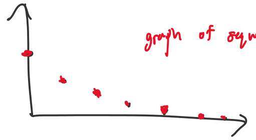
Ch11

Tuesday, 11 February 2025 8:52 PM

describe: 1. Equation 2. First few values 3. Words 4. Recurrence relation

11.1 What is a sequence?

A sequence is a $\boxed{\text{function with domain } \cancel{\mathbb{N}}}$. $\{n \in \mathbb{Z} \mid n \geq n_0\} = \{n_0, n_0+1, n_0+2, \dots\}$ not start from 0.



graph of sequence

11.2. The limit of a sequence

$\boxed{\lim_{n\to\infty} a_n = L}$ → $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow L - \varepsilon < a_n < L + \varepsilon$

$\boxed{\{a_n\}_{n=0}^{\infty} \text{ converges to } L \in \mathbb{R}}$

(Every open interval centered at L contains a tail of sequence)
('Tail' = all terms of sequence after the first few ones)

A sequence is...
- convergent when it has limit
- divergent when it doesn't.

→ Convergent $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \dots\}$

Divergent:
- ∞ $\{n^2\} = \{\dots\}$
- $-\infty$ $\{-n\} = \{\dots\}$
- "Oscillating" ∿

11.3. Properties of limits of sequences

- Limit of sequence vs functions
  1. Limit laws
  2. The Squeeze Theorem
  3. ✗ L'Hopital's Rule (cannot $\frac{d}{dx}$ sequence)

- When sequence "comes from a function"
  - Define $\{a_n\}_{n=c}^{\infty}$ by $a_n = f(n)$ → already defined

$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$

$\lim_{x\to\infty} f(x) = \infty \Rightarrow \lim_{n\to\infty} a_n = \infty$

$\lim_{x\to\infty} f(x)$ DNE $\Rightarrow \lim_{n\to\infty} a_n$ may or may not exist

Compute $\lim_{n\to\infty} e^{\frac{1}{n!}}$

Theorem

Let $\{a_n\}$ be a sequence. Let f be a function. Let $L \in \mathbb{R}$.

If $\begin{cases} a_n \to L \\ f \text{ is continuous at } L. \end{cases}$

$\Rightarrow f(a_n) \to f(L)$

① $\frac{1}{n!} \to 0$
② e^x is continuous

$\Bigg\}\ e^{\frac{1}{n!}} \to e^0 = 1$

11.4 Monotonic and bounded sequences

A sequence

① • increasing when $\forall n \in \mathbb{N}$, $a_n \leq a_{n+1}$ ③ non-decreasing

② • decreasing when $\forall n \in \mathbb{N}$, $a_n \geq a_n$ ④ non-increasing

• Any four : Monotonic

Example : Is $\{n^3 e^{-n}\}_{n=0}^{\infty}$ monotonic

• comes from a function

• f is decreasing on $[3, \infty)$

* $\{n^3 e^{-n}\}_{n=3}^{\infty}$ is decreasing.

* $\{n^3 e^{-n}\}_{n=0}^{\infty}$ is eventually decreasing. not worth to define.

$\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geq n_0 \Longrightarrow a_n > a_{n+1}$

• bounded below : $\exists A \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $A \leq a_n$.

• bounded above : $\exists B \in \mathbb{R}$ s.t. $\forall n \in \mathbb{R}$, $B \geq a_n$.

bounded = above + below

Theorem

1. Convergent $\longrightarrow$ bounded

2. (Eventually) monotonic + Bounded $\longrightarrow$ Convergent

   - eventually increasing + bounded above
   - eventually decreasing + bounded below

Monotone Convergence Theorem

3. (Eventually) monotonic + NOT Bounded $\longrightarrow$ Divergent to $\pm\infty$

Proof. : Video ☺

11.5. Every convergent sequence is bounded

Bounded • $\exists A, B$: $A \leq a_n \leq B$ (for $\forall n \in \mathbb{N}$)

Convergent • $\exists L \in \mathbb{R}$:

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \in \mathbb{N}$

$n \geq n_0 \Longrightarrow |a_n - L| < \varepsilon$

11.6. The monotone convergence theorem for sequences

• Eventually monotonic + bounded $\Longrightarrow$ Convergent
  (increasing / decreasing)

Let's prove :

It a sequence is INCREASING and BOUNDED ABOVE

THEN it is convergent.

WTS: . . .

... $\exists L \in \mathbb{R}$ s.t.

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t.

$\forall n \in \mathbb{N}$, $n \geq n_0 \implies L - \varepsilon < a_n < L + \varepsilon$.

1. Let $\{a_n\}_{n=0}^{\infty}$ be an increasing, bounded above sequence.
2. Let $A = \{a_n \mid n \in \mathbb{N}\}$
   - not empty and bounded above, so it has a supremum.
3. Take $L = \sup A$. I will prove $L = \lim_{n\to\infty} a_n$
4. Let $\varepsilon > 0$.
5. By def. of supremum, $\exists n_0 \in \mathbb{N}$ s.t. $L - \varepsilon < a_{n_0}$.
   Take n_0 to be such value.
6. Fix $n \in \mathbb{N}$. Assume $n \geq n_0$. WTS $L - \varepsilon < a_n < L + \varepsilon$
   We know $L - \varepsilon < a_{n_0}$.
   Since sequence is increasing, $a_{n_0} \leq a_n$.
   By def. of supremum, $a_n \leq L$.

$\therefore \underline{L - \varepsilon} < a_{n_0} \leq \underline{a_n} \leq L < \underline{L + \varepsilon}.$

11.7. The Big Theorem

$a_n << b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

"a_n is much smaller than b_n"

"b_n is much larger than a_n"

$$\ln n << \underset{(a>0)}{n^a} << \underset{(c>1)}{c^n} << n! << n^n$$

$\boxed{1^2, 2^2, n^2.}$ $\boxed{2^1, 2^2, 2^n}$

11.8. Proof of the "Big Theorem"

$\ln n << n^a << c^n << n! << n^n$

for every $a > 0$, $c > 1$

1. $\lim_{n\to\infty} \frac{\ln n}{n^a} = 0$
2. $\lim_{n\to\infty} \frac{n^a}{c^n} = 0$

Use L'Hopital's Rule

3. $\lim_{n\to\infty} \frac{c^n}{n!} = 0$
4. $\lim \frac{n!}{}$

4. $\lim_{n\to\infty} \frac{\cdots}{n^n} = 0$

Formal Definition

The sequence is

$|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

(a) Convergent $(\exists L,)\ \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $n > n_0 \Rightarrow |a_n - L| < \varepsilon$

Negation

(b) Divergent $\forall L \in \mathbb{R}, \exists \varepsilon > 0$ s.t. $\forall n_0 \in \mathbb{N}$, $n > n_0$ AND $|a_n - L| \geq \varepsilon$

(c) Divergent to ∞ $\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{R}$ s.t. $n > n_0 \Rightarrow a_n \geq M$

(d) Divergent to $-\infty$ $\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{R}$ s.t. $n > n_0 \Rightarrow a_n \leq M$.

(e) Bounded above: $\exists B \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $a_n \leq B$

Bounded below: $\exists A \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $b_n \geq A$

Bounded: $\exists A, B \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $A \leq a_n \leq B$.