Ch11 Definition

Monday, 21 April 2025 1:08 AM

Definition 11.1 Dot Product $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ $\vec{v} = \begin{pmatrix} \vec{v_1} \\ \vdots \\ \vec{v_n} \end{pmatrix}$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$
$$= (\vec{u})^T \vec{v}$$

Definition 11.4 Norm

$$\|\vec{u}\| := \sqrt{\vec{u} \cdot \vec{u}}$$

Definition 11.5 Distance

$$\|\vec{u} - \vec{v}\| = \sqrt{(\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v})}$$

Definition 11.7 Orthogonal vectors

$$\theta = \arccos \frac{a \cdot b}{\|a\|\|b\|}$$

- $\vec{u}, \vec{v}$ orthogonal if $\vec{u} \cdot \vec{v} = 0$

Definition 11.9 Orthogonal basis, Orthonormal basis

Orthogonal: • $\vec{v_i} \cdot \vec{v_j} = 0$ for $i \neq j$

Orthonormal: • AND $\|\vec{v_i}\| = 1$ for every i.

Definition 11.12 Orthogonal matrix (orthonormal!)

- If $Q^{-1} = Q^T$. Then Q is orthogonal
- Column of Q forms orthonormal basis.

Proposition 11.2

(1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

$(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

Proposition 11.10

- If β is orthoNORMAL basis of $\mathbb{R}^n$, $x, y \in \mathbb{R}^n$

e.g. $B = \{\vec{v_1}, \dots, \vec{v_n}\}$ $[\vec{x}]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $[\vec{y}]_B = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

- THEN $[\vec{x}]_B \cdot [\vec{y}]_B$

$$= \vec{x} \cdot \vec{y}$$
$$= (x_1 v_1 + \dots + x_n v_n) \cdot (y_1 \vec{v_1} + \dots + \vec{v_n})$$
$$= x_1 y_1 (v_1 \cdot v_1) + \dots + (x_n y_n)(v_n \cdot v_n) \quad (\text{e.g. } v_1 \cdot v_2 = 0 \text{ (orthonormal)})$$
$$= x_1 y_1 + \dots + x_n y_n$$

Theorem 11.14 For any $\vec{v}, \vec{w} \in \mathbb{R}^n$,

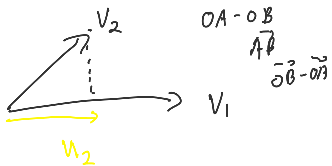
$$Q\vec{v} \cdot Q\vec{w} = \vec{v} \cdot \vec{w}$$

Definition 11.16 Orthogonal projection a onto b

$$\mathrm{proj}_b\, a = \frac{a \cdot b}{b \cdot b} b$$

Definition 11.17 The Gram-Schmidt Process

- Every vector space has an orthogonal basis



- $\vec{u_1} = \vec{v_1}$
- $u_2 = \vec{v_2} - \mathrm{proj}_{u_1} \vec{v_2}$
- $u_3 = \vec{v_3} - \mathrm{proj}_{u_1} \vec{v_3} - \mathrm{proj}_{u_2} \vec{v_3}$

$\vdots$

Definition 11.19 Orthogonally diagonalizable matrix

- if exist
  - orthogonal matrix Q
  - diagonal matrix D
- s.t. $\Rightarrow Q^T A Q = D$ $\quad A = Q D Q^{-1}$ (↑ the orthonormal basis of the eigenvectors)

$\Rightarrow A = Q D Q^T$

✗ Q is simply P in eigendecomposition, but each column normalized!

- Symmetric matrices have orthogonal eigenvectors!

Theorem 11.25 The Singular Value Decomposition

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{Q^T} \qquad (A = P^{-1} D P \quad P = A^TA,\ P^{-1} = (A^TA)^{-1} \neq A)$$

- U: orthogonal $m\times m$ matrix U (eigenvectors of AA^T)
- V: orthogonal $n\times n$ matrix V (eigenvectors of A^TA)

Σ : block diagonal matrix (no need $n\times n$) $(\sigma_i = \sqrt{\lambda_i})$

e.g. $\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $m>n$, $n>m$

Definition 11.26

Let A be $m\times n$ matrix.

Let $\vec{v_1}, \dots, \vec{v_n}$ be an orthonormal basis of $\mathbb{R}^n$ of eigenvectors for A^TA.

Singular values of A $\sigma_i := \|A\vec{v_i}\|$

$= \sqrt{\lambda_i}$

- $\vec{n_m} = \vec{v_m} - \text{proj}_{\vec{n_1}} \vec{v_m} - \dots - \text{proj}_{\vec{n_{m-1}}} \vec{v_m}$
- $\{\vec{n_1}, \dots, \vec{n_m}\}$ is orthogonal. → divide by norm gives orthonormal.

Theorem 11.21 (Spectral Theorem)

- Orthogonally diagonalizable if and only if $A = A^T$

✗ Not necessarily square.

Proposition 11.23

A: $m\times n$ matrix

e.g. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$

$A^TA = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$

$(5-\lambda)(4-\lambda) = 0$

$\lambda = 5$ or $\lambda = 4$.

$\sigma_1 = \sqrt{5}$ $\quad \sigma_2 = 2$

It means matrix A stretches some vector/space in direction of A^TA's eigenvector by $\sqrt{\lambda_i}$.
(↑ A^TA's λ.)

$\uparrow$ eigenvalue of $A'A$