# Ch13

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## 13.2 The definition of infinite sum

- A series is an infinite sum:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n$$

- A sequence is an infinite list

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots$$

1. First, construct the sequence

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_k = a_1 + a_2 + \cdots + a_k$

$= \sum_{n=1}^{k} a_n$

2. Then compute limit

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} S_n$$

## 13.3 A telescopic series

- a series which most term cancel out.

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

$S_1 = \frac{1}{2}$, $S_2 = \frac{2}{3}$, $S_3 = \frac{3}{4}$.

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+n} = \sum_{n=1}^{k} \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{k} - \frac{1}{k+1}$$

$$= 1 - \frac{1}{k+1}$$

$$= \frac{k}{k+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

## 13.4 Example of divergent series

1. $S = \sum_{n=1}^{\infty} 1 = \infty$

2. $S = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 \ldots$ $\quad S_n = \begin{cases} 0 & \text{if } k \text{ odd} \\ 1 & \text{if } k \text{ even} \end{cases}$

13.5 Geometric series

$\sum_{n=0}^{\infty} x^n$ convergent if $-1 < x < 1$

divergent if otherwise

13.6. Series are linear

1. If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent

Then $\sum_{n=0}^{\infty} (a_n + b_n)$ is also convergent

and $\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$

13.7 The tail of a series

$\sum_{n=0}^{\infty} a_n$ is convergent $\Longleftrightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent $\left( \sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n \right)$

13.8 A necessary condition for convergence of series

If $\sum_{n=0}^{\infty} a_n$ is convergent,

Then $\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} a_n = 0 \Rightarrow$ CONV or DIV.

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ DIV.

13.9 Positive series

a series $\sum_{n=0}^{\infty} a_n$ positive / negative / non-negative when $\forall n \in \mathbb{N}$, $a_n > 0$. — all terms are...

A positive series only may be convergent / divergent to $\infty$.

- To prove a positive series is CONVERGENT, we only have to prove it does not diverge to $\infty$.

13.10 Integral test.

Let $a \in \mathbb{R}$.
Let $f$ be a continuous, positive, decreasing function on $[a, \infty)$

Then $\int_a^\infty f(x)\,dx$ is convergent $\iff \sum_n^\infty f(n)$ is convergent.

Example 1.

For which p-value of $p>0$ is the series $\sum_{n=1}^\infty \frac{1}{n^p}$ convergent?

Let $f(x) = \frac{1}{x^p}$

For $x \geq 1$, $f$ is continuous, positive, decreasing.

By integral test, $\underbrace{\int_1^\infty \frac{1}{x^p}\,dx}_{p>1,\ \text{conv.}} \sim \underbrace{\sum_{n=1}^\infty \frac{1}{n^p}}_{p>1,\ \text{conv.}}$

$p>1$, conv. $\longrightarrow$ $p>1$, conv.

Example 2.

$\sum_{n=2}^\infty \frac{1}{n \ln n}$ convergent?

$$\sim \int_2^\infty \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx$$

$$= \lim_{b\to\infty} \left[\ln(\ln x)\right]_2^b$$

$$= \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2)$$

$$= \infty$$

$\therefore \sum_{n=2}^\infty \frac{1}{n\ln n} = \infty$

13.12 Comparison test for series (some)

BCT: $0 \leq a_n \leq b_n$,

1. If $\sum_n^\infty a_n = \infty$, Then $\sum_n^\infty b_n = \infty$
2. If $\sum_n^\infty b_n < \infty$, Then $\sum_n^\infty a_n < \infty$

LCT:

Let $\sum_n^\infty a_n$ and $\sum_n^\infty b_n$ be 2 positive series.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ exist and $L>0$.

Then $\sum_n^\infty a_n$ and $\sum_n^\infty b_n$ are both CONV or DIV

13.13 Alternating series

[ · A series is alternating when $\forall n$, $a_n a_{n+1} < 0$

· If sequence of even and odd term

$\{c_{2n}\}_n^\infty$ and $\{c_{2n+1}\}_n^\infty$ are convergent to same limit.

· Then full sequence $\{c_n\}^\infty$ ... convergent to same limit

... $c \cdot a_n$ is also convergent to same limit.

Theorem: Alternating Series Test

Consider series in $\sum_{n}^{\infty} (-1)^n b_n$ or $\sum_{n}^{\infty} (-1)^{n+1} b_n$

If 1) $\forall n,\ b_n > 0$

2) Sequence $\{b_n\}_n^{\infty}$ is decreasing

3) $\lim_{n\to\infty} b_n = 0$

Then the series is convergent

e.g. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conv.

13.14. Estimate the value of Alternating series

Alternate Series Theorem Part 2:

If xxx (3 hypotheses)

Then $|S - S_k| < b_{k+1} \leq$ Error.

13.15 Absolute convergence and conditional convergence

| | $\sum_{n}^{\infty} \lvert a_n \rvert < \infty$ | $\sum_{n}^{\infty} \lvert a_n \rvert = \infty$ |
|---|---|---|
| $\sum_{n}^{\infty} a_n$ convergent | Absolutely convergent | Conditionally convergent |
| $\sum_{n}^{\infty} a_n$ divergent | Impossible | divergent |

13.17. Infinite sums are not commutative

- Absolute conv: can reorder terms
- Conditionally conv: cannot reorder terms

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Formal Definition.

A series

- $\boxed{\text{convergent}}$ if its partial sums limit...

Estimate

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$ with error smaller than 0.001

Actual value $= S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \lim_{k\to\infty} S_k$

Error of estimation:

$$|S - S_k| < \frac{1}{(k+1)^4}$$

Need $k$ s.t. $\frac{1}{(k+1)^4} < 0.001$

$k = 5$ works

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Example.

Is $\sum_{n}^{\infty} \frac{\sin n}{n^2}$ convergent?

- $0 \leq \frac{|\sin n|}{n^2} < \frac{1}{n^2}$, $\sum_{n}^{\infty} \frac{1}{n^2}$ is convergent,

$\Rightarrow \sum_{n}^{\infty} \frac{|\sin n|}{n^2}$ convergent

$\Rightarrow \sum_{n}^{\infty} \frac{\sin n}{n^2}$ convergent

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Example

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots$ is CONVERGENT (Alternating Series Test)

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent

$\lim_{n\to\infty} S_n = L$ (if $L$ exist)

- $\boxed{\text{Divergent}}$

  $\lim_{n\to\infty} s_n = \pm\infty$.

  OR $\lim_{n\to\infty} a_n \neq 0$.

- $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is DIVERGENT (p serie ; p = 1)

| 
- $\sum_{n=1}^{\infty} \frac{(-1)^{\cdots}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} \cdots$ is CONVERG

- $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots$