Ch10

Friday, 28 March 2025 11:37 AM

$A = PDP^{-1}$

P = the columns with eigenvector of A

D = The diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ & & \ddots \end{pmatrix}$

$P_1 = [v_1\ v_2]$, $D_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
$P_2 = [v_2\ v_1]$ $D_2 = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}$

Both can be used for diagonalization

- A and B are similar

$\Rightarrow B = P^{-1}AP$.

Characteristic polynomial:

$\chi_A(\lambda) = \det(A - \lambda I)$

$B\vec{x}_{new} = P^{-1}AP\vec{x}_{new}$.

① $P\vec{x}_{new}$: transform form: new basis to standard basis

$A(P\vec{x}_{new})$: Apply linear transformation under standard basis

$P^{-1}(AP\vec{x}_{new})$: Convert form: standard basis back to new basis

2 ways to check diagonalizable.

① Number of distinct λ = n.

② $GM(\lambda) = AM(\lambda)$ (e.g. $\lambda^2 = 0$ (repeated) AM=2, GM=2 v_1, v_2; $\lambda = 3$ v_3 AM=1 GM=1, 3 eigenvectors = n)

GM: dimension of Eigenspace of a specific λ value.

AM: no. of λ appearing as roots.

Exercise 10.1

(a) Let λ_1 and λ_2 be distinct eigenvalues of matrix A.
Suppose $\vec{v_1} \in E_{\lambda_1}$ and $\vec{v_2} \in E_{\lambda_2}$.
Show if $x_1\vec{v_1} + x_2\vec{v_2} = \vec{0}$, then

$x_1\lambda_1\vec{v_1} + x_2\lambda_2\vec{v_2} = \vec{0}$
$x_1\lambda_1\vec{v_1} + x_2\lambda_1\vec{v_2} = \vec{0}$

- We know: · A is n×n.
  - λ_1 + λ_2, are eigenvalues of A.
  - $\vec{v_1} \in E_{\lambda_1}$, $A\vec{v_1} = \lambda$
  - $\vec{v_2} \in E_{\lambda_2}$, $A\vec{v_2} = \lambda_2\vec{v_2}$

Proof.

1. $x_1\vec{v_1} + x_2\vec{v_2} = 0$
$A(x_1\vec{v_1} + x_2\vec{v_2}) = 0$ (As $A(\vec{0}) = \vec{0}$)
$x_1A\vec{v_1} + x_2A\vec{v_2} = 0$ (Linearity)
$x_1(\lambda_1\vec{v_1}) + x_2(\lambda_2\vec{v_2}) = 0$ (As $A\vec{v_1} = \lambda_1\vec{v_1}$, $A\vec{v_2} = \lambda_2\vec{v_2}$)
$\boxed{x_1\lambda_1\vec{v_1} + x_2\lambda_2\vec{v_2} = \vec{0}}$

2. $x_1\vec{v_1} + x_2\vec{v_2} = \vec{0}$
$\lambda_1(x_1\vec{v_1} + x_2\vec{v_2}) = \lambda_1 \cdot \vec{0}$
$\boxed{x_1\lambda_1\vec{v_1} + x_2\lambda_1\vec{v_2} = \vec{0}}$

(b) Take the difference of the equalities above to show $x_2 = 0$.

$x_1\lambda_1\vec{v_1} + x_2\lambda_2\vec{v_2} - x_1\lambda_1\vec{v_1} - x_2\lambda_1\vec{v_2} = 0$
$(\lambda_2 - \lambda_1)x_2\vec{v_2} = 0$
$(\lambda_2 x_2)\vec{v_2} = (\lambda_1 x_2)\vec{v_2}$
Since $\lambda_1 \neq \lambda_2$, $\vec{v_2} \neq \vec{0}$, $\therefore x_2 = 0$.

(c) Use similar argument to show $x_1 = 0$.

$\cdot\ x_2 = 0$ into $x_1 v_1 + x_2 v_2 = 0$

$x_1 \vec{v_1} = 0 \quad (\vec{v_1} \neq 0)$

$x_1 = 0$

Since $x_1 \vec{v_1} + x_2 \vec{v_2} = 0$ if and only if $x_1 = x_2 = 0$

$\therefore \vec{v_1}$ and $\vec{v_2}$ are linearly independent.

(d) Suppose that $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are eigenvector of matrix A with corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

If $\lambda_i \neq \lambda_j$, for any $i \neq j$. (all λ_i are distinct real no.)

Show $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a linearly independent set.

WTS: $x_1 \vec{v_1} + x_2 \vec{v_2} + x_3 \vec{v_3} = 0$ only has trivial solution.

$\cdot$ We know $A\vec{v_1} = \lambda_1 \vec{v_1}$

$A\vec{v_2} = \lambda_2 \vec{v_2}$

$A\vec{v_3} = \lambda_3 \vec{v_3}$

1. $A(x_1 \vec{v_1} + x_2 \vec{v_2} + x_3 \vec{v_3}) = 0 \quad (A(\vec{0}) = \vec{0})$

$x_1(A\vec{v_1}) + x_2(A\vec{v_2}) + x_3(A\vec{v_3}) = 0 \quad$ (From $\bullet$)

$x_1 \lambda_1 \vec{v_1} + x_2 \lambda_2 \vec{v_2} + x_3 \lambda_3 \vec{v_3} = 0 \quad - (1)$

2. $x_1 \vec{v_1} + x_2 \vec{v_2} + x_3 \vec{v_3} = 0$

$x_1 \lambda_3 \vec{v_1} + x_2 \lambda_3 \vec{v_2} + x_3 \lambda_3 \vec{v_3} = 0 \quad - (2)$

3. $(1)-(2)$: $x_1 \lambda_1 \vec{v_1} + x_2 \lambda_2 \vec{v_2} - x_1 \lambda_3 \vec{v_1} + x_2 \lambda_3 \vec{v_2}$

$(\lambda_1 - \lambda_3) x_1 \vec{v_1} + (\lambda_2 - \lambda_3) x_2 \vec{v_2} = 0$

Since $\lambda_1 \neq \lambda_2 \neq \lambda_3$, $x_1 = 0$, $x_2 = 0$.

