Ch2 Topology

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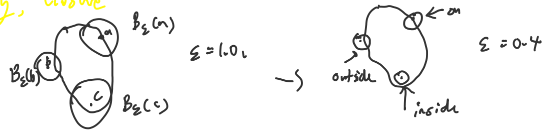
2.1 Interior, Boundary, Closure

need little ball
(fully inside set



- Interior

- $A \subseteq \mathbb{R}^n$
- $p \in \mathbb{R}^n$ is an interior point of A if exist $\varepsilon > 0$ s.t. $B_\varepsilon(p) \subseteq A$

- Interior of $A \subseteq \mathbb{R}^n$,
  - A° or $\text{int}(A)$,
  - set of interior points of A.

1D.

Proof is interior point
- $p = 2$
- $A = [1,4)$

✓ is interior set since $B_{\frac{1}{2}}(2) = (1.5, 2.5) \subseteq A$

Proof not interior point
- $p = 5$
- $A = [1,4)$
- Proof negation: $\forall \varepsilon > 0$, $B_\varepsilon(5) = (5-\varepsilon, 5+\varepsilon) \subseteq [1,4) = A$.
- Fix $\varepsilon > 0$.
- Take $x = 5$.
- $x \in (5-\varepsilon, 5+\varepsilon)$ but $x \notin [1,4)$.
- $\therefore B_\varepsilon(5) \subseteq [1,4)$ so 5 is not an interior point of A.

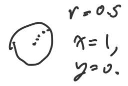
2D

$p = (1,0)$

$A = \{(x,y) \in \mathbb{R}^2 : x \leq 2\}$

- Prove $\exists \varepsilon > 0$ s.t.
  $(x,y) \in B_\varepsilon((1,0)) \Rightarrow (x,y) \in A$

- Proof
- Take $\varepsilon = 0.5$.
- Let $(x,y) \in B_{0.5}((1,0))$
  $\rightarrow (x-1)^2 + y^2 < 0.5^2$



- Since $y^2 \geq 0$, $0 \leq y^2$
  $(x-1)^2 < \left(\frac{1}{2}\right)^2$ (since L.H.S. get smaller only)
  $x - 1 < \frac{1}{2}$
  $x < 1 + \frac{1}{2} < 2$

$\therefore x \in A = \{(x,y) \in \mathbb{R}^2 : x \leq 2\}$

Set in set
- TBD

2.12. Boundary.

Definition:

$p \in \mathbb{R}^n$ is a boundary point of $A \in \mathbb{R}^n$ if for $\forall \varepsilon > 0$, $B_\varepsilon(p) \cap A \wedge B_\varepsilon(p) \cap A^c$ are $\boxed{\text{both non-empty.}}$

- no matter how much you zoom into p, you can see points nearby inside A and outside A.

- Topological boundary
  - ∂A, set of boundary points of A.
  - $A = [1,4)$, $\partial A = \{1,4\}$
    $A = \{(x,y) \in \mathbb{R}^2 : x \leq 2\}$, $\partial A = \{(2,y) : y \in \mathbb{R}\}$

1D

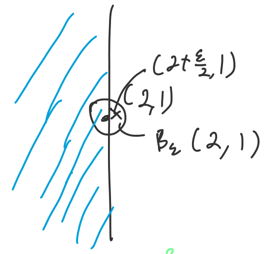
Proof.
- $p = 1$
- $A = [1,4)$.
- Let $\varepsilon > 0$.
- Take $x = 1$, $y = 1 - \frac{\varepsilon}{2}$.
  $x \in [1,4)$ $y \notin [1,4)$

$x, y \in B_\varepsilon(1) = (1-\varepsilon, 1+\varepsilon)$

$\therefore$ $x \in B_\varepsilon(1) \cap A$
$y \in B_\varepsilon(1) \cap A^c$

2D

- $p = (2,1)$
- $A = \{(x,y) \in \mathbb{R}^2 : x \leq 2\}$.



Proof.

Let $\varepsilon > 0$.

