Ch5 Linear Transformation

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• 5.1 Linearity → or, 'transformation'

Function F is called linear if

• (1) $F(\vec{x}+\vec{y}) = F(\vec{x}) + F(\vec{y})$

• (2) $F(c\vec{x}) = cF(\vec{x})$

Linear combination:

$$\begin{bmatrix} \text{Final } x \\ \text{Final } y \end{bmatrix} = x \begin{bmatrix} \text{Transformed } \hat{i}\text{'s } x \\ \text{Transformed } \hat{i}\text{'s } y \end{bmatrix} + y \begin{bmatrix} \text{Transformed } \hat{j}\text{'s } x \\ \text{Transformed } \hat{j}\text{'s } y \end{bmatrix}$$

2x2 Matrix

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} x + \begin{bmatrix} -2 \\ 2 \end{bmatrix} y$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

$\hat{i}$ coord, $\hat{j}$ coord

5.2 Matrix Transformation (multiply vector by matrix)

$$\underset{\mathbb{R}^m}{y} = \underbrace{A}_{m\times n \text{ matrix}} \underset{\mathbb{R}^n}{\vec{x}}$$

Definition 5.2.
Matrix transformation associated to A is

$$\boxed{T_A(\vec{x}) := A\vec{x}.}$$

Proposition 5.3. Every matrix transformation is a linear transformation.

$$T_A(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) = T_A(\vec{x}) + T_A(\vec{y})$$

Activity 5.1

(1) $F(\vec{x}) = 2\vec{x}$
Since $2(\vec{x}+\vec{y}) = 2\vec{x}+2\vec{y} = F(\vec{x}) + F(\vec{y})$
$2(c\vec{x}) = c(2\vec{x}) = cF(\vec{x})$

∴ Linear

(2) $G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$

No. $G\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} = \begin{pmatrix} (x_1+x_2)^2 \\ (y_1+y_2)^2 \end{pmatrix} \neq \begin{pmatrix} x_1^2+x_2^2 \\ y_1^2+y_2^2 \end{pmatrix}$

$G(c(x,y)) = \begin{pmatrix} (cx)^2 \\ (cy)^2 \end{pmatrix} = c^2\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$

(3) $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+1 \\ y-1 \end{pmatrix}$

$$T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} = \begin{pmatrix} (x_1+x_2)+1 \\ (y_1+y_2)-1 \end{pmatrix}$$

5.2 $F: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.
Recall a line in $\mathbb{R}^2$ can be described by

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : y = mx+b \right\}$$

Show that $F(S) := \{F(\vec{v}) : \vec{v} \in S\}$ is either a line or a point.

Activity 5.3

$F(\vec{e_1}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $F(\vec{e_2}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(1) Find $F\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ and $F\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)$.

Theorem 5.4.

Every linear transformation is a matrix transformation.

if $F: \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $F = T_A$ where
(columns of A are $T(e_1)$ $T(e_2)$ $T(e_3)$ implies $T(x) = Ax$) if linear

$$A = \underbrace{\left(F(\vec{e_1}) \;\; F(\vec{e_2}) \;\; \cdots \;\; F(\vec{e_n})\right)}_{\text{unit vectors}}$$

Proof
Suppose that $F: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. $F = T_A$. (Function = Transformation with A.)
Let A be define as $A: (F(\vec{e_1}) \;\; F(\vec{e_2}) \;\; \dots \;\; F(\vec{e_n}))$

$(\vec{e_1}, \vec{e_2}, \dots, \vec{e_n})$ $\mathbb{R}^n$

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Definition 5.5

- Given a matrix A.
- $\boxed{T_A}$ is the $\boxed{\text{MATRIX TRANSFORMATION}}$ correspond to matrix A.
- call $m \times n$ matrix $\boxed{A_F}$ as $\boxed{\text{DEFINING MATRIX}}$ of transformation F.

$T(x) = Ax + \overset{\neq 0}{b}$ (Affine)

$T(x) = Ax$ (Linear)

Activity 5.4: Find the defining matrices for following linear transformation

(1) $\vec{x} \mapsto 2\vec{x}$

• Basis of $\mathbb{R}^3$ $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$F(e_1) = 2e_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$F(e_2) = 2e_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$F(e_3) = 2e_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

Defining matrix A

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(2)

Activity 5.5 $T: \mathbb{R}^3 \to \mathbb{R}^2$, relationship of basis and domain.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+1 \\ y-1 \end{pmatrix}$$

Not linear Transformation
- $T(0) \neq 0$.
- Shifts the origin → not linear

Proof.
We have $T(\vec{e_1}) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$T(\vec{e_2}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$T(\vec{e_3}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$T(\vec{x}) = A\vec{x}$

$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix}$

$G : \mathbb{R}^2 \to \mathbb{R}^3$.

$F(e_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$F(e_2) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$

By Proposition 5.3, every matrix transformation is linear.
So T is a linear transformation.

Activity 5.6.
Function $H : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$H\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x+y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} y = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\therefore$ We have $H(x, y) = A\begin{pmatrix} x \\ y \end{pmatrix}$ for $(x, y) \in \mathbb{R}^2$

Since any transformation of the form $T(x) = Ax$ is linear,

5.3 Function Composition and the Matrix Product

$f : A \to B$ $\qquad f \circ g \neq g \circ f$

$g : B \to C$

Composite function $g \circ f : A \to C$ is defined by

$$(g \circ f)(a) = g(f(a))$$

$T_A : \mathbb{R}^k \to \mathbb{R}^m,\ \vec{x} \mapsto A\vec{x}$

$T_B : \mathbb{R}^n \to \mathbb{R}^k,\ \vec{x} \mapsto B\vec{x}$

$\therefore\ T_A \circ T_B : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$(T_A \circ T_B)(\vec{x}) = \underbrace{A(B}_{C = AB}\vec{x})$$

$$T_A \circ T_B = T_C$$

Basically Matrix Multiplication.

Definition 5.6.
Let A be an $m \times k$ matrix.
Let B be a $k \times n$ matrix.
MATRIX PRODUCT of A and B is $m \times n$ matrix C.

$$T_A \circ T_B = T_C.$$

Then $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $B = (\vec{b_1} \cdots \vec{b_n})$

We have $A(B\vec{x}) = A(x_1\vec{b_1} + x_2\vec{b_2} + \dots + x_n\vec{b_n})$

$= C\vec{x}.$

$(A\vec{b_1}\ \ A\vec{b_2}\ \cdots\ A\vec{b_n})$

Example 5.7

Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix}$

2×3 $\qquad$ 3×2

$\therefore$ C is 2×2.

5.4. Geometric Rank-Nullity

Recall $F : A \to B$

Domain $\qquad$ CODOMAIN

Definition 5.10
Given a function $F : \mathbb{R}^n \to \mathbb{R}^m$.

1. Kernel of F is subset of $\mathbb{R}^n$

$$\ker(F) := \{\vec{x} \in \mathbb{R}^n \mid F(\vec{x}) = \vec{0}\}$$

(i.e. all the vectors that becomes origin $\vec{0}$ after transformation)

2. Image of F is subset of $\mathbb{R}^m$

$$\operatorname{im}(F) := \{\vec{y} \in \mathbb{R}^m \mid F(\vec{x}) = \vec{y} \text{ for some } \vec{x} \in \mathbb{R}^n\}$$

span!!

