Ch6

Wednesday, 20 November 2024 5:28 PM

6.5 Intermediate Forms. $\frac{0}{0}$ is a interdeterminate form. (LZMII)

$$\boxed{\text{If } \begin{cases} \lim_{x\to a} f(x) = 0 \\ \lim_{x\to a} g(x) = 0 \end{cases} \quad \text{THEN} \quad \text{can't draw conclusions about} \quad \lim_{x\to a} \frac{f(x)}{g(x)}}$$

- does not mean undefined / DNE

Common forms

$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, $1^{\pm\infty}$ ($\frac{1}{0}$ is NOT included)

6.6. L'Hôpital's Rule.

$\lim_{x\to a} \frac{f(x)}{g(x)}$.

If
1. f, g differentiable as $x \to a$
2. g, g' never 0 as $x \to a$
3. $\lim_{x\to a} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.
4. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exist or $+\infty$ or $-\infty$

$$\Rightarrow \boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

6.7 L'Hôpital's Rule: Examples.

Eg. 1 $\lim_{x\to\infty} \frac{x}{\ln x}$ $\left(\frac{\infty}{\infty}\right)$

$$\overset{(L'H)}{\underset{*}{=}} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty$$

Eg. 2 $\lim_{x\to 0} \frac{\cos x - \cos 2x}{xe^x - x}$ $\left(\frac{0}{0}\right)$

$$\overset{L'H}{\underset{*}{=}} \lim_{x\to 0} \frac{-\sin x + 2\sin 2x}{e^x + xe^x - 1} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{\underset{*}{=}} \lim_{x\to 0} \frac{-\cos x + 4\cos 2x}{1 + e^x + xe^x} = \boxed{\frac{3}{2}}$$

Eg. 3 $\lim_{x\to 1} \frac{x^3 - 2x + 1}{x^2 + 3x + 2} = \frac{0}{6} = 0$ (NOT ALLOWED TO USE!)

6.8 When L'Hopital's Rule goes wrong

L'H lim 1+cos x … = DNE (periodic function!)

$$\lim_{x\to\infty} \frac{x+\sin x}{2x+\cos x} = \lim_{x\to\infty} \frac{\ldots}{2-\sin x} = DNE \text{ (periodic function)}$$

$$= \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{2+\frac{\cos x}{x}}$$

$$= \boxed{\frac{1}{2}}$$

By Squeeze Thm.
$-1 \le \sin x \le 1$
$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$
$0 \le \lim_{x\to\infty} \frac{\sin x}{x} \le 0$

6.9. $\boxed{0\cdot\infty\text{: TURN INTO QUOTIENT}}$

$$\lim_{x\to\infty} x\left(1-e^{\frac{2}{x}}\right) \quad (\infty\cdot 0)$$

$$= \lim_{x\to\infty} \frac{1-e^{\frac{2}{x}}}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$\overset{LH}{\underset{*}{=}} \lim_{x\to\infty} \frac{-e^{\frac{2}{x}}\cdot\left[-\frac{2}{x^2}\right]}{-\frac{1}{x^2}} = \lim_{x\to\infty}\left[-2e^{\frac{2}{x}}\right] = -2.$$

6.10. $\boxed{\infty-\infty\text{ : M1: Common factor into } \infty\cdot 0 \text{ form ; M2: Conjugate}}$

M1: $\lim_{x\to\infty}\left[\sqrt{x^2-x}-x\right] \quad (\infty-\infty)$

$$= \lim_{x\to\infty}\left(x\sqrt{1-\tfrac{1}{x}}-x\right)$$

$$= \lim_{x\to\infty} x\left(\sqrt{1-\tfrac{1}{x}}-1\right) \quad (\infty\cdot 0)$$

