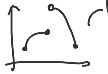
Test 2 ch2.11-6.2

Thursday, 21 November 2024 1:09 AM

Remember define the function on $[a,b]$,
Let $a < b, \in \mathbb{R}$,


No Max.
For c, always use open interval (a,b)

★ Extreme Value Theorem (EVT)

1. If continuous on $[a,b]$,
2. Then has a maximum and minimum on $[a,b]$.

★ The Intermediate Value Theorem

1. If $f(a) < M < f(b)$
2. If f is continuous on $[a,b]$
3. Then there exist c s.t. $f(c) = M$

1. If $f(a) < 0$, $f(b) > 0$
2. If f is continuous on $[a,b]$
3. Then there exist c s.t. $f(c) = 0$.

★ Differentiable implying continuous.

- $\lim_{x\to c} [f(x) - f(c)] = \lim_{x\to c} \left[\frac{f(x)-f(c)}{x-c} \cdot (x-c)\right]$

$= f'(c) \cdot 0$

$= 0$

★ Prove Product Rule

$h(x) = f(x)g(x)$

f, g differentiable at $a \implies h'(a) = f'(a)g(a) + f(a)g'(a)$ (h differentiable at a)

$$h'(x) = \lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x-a}$$

$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a} g(x) + \frac{g(x)-g(a)}{x-a} f(a)\right)$$

$$= f'(a)g(a) + g'(x)f(a)$$

★ Proof of the power rule

$$\frac{d}{dx} x^c = c x^{c-1}$$

Proof by induction on c,

- BASE CASE ($c = 1$) WTS $\frac{d}{dx}[x] = 1 \cdot x^0 = 1$

Call $f(x) = x$

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{x+h-x}{h}$

$= \lim_{h\to 0} \frac{h}{h}$

$= 1$

- INDUCTION STEP

Fix $c \geq 1$

Assume $\frac{d}{dx} x^c = c x^{c-1}$

WTS $\frac{d}{dx} x^{c+1} = (c+1) x^c$

$\frac{d}{dx} x^{c+1} = \frac{d}{dx}(x^c \cdot x)$ (by product rule)

EVT: continuous $[a,b]$ $\implies$ has min/max

IVT: continuous $[a,b)$ + $f(a) < M < f(b)$ $\implies$ exist $f(c) = M$.

Local EVT: local extremum at c + interior $\implies$ $f'(c) = 0$ / DNE

Critical point: $f' = 0$ / DNE + interior

Rolle's Theorem: cont. diff. $f(a) = f(b)$ $\implies$ $f'(c) = 0$, $c \in (a,b)$

Mean Value Theorem: Cont. diff. $f'(c) = \frac{f(b) - f(a)}{b-a}$, $c \in (a,b)$

Zeroes:

Rolles' • f' zeros + 1 $\geq f$ zeroes. (AT MOST)

IVT • AT LEAST.

Max, local max

- f defined on at least interval centred at a.
- $\forall x \in I$, $f(x) \leq f(c)$
- $\exists \delta > 0$, $|x - c| < \delta \implies f(x) \leq f(c)$

Inverse:

- $g^{-1}: B \to A$ defined by.
- $\forall x \in A, \forall y \in B$ $y = g(x) \iff x = g^{-1}(y)$

Graph

- tan / arctan $\implies$ Always $x \in \mathbb{R}$.
- arctan(tan(x)) remb discontinuity holes.

f is diff at a.

→ Let f . defined, $\boxed{\text{at least on an interv}}$

→ $\lim_{x\to a} \frac{f(x) - f(a)}{x-a}$ exists.

f is dif at a

EVT

IVT

Local EVT

Critical pt

Rolle's Theorem

Mean Value Theo

Max, local max

Inverse

$$= \left(\frac{d}{dx} x^c\right) \cdot x + x^c \left(\frac{d}{dx} x\right) \quad \text{(by induction hypothesis)}$$

$$= (c x^{c-1}) \cdot x + x^c \cdot 1$$

$$= (c+1) x^c$$

- Non-differentiable ...
  1. side limits are different (corner) $\rightarrow \frac{|x|}{x}$ $\boxed{\begin{array}{ll} x \to 0^+ & = 1 \\ x \to 0^- & = -1 \end{array}}$
  2. limit is $\pm\infty$ (Vertical Tangent)

