Ch14

Sunday, 23 March 2025 5:08 PM

Quick Recap

- Power series is $\sum_{n=0}^{\infty} c_n x^n$ (i.e. $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$)

  or $\sum_{n=0}^{\infty} c_n (x-a)^n$

- Ratio test = $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$ absolutely convergent

Taylor series : $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ ← Taylor series centred at a

14.1 Power Series

- Find domain of $g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

  => $\{x \in \mathbb{R} \mid$ the series $g(x)$ is convergent$\}$
  ↳ value exist

※ Ratio Test! Part 1:

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)3^{n+1}}}{\frac{|x|^n}{n3^n}} = \lim_{n\to\infty} \frac{|x|\,n}{3(n+1)} = \frac{|x|}{3} < 1.$$

$|x| < 3$

Part 2: Check endpoints

$$g(3) = \sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

$$g(-3) = \sum_{n=1}^{\infty} \frac{(-3)^n}{n(3^n)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \text{convergent (By AST)}$$

∴ Domain $g = [-3, 3)$ = Interval of convergence,

Radius of convergence = 3. (Half of the interval)

14.2 Power Series: The main theorem

Let $a \in \mathbb{R}$,

A power series centered at a is a function f defined by an equation like is power series at a:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where $c_1, c_2, \ldots \in \mathbb{R}$. Domain $f = \{ x \in \mathbb{R} \mid f(x) \text{ is convergent} \}$

- The domain of f is an interval centered at a:
  - $(a-R, a+R)$
  - $(a-R, a+R]$
  - $[a-R, a+R]$
  - $[a-R, a+R)$
  - $\mathbb{R}$
  - $\{a\}$

  Possible domains. Also i.e. Interval of Convergence

  (R is the radius of convergence, $0 \le R \le \infty$)

- Interior of IC: Series is ABSOLUTELY CONVERGENT

  Exterior of IC: Series is DIVERGENT

  At Endpoints: Anything can happen.

- In interior of IC:
  - power series can be treated like polynomials
    - Addition, multiplication, composition, differentiated, integrated
      → Does NOT change the radius of convergence

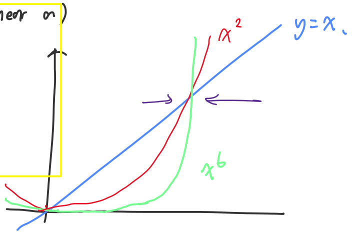
14.3 Taylor polynomials (1) - The Definition with the limit

- Goal: Approximate functions with polynomials. (Want $P(x) \approx f(x)$, for x near a)

  $a \in f(x)$, $P(x)$

- $R(x)$: "remainder" or "error". We want $\lim_{x \to a} R(x) = 0$ fast.

  $R(x) = f(x) - P(x)$



→ $y = x^6$ get closer to $f(1)$ much faster than $y = x^2$

Let f, g be continuous function at 0.

Let $n \in \mathbb{N}$.

- g is an approximation for f near 0 of order n when

$$\lim_{x \to 0} \frac{f(x) - g(x)}{x^n} = 0$$

← The polynomial $P(x)$

- g is an approximation for f near a of order n when

$f(x) - g(x)$

- Implies: $f(x) = g(x) + R(x)$.
  As $x \to 0$, $R(x) \to 0$ faster than $x^n \to 0$.

x^n is just a benchmark of how well the function $g(x)$ estimates $f(x)$

- If limit is zero, $\boxed{f(x) - g(x)}$ shrinks $\boxed{\text{FASTER THAN}}$ $\boxed{x^n}$ as $x \to a$.
  → $g(x)$ is a very good estimation of $f(x)$ up to order n.

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$$

As $x \to a$, $R(x)$ goes faster to 0 than $(x-a)^n \to 0$

- First Definition of Taylor polynomial
  - Let $a \in \mathbb{R}$.
  - Let f be a continuous function defined at and near a.
  - Let $n \in \mathbb{N}$.
  - The n-th Taylor polynomial for f at a is polynomial P_n.

Approximation function.

P_n is an approximation for f near a of order n

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$$

, with degree at most n.

14.4 Taylor polynomials (2) - The definition with the derivatives.

A function f is called ...

- C^0 when f is continuous
- C^1 when f' exists and is continuous
- C^2 when f', f'' exist and are continuous

⋮

- C^n when $f', f'', \dots, f^{(n)}$ exist and are continuous
- C^∞ when all derivatives exist and are continuous

- Second definition of Taylor polynomial
  - Let $a \in \mathbb{R}$
  - Let $n \in \mathbb{N}$
  - Let f be a C^n function at a.

The n-th Taylor polynomial for f at a is

- P_n s.t. $P_n(a) = f(a)$,
  $P_n'(a) = f'(a)$
  ...
  $P_n^{(n)}(a) = f^{(n)}(a)$ with degree at most n.

