Ch6 Injective and Surjective Functions

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Definition 6.1

- Let $f: X \to Y$ be a function for sets X and Y.

- Injective : ONE - TO - ONE (All element in X maps to a unique element in Y)
- Surjective : ONTO (All element in Y mapped to at least one X)
- Bijective : Both

Example 6.2

Bijective : $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x+1$

Neither : $g: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$

Theorem 6.3.

Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with defining matrix A_F. Then

(1) F is injective if and only if every column in rref(A_F) has a pivot.

(2) F is surjective if and only if every row in rref(A_F) has a pivot.

$$\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Activity 6.1

(1) $T_A: \mathbb{R}^2 \to \mathbb{R}^2$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Activity 6.2 Determine injective / surjective / bijective

(1) $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

(2) $G: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x-y \\ y+z \\ x+z \end{pmatrix}$

- (1) $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}y + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}z$

– only every column has pivot. $\Rightarrow$ $\boxed{\text{injective !}}$

(2) $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x-y \\ y+z \\ x+z \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}x + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}y + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}z$

$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

– not pivot in every column + row.

=) IVT, the

6.2 ISOMORPHISMS.

- Preposition 6.4.
  - Suppose $F: \mathbb{R}^n \to \mathbb{R}^m$
  - ※ If F is BIJECTIVE, then n=m.

- Definition 6.5
  - Let V, W be subspace of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively
  - ☆ ISOMORPHISM between V and W is any
  - linear bijective map $F: V \to W$.
  - V and W are isomorphic, $\cong$
  - $V \cong W$

- Theorem 6.6

  Let V and W be vector subspaces
  - $V \cong W \iff \dim(V) = \dim(W)$ (n=m)

Example 6.7

$V = \mathrm{Span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$ ※ $\dim(V) = \dim(W)$

$W = \mathrm{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$ $\Rightarrow V \cong W$.

6.3. Matrix Inverses

- Definition 6.8
  - Let X be a set.
  - IDENTITY on X is $id_X : X \to X$,
    defined by $id_X(x) = x$ for all $x \in X$

Definition 6.9. Let $f: X \to Y$ be a function.

- Inverse of f:

  $f^{-1}: Y \to X$

  - $f^{-1} \circ f = id_X$
  - $f \circ f^{-1} = id_Y$

Example 6.10

- If $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2x+1$
- Then $f^{-1}: \mathbb{R} \to \mathbb{R}$ exist and

  $f^{-1}(y) = \frac{y-1}{2}$

Proposition 6.11

- If $T: \mathbb{R}^n \to \mathbb{R}^n$ is bijective linear transformation
- Then it is 1. invertible

  2. $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is also LINEAR TRANSFORMATION

Definition 6.12

- IDENTITY MATRIX I_n

  = defining matrix of the identity transformation $id_{\mathbb{R}^n}: \mathbb{R}^n \to \mathbb{R}^n$, $id_{\mathbb{R}^n}(\vec{x}) = \vec{x}$

Definition 6.13 Inverse Matrix, Geometric Definition. ($n \times n$ matrix)

- inverse of A, if it exists, is defining matrix of the inverse transformation T_A^{-1}
- If B is defining matrix of T_A^{-1},

  $$\left.\begin{aligned} T_A \circ T_B &= id_{\mathbb{R}^n} \\ T_B \circ T_A &= id_{\mathbb{R}^n} \end{aligned}\right\} \quad \boxed{AB = BA = I_n}$$

Definition 6.14 (Inverse Matrix, algebraic definition) ($n \times n$ matrix)

If A is $n \times n$ matrix

- Then inverse of A, if exists, is $n \times n$ matrix B satisfying

  $$AB = BA = I_n$$

- If B exists A is invertible $\boxed{B = A^{-1}}$

$\boxed{T_{A^{-1}} = T_A^{-1}}$

Example 6.15

7. find inverse of matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$...

1. We know $AA^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. Let $A^{-1} = \begin{pmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{pmatrix}$,

3. We know $A\vec{b_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $A\vec{b_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $A\vec{b_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

4. $\therefore \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ $\therefore x=1, y=0, z=0$

   $\therefore \vec{b_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

   $\therefore \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$ $\therefore x=-1, y=-1, z=2$

   $\vec{b_2} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

   $\therefore \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$ $\therefore \vec{b_3} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

OR, just

$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right)$

$= B = A^{-1}$

Lemma 6.16

1. For $n \times n$ matrices A, B, if $AB = I_n$, then $B = A^{-1}$.

Theorem 6.17

If $(A \mid I_n)$ is row equivalent to $(I_n \mid B)$ for an $n \times n$ matrix B, then A is invertible with $A^{-1} = B$

6.4. Elementory Matrices

Definition 6.18

- $n \times n$ matrix is elementary if it could be obtained by performing exactly one row operation to the identity matrix.

① Row-switching matrices

S_{ij} : swap ith and jth row of I_n

$$S_{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{array}{l} \leftarrow 3rd \\ \\ \leftarrow 1st \end{array}$$

② Row-multiplying matrix

$M_i(c)$: multiplying ith row by constant c.

$$M_2(5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

③ Row-addition matrix

$A_{i,j}(c)$: adding c times of jth row to the ith row of I_n.

$$A_{1,2}(5) = \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times 5, \text{add!}$$

Preposition 6.19

If E is elementary matrix obtained by performing row operation to I_n, EB is obtained by performing the same elementary row operation to B.

i.e. ①, ②, ③ works.

Preposition 6.20

- Every elementary matrix is invertible.

- Inverse of elementary matrix is an elementary matrix.