$$= \lim_{x\to\infty} \frac{\sqrt{1-\frac{1}{x}}-1}{\frac{1}{x}}$$

6.11 Why is 1^∞ an indeterminate form?
... 1.01^∞ ? 0.99^∞ ... ?

6.12. $\boxed{1^\infty\text{ : Use logarithms}}$

$f(x) = (1-x)^{\frac{1}{x}}$
$\ln f(x) = \frac{1}{x}\ln(1-x)$

1) $\lim_{x\to 0^+} \ln f(x)$

① $\boxed{\text{Find } \ln f(x)}$

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) \quad (\infty-\infty)$$

$$= \lim_{x\to 1^+} \frac{(\ln x)x - x + 1}{(x-1)\ln x}$$

$$= \lim_{x\to 1^+} \frac{1+\ln x - 1}{1+\ln x - \frac{1}{x}}$$

$$= \lim_{x\to 1^+} \frac{\ln x}{1+\ln x-\frac{1}{x}}$$

$$= 1$$

$$\lim_{x\to 0^+} x^x$$

$\lim_{x\to 0^+} \ln f(x)$
$= \lim_{x\to 0^+} x\ln x \quad (0\cdot-\infty)$

f(
f

$= \lim_{x \to 0^+} \frac{\ln(1-x)}{x}$ $\left(\frac{0}{0}\right)$

$= -1$

2) $\lim_{x \to 0^+} f(x)$ -1

$= \lim_{x \to 0^+} e^{\ln f(x)}$

$= e^{-1}$

② $\boxed{\text{find } f(x) \text{ by } e^{\ln f(x)}}$

$= \lim_{x \to 0^+} e^{x \ln x}$

$= e^{-0}$

$= 1$

$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$

$= \frac{\frac{1}{x}}{\frac{-1}{x^2}}$

$= -\frac{x^2}{x}$

$= -x$

$= -0$

x^{-1}

$-1x^{-2}$

6.13 Definitions of Concavity.

f concave up when f' is increasing

f concave down when f' is decreasing.

Inflection point: when f changes concavity.

Theorem 1.

- If $\forall x \in I$, $f''(x) > 0$, then f concave up on I.
  $f''(x) < 0$, then f concave down on I.

Theorem 2.

- If f has inflection point at c,
  $\Rightarrow f''(c) = 0 \ / \ f''(c) = \text{DNE}$.

6.14 Example.

6.15 Asymptotes

Let L be a line, C be a curve in the plane

L is an asymptote for C when

"L and C become arbitrarily close as we move away from the origin in one direction"

Vertical :

$$\lim_{x\to a^{\pm}} f(x) = \pm\infty$$

Horizontal :

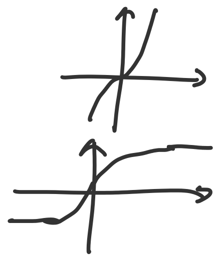
$$\lim_{x\to\pm\infty} f(x) = L.$$

Slant asymptote:

$$\lim_{x\to\pm\infty} f(x)-(mx+b) = 0$$

6.18 Asymptotes : A hard example

arctanx



$h(x) = x\arctan x$

$$\lim_{x\to\infty}\left[x\arctan x - \frac{\pi}{2}x\right]$$

$$= \lim_{x\to\infty} x\left(\arctan x - \frac{\pi}{2}\right)$$

$$= \lim_{x\to\infty} \frac{\arctan x - \frac{\pi}{2}}{\frac{1}{x}}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{-x^2}{1+x^2}$$

$$= \lim_{x\to\infty} \frac{-1}{\frac{1}{x^2}+1}$$

$$= -1$$

$$\therefore \lim_{x\to\infty} x\arctan x - \boxed{\left(\frac{\pi}{2}x - 1\right)} = 0$$

Asymptote.

$$\lim_{x \to \infty} \frac{e^{3x}}{x}$$

$$= \lim_{x \to \infty} \frac{e^{3x} \cdot 3}{1}$$

$$= \infty$$

#4.

$$\lim_{x \to 0} \frac{\ln(\cos x)}{\ln(\cos 7x)}$$

$$= \lim_{x \to 0} \frac{\frac{-\sin x}{\cos x}}{-\frac{7\sin 7x}{\cos 7x}}$$

$$= \lim_{x \to 0} \frac{-\tan x}{-7\tan 7x}$$

$$\frac{1}{\cos x}$$

$$= \lim_{x \to 0} \frac{-\sec^2 x}{-7\sec^2 7x \cdot 7}$$

$$= \frac{-\frac{1}{\cos^2 x}}{-49\frac{1}{\cos^2 7x}}$$

$$= \frac{1}{49}$$

#5. (A) $\lim_{x \to 0} \frac{\sin 6x}{\sin(-7x)}$

$$\overset{\text{L.H.}}{=} \lim_{x\to 0} \frac{6\cos 6x}{-7\cos(-7x)}$$

$$= -\frac{6}{7}$$

(B) $\lim_{x\to 1} \frac{\ln(x)}{x^{11}-1}$

$$\overset{\text{L.H.}}{\underset{*}{=}} \lim_{x\to 1} \frac{\frac{1}{x}}{11x^{10}}$$