Prove.

Want $\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$.

If $f(a) - g(a) \neq 0$, then "$L = \frac{\text{not } 0}{0} = \pm\infty$"

Assume $f(a) = g(a)$. $L = \frac{0}{0}$, use L'Hôpital's.

$$L \overset{(*)}{=} \lim_{x \to a} \frac{f'(x) - g'(x)}{n(x-a)^{n-1}}$$

$$\overset{(*)}{=} \lim_{x \to a} \frac{f''(x) - g''(x)}{n(n-1)(x-a)^{n-2}}$$

$$\overset{(*)}{\vdots} \ \frac{f^{(n)}(a) - g^{(n)}(a)}{n!}$$

$\to$ $L = 0 \iff \begin{cases} f(a) = g(a) \\ f'(a) = g'(a) \\ \dots \\ f^{(n-1)}(a) = g^{(n-1)}(a) \end{cases}$

- $y = P_1(x)$ is equation of tangent of $f(x)$!

Valid as f, g are C^n.

$f^{(n)}(a) = g^{(n)}(a)$

14.5 Taylor Polynomials (3) - The formula.

$P_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$

$P_n^{(k)}(0) = (k!) \cdot c_k = f^{(k)}(0)$

$c_k = \frac{f^{(k)}(0)}{k!}$

$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$

Third definition of Taylor Polynomial

- Let $a \in \mathbb{R}$.
- Let $n \in \mathbb{N}$.
- n-th Taylor polynomial $P_n(x)$ for f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

- degree $P_n \leq n$
- Taylor polynomials of a function are unique.

Let $a \in \mathbb{R}$.
Let f be a C^∞ function at a.
The Taylor Series for f at a is the power series

$$S(x) = \sum_{n=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

- " $\forall k \in \mathbb{N}, \ S^{(k)}(a) = f^{(k)}(a)$ "
- Analytic Function: $f(x) = S(x)$
- Maclaurin Series: Taylor series at 0, $\boxed{a=0}$

14.6 The four main Maclaurin Series

Example 1: Maclaurin series for $f(x) = e^x$.

- $\forall k \in \mathbb{N}, \ f^{(k)}(x) = e^x$

$f^{(k)}(0) = 1$

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Taylor series $f(x) = e^x$ at c.

$u = x - c$.

$$e^x = e^{c+u} = e^c \cdot e^u = e^c \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!} (x-c)^n$$

Example 2: Maclaurin Series for $f(x) = \sin x$

$k=0, \ g^{(0)}(0) = 0$

$k=1, \ g^{(1)}(0) = 1$

$k=2, \ g^{(2)}(0) = 0$

$k=2$, $g^{(2)}(0) = 0$

$k=3$, $g^{(3)}(0) = -1$

$k=4$, $g^{(4)}(0) = 0$

$$S(x) = \sum_{n=0}^{\infty} g^{(n)}(0) \frac{x^n}{n!}$$

$$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

Analytic

Four Main Maclaurin Series.

For all x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

For $|x|<1$: $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

14.7 Analytic functions and the Remainder Theorems

Analytic Functions:

Let f be a C^∞ function defined on an open interval I.

- Let $a \in I$.
- Let $S_a(x)$ be Taylor Series of f at a.

▶ $\boxed{f \text{ is analytic at } a}$ when

$\exists$ an open interval J_a centred at a s.t.

$\forall x \in J_a$, $f(x) = S_a(x)$

▶ $\boxed{f \text{ is analytic}}$ when

$\forall a \in I$, f is analytic at a.

1. Polynomials are analytic

2. Sums, products, quotients, composition, derivatives, antiderivatives of analytic functions are analytic.

3. Interior of interval of convergence: power series can be manipulated like a polynomial

4. Taylor series of a function at a point is unique

- Analytic.

Definition of Taylor polynomial: $\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0$

Analytic function: $\lim_{n \to \infty} R_n(x) = 0$

- Three versions of remainder theorem

1. Lagrange's form
2. Cauchy's form
3. Integral form

1. Lagrange's Remainder Theorem.

Let I be an open interval.
Let $a \in I$.
Let f be a C^{n+1} function on I.
Let $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ be its nth Taylor polynomial
Let $R_n(x) = f(x) - P_n(x)$ be the remainder.

Then

value

$\exists\, \xi$ between a and x s.t.

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Useful..

1. Prove a function is analytic
$\lim_{n \to \infty} R_n(x) = 0$

2. To estimate:
Approximate $f(x)$ by $P_n(x)$ and bound the error.

14.8 A proof that the exponential function is analytic.

W/S: $f(x) = e^x$ is analytic at 0.

