Ch7

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7.3 Supremum and infimum.

Let $A \subseteq \mathbb{R}$.
Let $c \in \mathbb{R}$.

- c is upper bound of A => no values in A is greater than c. $\forall x \in A, x \leq c$.
- c is least upper bound / supremum of A
  - => c is upper bound of A
  - => if b is upper bound of A => $c \leq b$.

=> If supremum of A in A, then it is maximum

- A is bounded above = has at least 1 upper bound.

Least Upper Bound Principle (LUB)

- A is bounded above
- A is not empty

=> has a least upper bound.

(bounded above = have supremum)

⇓

By substituting the range of a function into A,

- If f is bounded above on I
  => has supremum on I.

(not

7.4 Supremum & Infimum of Function.

Supremum $\{f(x) \mid x \in I\}$

$= \sup_{x \in I} f(x)$ = the supremum of its range.

EVT.

- If f is continuous on $[a,b]$
  => f has max/min on $[a,b]$.

7.5 Definition of integral

f is bounded function on $[a,b]$

"f is integrable" 6 marks.

1. A partition P of $[a,b]$ is a FINITE SET of POINTS that include endpoints a, b.
   (a finite subset $P \subseteq [a,b]$, $a, b \in P$.)

2. $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$

3. Denote $m_j = \inf_{x \in [x_{i-1}, x_i]} f(x)$, $M_j = \sup_{x \in [x_{i-1}, x_i]} f(x)$,

4. Then $U_P(f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$, $L_P(f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$

for each $i = 1, \dots, n$,
m_i = inf of f on $[x_{i-1}, x_i]$
M_i = sup of f on $[x_{i-1}, x_i]$
$\Delta x_i = x_i - x_{i-1}$

5. Integrable if

$$\inf U_P(f) = \sup L_P(f)$$
$$\int_a^b f(x)\,dx = \overline{I}_a^b(f) = \underline{I}_a^b(f)$$

- Every lower sum < upper sum.

$\underline{I}_a^b(f) < \overline{I}_a^b(f)$.
$\Rightarrow f$ is non-integrable on $[a,b]$.
$\Rightarrow \int_a^b f(x)\,dx$ is undefined.

If f is a continuous on $[a,b]$
$\Rightarrow f$ is integrable on $[a,b]$.

7.6. Properties of Lower and Upper sums.

1. $L_P(f) \leq U_P(f)$.
2. If $P \subseteq Q$.
   - $L_P(f) \leq L_Q(f)$ ← more points, closer to actual func
   - $U_Q(f) \leq U_P(f)$
3. $L_P(f) \leq U_Q(f)$
4. $L_P(f) \leq L_R(f) \leq U_R(f) \leq U_Q(f)$,

   where $P \cup Q = R$.

7.8.

$$g(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$$

inf of g on any $[x_{i-1}, x_i] = 0$.

sup of g on any $[x_{i-1}, x_i] = 1$.

$L_P(f) = 0 \neq U_P(f) = 1 \Rightarrow \int_0^1 g(x)\,dx$ is undefined.

7.9 Integrals as limits.

Let $P = \{x_0, x_1, x_2, \ldots x_n\}$ be partition of $[a,b]$.

For each i, let $\Delta x_i = x_i - x_{i-1}$

The norm of P is

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}$$

☆ • $\underline{I}_a^b(f) = \sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}$

☆ • $\underline{I}_a^b(f) = \lim_{\|P\| \to 0} L_P(f)$ (i.e. $\max \Delta_i \to 0$)

$\Rightarrow$ $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall$ partition P of $[a,b]$

$$\|P\| < \delta \implies \left| \underline{I}_a^b(f) - L_P(f) \right| < \varepsilon$$

Pick sequence of partitions $P_1, P_2, P_3 \dots$

$\lim_{n\to\infty} \|P_n\| = 0$ e.g. P_n: break interval $[a,b]$ into n subintervals of equal length.

$\Rightarrow$ Then $\underline{I}_a^b(f) = \lim_{n\to\infty} L_{P_n}(f)$ $(= \sup L_P(f))$

$\overline{I}_a^b(f) = \lim_{n\to\infty} U_{P_n}(f)$ $(= \inf U_P(f))$

7.10 Riemann Sums.

• Let f be bounded function on $[a,b]$.

• Let $P = \{x_0, x_1, \dots, x_n\}$ be partition of $[a,b]$

For each $i = 1, 2, \dots, n$.

- let $\Delta x_i = x_i - x_{i-1}$
- choose $x_i^* \in [x_{i-1}, x_i]$

Then

$$S_P^*(f) = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i.$$

• Let f be bounded interval $[a,b]$.
• Assume f is integrable on $[a,b]$

Proof
f is integrable on $[a,b]$. ☆☆☆

- Pick sequence of partitions $P_1, P_2, \dots, P_n$ of $[a,b]$ s.t.

$$\lim_{n\to\infty} \|P_n\| = 0$$

- On each subinterval, pick $x_i^* \in [x_{i-1}, x_i]$ (e.g. $x_i^* = x_i$)

Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_{P_n}^*(f) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot\Delta x_i$$

$$\Rightarrow \lim_{x\to\infty} L_{P_n}(f) = \underline{I}_a^b(f) = \int_a^b f(x)\,dx$$

$$\Rightarrow \lim_{x\to\infty} U_{P_n}(f) = \overline{I}_a^b(f) = \int_a^b f(x)\,dx$$

$$L_{P_n}(f) \le S_{P_n}^*(f) \le U_{P_n}(f)$$

Squeeze Theorem, $\lim_{n\to\infty} S_{P_n}^*(f) = \int_a^b f(x)\,dx$

Calculate $\int_0^1 x\,dx$ using Riemann Sum.

① $f(x) = x$ is continuous
$\Rightarrow$ integrable
$\therefore \int_0^1 x\,dx = \lim_{n\to\infty} S_{P_n}^*(f)$

② Choose $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$
(break $[0,1]$ into n equal subintervals)

③ Choose right endpoint.

$$x_i^* = \frac{i}{n}$$

$$S_{P_n}^* = \sum_{i=1}^{n} \underbrace{\frac{i}{n}}_{f(x_i)} \cdot \underbrace{\frac{1}{n}}_{\Delta x}$$

7.11 Five properties of definite integrals.

If f, g are integrable on $[a,b]$.
Then $f+g$ is integrable on $[a,b]$.

If f is integrable on $[a,b]$ and $[b,c]$,

Then f is integrable on $[a,c]$