\
&= \mathbf{P}_1\mathbf{H}^T\left\{\mathbf{I}-(\mathbf{P}_2+\mathbf{H}\mathbf{P}_1\mathbf{H}^T)^{-1}\mathbf{H}\mathbf{P}_1\mathbf{H}^T\right\}\mathbf{P}_2^{-1}\\
&= \mathbf{P}_1\mathbf{H}^T(\mathbf{P}_2+\mathbf{H}\mathbf{P}_1\mathbf{H}^T)^{-1}\mathbf{P}_2\mathbf{P}_2^{-1}\\
&= \mathbf{P}_1\mathbf{H}^T(\mathbf{P}_2+\mathbf{H}\mathbf{P}_1\mathbf{H}^T)^{-1} = \mathbf{K}\\
\mathbf{X} &= \mathbf{P}(\mathbf{P}_1^{-1}\mathbf{X}_1+\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{X}_2)\\
&= \mathbf{P}\mathbf{P}_1^{-1}\mathbf{X}_1+\mathbf{P}\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{X}_2\\
&= (\mathbf{I}-\mathbf{K}\mathbf{H})\mathbf{X}_1+\mathbf{K}\mathbf{X}_2 = \mathbf{X}_1+\mathbf{K}(\mathbf{X}_2-\mathbf{H}\mathbf{X}_1)\\
\mathbf{P}_i &= \mathbf{P}\left(\mathbf{P}_1^{-1}\mathbf{P}_{1i}\mathbf{P}_1^{-1}+\mathbf{H}^T\mathbf{P}_2^{-1}\mathbf{P}_{2i}\mathbf{P}_2^{-1}\mathbf{H}\right)\mathbf{P}\\
&= (\mathbf{I}-\mathbf{K}\mathbf{H})\mathbf{P}_{1i}(\mathbf{I}-\mathbf{K}\mathbf{H})^T+\mathbf{K}\mathbf{P}_{2i}\mathbf{K}^T\\
\mathbf{P}_d &= \mathbf{P}-\mathbf{P}_i.
\end{aligned}$$

Therefore, we have (6). □

Let $\mathbf{P}_{1d}$ and $\mathbf{P}_{2d}$ be zero. Then, the split CIF in (6) will become the same to the formalism of the Kalman filter for the partial observation case.

III. Decentralized Vehicle Localization Using the Split CIF

To present a concrete application example of the split CIF, we describe a general architecture of decentralized vehicle localization, which is based on our previous works [30], and a new strategy of using the split CIF to handle an unknown temporal correlation among APMs. Here, we only briefly review the part of the previous works [30] in order to keep the integrity of the presented architecture. On the other hand, we detail more about the new strategy, including the reason why we had to take into account the temporal data correlation.

A. Basic Functionalities for Decentralized Vehicle Localization

For a generic vehicle, the following functionalities are assumed available, which are abstracted from field practice based on their feasibility in reality. These functionalities enable a vehicle to perform localization by itself or to perform cooperative localization with its neighboring vehicles.

Absolute positioning function: A vehicle can obtain its position measurements with respect to an absolute reference, i.e., APMs. This function can be realized in different ways, such as via GPS.

Relative positioning function: A vehicle can estimate the relative pose of neighboring vehicles. This function can be realized based on perceptive sensors such as laser scanners. Here, the term *neighbor* indicates being within the perception range.

Motion monitoring function: A vehicle is equipped with motion sensors that output measurements on its motion state. For example, motion data can be provided by odometers, accelerometers, gyroscopes, etc.

Communication function: Data can be shared among neighboring vehicles. An effective communication range is usually much larger than an effective perception range; therefore, the aforementioned term *neighbor* has an additional meaning of being able for communication.

Timestamping function: A vehicle can timestamp their data according to an absolute time reference. For example, the GPS universal time, even provided by a low-cost GPS, can fairly assume the role of an absolute time reference.