$$= \lim_{x\to 1} \frac{1}{11x^{11}}$$

$$= \frac{1}{11}$$

(C) $\lim_{x\to 1} \frac{x^4-1}{x^3-1}$

$$= \frac{\cancel{(x^2-1)}(x^2+1)(x+1)}{\cancel{(x-1)}(x^2+x+1)}$$

$$= \frac{2\cdot 2}{3}$$

$$= \frac{4}{3}$$

#6

$$\lim_{x\to -\infty} 4x^2 - \sqrt{15x^4-1}$$

$$\sqrt{x^4(15-\frac{1}{x^4})}$$

$$x^2\sqrt{15-\frac{1}{x^4}}$$

$$= \lim_{x\to-\infty} x^2 \left(4 - \sqrt{15 - \frac{1}{x^4}}\right)$$

$$= \lim_{x\to-\infty} \frac{4 - \sqrt{15 - \frac{1}{x^4}}}{\frac{1}{x^2}}$$

$$= \frac{4 - \sqrt{15}}{\frac{1}{\infty}}$$

#7. $\lim_{x\to-\infty} x^2\sqrt{1+\frac{6}{x}} - x^2\sqrt{1-\frac{10}{x}}$

$$= \lim_{x\to-\infty} \frac{\sqrt{1+\frac{6}{x}} - \sqrt{1-\frac{10}{x}}}{\frac{1}{x^2}}$$

$$\frac{\frac{1}{2}\left(1+\frac{6}{x}\right)^{-\frac{1}{2}} \cdot (-6x^{-2}) - \frac{1}{2}\left(1-\frac{10}{x}\right)^{-\frac{1}{2}}(10x^{-2})}{-2x^{-3}}$$

x^{-2+3}

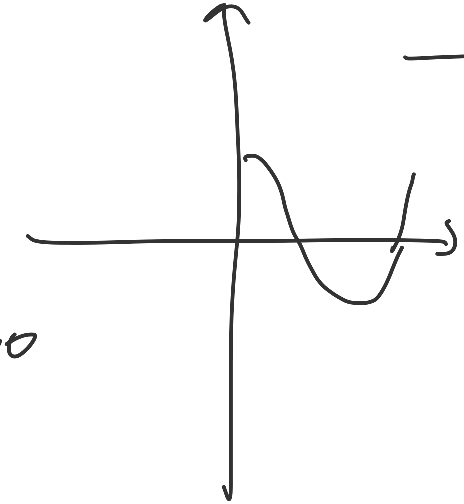
$$= \frac{x(-3(1) - 5(1-0))}{-2}$$

$$\frac{-8}{-2} = -4$$

#9 $\lim_{x\to\infty} 9x\left((\ln x+4) - (\ln x-4)\right)$

$$= \lim_{x \to \infty} \frac{9(\ln(x+4) - \ln(x-4))}{\frac{1}{x}}$$

$$= \lim_{x \to \infty}$$



#10. $\lim_{x \to \frac{\pi}{2}^-} \tan 3x^{\cos 5x}$

$f(x) = \ldots$ $\quad +\infty^{+0}$

$$\lim_{x \to \frac{\pi}{2}} \ln f(x) = \cos 5x (\ln \tan 3x)$$

$$= \frac{\ln \tan 3x}{\frac{1}{\cos 5x}}$$

$$= \frac{\frac{1}{\tan 3x} \cdot \sec^2 3x \cdot 3}{-(\cos 5x)^{-2} \cdot 5}$$

$$\lim_{x \to \frac{\pi}{2}^-} \tan 3x^{\cos 5x}.$$

Let $f(x) = \tan 3x^{\cos 5x}$

$\ln f(x) = \cos 5x \ \ln(\tan 3x)$

Take limit,

$$\lim_{x \to \frac{\pi}{2}^-} \cos 5x \ln(\tan 3x) \quad (0 \cdot \infty)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\ln \tan 3x}{\frac{1}{\cos 5x}}$$

$$\overset{LH}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{1}{\tan 3x} \cdot \sec^2 3x \cdot 3}{\frac{-5 \sin 5x}{(\cos 5x)^2}}$$

$*$

$\cancel{\frac{\cos}{\sin}} \cdot \frac{1}{\cos^2 x}$

$\frac{1}{\sin \cos x}$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{3}{\sin 3x \cdot \cos 3x}}{\frac{-5 \sin 5x}{}}$$

?

$$\# \quad \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{\frac{6}{x^2}} \leftarrow \infty$$

$$1.$$

Let $\ln f(x) = \frac{6}{x^2} \ln \left(\frac{\sin x}{x} \right)$

$$\ln f(x) = \frac{\frac{6}{x^2}}{\frac{1}{\ln\left(\frac{\sin x}{x}\right)}}$$

$$\frac{6}{x^2} \ln |$$

$$= \frac{-12x^{-3}}{\left(\ln\left(\frac{\sin x}{x}\right)\right)^{-2} \cdot \frac{x}{\sin x} \cdot \frac{x\cos x - \sin x}{x^2}}$$

