Ch4 Fundamental Subspace

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4.1 Three Fundamental Subspace

$$A = \underbrace{\begin{pmatrix} x_1 & y_1 & z_1 \cdots \\ x_2 & \cdots & \\ \vdots & & \\ x_m & & \end{pmatrix}}_{n.} \Big\} m$$

Each vector $\vec{x}$ in $\mathbb{R}^n$, $A\vec{x}$ yields vector $\mathbb{R}^m$.

Definition 4.1.

Let A be $m \times n$ matrix

$A = (\vec{v}_1 \cdots \vec{v}_n)$

1st:

- Column Space of A is subspace of R^m given by

$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \to \begin{pmatrix} x \\ x \\ x \end{pmatrix}$

$$\boxed{Col(A) := Span(\vec{v}_1, \dots, \vec{v}_n)}$$

$Col(A) \subseteq R^m$ ← No. of rows

2nd:

- Null Space of A is subspace of $\mathbb{R}^n$ given by

$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \to \begin{pmatrix} x \\ x \end{pmatrix}$

$$\boxed{Nul(A) := \{\vec{x} \in \mathbb{R}^m \mid A\vec{x} = \vec{0}\}}$$

$Nul(A) \subseteq R^n$ ← No. of columns

Activity 4.1

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

① Find $\vec{v} \in Col(A)$

- $\vec{v} = c_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$
- Any c_1, c_2 yields the required $\vec{v}$.

② Find $\vec{w} \in Nul(A)$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 + 2w_2 \\ 3w_1 + 4w_2 \\ 5w_1 + 6w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\therefore w_1 = 0, \ w_2 = 0.$

$\therefore \vec{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Activity 4.2 Show $Col(A)$, $Nul(A)$ are vector spaces

– addition, multiplication, zero vector...

Activity 4.2

Let $A = (\vec{v}_1 \ \dots\dots \ \vec{v}_n)$

$Nul(A)$ is a subspace.

- non-empty: $\vec{0} \in Nul(A)$

- Closure under addition:

$\vec{x}, \vec{y} \in \vec{A}$

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\begin{aligned} A(\vec{x} + \vec{y}) &= (x_1 + y_1)\vec{v}_1 + \dots\dots + (x_n + y_n)\vec{v}_n \\ &= \vec{x}_1\vec{v}_1 + \dots + \vec{x}_n\vec{v}_n + \vec{y}_1\vec{v}_1 + \dots + \vec{y}_n\vec{v}_n \\ &= A\vec{x} + A\vec{y} \end{aligned}$$

$$\begin{aligned} A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad &= 0 \ \text{to} \\ &= \vec{0} \end{aligned}$$

$\therefore \vec{x} + \vec{y} \in Nul(A)$

Closure under multiplication

$\vec{x} \in Nul(A)$, $\lambda \in \mathbb{R}$.

$$\begin{aligned} A(\lambda\vec{x}) &= (\lambda x_1)\vec{v}_1 + \dots + (\lambda x_n)\vec{v}_n \\ &= \lambda(x_1\vec{v}_1 + \dots + x_n\vec{v}_n) \quad \text{Since } \vec{x} \in Nul(A) \\ &= \lambda(\vec{0}) \\ &= \vec{0} \end{aligned}$$

$\therefore \lambda\vec{x} \in Nul(A)$

Definition 4.3.

- $rank(A) := dim(Col(A))$
- $nullity(A) := dim(Nul(A))$

Activity 4.3 Find the rank and nullity

(1) $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Rank:

$$\begin{aligned} Col(A) &= c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= c_4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ where } c_3, c_4 \in \mathbb{R}. \end{aligned}$$

$\therefore Col(A) = Span\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$

Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has two pivots $\Rightarrow dim(Col(A)) = 2$

Nullity:

$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0$, where $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$w_1 - w_2 = 0$, $w_1 = w_2$.

$w_3 = 0$

$\therefore \vec{w} = \begin{pmatrix} w_1 \\ w_1 \\ 0 \end{pmatrix}$, where w_1 is any $\mathbb{R}$.

(2) $B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 3 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$

Rank:

$Col(B) = c_1\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} + c_4\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ Pivot columns are vectors that are linearly independent.

$\therefore Col(B) = Span\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}$

$Rank(B) = dim(Col(B)) = 2$. (no. of independent vectors in Span)

Nullity: $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$

$\therefore Nul(A) = \{ t \begin{pmatrix} 1 \\ 0 \end{pmatrix}, t \in \mathbb{R} \}$

↑ only one vector direction

$\therefore Nullity(A) = \dim \{ t \begin{pmatrix} 1 \\ 0 \end{pmatrix}, t \in \mathbb{R} \}$

$= 1.$

$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_3 \\ w_4 \end{pmatrix}$

$w_1 + w_2 + w_4 = 0$

$w_2 + w_3 - w_4 = 0$

Consider $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array} \right)$

Let w_3, w_4 be free variables.

Then $w_2 = w_4 - w_3$

$w_1 = -w_4 + w_3 - w_4$

$w_1 = -2w_4 + w_3$

$\therefore Nul(B) = span \left\{ \begin{pmatrix} x - 2y \\ -x + y \\ x \\ y \end{pmatrix} \right\}$

$= span \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} y \text{ where } x, y \in \mathbb{R}. \right\}$

$= Span \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$Nullity = \dim span \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$= 2.$ (or no. of columns minus rank)

Definition 4.4

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Definition 4.6 $A = m \times n$ matrix,

3rd Subspace: $Row(A) = Col(A^T)$

↓

$\mathbb{R}^n$.