$f(x) = e^x$, $a = 0$

$$e^x = P_n(x) + R_n(x)$$

$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ is n-th Taylor polynomial.

$R_n(x)$ is the n-th remainder.

Need to prove: $\forall x \in \mathbb{R}$, $\lim_{n\to\infty} R_n(x) = 0$

Proof.

Fix $x \in \mathbb{R}$.

Use Lagrange's Remainder Theorem for $f(x) = e^x$ and $a = 0$.

$\exists c$ between 0 and x s.t.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{e^c}{(n+1)!} x^{n+1}$$

For $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \left[e^c \cdot \frac{x^{n+1}}{(n+1)!} \right]$

Case 1: $x > 0$.

Then $0 < c < x$ so.

$$0 \le R_n(x) = e^c \frac{x^{n+1}}{(n+1)!} \le e^x \frac{x^{n+1}}{(n+1)!}$$

Big Theorem: $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$.

Squeeze Theorem: $\lim_{n\to\infty} R_n(x) = 0$

Case 2: $x < 0$ (exercise)

⋮

$\therefore e^x = \sum_{n=0}^{\infty} \frac{x^k}{k!}$

14.9 How to write functions as power series quickly.

Slow method

1. Start with a C^∞ function f

Ex 1. $f(x) = e^{-x}$ as power series at 0.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

Ex 2 Write $f(x) = x^3 \sin x$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1}{(2n}$

$\sin x^2 = \sum \frac{(}{}$

2. Obtain the Taylor Series

$$S(x) = \sum_{n=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

3. Use Remainder Theorem to prove

$$\lim_{n\to\infty} R_n(x) = 0$$

4. Then $f(x) = S(x)$

$$e^{-x} = \sum_{n=0} \frac{(-x)}{n!} = \sum_{n=0} \frac{\cdots}{n!} x^n \quad \text{for all } x.$$

$$f(x) = x^3 \sin x^2 = \sum_{n=0}^{\infty}.$$

$n=0$ (2

14.10 Logarithm as a power series

- Write $f(x) = \ln(1+x)$ as power series at 0.

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{for } |x| < 1.$$

$$f(x) = \int \sum_{n=0}^{\infty} (-1)^n (x^n)\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^n\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

for $|x| < 1$, as taking derivatives/antiderivatives in interval of convergence does not change radius of convergence. ($x=1$, $x=-1$ may change behavior!)

Evaluate at $x=0$: $0 = f(0) = 0 + C$

$\therefore C = 0$

$$\therefore \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } |x| < 1$$

Exercise: arctanx as power series at 0.

14.11 Taylor applications: Estimation

Exercise:

Estimate $\sqrt{e}$ with error < 0.001.

- Let $f(x) = e^x$.
- Want $f(\frac{1}{2})$.
- $f(\frac{1}{2}) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{2^k} = \underbrace{P_n(\frac{1}{2})}_{\text{Estimation}} + \underbrace{R_n(\frac{1}{2})}_{\text{Error}}$

- Want $|R_n(\frac{1}{2})|$ error < 0.001

▷ Lagrange's Remainder Theorem.

$\exists c \in (0, \frac{1}{2})$ s.t.

$$R_n(\frac{1}{2}) = \frac{f^{(n+1)}(c)}{(n+1)!} \left(\frac{1}{2} - 0\right)^{n+1}$$

e^c

$\Downarrow$

$$0 < R_n(\frac{1}{2}) < \frac{e^{\frac{1}{2}}}{(n+1)!} \cdot \frac{1}{2^{n+1}} < \frac{2}{2^{n+1}(n+1)!} < 0.001$$

Need $2^n(n+1)! > 1000$

$n = 4$ works

$$\therefore P_4(\frac{1}{2}) = \sum_{k=0}^{4} \frac{1}{k!2^k} = \frac{211}{128}$$

14.12 Taylor applications: Integrals

Ex1 Compute $I = \int_0^3 e^{-x^2} dx$

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

Ex2 $\int \frac{1}{1-x^8} dx$

$$= \int \sum_{n=0}^{\infty} x^{8n} dx$$

$$= \sum_{n=0}^{\infty} \frac{x^{8n+1}}{8n+1} + C \quad \text{for } |x|<1$$

$$I = \int_0 \sum_{n=0} \frac{\quad}{n!} \quad dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^3 x^{2n} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{x^{2n+1}}{2n+1} \right]_0^3$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{3^{2n+1}}{2n+1}$$

14.13 Taylor applications: Limits

☆ Trick: only care the smallest non-zero term.