Function
- domain $\longrightarrow$ codomain (Range : Actual Output)

  {Input} {Potential Output}

Inverse
- $f^{-1} : B \to A$

  $x = f^{-1}(y) \iff y = f(x)$

> Inverse : 2 marks
> g's inverse is another function $g^{-1} : B \to A$ defined by:
> $\forall x \in A, \forall y \in B, \; y = g(x) \iff g^{-1}(y) = x$

- Surjective : Range = Codomain
- Inverse : $x_1 \neq x \implies f(x_1) \neq f(x_2)$

  If 1. f has an inverse
  2. f is differentiable
  3. For all $x \in I$, $f'(x) \neq 0$

  Then f^{-1} is differentiable.

$$\frac{d}{dy} f(f^{-1}(y)) = \frac{d}{dy} y$$

$$f'\underbrace{f^{-1}(y)}_{x} \cdot (f^{-1})'\underbrace{(y)}_{f(x)} = 1$$

$$f'(x) \cdot (f^{-1})'(y) = 1$$

$$f^{-1}(y) = \frac{1}{f'(x)}$$

$\underset{[-1,1]}{\arcsin x} = \underset{[-\frac{\pi}{2}, \frac{\pi}{2}]}{y}$

$\underset{[-1,1]}{\arccos x} = \underset{[0, \pi]}{y}$

$\underset{(-\infty,\infty)}{\arctan x} = \underset{(-\frac{\pi}{2}, \frac{\pi}{2})}{y}$

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

ARCSIN

$y = \sin x \iff x = \arcsin y$

cannot outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$

cannot outside $(-1, 1)$

$x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y \in [-1, 1]$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1+x^2}}$$

Call $\theta = \arcsin x$

Know $\sin\theta = x$

Want $\cos\theta$

$\cos^2\theta = 1 - \sin^2\theta$

$\cos\theta = \pm\sqrt{1-x^2}$

$\cos\theta \geq 0$ $\left(\begin{array}{c} -1 \leq x \leq 1, \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{array} \right)$

$\frac{d}{dx}\sin(\arcsin x) = \frac{d}{dx} x$

$\cos(\arcsin x) \cdot \frac{d}{dx}\arcsin x = 1$

$\frac{d}{dx}\arcsin x = \frac{1}{\cos(\arcsin x)}$

- Local Extreme Value Theorem
  1. f has a local extremum at c, and
  2. c is an interior point to I (not an end-point)

  $\Rightarrow f'(c) = 0$ or DNE

- Critical Point
  1. c is interior point of domain of f
  2. $f'(c) = 0$ or DNE

EVT + Local EVT

$\searrow \swarrow$

Rolle's Theorem

$\downarrow$

Mean Value Theorem

- Maximum $\forall x \in I,\ f(x) \leq f(c)$
- Local Maximum :

  $\exists \delta > 0$ s.t. $|x-c| < \delta \Rightarrow f(x) \leq f(c)$

- Rolle's Theorem

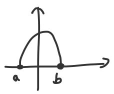
Let $a<b$. Let f be a function defined on $[a,b]$.

If ① f is continuous $[a,b]$

② f is differentiable (a,b)

③ $f(a) = f(b)$

Then ① $\exists c \in (a,b)$ s.t. $f'(c) = 0$



☆ ① EVT $\Rightarrow$ f has a max / min on $[a,b]$

② Case 1 : max and min in $c \in (a,b)$ $\Rightarrow$ Local EVT : $f'(c) = 0$. ☆ interior

③ Case 2 : max and min at endpoints $\Rightarrow$ f must be constant (as given $f(a) = f(b)$)

- Number of Zeroes
  1. Use IVT prove at least n.
  2. Use Rolle's theorem prove at most n.

$g(-2) = 64$

$x_1 \to g(0) = -2$

$x_2 \to g(1) = 1$

1. $f(x_1), f(x_2) = 0$.

2. There exist $\alpha_1 < a < x_2 \Rightarrow f'(a) = 0$ (Rolle's Theorem)

① (Between 2 zeroes of f, have at least one zero of f')

3. Assume $x_1 < x_2 < x_3$ are zeroes of f.
- exist $f'(a) = 0$ $(x_1 < a < x_2)$
- exist $f'(b) = 0$ $(x_2 < b < x_3)$

can have more!

