Ch1 Maps

Thursday, 4 September 2025 4:21 PM

1.1 Parametric Values $\mathbb{R} \to \mathbb{R}^n$

- parameter is input, often referred as time
- output is POSITION at time t
- Unit circle $\gamma_2(t) = (\cos(t), \sin(t))$
- Helix $\gamma_3(t) = (\cos(t), \sin(t), t)$

Velocity: $\gamma'(t) = \lim_{h\to 0} \dfrac{\gamma(t+h) - \gamma(t)}{h}$

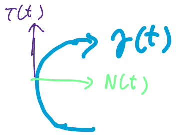
Speed (magnitude of velocity): $\|\gamma'(t)\| = \sqrt{\gamma_1'(t)^2 + \dots + \gamma_n'(t)^2}$

Unit Tangent Vector: $T(t) = \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$

Acceleration: $\gamma''(t) = \lim_{h\to 0} \dfrac{\gamma'(t+h) - \gamma'(t)}{h} = (\gamma_1''(t), \dots, \gamma_n''(t))$

- Derivative of $\gamma(t)$ is calculated by taking each component's derivative.

- Fronet frame in 3D.
  - $N(t) = \dfrac{T'(t)}{\|T'(t)\|}$ (T and N are orthogonal)



→ $T(t), N(t)$ are linearly independent.

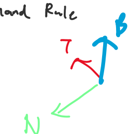
$a \times b = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$

$B = T \times N$ (cross product)

- Binormal unit vector B s.t. $\{T, N, B\}$ form positively-oriented order orthogonal basis in $\mathbb{R}^3$.

  Fronet Frame

- Right-hand Rule



- Trace of parametric curve is image of γ.

$\therefore \gamma(I)$, for $\gamma: I \to \mathbb{R}^n$.

1.2 Real-valued functions $\mathbb{R}^n \to \mathbb{R}$

physics: scalar fields
scalar functions
scalar potentials.

- aka. real-valued functions of n variables.

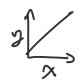
- Real-valued functions that ≥ 0 are called 'densities'
$\rho = \frac{m}{V}$

Examples: 1. multivariable polynomial
2. piecewise.
3. norm function.

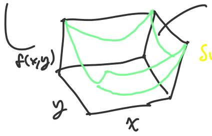
Definition 1.2.11 Graph of function
$f: A \to \mathbb{R}$ is the set in $\mathbb{R}^{n+1}$ given by
let $A \subseteq \mathbb{R}^n$ $\{(x, f(x)) : x \in A\}$

$(x, f(x))$ belongs to $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$,
since $x \in \mathbb{R}^n$ and $f(x) \in \mathbb{R}$.

just like $f(x) = y$.



2D!
Input: 1D
Output: 2D



bowl shaped.
Surface plot.

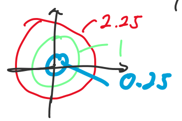
Definition 1.2.14 k-level set (less 1 dimension than graph)
Let $A \subseteq \mathbb{R}^n$ and $f: A \to \mathbb{R}$ be real-valued function.
Fix $k \in \mathbb{R}$.
k-level-set: $\{x \in \mathbb{R}^n : f(x) = k\}$ (same dimension as input)
- a level set in $\mathbb{R}^2$ is called a contour.

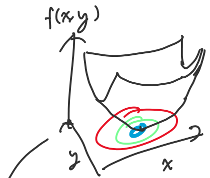
Example
$f: \mathbb{R}^2 \to \mathbb{R}$
- $f(x,y) = x^2 + y^2$

1-level set unit circle.

(−1)-level-set = $\emptyset$ empty.