$\begin{pmatrix} \overrightarrow{1 \; 2} \\ 3 \; 4 \\ 5 \; 6 \end{pmatrix} \to \begin{pmatrix} x \\ x \end{pmatrix}$

4.2 Rank - Nullity,

~~Activity. 4.4. Let A = 3x3 matrix.~~

$A = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$ or $(v_1 \; v_2 \; v_3)$

(1) • Suppose A has 2 pivot columns. (Col 1, 3)

Then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ are linearly independent

$\therefore Col(A) = Span \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right\}$

$Rank\, A = \dim Col(A) = 2.$

$Nul\, A$: Since column 2 has no pivot, variable for such column is free.

The vector of column 1 and column 3 determined

$\therefore \vec{v_1} = k_1 \vec{v_2}, \quad \vec{v_3} = k_2 \vec{v_2}.$

$\therefore Nul\, A = \{ k v_2 \mid k \in \mathbb{R} \}$

$\therefore Nullity = \dim(Nul(A)) = 1.$

(2) • Suppose A has 1 pivot column in column 1.

Rank: $\text{Col } A = \text{Span}\{v_1\}$, as only v_1 is linearly independent.

$\text{Rank } A = \dim(\text{Span}\{v_1\}) = 1.$

Nullity: Consider $(\vec{v_1} \ \vec{v_2} \ \vec{v_3})\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0.$

Since there is only one pivot, in column 1,

Let $\text{rref}(A) = \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $\begin{pmatrix} 1 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0$

$w_1 + bw_2 + cw_3 = 0$

$w_1 = -bw_2 - cw_3$

$\therefore \vec{x} = \begin{pmatrix} -bw_2 - cw_3 \\ w_2 \\ w_3 \end{pmatrix}$

$\vec{x} = \begin{pmatrix} -b \\ 1 \\ 0 \end{pmatrix} w_2 + \begin{pmatrix} -c \\ 0 \\ 1 \end{pmatrix} w_3$

$\therefore \text{Nul}\left\{\begin{pmatrix} -b \\ 1 \\ 0 \end{pmatrix} w_2 + \begin{pmatrix} -c \\ 0 \\ 1 \end{pmatrix} w_3 \ \middle|\ w_2, w_3 \in \mathbb{R}\right\}$

$= \text{Span}\left\{\begin{pmatrix} -b \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -c \\ 0 \\ 1 \end{pmatrix}\right\}$

$\text{Nullity} = \dim\left(\text{span}\left\{\begin{pmatrix} -b \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -c \\ 0 \\ 1 \end{pmatrix}\right\}\right)$

$= \boxed{2}$

Theorem 4.7.

Let A = $m \times n$ matrix with r pivot columns.

- $\text{rank}(A) = r$
- $\text{nullity}(A) = n - r.$ $\left(\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \text{Nul}(A) \text{ if } x_1\vec{v_1} + \cdots + x_m\vec{v_m} = 0\right)$

Collary 4.8.

The Rank-Nullity Theorem

$\text{rank}(A) + \text{nullity}(A) = n.$ ← no. of columns.

4.3 Homogenous Systems and the Geometry of Systems.

Definition 4.9.

HOMOGENOUS

- A system of linear equations
  - ALL constants = 0.

$\begin{cases} 2x - y = 0 \\ x + 3y = 0. \end{cases}$ → homogenous.

Theorem 4.10.

Solution set to any homogenous system of equations

- is a vector space
- solution set = $\text{Nul}(A)$ ← coefficient matrix

As $A\vec{x} = 0$, set of solutions = $\text{Nul}(A)$

Example 4.11 $\vec{x}$

$$\begin{cases} x + 2y + 4z = 0 \\ x + y - z = 0 \\ \quad\ \ y + 5z = 0 \end{cases} \to \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & -1 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & -1 \\ 0 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $z = t$.

Then $y = -5t$. $x = 6t$.

$\therefore$ Solution set $= (x, y, z) = (6t, -5t, t)$

$$\therefore \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ -5 \\ 1 \end{pmatrix} = 0.$$

$$\therefore \text{Nul}(A) = \left\{ \begin{pmatrix} 6t \\ -5t \\ t \end{pmatrix} \middle| \ t \in \mathbb{R} \right\}$$

$$= \text{Span}\left(\begin{pmatrix} 6 \\ -5 \\ 1 \end{pmatrix} \right)$$

$$= 1.$$

Theorem 4.13.

Solution set to a consistent system with coefficient matrix A.

$= \vec{p} + \text{Nul}(A)$

$= \{ \vec{p} + \vec{v} \mid \vec{v} \in \text{Nul}(A) \}$

where $\vec{p}$ is any particular vector solution to the system of linear equations

Remark 4.14

- Homogenous

$\rightarrow \vec{p} + \text{Nul}(A) = \vec{0} + \text{Nul}(A)$

$= \text{Nul}(A)$

Remark 4.15

$\vec{p} + V$

- Translated Vector Spaces / Translated Spans
- A vector space translated by some vector $\vec{p}$.

$$\begin{cases} x + 2y + 4z = 1 \\ x + y - z = 2 \\ \quad\ \ y + 5z = -1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 1 & 1 & -1 & 2 \\ 0 & 1 & 5 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & -1 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = 1 + 5z$

$x = 1 - 4z - 2 - 10z$

$x = -1 - 14z$

$$\therefore \begin{pmatrix} -1 - 14z \\ 1 + 5z \\ z \end{pmatrix}$$

$\therefore$ Rank $= 2$,

Nullity $= 1$

$\therefore$ Line.

Span(