Then $B_\varepsilon(p) = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + (y-1)^2 < \varepsilon^2\}$

$\therefore$ $(2,1) \in B_\varepsilon(p) \wedge (2+\frac{\varepsilon}{2}, 1) \in B_\varepsilon(p)$
$(2,1) \in A \wedge (2+\frac{\varepsilon}{2}, 1) \in A^c$

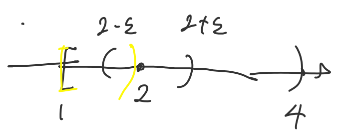
$\therefore$ $B_p(p) \cap A$, $B_\varepsilon(p) \cap A^c$
contain $(2,1)$ $(2+\frac{\varepsilon}{2}, 1)$
non-empty.

2.13 Closure.

- Limit point
  - $p \in \mathbb{R}^n$ is limit point of $A \subseteq \mathbb{R}^n$ if
    $\forall \varepsilon > 0$, $B_\varepsilon(p) \setminus \{p\}$ contains points in A.

- Closure
  - closure of A, $\bar{A}$ or $cl(A)$,
  - is union of set A and set of limit points of A (A^*)
    $A \cup A^*$

1D.



- $p = 2$
- $A = [1,4)$
- Let $\varepsilon > 0$.
- Take $x = \max\{1, 2 - \frac{\varepsilon}{2}\}$ (want x in $(2-\varepsilon, 2+\varepsilon)$, while $x \in A$.
  $\in B_\varepsilon(2) = (2-\varepsilon, 2+\varepsilon)$
- Then, $B_\varepsilon(2) \setminus \{2\}$
  has points in $[1,4)$.

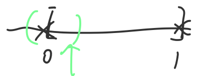
$\Rightarrow$ 2 is limit point of $[1,4)$

e.g. $A = (0,1)$

each pt is a
limit point because every open interval

Limit points $= [0,1]$ around x, you find other points of $(0,1)$

Closure $= (0,1) \cup [0,1]$
$= [0,1]$



always find some number $(0-\varepsilon, 0+\varepsilon)$ that is in $(0,1)$

Theorem 2.1.32.

- $A^\circ \subseteq A \subseteq \bar{A}$
- $A^\circ \cap \partial A = \emptyset$
- $\bar{A} = A^\circ \cup \partial A$
- $\partial A = \bar{A} \setminus A$

Interior A°: all points are not on edge
Boundary ∂A: edge, every neighborhood touches set and outside.
Closure $\bar{A}$: set itself plus limit points.
Limit point: a point can get arbitrarily close to using points from the set.

Interior: all inside, no edge
Boundary: just the edge
Closure: set + edge + accumulation
Interior $\cup$ Boundary

2.2. Sequences.

Definition 2.2.1 Sequence. in $\mathbb{R}^n$ is

- a function with domain $\{k \in \mathbb{Z} : k \geq k_0\}$ for some fixed $k_0 \in \mathbb{Z}$ and codomain $\mathbb{R}^n$.
- notations: $(x(k))$ $\quad (x(k))_k$ $\quad (x(k))_{k=k_0}^{\infty}$

$\{x(k)\}$ $\quad \{x(k)\}_k$ $\quad \{x(k)\}_{k=k_0}^{\infty}$

e.g. $x(k) = (\cos k, \sin k)$ in $\mathbb{R}^2$.
$x(1) = (\cos 1, \sin 1)$, ..., $x(w) = (\cos w, \sin w)$

Definition 2.2.4 Subsequence

- $x : \mathbb{N}^+ \to \mathbb{R}^n$ (sequence)
- $m : \mathbb{N}^+ \to \mathbb{N}^+$ strictly increasing function.

$\to$ $\{x(m(k))\}_{k=1}^{\infty}$ is a subsequence of $\{x(k)\}_{k=1}^{\infty}$.

e.g. $m(k) = 2k$. Let $x(k) = k^2$.
Then $\{x(m(k))\} = \{x(2k)\} \subseteq \{x(k)\}$
$\{4k^2\} \subseteq \{k^2\}$

2.2.1. