Ch7 Determinant

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7.1 Determinants in $\mathbb{R}^2$

Definition 7.1

The unit square is subset of $\mathbb{R}^2$:

$$S := \{ \alpha_1 \vec{e_1} + \alpha_2 \vec{e_2} : 0 \le \alpha_1, \alpha_2 \le 1 \}$$

Proposition 7.2.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.

- Then $F(S)$ is a parallelogram given by $F(\vec{e_1})$ and $F(\vec{e_2})$
- $\mathbb{R}^2$ coordinate grid transformed into a grid with axes in $F(\vec{e_1})$ and $F(\vec{e_2})$

Proof.

$$F(S) = \{ F(\alpha_1 \vec{e_1} + \alpha_2 \vec{e_2}) : 0 \le \alpha_1, \alpha_2 \le 1 \}$$
$$= \{ \alpha_1 F(\vec{e_1}) + \alpha_2 F(\vec{e_2}) : 0 \le \alpha_1, \alpha_2 \le 1 \}$$

Definition 7.3

An ordered basis $\{\vec{b_1}, \vec{b_2}\}$ for $\mathbb{R}^2$ is called POSITIVELY ORIENTED if we can rotate $\vec{b_1}$ less than $180°$ counterclockwise to reach $\vec{b_2}$. Else, basis is negatively oriented.

Definition 7.4

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.

Determinant of F, $\det(F)$, is the oriented area of $F(S)$.

For $a(F(S))$ being area of $F(S)$:

$$\det(F) := \begin{cases} a(F(S)) & \text{if } \{F(\vec{e_1}), F(\vec{e_2})\} \\ -a(F(S)) & \text{if } \{F(\vec{e_1}), F(\vec{e_2})\} \\ 0 & \text{if } a(F(S)) = 0 \end{cases}$$

Proposition 7.5.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

7.2 Determinants in $\mathbb{R}^3$

Definition 7.6

UNIT CUBE

$C := \{\alpha_1 \vec{e_1} + \alpha_2 \vec{e_2} + \alpha_3 \vec{e_3} : 0 \le \alpha_1, \alpha_2, \alpha_3 \le 1\}$

Definition 7.7.

$\{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$ positively oriented if follows right-hand rule.

Definition 7.8. determinant of F is oriented volume of $F(C)$.

Let $v(F(C))$ be volume.

$$\det(F) := \begin{cases} v(F(C)) & \text{if } \{F(\vec{e_1}), F(\vec{e_2}), F(\vec{e_3})\} \text{ is positively oriented} \\ -v(F(C)) & \text{if } \{F(\vec{e_1}), F(\vec{e_2}), F(\vec{e_3})\} \text{ is negatively oriented.} \\ 0 & \text{if } v(F(C)) = 0 \end{cases}$$

Preposition 7.9.

$$\det\begin{pmatrix} c & 0 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = c \det\begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix}$$

Preposition 7.10

$$\det(\vec{v_1}\ \vec{v_2}\ \vec{v_3}) = -\det(\vec{v_2}\ \vec{v_1}\ \vec{v_3})$$

Preposition 7.11 row linearity

$$\det\begin{pmatrix} \alpha & \beta & \gamma \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \det\begin{pmatrix} \alpha & 0 & 0 \\ & m & \end{pmatrix} + \det\begin{pmatrix} 0 & \beta & 0 \\ & m & \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & \gamma \\ & m & \end{pmatrix}$$

Example 7.12 Calculate determinant of 3x3 matrix

$$A = \begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\det(A) = \det\begin{pmatrix} a & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \det\begin{pmatrix} 0 & b & 0 \\ 1 & 2 & 3 \\ \dots \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & c \\ 1 & 2 & 3 \\ \dots \end{pmatrix}$$

$$= \det\begin{pmatrix} a & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} - \det\begin{pmatrix} b & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \det\begin{pmatrix} c & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 3 & 6 \end{pmatrix}$$

$$= a\det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} - b\det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} + c\det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$$

$$= -3a + 6b - 3c \,//$$

7.3 Cofactor Expansion and Determinant in $\mathbb{R}^n$.

Definition 7.13

- For an $n \times n$ matrix $A = (a_{ij})$
- ij - MINOR of A is defined to be the

$(n-1) \times (n-1)$ matrix of A_{ij}, with ith and jth row deleted

Definition 7.14

Let A be $n \times n$ matrix with ij-entry equal to a_{ij}.

$$\det(A) := a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{n+1} a_{1n}\det(A_{1n})$$

Definition 7.16.