$\mathrm{Nul}(A_F) = \ker(F)$ = set of vectors that get mapped to zero

$\mathrm{Col}(A_F) = \operatorname{im}(F)$ = the span of its column vectors

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Activity 5.7

Consider $F : \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x - y \\ z \end{pmatrix}$

1. Find a vector $\vec{x}$ in $\ker(F)$.

$F\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0 \end{pmatrix} = \vec{0}$ $\qquad$ OR $\qquad$ $F\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 0 \end{pmatrix} = \vec{0}$

$\therefore \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. (in $\mathbb{R}^3$) $\qquad$ $\therefore \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (in $\mathbb{R}^3$)

2. Find a vector $\vec{y}$ in $\operatorname{im}(F)$

We want any $(a, b) \in \mathbb{R}^2$ written as $(x-y, z) \in \mathbb{R}^3$.

Choose $(a, b) = (1, 2)$

Then pick preimage $= (3, 2, 2)$

$v_1 \quad v_2 \quad v_3$

$F\begin{pmatrix}3\\3\\3\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$\uparrow$

$y = \begin{pmatrix}1\\2\end{pmatrix}$ is in $im(F)$

Activity 5.8

Show that 1. $ker(F) = Nul(A_F)$,

2. $im(F) = Col(A_F)$

※ ① Recall.

$ker(F) = \{\vec{x} \in \mathbb{R}^m \mid F(\vec{x}) = 0\}$

$Nul(A) := \{\vec{x} \in \mathbb{R}^m \mid A\vec{x} = 0\}$

Since $F(\vec{x}) = A\vec{x}$

Definition 5.11

- RANK of linear transformation F is the dimension of $im(F)$.
- NULLITY of linear transformation F is dimension of $ker(F)$

Activity 5.9.

Find rank and nullity of $F: \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix}x\\y\\z\end{pmatrix} \longmapsto \begin{pmatrix}x-y\\z\end{pmatrix}$

- $im(F)$: To find $im(F)$, we want to find the no. of independent vectors in A_F.

$$\begin{pmatrix}1 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}x + \begin{pmatrix}-1\\0\end{pmatrix}y + \begin{pmatrix}0\\1\end{pmatrix}z$$

- All vectors : $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, only 2 are independent.

Rank = $\dim(Col(A)) = \dim(im(F)) = 2$

Nullity = $\dim(Nul(A)) = \dim(ker(F))$ = Number of columns of $A_F - 2 = 1$

Pbset.

P.5.2

Let V be a vector subspace of $\mathbb{R}^n$ and suppose that $F: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Show that $F(V) := \{f(v) \mid v \in V\}$ is a vector subspace of $\mathbb{R}^m$.

1. Since V is a subspace, $\vec{0} \in V$.
   Since F is linear transformation,
   $F(\vec{0}) = \vec{0}$
   $\therefore \vec{0} \in F(V)$
2. Since ..., let $v_1, v_2 \in V$, $v_1 + v_2 \in V$.
   $F(v_1+v_2) = F(v_1) + F(v_2) \in F(V)$

Proof ① $\vec{0} \in F(V)$

② Closed under addition

③ Closed under multiplication of scalar.

Proof.

① $\vec{0} \in V$ as V is a subspace

Since F is linear transformation, $F(\vec{0}) = \vec{0}$
$\therefore \vec{0} \in F(V)$

② Let $v_1, v_2 \in V$.
Let $F(v_1) = w_1$, $F(v_2) = w_2$.

V is subspace ; $v_1 + v_2 \in V$.

$F(v_1 + v_2) = F(v_1) + F(v_2) = w_1 + w_2$

$\therefore$ $w_1 + w_2 \in F(V)$

③ Let $v \in V$.
Let $F(v) = w$

V is subspace : $cv \in V$ (c is constant)

$F(cv) = cF(v) = cw \in F(V)$

P. 5.8

Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
If $n < m$, show that F cannot be surjective.

Let $A\vec{x} = \vec{b}$

- F is surjective if all its rows has a pivot.

Assume $n=2$, $m=3$. Then the Transformation matrix is in form $\begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix}$. Since each column can at most have one pivot, there is at most 2 pivots, but we have 3 rows $\therefore$ Not all of the rows has a pivot.

$\therefore$ F cannot be surjective.