- Ex0: $\lim_{x\to 0} \frac{2x^3 + x^4 + 11x^6}{5x^3 + x^5 - 7x^6}$

$$= \lim_{x\to 0} \frac{\cancel{x^3}}{\cancel{x^3}} \cdot \frac{(2 + x + 11x^3)}{(5 + x^2 - 7x^3)}$$

(☆ Not divide by largest term. We $x \to 0$, not $x \to \infty$! Factor the smallest term)

$$= \frac{2}{5}.$$

- Ex1: $\lim_{x\to 0} \frac{\sin x - x}{x^3}$ (can use L'Hopital's Rule)

$$= \lim_{x\to 0} \frac{[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots] - x}{x^3}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}{x^3}$$ → Only the smallest term matters.

$$= \lim_{x\to 0} -\frac{1}{6} + \frac{1}{120}x^2 + \cdots$$

$$= -\frac{1}{6}.$$

Ex2 $\lim_{x\to 0} \frac{3x^2 - e^{x^2} + \cos 2x}{x \sin x - \ln(1 + x^2)}$

$$= \lim_{x\to 0} \frac{\frac{1}{6}x^4 + (\text{higher order terms})}{\frac{1}{3}x^4 + (\text{higher order terms})}$$

Numerator

(+) $3x^2$.

(−) $e^{x^2} = 1 + x^2 + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \cdots$

(+) $\cos 2x = 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6 + \cdots$

$$3x^2 - e^{x^2} + \cos 2x = (-1 + 1) + \left(3 - 1 - \frac{4}{2}\right)x^2 + \left[-\frac{1}{2} + \frac{2^4}{4!}\right]x^4 + \cdots = \frac{1}{6}x^4 + (\ldots$$

$= \dfrac{\frac{1}{6}}{\frac{1}{3}}$

$= \boxed{\frac{1}{2}}$

Denominator.

$(+)\ x\sin x = x[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots]$

$(-)\ \ln(1+x^2) = (x^2) - \frac{1}{2}(x^2)^2 + \frac{1}{3}(x^2)^3 - \dots$

$x\sin x - \ln(1+x^2) = [1-1]x^2 + [-\frac{1}{6}+\frac{1}{2}]x^4 + \dots = \frac{1}{3}x^4 + (\text{higher order terms})$

14.14 Taylor applications: Adding series.

Ex 1: Compute $A = \sum_{n=1}^{\infty} \frac{n}{2^n}$

Want $\sum_{n=1}^{\infty} n x^n$ when $x = \frac{1}{2}$.

- Start with $\sum_{n=0}^{\infty} x^n$.

Take $\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\frac{1}{1-x}$

$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$

Evaluate at $x = \frac{1}{2}$,

$\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$

Ex 2: Compute $B = \sum_{n=0}^{\infty} \frac{2^n}{(n+2)n!}$

Want $\sum_{n=0}^{\infty} \frac{x^n}{(n+2)n!}$ when $x = 2$.

Know $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

$\int \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)} + C$

$\int xe^x\,dx = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} + C$

$(x-1)e^x = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} + C$

Evaluate at $x=0$: $-1 = 0 + C$

$C = -1$

$\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} = (x-1)e^x + 1$

$\sum_{n=0}^{\infty} \frac{x^n}{(n+2)n!} = \frac{(x-1)e^x + 1}{x^2}$

Evaluate at $x=2$: $\sum^{\infty} 2^n \quad e^2 + 1$

$\sum_{n=0} \frac{}{(n+2)n!} \cdot \frac{}{4}$

14.15 Taylor Series Applications: Physics

- Kinetic energy of a particle
- Classical Physics

$T = \frac{1}{2} m_0 v^2$

m_0 = mass at rest
v = velocity
c = speed of light
m = mass

- Relativity ; $\frac{v}{c}$ "small"

$T = mc^2 - m_0 c^2$

$= \frac{m_0 c^2}{\sqrt{1-(\frac{v}{c})^2}} - m_0 c^2$

$= m_0 c^2 \left(\boxed{\frac{1}{\sqrt{1-(\frac{v}{c})^2}}} - 1 \right)$

- Let $f(x) = (1-x)^{-\frac{1}{2}} \approx f(0) + \boxed{f'(0) \cdot x} + \frac{f''(0)x^2}{2!}$ (1st order taylor polynomial) 2nd order Taylor polynomial

$= 1 + \boxed{\frac{1}{2} x} + \frac{3}{8} x^2$

$\therefore \frac{1}{\sqrt{1-x^2}} \approx \boxed{1 + \frac{1}{2} x^2} + \frac{3}{8} x^4$

$\therefore$ Kinetic Energy $T = m_0 c^2 \left[\frac{1}{2} \left(\frac{v}{c}\right)^2 + \frac{3}{8} \left(\frac{v}{c}\right)^4 \right]$

$= \boxed{\frac{1}{2} m_0 v^2} + \boxed{\frac{3 m_0 v^4}{8 c^2}}$

$\frac{1}{2} m v^2$ higher accuracy.