The unit n-cube C_n is the subset of $\mathbb{R}^n$ defined by

$$C_n = \{ a_1\vec{e_1} + \cdots a_n\vec{e_n} : 0 \leq a_i \leq n \}$$

Proposition 7.17

Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

- image of the unit n-cube under F is a parallelepiped in $\mathbb{R}^n$.

$$F(C_n) = \{ a_1 F(e_1) + \cdots + a_n F(\vec{e_n}) : 0 \leq a_1, \ldots, a_n \}$$

$$\text{Volume} = v(F(C_n)) = |\det(F)|$$

7.4 Properties of the Determinant

Theorem 7.18.

- Let A be an $n \times n$ matrix.
- A is only invertible if and only if $\det A \neq 0$

Lemma 7.19

Let $A = (a_{ij})$ be an $n \times n$ matrix with ij-entry equal to a_{ij}.

$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots + (-1)^{n+1} a_{n-1} \det(A_{n1})$

(we can perform cofactor expansion by along first column instead of row)

Theorem 7.20 Let A be an $n \times n$ matrix.

(1) If B is obtained by interchanging 2 rows of A,

$\det(B) = -\det(A)$

(2) If B is obtained by multiplying one row of A by a constant c,

$\det(B) = c\det(A)$

(3) If B is obtained by replacing a row of A by that row and a scalar multiple of another row of A, then

$\det(B) = \det(A)$

Corollary 7.21 If A is not invertible, $\det(A) = 0$

Lemma 7.22 Let E be an $n \times n$ elementary matrix, and B be any $n \times n$ matrix.

$\det(EB) = \det(E)\det(B)$

Proposition 7.23

- $\det(A) = \det(A^T)$

· $\det(AB) = \det(A)\det(B)$

P.7.3 Show that if A is not invertible, then A^T is not invertible. Conclude that if A is not invertible, then $\det(A^T) = 0$

Let $\det(A) = 0$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

· Assume A is not invertible.

Then there exist $\vec{y} \neq 0$ s.t. $A\vec{y} = \vec{0}$

Taking transpose on both sides,

$$(A\vec{y})^T = 0^T$$

$$\vec{y}^T A^T = 0$$

Since $\vec{y} \neq 0$, $\vec{y}^T$ is also not zero.

$\therefore$ There exist non-trivial solution, thus y^T is not inv

Show that if A or B is not invertible, then AB is not invertible.

P.7.4. Let A be an invertible matrix.

Show $\det(A^{-1}) = \dfrac{1}{\det(A)}$

- Assume A is invertible.

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det(I)$$

$$\det(A)\det(A^{-1}) = \det(I)$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

P. 7.5 True or False:

- For any $n\times n$ matrix A, $\det(-A) = -\det(A)$

- False.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\det(A) = 1$.

$$-A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\det(-A) = \det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1 \neq -1.$$

P. 7.9. An $n\times n$ matrix A is called skew-symmetric if $A = -A^T$.

Show that when n is odd, any $n\times n$ skew-symmetric matrix is not invertible.

$$\det A = \det(-A^T)$$

$$\det A = (-1)^n \det(A^T)$$

$$\det(A) = -\det(A)$$

$$2\det A = 0$$

$$\det A = 0$$

$\therefore$ Not invertible.

P8.1 (4, 5)

- Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation which rotates every vector in $\mathbb{R}^2$ counterclockwise by an angle θ

(a) Show that $A_F = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

- $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\therefore F(e_1) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$
- $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\therefore F(e_2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

$$A_F = \begin{pmatrix} F(e_1) & F(e_2) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(b) Show that A_F does not have any real eigenvalues unless θ is an integer multiple of $180°$.

1. $\det(A_F - \lambda I) = 0$

$$\det\begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}$$

$$= (\cos\theta - \lambda)(\cos\theta - \lambda) - (-\sin\theta)(\sin\theta)$$

$$= \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta$$

$$= \lambda^2 - 2\lambda\cos\theta + 1$$

$$\Delta = (-2\cos\theta)^2 - 4(1)(1)$$

$$= -4\sin^2\theta$$

$\sin^2\theta \geq 0$, $\therefore$ only $-4\sin^2\theta = 0$

$\sin\theta = 0$

$\theta = k\pi$.

P.8.2. Show that for any $n \times n$ matrix A we have $\chi_A = \chi_{A^T}$.

Conclude A and A^T hv same eigen values.

1. $\chi_{A^T} = \det(A^T - \lambda I)$

$$= \det((A - \lambda I)^T) \quad (\text{as } (A - \lambda I)^T = A^T - \lambda I)$$

$= \det(A - \lambda I)$

$= \chi_A$.

$\Rightarrow$ Same eigenvalue.