② f' zeroes $\geq$ f zeroes $- 1$ (f' zeroes AT LEAST)

③ f' zeroes $+1 \geq$ f zeroes (f zeroes AT MOST)

1. $g(x) = x^6 + x^2 + x - 2$

$g'(x) = 6x^5 + 2x + 1$

$g''(x) = 30x^4 + 2$ (ALWAYS POSITIVE)

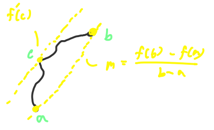
$\Rightarrow g''$ 0 zeroes.

$\Rightarrow g'$ AT MOST 1 zero.

$\Rightarrow g'$ AT MOST 2 zero.

$\boxed{\text{IVT, Rolle's Both confirm 2 zeroes.}}$

- The Mean Value Theorem



If ① f continuous on $[a, b]$

② f differentiable on (a, b)

Then ① $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Proofs with MVT, Rolle's

1. Zero derivative implies constant

$f'(0) \Rightarrow$ Constant

WTS: $\forall x_1, x_2 \in [a, b], f(x_1) = f(x_2)$

$\rightarrow$ Verify x_1, x_2 continuous, differentiable,

By MVT, $\exists c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad f(x_2) - f(x_1) = 0$$

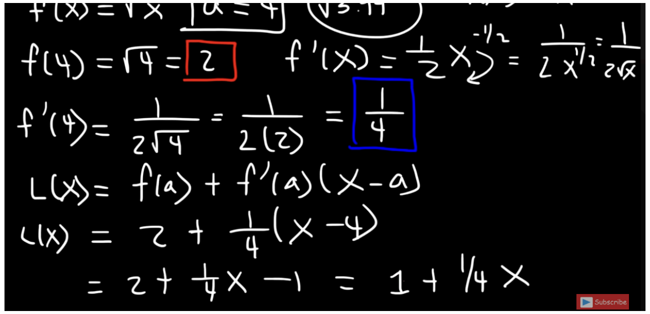
Assumed zero.

巻 5.11, 5.12, 6.1 - 6.2,

Linear Approximation


1. Find the linearization L(x) of the function at the given value shown below:

$f(x) = x^{1/2}$



Notes on Question Types.

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For point $(0, 2)$

- Remember $(f^{-1})'(\underset{2}{y}) = \dfrac{1}{f'(\underset{0}{x})}$.

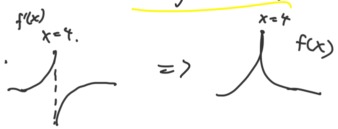
  ↖ Given tangent at 0,
  $y = 3x + 2$
  $f'(0) = 3$!

- Chain rule. $\arctan\left(\frac{1+x}{2}\right) \Rightarrow \dfrac{1}{1+\left(\frac{1+x}{2}\right)^2} \cdot \left(\frac{1}{2}\right)$
  $\Rightarrow \frac{1}{4}$

- $\boxed{\frac{\pi}{4} = 45°.}$

3b. For f 'strictly decreasing', it has to be CLOSED BRACKETS.

3d.



4. Period of tangent is π!

$\tan(x + \pi)$
$= \tan(x + \pi + \pi)$
$= \tan(x + \dots n\pi)$.

5. Find a formula for a '$\sin(\arctan x)$' expression
   1. Let $\arctan x = \theta$
   2. Obtain $\cos\theta$, $\pm\sqrt{1-\cos^2\theta}$
   3. obtain $\sin\theta$ by considering when $\sin\theta \leq 0$ / $\sin\theta \geq 0$ (when θ ...)

6. Linear approximation = f(original) + (new − original) f′(original)

Remember to "use" the limit law of product.

7. Facts:

False: f diff at a ⇒ f′ continuous at a

$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

False: f diff at a ⇒ f∘f dif at a.