B. Decentralized Vehicle State

The distributed formalism for each vehicle is the same; therefore, the formalism will be described only from the perspective of one single vehicle (referred to as *ego-vehicle*). The ego-vehicle state is denoted in the following split form:

$$\mathbf{E} = \{\mathbf{X}_E, \mathbf{P}_{dE} + \mathbf{P}_{iE}\}.$$

Here, $\mathbf{X}_E = [x_e, y_e, \theta_e]^T$ denotes the pose of the ego-vehicle and (x, y) and θ denote the position and the heading angle of the vehicle in the absolute reference, respectively. Subscripts d and i denote the correlated component and the independent component of the covariance $\mathbf{P}_E$, respectively.

C. State Evolution

The vehicle motion is modeled according to the kinematic bicycle model, which is compactly represented by function G. The state of the ego-vehicle $\mathbf{X}_{E(t)}$ can be evolved using its motion measurements $\mathbf{u}_{E(t)}$ as follows (see [30] for more details):

$$\mathbf{X}_{E(t)} = G(\mathbf{X}_{E(t-1)}, \mathbf{u}_{E(t)}). \tag{7}$$

$\mathbf{u}_{E(t)}$ is assumed to follow the Gaussian distribution $\mathrm{N}(\mathbf{0}, \mathbf{\Sigma}_u)$. The state covariance evolution is given as

$$\begin{aligned}\mathbf{P}_{E(t)} &= \mathbf{G}_{Xe}\mathbf{P}_{E(t-1)}\mathbf{G}_{Xe}^T + \mathbf{G}_u\mathbf{\Sigma}_u\mathbf{G}_u^T + \mathbf{R}_{E(t)}\\ \mathbf{P}_{iE(t)} &= \mathbf{G}_{Xe}\mathbf{P}_{iE(t-1)}\mathbf{G}_{Xe}^T + \mathbf{G}_u\mathbf{\Sigma}_u\mathbf{G}_u^T + \mathbf{R}_{iE(t)}.\end{aligned} \tag{8}$$

$\mathbf{G}_{Xe}$ and $\mathbf{G}_u$ are the Jacobian matrices of function G with respect to $\mathbf{X}_E$ and $\mathbf{u}_E$, respectively. $\mathbf{R}_{E(t)}$ and $\mathbf{R}_{iE(t)}$ characterize the motion model error. Since the split CIF is used, the total covariance $\mathbf{P}_{E(t)}$ and the independent part $\mathbf{P}_{iE(t)}$ are evolved.

D. State Update With APMs

Let the APM for the ego-vehicle be denoted as $\mathbf{Z}_A = (x_A, y_A)$. The measurement model is described as $\mathbf{H}_{A(2\times 3)} = [\mathbf{I}_{2\times 2}\ \mathbf{0}]$, i.e.,

$$\mathbf{Z}_{A(t)} = \mathbf{H}_{A(2\times 3)}\mathbf{X}_{E(t)} + \mathbf{R}_A.$$

As explained earlier, there might be certain temporal correlation among APMs in a long enough time interval. In order to show that this temporal data correlation is not negligible, we can perform an imaginary experiment as follows.

Given a long enough time interval, suppose that APM error $\mathbf{R}_A$ in this time interval consists of bias vector $\mathbf{b}$ (unknown) and Gaussian white noise $\mathbf{n}$. Let the vehicle be strictly stationary. Then, we only examine the position component. Let vehicle localization be carried out using the Kalman filter that assumes APM independence (i.e., neglecting the temporal data correlation) as follows.

(*State evolution*)

$$\mathbf{X}_{E(t)} = \mathbf{X}_{E(t-1)} \qquad \mathbf{P}_{E(t)} = \mathbf{P}_{E(t-1)}.$$

(*State update with APM*)

$$\begin{aligned}\mathbf{K} &= \mathbf{P}_{E(t)}(\mathbf{P}_{E(t)} + \mathbf{\Sigma}_A)^{-1}\\ \mathbf{X}_{E(t)} &= \mathbf{X}_{E(t)} + \mathbf{K}(\overline{\mathbf{X}} + \mathbf{b} + \mathbf{n}_{(t)} - \mathbf{X}_{E(t)})\\ \mathbf{P}_{E(t)} &= (\mathbf{I} - \mathbf{K})\mathbf{P}_{E(t)}.\end{aligned}$$