6.5 The Inverse Matrix Theorem

Theorem 6.21 (Invertible Matrix Theorem) (A be $n \times n$ matrix)

(1) A is invertible

= (2) $A\vec{x} = b$ has unique solution for any $\vec{b} \in \mathbb{R}^n$

= (3) $\text{rref}(A) = I_n$

= (4) A is a product of elementary matrices

Lemma 6.22

$$\boxed{\text{For } n \times n \text{ invertible matrices } A, B \quad (AB)^{-1} = B^{-1}A^{-1}}$$

P. 5, 9.

Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
If $n > m$, show F cannot be injective.

Assume $n > m$.
Then the transformation matrix is in form of $m \times n$.
Since each row can at most have one pivot,

and number of columns (m) > number of rows (n)

m > maximum no. of pivots.

$\therefore$ Some columns does not have a pivot.

$\therefore$ Since injectivity requires all columns to have a pivot,

F cannot be injective.

P. 5.10

Let $F: V \to W$ be a linear transformation between vector spaces V and W.

Show that F is injective if and only if $\ker(F) = \{\vec{0}\}$

- Let $F = T_A$, where A is the transformation matrix.

1. Assume F is injective.

Then every column in A has a pivot.

Then every column in A represents an independent vector.

$\ker(F) = \mathrm{Nul}(A) = \{A\vec{x} = 0 \mid \vec{x} \in V\}$

Let $A = (\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_n)$.

Then there only exist trivial solution to

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = 0$$

$\therefore \vec{x} = \vec{0}$. only.

$\therefore \ker(F) = \{\vec{0}\}$

2. Assume $\ker(F) = \{\vec{0}\}$

Then the only solution to $A\vec{x} = 0$ is trivial, i.e. $\vec{x} = 0$.

Since the system of linear equation only has one exact solution, there is pivot in all columns of the transformation matrix A.

$\therefore T_A = F$ is injective.

P.5.11

Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and

$\{v_1 \quad \dots \quad v_n\}$ ... basis for $\mathbb{R}^n$

$\{\vec{v}_1, \dots, v_n\}$ a basis for $\mathbb{R}^n$.

Show that F is injective if and only if the set $\{F(\vec{v_1}), \dots, F(\vec{v_n})\}$ is linearly independent.

1. Assume F is injective.
   Let $F = T_A$. A is the transformation matrix.
   Let $A = \big(F(v_1) \quad F(v_2) \quad \dots \quad F(v_n) \big)$
   Since F is injective, all columns of $\text{rref}(A)$ has a pivot.
   $\therefore \{F(v_1) \ \dots \ F(v_n)\}$ is linearly independent.

2. Assume $\{F(v_1), \dots, F(v_n)\}$ is linearly independent.
   Then all columns in $A = (F(v_1) \ \dots \ F(v_n))$ has a pivot.
   Then we know that $F = T_A$ is injective.

P5.13 Let $F: \mathbb{R}^5 \to \mathbb{R}^4$ be a linear transformation with $\text{nullity}(F) = 2$.
Show $\text{im}(F)$ is isomorphic to $\mathbb{R}^3$.

- Since $\text{nullity}(F) = 2$, $\text{rank}(F) = \dim(\mathbb{R}^5) - 2 = 3$.
- $\dim(\text{im}(F)) = \text{rank}(F) = 3$

$\to$ there exists a basis for $\text{im}(F)$ with 3 independent vectors.

$\therefore \text{im}(F)$ is isomorphic to $\mathbb{R}^3$.

P.6.1 Prove

Suppose $F: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.
If F is bijective, then $n = m$.

- Assume F is bijective.

Then all rows and columns in A has a pivot.

number of columns

∴ Number of pivots = number of rows = number of ...

∴ $m = n$

P.6.2.

Is it possible for an $m \times n$ matrix A to have an inverse when $m \neq n$? Explain why or not.

No. By invertible matrix Theorem, A is invertible if and only if $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b} \in \mathbb{R}$. However, A will not have exactly one solution since there must be at least one row or column that does not have a pivot.

∴ A is not invertible

P.6.5. Prove that the following conditions are equivalent.

Use Invertible Matrix Theorem.

(1) A is invertible

(2) The matrix-vector equation $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b} \in \mathbb{R}^n$

(3) $\mathrm{rref}(A) = I_n$

(4) A is a product of elementary matrices.

(a) A is invertible.

(b) $\mathrm{rref}(A)$ has n pivots

- A is invertible
- $F = T_A$ is bijective.
- F is injective (every column has a pivot) (n columns)

(c) $\mathrm{Nul}(A) = \{\vec{0}\}$

- $F = T_A$ is bijective
- A is linearly independent and
- there exist only trivial solution s.t. $A\vec{x} = \vec{b}$ for any $\vec{b} \in \mathbb{R}$.
- ∴ $\mathrm{Nul}(A) = \{\vec{0}\}$

(d) $Col(A) = \mathbb{R}^n$

- $T = T_A$ is bijective $\rightarrow$ injective
- A is linearly dependent
- $Col(A) = Span\{v_1\ v_2\ v_3 \dots v_n\}$ for $A = (v_1\ v_2\ \dots\ v_n)$

(e)
- $T = T_A$ is bijective $\rightarrow$ injective
- A is linearly independent

(f)
- T_A is an isomorphism.

Since A is invertible, T_A must be a bijective linear transformation.

$\Rightarrow T_A$ is an isomorphism.

(g) T_A is injective.
- invertible $\rightarrow$ injective a
- $rref(A) = I_n$, and I_n has pivot in all columns

(h) T_A is surjective
- $rref(A) = I_n$, and I_n has pivot in all rows.

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