$f(x) = |x-1|$

8. Define $g(x) = \boxed{(f(x))^2 - x}$ on $[0,1]$.

$g(1) \leq 0$

$g(0) \geq 0$

If $f(0) = 0$, find $c = 0$ s.t. $g(c) = 0 \Rightarrow (f(c))^2 - c = 0$

$(f(c))^2 = c.$

If $f(1) = 1$, find $c = 1$ s.t. $g(c) = 1$

$(f(c))^2 = c$

If $f(0) \neq 0$, $f(1) \neq 1$,

$g(0) > 0$ AND $g(1) < 0$

$g(1) < 0 < g(0)$

$\exists c \in (0,1)$ s.t. $g(c) = 0$.

$(f(c))^2 = c$

$\therefore$ $c \in [0,1]$ s.t. $(f(c))^2 = c$.

12a. Why $g(x) = \frac{1}{f(x)}$ is defined.

✱ Given $f(a) > 0$.

→ f is continuous, $f(a) > 0$

→ Then exist $\delta > 0$, s.t. $\forall x \in (a-\delta, a+\delta)$ $f(x) > 0$

$\therefore$ $g(x)$ defined for all x $(a-\delta, a+\delta)$

✱ Remember: To split limits explaining why exist

- limit laws of constant
- limit laws for products → conclude diff.

7b. Decreasing Quadratic Type.

① Let $x_1, x_2 \in (a,b)$, $x_1 < x_2$.

② f diff $\to$ f cont.

③ $(x_1, x_2) \subseteq (a,b) \Rightarrow$ f diff on (x_1, x_2)

$[x_1, x_2] \subseteq (a,b) \Rightarrow$ f cont on (x_1, x_2)

④ Mean Value Theorem

$c \in (x_1, x_2)$ implies $c \in (a,b)$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$$

$\Rightarrow f(x_2) \leq f(x_1)$

1819.

2. When given something like $g(a+b) = \dots$

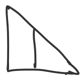
- USE as $\dfrac{g(x+h) - g(x)}{h}$!!

When asked (prove ... is differentiable at a and find g',)

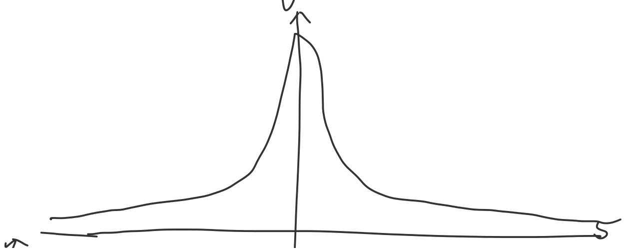
$\Rightarrow$ Use limit law for sum / products

$\Rightarrow$ limit ... exist for $x \in \mathbb{R}$.

$\Rightarrow g(x)$ diff for $x \in \mathbb{R}$.

4. When modelling, remember , similar Δ.

6. A function may not have a minimum.



never ending small

8 • restriction of f is one to one, and its inverse has vertical tangent line at 2

$$(f^{-1})'(2) = \frac{1}{f'(x)} = \infty$$

y

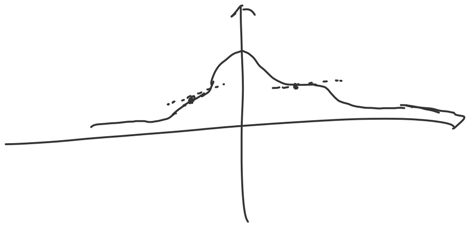
(2, ?)

$$f'(?) = 0.$$

• restriction of f is one-to-one, and its INVERSE has derivative 2 at 2.

$$(f^{-1})'(2) = \frac{1}{f'(x)} = 2$$

$$f'(x) = \frac{1}{2}$$



When defining function, try to do

$$F(x) = f(x) - (\text{something}) \quad / \text{ OR}$$
$$= f(x) - x$$

Pbset.

If $\theta_1 + \theta_2 \in [-\pi, -\frac{\pi}{2})$

• $\sin(\theta_1 + \theta_2) = -\sin(\pi + \theta_1 + \theta_2)$

Remember!

$\sin(\pi + \text{anything}) = -\sin(\text{anything})$ | $\sin(\pi - x) = \sin x$

$\cos(\pi + \text{anything}) = -\cos(\text{anything})$ | $\cos(\pi - x) = -\cos x.$

$\tan(\pi + \text{anything}) = \tan(\text{anything})$ | $\tan(\pi - x) = -\tan x.$