Closed-form expression of the state estimate can be derived from the above recursive Kalman filter procedures, i.e.,

$$\begin{aligned}\mathbf{X}_{E(t)} &= (t\mathbf{I} + \mathbf{\Sigma}_A\mathbf{P}_0^{-1})^{-1}\\ &\qquad \left\{\mathbf{\Sigma}_A\mathbf{P}_0^{-1}\mathbf{X}_0 + \sum_{i=1}^{t}(\overline{\mathbf{X}} + \mathbf{b} + \mathbf{n}_{(i)})\right\}\\ \mathbf{P}_{E(t)} &= (t\mathbf{I} + \mathbf{\Sigma}_A\mathbf{P}_0^{-1})^{-1}\mathbf{\Sigma}_A.\end{aligned}$$

As t increases, then, we have

$$\lim_{t\to\infty}[\widetilde{\mathbf{X}}_{E(t)}\widetilde{\mathbf{X}}_{E(t)}^T] = \mathbf{b}\mathbf{b}^T$$
$$\lim_{t\to\infty}\mathbf{P}_{E(t)} = \mathbf{0}.$$

In other words, no matter how the initial value $\{\mathbf{X}_0, \mathbf{P}_0\}$ and APM covariance $\mathbf{\Sigma}_A$ are set, the estimated covariance matrix will asymptotically converge to zero, whereas the true covariance of the estimated state vector will asymptotically converge to a semi-positive matrix. Apparently, this estimate is unreasonably overconfident. Therefore, we had better to take into account the temporal data correlation.

The split CIF enjoys a natural mechanism to handle unknown temporal correlation among APMs. The new strategy is as follows. Let the covariance for the error $\mathbf{R}_A$ be represented as $(\mathbf{\Sigma}_{Ad} + \mathbf{\Sigma}_{Ai})$. $\mathbf{\Sigma}_{Ai}$ represents the degree of error independence, and $\mathbf{\Sigma}_{Ad}$ represents the degree of temporal data correlation. The split CIF is carried out as

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{P}_{dE(t)}/w + \mathbf{P}_{iE(t)} \\ \mathbf{P}_2 &= \mathbf{\Sigma}_{Ad}/(1-w) + \mathbf{\Sigma}_{Ai} \\ \mathbf{K} &= \mathbf{P}_1\mathbf{H}_A^T(\mathbf{H}_A\mathbf{P}_1\mathbf{H}_A^T + \mathbf{P}_2)^{-1} \\ \mathbf{X}_{E(t)} &= \mathbf{X}_{E(t)} + \mathbf{K}(\mathbf{Z}_{A(t)} - \mathbf{H}_A\mathbf{X}_{E(t)}) \\ \mathbf{P}_{E(t)} &= (\mathbf{I} - \mathbf{K}\mathbf{H}_A)\mathbf{P}_1 \\ \mathbf{P}_{iE(t)} &= (\mathbf{I} - \mathbf{K}\mathbf{H}_A)\mathbf{P}_{iE(t)}(\mathbf{I} - \mathbf{K}\mathbf{H}_A)^T + \mathbf{K}\mathbf{\Sigma}_{Ai}\mathbf{K}^T \\ \mathbf{P}_{dE(t)} &= \mathbf{P}_{E(t)} - \mathbf{P}_{iE(t)}. \end{aligned} \tag{9}$$

Let $\mathbf{\Sigma}_{Ad}$ be always zero. Then, (7)–(9) will be reduced to the formalism of the Kalman-filter-based (single) vehicle localization. In other words, the formalism of (7)–(9) is a generalization of the Kalman-filter-based vehicle localization, which enables a mechanism to handle unknown temporal correlation among APMs.

E. State Update With Relative Positioning Estimates and the States of Other Vehicles

The ego-vehicle can receive extra data from its neighboring vehicles. Without loss of generality, let a generic neighboring vehicle be referred to as *communicated ego-vehicle*, whose state is denoted as $\{\mathbf{X}_{CE}, \mathbf{P}_{dCE} + \mathbf{P}_{iCE}\}$ from the perspective of the ego-vehicle. (Subscript C is used here to distinguish between the state estimate imported and that of the ego-vehicle itself.) $\mathbf{X}_{CE} = [x_{ce}, y_{ce}, \theta_{ce}]^T$ denotes the pose of the communicated ego-vehicle.

