Ch12

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12.1 Improper integral.

- Type 1.

Let $a \in \mathbb{R}$. Let f be a continuous function on $[a, \infty)$

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

- Convergent when limit exist
- Divergent when limit DNE

Example

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \left[\arctan b - \arctan 0\right]$$

$$= \frac{\pi}{2}$$

12.4 Improper Integral - p-function

$$\int_1^\infty \frac{1}{x^p}\,dx$$

- $p > 1$: convergent
- $p \le 1$: divergent ∞.

12.3 Improper integral : vertical asymptote.

Let $a < b$. Let f be a continuous function on $(a, b]$.

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

- Convergent if limit exist
- Divergent if limit DNE

$$\int_0^1 \ln x\,dx$$

$$= \lim_{c\to 0^+} \int_c^1 \ln x\,dx$$

$$= \lim_{c\to 0^+} \left[x\ln x - x\right]_c^1$$

$$= \lim_{c\to 0^+} (1\cdot\ln 1 - 1) - (c\ln c - c)$$

$$= -1$$

L'Hopital's

12.6 Doubly improper integral

both limits exist

- If each pieces are convergent, then the full integral also convergent,
- If one piece divergent, then full integral must diverge.

$$\int_{-\infty}^{\infty} x\,dx = \int_0^\infty x\,dx + \int_{-\infty}^0 x\,dx = \underbrace{\lim_{R\to\infty}\int_0^R x\,dx}_{DNE} + \underbrace{\lim_{R\to\infty}\int_{-R}^0 x\,dx}_{DNE}$$

12.7 Basic Comparison Test

$\forall x \in \mathbb{R},\ x \geq a$:

IF $0 \leq f(x) \leq g(x)$

Then

1. If $\int_a^\infty f(x)\,dx = \infty$, $\int_a^\infty g(x) = \infty$
2. If $\int_a^\infty g(x)\,dx < \infty$, $\int_a^\infty f(x) < \infty$.

Example 1.

$$\int_1^\infty \frac{\sin^2 x}{x^2}\,dx < \infty \quad \left(\because\ 0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}\right) \quad \left(\int_1^\infty \frac{1}{x^2}\,dx < \infty\right)$$

Example 2

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

(Big Theorem $\ln x \ll x^{\frac{1}{2}}$

· For large x, $\frac{\ln x}{x^{\frac{1}{2}}} < 1$, $\ln x < x^{\frac{1}{2}}$)

$$\therefore\ 0 \leq \frac{\ln x}{x^2} < \frac{x^{\frac{1}{2}}}{x^2} = \frac{1}{x^{\frac{3}{2}}} \rightarrow \int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx < \infty, \Rightarrow \int_1^\infty \frac{\ln x}{x^2}\,dx < \infty$$

12.9. Limit-Comparison Test for improper integrals.

Let $a \in \mathbb{R}$. Let f and g be positive, continuous functions on $[a, \infty)$

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ exists and $L > 0$,

THEN, $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are either both convergent or divergent.

Ex1.

$$f(x) = \frac{x^2+3x}{\sqrt{x^5+1}}\,dx \underset{x\to\infty}{\sim} \frac{x^2}{x^{\frac{5}{2}}} = \frac{1}{x^{\frac{1}{2}}} = g(x)$$

$$\lim_{x\to\infty} \frac{\frac{x^2+3x}{\sqrt{x^5+1}}}{\frac{1}{x^{\frac{1}{2}}}} = \cdots = 1.$$

By LCT,

$$\int_1^\infty f(x)\,dx = \infty$$

$$\because \int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^{\frac{1}{2}}}\,dx = \infty, \therefore$$

Ex2

$$\int_1^\infty \sin\frac{1}{x^2}\,dx$$

Let $f(x) = \sin\frac{1}{x^2}$.

Let $g(x) = \frac{1}{x^2}$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sin\frac{1}{x^2}}{\frac{1}{x^2}}$$

$\int_1 \frac{1}{x^2}\,dx < \infty, \quad = 1.$

$\therefore \int_1^\infty \sin\frac{1}{x^2}\,dx$ is convergent.