(2.25)-level-set



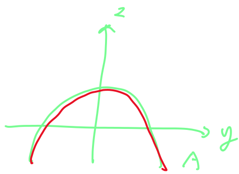


Definition 1.2.19
Let $A \subseteq \mathbb{R}^2$ and $f: A \to \mathbb{R}$ be real-valued function.
- For fixed $a \in \mathbb{R}$, $b \in \mathbb{R}$
  - A-slice at a of graph of f is set
  $\{(y, z) \in \mathbb{R}^2 : (a, y) \in A,\ z = f(a, y)\}$
  - y-sliced at b of graph of f is set
  $\{(x, z) \in \mathbb{R}^2 : (x, b),\ z = f(x, b)\}$

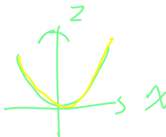
For 3D...
For fixed $c \in \mathbb{R}$:
z-slice at c of graph f is set
$\{(x, y, w) \in \mathbb{R}^3 : (x, y, c) \in A,\ w = f(x, y, c)\}$

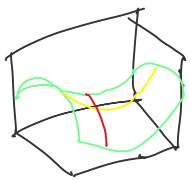
1. replace slice variable with slice function
2. replace with fixed value
3. replace any 'z' with constant c.

$f(x,y) = x^2 - y^2$.
- x-slice at $x = -1$
  - replace $z = (-1)^2 - y^2$



- x-slice at $y = 0$
replace $z = x^2 - 0^2$





Summary:
Graph: X constant
Point: All constant
Contour Plot: hold constant of output
Slice: hold constant one of the inputs.

1.3 Vector fields $\mathbb{R}^n \to \mathbb{R}^n$

Definition 1.3.1 n-dimensional vector fields

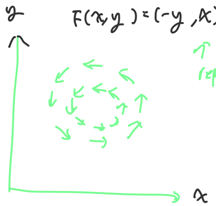
- is a function F with domain, codomain in $\mathbb{R}^n$.

$F(x) \in \mathbb{R}^n$ is a vector at the point $x \in \mathbb{R}^n$.

- Reason why vector fields have arrows, is because on a coordinate, value is in (x, y). plot as vector!

- Equivalent notations.
  - $F(x,y,z) = (x^2, yx, -z)$
    $= \langle x^2, yx, -z \rangle$
    $= [x^2, yx, -z]$
    $= x^2\hat{i} + yx\hat{j} - z\hat{k}$

$F(x,y) = (-y, x)$ — length is $\sqrt{(-y)^2 + (1)^2}$; represents arrow



1.4 Coordinate transformations

- Transformation: any map within domain and codomain in $\mathbb{R}^n$.
- Coordinate transformation: usually bijective
- $id(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$

1.4.10 Polar Coordinates

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(r, \theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$

r: Radius — can be negative; θ: polar angle

$\therefore$ If $r = 2$,

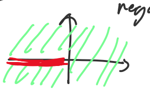
$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$
$= r^2(\sin^2\theta + \cos^2\theta)$
$= 2^2$
$= 4$

- **Lemma 1.4.10. Restriction of domain of polar coordinate**

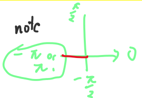
  $T(r, \theta) = (r\cos\theta, r\sin\theta)$

  Domain $(0, \infty) \times (-\pi, \pi)$ to codomain $\mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\}$ — remove

it means remove negative x-axis



note

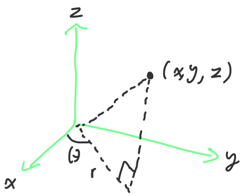


1.4.2 Cylindrical coordinates

$T(r, \theta, z) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y}, z)$ — 'usual rectangular coordinate'

r: polar radius, θ: angle

$T(r, \theta, z)$: In polar, — $(r\cos\theta, r\sin\theta, z)$: In rectangular,



a or b — $\times$ a and $\times$ b — $\times$ b have c.

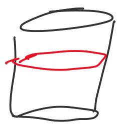
$\sim A$ and $\sim B$ or c and ~~not b.~~ $\sim B$

$\sim A$ and $\sim B$ or c and $\sim B$

If $r=2$, $T(A) = \{(2\cos\theta, 2\sin\theta, z) : \theta \in \mathbb{R}, z \in \mathbb{R}\}$

↳ z-slice is $(2\cos\theta, 2\sin\theta) : \theta \in \mathbb{R} \subseteq \mathbb{R}^2$.