According to the relative positioning function assumption, the relative pose between the ego-vehicle and the communicated ego-vehicle (denoted as $\mathbf{Z}_R = [\Delta x_R, \Delta y_R, \Delta\theta_R]^T$) can be estimated. The state of the ego-vehicle can be indirectly computed by compounding the relative pose estimate and the state estimate of the communicated ego-vehicle. This indirect estimate of the ego-vehicle state is denoted as $\mathbf{X}_{EI}$ with covariance $\mathbf{P}_{EI}$, i.e.,

$$\begin{aligned} \mathbf{X}_{EI} &= \mathbf{X}_{CE} \oplus \mathbf{Z}_R \\ \mathbf{P}_{EI} &\approx \left(\frac{\partial \mathbf{X}_{EI}}{\partial \mathbf{X}_{CE}}\right)\mathbf{P}_{CE}\left(\frac{\partial \mathbf{X}_{EI}}{\partial \mathbf{X}_{CE}}\right)^T \\ &\quad + \left(\frac{\partial \mathbf{X}_{EI}}{\partial \mathbf{Z}_R}\right)\mathbf{\Sigma}_R\left(\frac{\partial \mathbf{X}_{EI}}{\partial \mathbf{Z}_R}\right)^T \\ \mathbf{P}_{iEI} &\approx \left(\frac{\partial \mathbf{X}_{EI}}{\partial \mathbf{X}_{CE}}\right)\mathbf{P}_{iCE}\left(\frac{\partial \mathbf{X}_{EI}}{\partial \mathbf{X}_{CE}}\right)^T \\ &\quad + \left(\frac{\partial \mathbf{X}_{EI}}{\partial \mathbf{Z}_R}\right)\mathbf{\Sigma}_{iR}\left(\frac{\partial \mathbf{X}_{EI}}{\partial \mathbf{Z}_R}\right)^T. \end{aligned}$$

$\partial\mathbf{X}_{EI}/\partial\mathbf{X}_{CE}$ and $\partial\mathbf{X}_{EI}/\partial\mathbf{Z}_R$ are the Jacobian matrices of $\mathbf{X}_{EI}$ with respect to $\mathbf{X}_{CE}$ and $\mathbf{Z}_R$ (see [30]). Then, $\mathbf{X}_E$ is fused with $\mathbf{X}_{EI}$ using the split CIF as follows (the w is determined by minimizing the determinant of the new covariance):

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{P}_{dE(t)}/w + \mathbf{P}_{iE(t)} \\ \mathbf{P}_2 &= \mathbf{P}_{dEI(t)}/(1-w) + \mathbf{P}_{iEI(t)} \\ \mathbf{K} &= \mathbf{P}_1(\mathbf{P}_1 + \mathbf{P}_2)^{-1} \\ \mathbf{X}_{E(t)} &= \mathbf{X}_{E(t)} + \mathbf{K}(\mathbf{X}_{EI(t)} - \mathbf{X}_{E(t)}) \\ \mathbf{P}_{E(t)} &= (\mathbf{I} - \mathbf{K})\mathbf{P}_1 \\ \mathbf{P}_{iE(t)} &= (\mathbf{I} - \mathbf{K})\mathbf{P}_{iE(t)}(\mathbf{I} - \mathbf{K})^T + \mathbf{K}\mathbf{P}_{iEI(t)}\mathbf{K}^T \\ \mathbf{P}_{dE(t)} &= \mathbf{P}_{E(t)} - \mathbf{P}_{iE(t)}. \end{aligned} \tag{10}$$

F. General Architecture

Vehicle localization is realized in a full decentralized (or distributed) manner. From the perspective of a generic vehicle, the localization procedures are as follows:

> *The vehicle evolves its state estimate (in split form) using its motion measurements, according to* (7) *and* (8).*When the vehicle has an absolute positioning measurement of its own, it can update its state estimate according to* (9).*When the vehicle receives data from a neighboring vehicle, it can update its state estimate according to* (10).