(e) Prove $\{\vec{v_1}, \dots, \vec{v_n}\}$ is linearly independent for distinct eigenvalues.

Induction.

Base Case: $n = 2$ (proved)

Inductive step:

Assume it holds for $n-1$.

Show it holds for n.

Assume $x_1 \vec{v_1} + x_2 \vec{v_2} + \dots + x_n \vec{v_n} = \vec{0}$

$A(x_1 \vec{v_1} + x_2 \vec{v_2} + \dots + x_n \vec{v_n}) = \vec{0}$

$x_1 \lambda_1 \vec{v_1} + x_2 \lambda_2 \vec{v_2} + \dots + x_n \lambda_n \vec{v_n} = 0 \quad - (1)$

Note $x_1 \lambda_1 \vec{v_1} + x_2 \lambda_1 \vec{v_2} + \dots + x_n \lambda_1 \vec{v_n} = 0$ (2) as $\lambda_1 (x_1 \vec{v_1} + \dots) = \lambda_1 \cdot 0$

- $(1)-(2)$:

$x_2(\lambda_2 - \lambda_1)\vec{v_n} + \dots + x_n(\lambda_n - \lambda_1)\vec{v_n} = 0$

Since $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$.

$\therefore x_2 = x_3 = \dots = x_n = 0$

$\therefore x_1 \vec{v_1} = 0$, and $x_1 = 0$.

$\therefore \{\vec{v_1}, \dots, \vec{v_n}\}$ is linearly independent

P10.2

- Show if A and B are similar matrices $B = P^{-1}AP$

then $\chi_A = \chi_B$.

- Conclude similar matrices have same eigenvalues.

1. Let $A, B \in \mathbb{R}^{n\times n}$.

- let $B = P^{-1}AP$ (similar). WTS $\chi_A(\lambda) = \chi_B(\lambda)$ $= \det(A - \lambda I)$

$\downarrow$

$\chi_B(\lambda) = \det(B - \lambda I)$

$= \det(P^{-1}AP - \lambda I)$

$= \det(P^{-1}AP - \lambda(P^{-1}IP))$

$= \det(P^{-1}AP - P^{-1}(\lambda I)P)$

$= \det(P^{-1}(A - \lambda I)P)$

$= \det(P^{-1})\det(A-\lambda I)\det(P)$

$= \det(A - \lambda I)$ (as $\det(P^{-1})\det(P) = \frac{1}{\det(P)} \cdot \det(P) = 1$)

$= \chi_A(\lambda)$

Since $\chi_A(\lambda) = \chi_B(\lambda)$

- they have same eigenvalues.

P10.6 True or False:

If A is diagonalizable, then A must be invertible.

False.

- Assume $\exists P$ s.t.

$A = PDP^{-1}$, where D is diagonal matrix, P is invertible.

- Invertible means $\det(A) \neq 0$, i.e. A has NO ZERO Eigenvalues.

☆ Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, 2x2 matrix

- diagonal as all non-diagonal entries are zero

- not invertible as $\det(A) = 0$

$\det(A - \lambda I) = 0$

$-\lambda(1-\lambda) = 0$

$\lambda = 0$ or $\lambda = 1$.

$\therefore$ has 2 eigenvectors = no. of columns of A.

$\Rightarrow$ Diagonalizable.

☆

10.7 A horizontal shear is function $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + my \\ y \end{pmatrix}$

A vertical shear is function $G: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$G\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y+mx \end{pmatrix}$

Show that the shear functions are not diagonalizable when $m \neq 0$.

- Let $A = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$

E1 $\det(A - \lambda I) = (1-\lambda)^2 = \lambda = 1.$ $(AM = 2)$

$(A - I)\vec{v} = 0$

$\left(\begin{array}{cc|c} 0 & m & 0 \\ 0 & 0 & 0 \end{array}\right)$

$my = 0$, $y = 0$.

$\therefore \begin{bmatrix} x \\ 0 \end{bmatrix}$ for $x \in \mathbb{R}$ is $\in E_{\lambda_1}$.

$\therefore$ Eigenspace $= \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, $GM = \dim(\text{eigenspace}) = 1$, .
$AM = 2$.

$\boxed{GM \neq AM \Rightarrow \text{Not diagonalizable}}$

E2 $\det(B - \lambda I) = (1 - \lambda^2) = 0$
$\lambda^2 = 1$
$\lambda = 1$ $(AM = 2)$

$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ m & 0 & 0 \end{array}\right)$

$mx = 0$
$x = 0.$ | y can be anything.
$\therefore \begin{pmatrix} 0 \\ y \end{pmatrix}$ for $y \in \mathbb{R}$, is $\in E_{\lambda_1}$.

$\therefore$ Eigenspace $E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$, $GM = \dim(E_{\lambda_1}) = 1$.

$\therefore$ Not diagonalizable.