#17.

$$\int \frac{\ln x}{x}\,dx$$

$$= \cancel{\frac{1}{x}} \int \ln x \;\; \frac{d \ln x}{\cancel{\frac{1}{x}}}$$

$$= \frac{1}{2}(\ln x)^2 + C.$$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx \qquad \text{let } u = \sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}}\,dx.$$

$$= \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$$

$$= \int \frac{\cos u}{\cancel{u}} \cdot \frac{2\cancel{u}}{1}\,du$$

$$= 2\int \cosh u \, du$$

$$= 2\sinh u + C.$$

$$\int \frac{x^2}{\sqrt{x+1}}\,dx$$

let $u = x+1$
$du = 1$

$$= \int \frac{(u-1)^2}{\sqrt{u}}\,du$$

$$= \int \frac{(u-1)^2}{\sqrt{u}}\,du$$

$$= \int \frac{(n^2 - 2n + 1)}{\sqrt{n}} \, dn$$

$$= \int n^{1.5} - 2n^{0.5} + n^{-\frac{1}{2}} \, dn$$

$$= \int n^{2.5} -$$

$$\int \sin^6 x \cos x \, dx$$

$= \int \sin^6 x \cdot d\sin x$

$\int x e^{5x^2}$

$= \int x e^{5x^2}$

$= \int \cancel{x} \, \frac{de^{5x^2}}{e^{5x^2} \cdot 10\cancel{x}}$

$= \frac{1}{10} \int \frac{1}{e^{5x^2}} \cdot de^{5x^2}$

$= \frac{1}{10} (1 - \int e$

$$\int_1^e \frac{11}{x\sqrt{1-(\ln x)^2}}\,dx \quad \int(-u+1)$$

$$u = 1-(\ln x)^2$$

$$du = -2(\ln x)\cdot\cancel{\tfrac{1}{x}}\,dx$$

$$= \int_1^e \frac{11}{u(-2)\left(\sqrt{1-u}\right)}$$

$$\int_1^e \frac{11}{x\sqrt{1-(\ln x)^2}}+8\,dx$$

$$e^u = x,$$

Let $u = \ln x$

$$du = \frac{1}{x}\,dx$$

When $x=e,\ u=1$

$x=1,\ u=0$

$$= \int_0^1 \frac{11}{\sqrt{1-u^2}}\,du$$

$$= 11\int_0^1 \frac{1}{\sqrt{1-u^2}}\,du$$

$= \quad " \left(\arcsin x \right)_0 + 8e - 8$

$= 11 \frac{\pi}{2} + 8e - 8$

$=$

$$\int_0^{\frac{\pi}{2}} \frac{6e}{\pi\sqrt{1 - \frac{x^2}{\pi^2}}} dx \quad \text{let } u = \frac{x}{\pi}.$$

$$du = \frac{1}{\pi} dx,$$

$$\frac{1}{1 - \frac{x^2}{\pi^2}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{6e}{\sqrt{1 - u^2}} du$$

$\sin x = \frac{1}{2}.$

$$= 6e \left(\arcsin\left(\frac{x}{\pi}\right) \right)_0^{\frac{\pi}{2}}$$

$6e \left(\arcsin \frac{1}{2} \right.$ $\quad \frac{\frac{\pi}{2}}{\cancel{\pi}}$

$6e\left(\frac{\pi}{6}\right.$

#20.

$$3\int \frac{\cancel{6x+18}}{\sqrt{x^2+6x+29}} \frac{d(x^2+6x+29)}{\cancel{2x+6}}$$

$$= \frac{3\,(x^2+6x+29)^{\frac{1}{2}}}{\frac{1}{2}}.$$

#21. $-\int_0^{\frac{\pi}{2}} \sin^2 x \cos^6 x \; d\cos x$

$= -\int_0^{\frac{\pi}{2}} (1-\cos^2 x)(\cos^6 x)\, d\cos x$

$= -\int_0^{\frac{\pi}{2}} \cos^6 x - \cos^8 x \; d\cos x$

$$= -\left[\cancel{\frac{\cos^7 x}{7}} - \cancel{\frac{\cos^9 x}{9}}\right]_0^{2}$$

$$0 \qquad \frac{1}{7} - \frac{1}{9}$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^6 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^4 x \sin^6 x \, d\sin x$$