The presented decentralized vehicle localization architecture is rather simple and enables a vehicle to enjoy full flexibility in handling new data. When the vehicle has some new data from itself or from another vehicle, it can simply use the new data to evolve or update its state estimate. No controlling of the data flow within vehicle networks is needed. Despite the simplicity of this architecture, the risk of overconfident estimation (or even overconvergence) due to the spatial correlation or the temporal correlation among the estimates can be essentially removed because the risk is removed directly by the split CIF during data fusion.

IV. SIMULATIONS

In this section, we evaluate the performance of the presented decentralized vehicle localization architecture using the split CIF. Here, we do not intend to focus on the absolute performance of the presented architecture, which in reality depends on ad hoc vehicle sensor configurations and environment conditions. Instead, we present a simulation-based comparative study, which is similar to the methodology of experimentation in [30]. This comparative study is to demonstrate the potential advantages of the split CIF and how it can be useful in concrete applications where unknown data correlation exists.

The advantages of using the split CIF to handle multivehicle spatial correlation in cooperative localization have been demonstrated in the comparative study in [30]. Therefore, in this section, we only focus on demonstrating how the new strategy (9), i.e., using the split CIF to handle unknown temporal correlation among APMs, can benefit single vehicle localization (SL) and how this new strategy can bring marginal improvements over the method in [30] in cooperative localization.

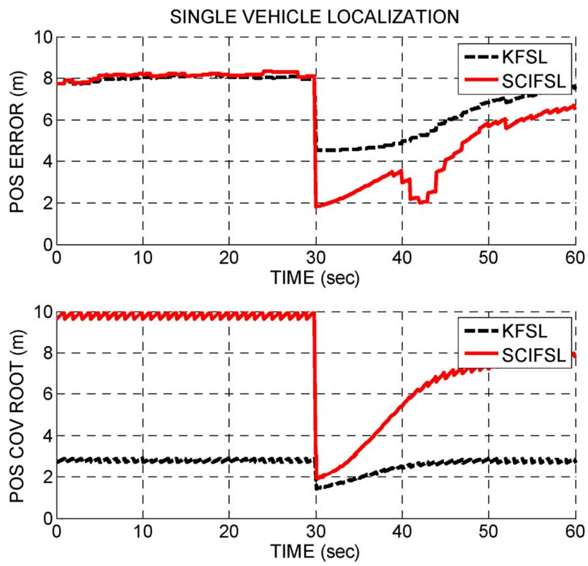

Fig. 1. (Top) Position error. (Bottom) Position covariance root.

Experiments have been carried out under different simulation conditions. Since APMs mainly affect vehicle localization performance, without loss of generality, we only adjust the condition of APMs while fixing other simulation conditions (the same to those in [30]). The different APM conditions used for tests will be specified, respectively, in the following.

A. SL Scenario

Given a vehicle moving at 50 km/h in simulation, two methods were executed simultaneously on the same synthetic data: 1) the split-CIF-based localization (SCIFSL), i.e., (7)–(9), and 2) the Kalman-filter-based localization (KFSL), i.e., the special case of (7)–(9) neglecting all correlated components.

The positioning errors of the two localization processes during a time interval of 60 s were recorded for comparison. The APM condition for this time interval is set as follows. Only at the 30th second, the vehicle absolute positioning standard error is momentarily set to be 2 m; at all other instants, the absolute positioning standard error is set to be 10 m with a bias subcomponent. A large number of independent trials (50 rounds) have been carried out. The standard magnitude of the bias error vector for all the trials is set to be 8 m, whereas the bias error vector is always unknown. Fig. 1 shows the result quadratic mean of the trials.

Fig. 1 (top) shows the positioning error of the two localization processes. Fig. 1 (bottom) shows the root of the estimated position covariance of the two localization processes. The horizontal coordinates indicate the time interval of 60 s. As shown in Fig. 1 (top), the KFSL and the SCIFSL have almost the same positioning errors during the first 30 s, whereas the SCIFSL has apparently smaller positioning errors than the KFSL during the last 30 s. The reason can be explained by Fig. 1 (bottom). The KFSL results in an apparently overconfident estimation; when a better APM is momentarily available at the 30th second, the KFSL cannot quickly adapt the estimate to the better measurement because it has unreasonable confidence in the historical estimate. In contrast, the SCIFSL can keep consistent estimation; thus, it can properly adapt the estimate to better measurement.

