Ch8

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8.1 Antiderivatives

- antiderivative of f is
  - $\underbrace{\text{ANY function } F}$ s.t. $F' = f$.
  - $\int f(x)\,dx$.

8.2. Functions defined as integrals.

FTC 1.

1. Let I be an interval. Let $a \in I$.
2. Let f be function on I.
3. Let $F(x) = \int_a^x f(t)\,dt$.
4. If f is continuous (integrable)
5. THEN F is differentiable and $F' = f$.

i.e. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

EG1

$$F(x) = \int_1^x e^{-t^2}\,dt.$$

$$F'(2) = \frac{d}{dx}\int_1^2 e^{-t^2}\,dt$$

$$= e^{-2^2}$$

$$= e^{-4}$$

EG2. Construct g s.t.

$$g'(x) = \frac{1}{1+x^2+x^{10}}$$

$$g(2) = 5$$

Proof.

WTS $F'(x) = f(x)$

Fix $x \in I$.

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\int_x^{x+h} f(t)\,dt\right)$$

WTS $= f(x)$.

Assume $h>0$.

M_h = sup of f on $[x, x+h]$

m_h = inf of f on $[x, x+h]$

Then $m_h \cdot h \le \int_x^{x+h} f(t)\,dt \le M_h \cdot h$

$$m_h \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M_h$$

- f is continuous $\Rightarrow$ EVT $\Rightarrow$ f has max on $[x, x+h]$
- M_h = max of f on $[x, x+h]$
  - $\forall h>0, \exists c_h \in [x, x+h]$ s.t. $M_h = f(c_h)$
- $x \le c_h \le x+h$

Same for min.

$$g(x) = 5 + \int_2^x \frac{1}{1+x^2+x^{10}}$$

EG3

$$G(x) = \int_{-4}^{x^2} \frac{\sin t}{t}\,dt. \quad \text{Find } G'(x)$$

Let $F(x) = \int_{-4}^{x} \frac{\sin t}{t}\,dt$

$$F'(x) = \frac{\sin x}{x}$$

$$G(x) = F(x^2)$$

$$G'(x) = F'(x^2)\cdot 2x$$

$$G'(x) = \frac{\sin x^2}{x^2}\cdot 2x = \frac{2\sin x^2}{x}$$

EG 4.

$$H(x) = \int_{x^3+1}^{x^2+2x} e^{-t^2}\,dt$$

$$H(x) = \int_0^{x^2+2x} e^{-t^2}\,dt - \int_0^{x^3+1} e^{-t^2}\,dt$$

$$H'(x) = e^{-(x^2+2x)^2}\cdot(2x+2) - e^{-(x^3+1)^2}(3x^2)$$

...

As $h\to 0$,

$c_h \to x$.

$M_h = f(c_h) \to f(x)$

$m_h = f(c_h) \to f(x)$

$$\therefore \quad m_h \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M_h$$

$\lim_{h\to 0} m_h = \lim_{h\to 0} M_h = f(x)$.

$\therefore \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$

FTC 2.

Let $a<b$.

Let f be continuous on $[a,b]$.

- Let G be any antiderivative of f.
- Th[illegible]

Proof.

$G' = f$.

- Define $F(x) = \int_a^x f(t)\,dt$.
- f is continuous $\Rightarrow F'(x) = f(x)$
- $F' = G'$, $\therefore F' - G' =$ constant.
- $\therefore F(x) = G(x) + C$ $(\exists C\in\mathbb{R})$

- Then $\int_a^b f(x)\,dx = G(b) - G(a)$

- Since $F(x) = \int_a^x f(t)\,dt$, put $x = a$.

$0 = F(a) = G(a) + C$

$\therefore C = -G(a)$

$\therefore F(x) = G(x) - G(a)$

$\therefore F(b) = G(b) - G(a)$.

8.7. Indefinite integral (Consequence of MVT)

- collection of all antiderivatives of f.

Theorem (Consequence of MVT)

Find one antiderivative G of f.

Then $\int f(x)\,dx = G(x) + C$