$$= \int_0^{\frac{\pi}{2}} (1 - \sin^2 x)^2 \sin^6 x \, d\sin x$$

$$= \int_0^{\frac{\pi}{2}} (1 - 2\sin^2 x + \sin^4 x) \sin^6 x \, d\sin x$$

$$= \int_0^{\frac{\pi}{2}} \sin^6 x - 2\sin^8 x + \sin^{10} x \, d\sin x$$

$$= \cancel{\int_0^{\frac{\pi}{2}}} \left[\frac{\sin^7 x}{7} - \frac{2\sin^9 x}{9} + \frac{\sin^{11} x}{11}\right]_0^{\frac{\pi}{2}}$$

$$\frac{1}{7} - \frac{2}{9} + \frac{1}{11}$$

$$\int_0^{\frac{\pi}{2}} \sin^2 6x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 12x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{1}{2}\cos 12x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}dx - \int_0^{\frac{\pi}{2}} \frac{1}{2\cdot 12} \cos 12x \, d12x$$

$$= \left[\frac{1}{2}x\right]_0^{\frac{\pi}{2}} - \frac{1}{24}\left[\cos 12x\right]_0^{\frac{\pi}{2}}$$

$$= \boxed{\frac{\pi}{4} + \frac{1}{24}}$$

$$\int_0^{\frac{\pi}{2}} \sin^2 6x \cos^2 6x \, dx$$

$$\int \tan^3 x \sec^3 x \, dx$$

$= \int \tan^3 x$

$\int_0^3 x^3 g'(x)\,dx$

$\frac{dx^3}{\cancel{dx}} = 3x^2 \cancel{dx}$

$= \int_0^3 x^3 \quad dg(x)$

dx^3

x^3

$= \left[x^3 g(x)\right]_0^3 - \int_0^3 g(x)\,3x^2\,dx$

$\approx 27(8.10) - 0.5(0.5 + 2.5 + 3.1 + 4.6 + 5.9 + 6.2)$

#24. $\int \sec^4 x \tan^3 x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx.$

$= \int (1+u^2)\,u^3\,du.$

$1 + \tan^2$

$= \int u^3 + u^5\,du$

$1 + u^2 = \sec^2 x.$

$$= \frac{u^4}{4} + \frac{u^6}{6} + C.$$

$$= \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C$$

#123

$$\cancel{2}\int \frac{1}{(x^2+1)^2} \frac{d(x^2+1)}{2x}$$

$$\int \frac{\cancel{2x}}{(x^2+1)^2} \frac{d(x^2+1)}{\cancel{2x}}$$

$$= \cancel{2 \cdot -2(x^2+1)^{-1} + C}$$

$$\int (x^2+1)^{-2}$$

$$-(x^2+1)^{-1} + C$$

$$\frac{7x+1}{\quad}$$

A B

$$\frac{\quad}{(x+2)(x+3)} = \frac{\quad}{(x+2)} + \frac{\quad}{(x+3)}$$

$$7x+1 = Ax + 3A + Bx + 2B$$

$$2A + 2B = 14$$

$$3A + 2B = 1$$

$$A = 1 - 14 = \boxed{-13}$$

$$A + B = 7$$

$$-13 + B = 7$$

$$B = 20$$

$$\int_3^7 \frac{3}{16+4x^2}\,dx$$

let $u = \frac{x}{2}$

$du = \frac{1}{2}\,dx$

$$= \int_3^7 \frac{3}{16\left(1+\frac{x^2}{4}\right)}\,dx$$

$$= \frac{2\cdot 3}{16}\int_3^7 \frac{1}{1+u^2}\,du$$

$$= \frac{3}{8}\arctan x$$

$$\frac{10}{(x-1)(x^2+9)}$$

$(x-1)(x^2+1)$

$= \frac{10}{(x-1)}$

$x^2+2x+4+17$

$(x+2)^2+17$

$6\left(17\left(\frac{(x+2)^2}{17}+1\right)\right)$

$6\int \frac{1}{(x+2)}$

$= 17 \cdot 6 \cdot \sqrt{17} \int \frac{}{u^2}$

$=$

$$6\int \frac{1}{x^2+2x+21}\,dx$$

$$(x^2+2x+1)+20$$

$$(x+1)^2+20$$

$$= 6\int \frac{1}{(x+1)^2+20}\,dx$$

let $x+1 = \sqrt{20}\,u$

$(x+1)^2 = 20u^2$

$dx = \sqrt{20}\,du$

$$= 6\int \frac{\sqrt{20}}{20(u^2+1)}\,\cancel{dx}\,du$$

$$= \frac{\sqrt{20}\,6}{20}\int \frac{1}{(u^2+1)}\,du$$