B. Cooperative Localization Scenario

The scenario is the same as that in [30]. Let a chain of eight vehicles (numbering in order) move on the same road in the same direction. Each vehicle is only able to observe and cooperate with its immediate neighboring vehicles; for example, the fourth vehicle can only cooperate with the third vehicle and the fifth vehicle (the network is decentralized).

Three methods were simultaneously executed on the same synthetic data: 1) the presented decentralized vehicle localization method (SCIFCL2), i.e., (7)–(10); 2) the method in [30] (SCIFCL), i.e., the special case of (7)–(10) neglecting temporal correlation among APMs; and 3) SL or noncooperative localization, i.e., (7)–(9).

Let all the vehicles have the same absolute positioning ability, except the first vehicle that has better absolute positioning ability. More specifically, the absolute positioning standard error for the first vehicle is set to be 0.5 m, whereas that for the second vehicle to the eighth vehicle is set to be 10 m with a bias subcomponent (set the same way as in Section IV-A). Fifty rounds of independent trials have also been carried out; the result quadratic mean of the trials is shown in Fig. 2.

For the second vehicle to the eighth vehicle, the SCIFCL positioning errors are much smaller than the SL positioning errors, which demonstrate the advantage of cooperative localization over SL. Although the third vehicle to the eighth vehicle cannot cooperate with the first vehicle directly, they can still indirectly benefit from the first vehicle. As explained in [30], the split-CIF-based cooperative localization does not rely on any controlling of the data flow; it enables good localization results to be naturally spread within a connective vehicle network. The SCIFCL2 further brings marginal improvement over the original SCIFCL, particularly noticeable for the vehicles comparatively far away from the first vehicle.

Some explanations are given for this marginal improvement. The SCIFCL suffers from a certain degree of estimation inconsistency, due to its neglect of the temporal correlation among APMs (see Section IV-A). It is worth noting that this inconsistency will not be amplified during circular reasoning among the vehicles because the split CIF itself will not add any extra confidence on the fusion estimate. In contrast, the SCIFCL2 that improves the SCIFCL by taking the temporal data correlation into account can prevent the second vehicle to the eighth vehicle from having overconfidence on their own data. Thus, the SCIFCL2 in general enables the second vehicle to the eighth vehicle to be directly or indirectly more responsive to the good estimates originating from the first vehicle, which accounts for the marginal advantage of the SCIFCL2 over the SCIFCL.

C. Discussion

In reality, on one hand, the scenarios of vehicle localization are not limited to the two given examples and can be