$\therefore r=2$ is cylinder in $\mathbb{R}^3$



Lemma 1.4.13 Cylindrical Coordinate Transformation Restriction.

open interval

$T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$

maps $(0,\infty) \times (-\pi,\pi) \times \mathbb{R}$

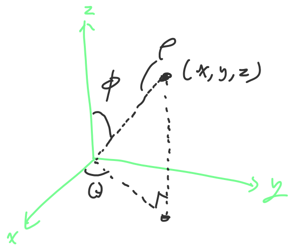
$\to \mathbb{R}^3 \setminus \{(x,0,z) : x \le 0, z \in \mathbb{R}\}$

1.4.3 Spherical Coordinates

$T(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$

$= (x, y, z)$

spherical radius, 'x' longitude, 'y' latitude



e.g. $(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}) = (-\sqrt{2}, \frac{\pi}{4}, \frac{3\pi}{2})$ in (ρ, θ, ϕ)

$(1, 1, 0)$

- $x^2 + y^2 + z^2 = \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi + \rho^2\cos^2\theta$
  $= \rho^2\sin^2\theta + \rho^2\cos^2\theta$
  $= \rho^2$
  $= 4$

Lemma 1.4.20

$T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$

maps $(0,\infty) \times (-\pi,\pi) \times (0,\pi)$ ← recall it is a line

$\to \mathbb{R}^3 \setminus \{(x,0,z) : x \le 0, z \in \mathbb{R}\}$

1.5 Parametric, explicit, and implicit form.

Parametric form with n-variables: $\mathbb{R}^n \to \mathbb{R}^m$ $(n<m)$

- $S \in \mathbb{R}^m$ can be in parametric form (with n-var.)
- If exist $A \subseteq \mathbb{R}^n$ and continuous map $g: A \to \mathbb{R}^m$ s.t.

$$S = \{ g(x) : x \in A \} = im(g)$$

set in $\mathbb{R}^m$

- S is parametised by g.

- not must n-manifold! e.g. $\mathbb{R}^2 \to \mathbb{R}^3$ could give image of a curve (a line bent), a point

Explicit form

Definition 1.58 Explicit form (in n-variables)

A set $S \in \mathbb{R}^m$ can be written in explicit form

if S is • a graph of continuous function $f: A \to \mathbb{R}^{m-n}$ where $A \subseteq \mathbb{R}^n$

Definition 1.56. Graph of f.

$(A \subseteq \mathbb{R}^n,\ f: A \to \mathbb{R}^{m-n}$ be continuous$)$

- $S = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^{m-n} : x \in A,\ y = f(x) \}$ is graph of f.

Implicit form $\mathbb{R}^n \to \mathbb{R}^m$ $(n > m)$

- A set $S \subseteq \mathbb{R}^n$ can be written in implicit form (with m equations)

if ① exist $c \in \mathbb{R}^m$
- set $A \subseteq \mathbb{R}^n$
- continuous $f: A \to \mathbb{R}^m$ s.t.

$$S = f^{-1}(\{c\}) = \{ x \in \mathbb{R}^n : f(x) = c \}$$

Summary.

① Parametric: generate points with parameters $(\mathbb{R}^2 \to \mathbb{R}^3)$

- $g(u,v) = \underbrace{(x, y, z)}_{\text{Set } S}$ $\qquad S = \{ g(\theta, \phi) : (\theta, \phi) \in \mathbb{R}^2 \}$

② Explicit: one variable ritten as function of others

- $z = f(x, y)$ $\qquad S = \{ (x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2 \}$

set S

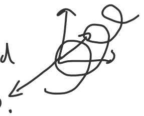
③ Implicit: set given as solution of an equation

- $F(x,y,z) = c$
- $x^2 + y^2 + z^2 = 1$

$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$

• Concept!

- a helix is 1D-manifold
- only one line, but exist 3D.



- degree of freedom / free variable is 1.

$(t, \cos t, \sin t)$

- only 1D-manifold, despite spanning $\mathbb{R}^3$.

1.6 Dimension Reduction

- $\Pi_3(x,y,z) = (x,y)$ = projection into xy-plane

Full Summary : $\mathbb{R}^n \to \mathbb{R}^m$

$n = 1$

- parametric curves

$m = 1$

- real-valued functions

$m = n$

- vector-fields, transformations, coordinate systems.

$n < m$

- parametric, explicit

$n > m$

- implicit form, dimension reduction