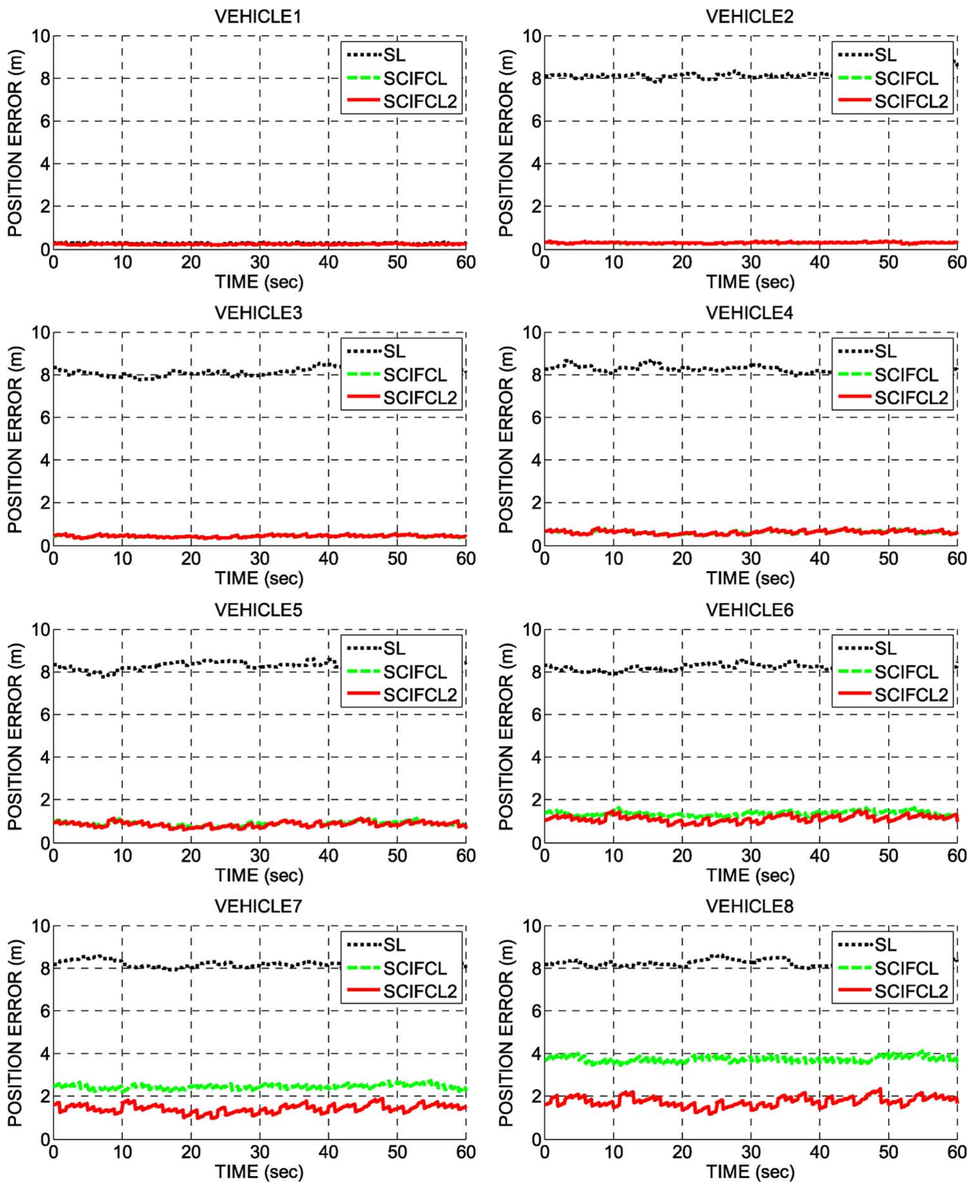

Fig. 2. Performance of the SL, the SCIFCL, and the SCIFCL2.

of infinite possibilities. On the other hand, the two given examples of simulation tests well demonstrate the features of the split CIF and how it can potentially benefit vehicle localization.

A point worth noting is that presenting the SL tests and the cooperative localization tests separately does not imply that the two kinds of vehicle localization should be separated clearly from each other. In the presented decentralized vehicle localization architecture, the two kinds of vehicle localization are not distinguished. When a vehicle has no neighboring vehicle for cooperation or wants to rely only on its own data, then the localization process of this vehicle is naturally reduced to SL (or noncooperative localization). When a vehicle has neighboring vehicles for cooperation, then cooperative localization can take place. This point reflects the full flexibility of the presented architecture using the split CIF. In one word, a vehicle can use all available data as preferred. (See [41] for details of some field tests in addition to the simulation demonstration.)

V. Conclusion

In this paper, we have provided a theoretical foundation for the split CIF. First, we have pointed out some limitations of the consistency definition in the original CIF and proposed the augmented version of the consistency definition, which is coined as the *split consistency*. Then, we have provided a theoretical proof for the fusion consistency of the split CIF. Moreover, we have derived the formalism of the split CIF for the partial observation case.

In addition to the description of the split CIF theory, we have presented a general architecture of decentralized vehicle localization, which serves as a concrete application example of the split CIF to demonstrate the advantages of the split CIF and how it can potentially benefit vehicle localization (noncooperative and cooperative). The presented architecture itself will be also a valuable guide for real implementation.

The split CIF is a useful tool for general data fusion. It can reasonably handle both known independent information and unknown correlated information in source data. It has the potential to be applied in a variety of tasks in the ITS field, as the Kalman filter does (noting that the Kalman filter can be regarded as a special case of the split CIF). Therefore, the works presented in this paper can serve as a baseline for prospective research works where the split CIF is intended to be applied.

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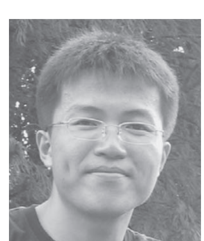

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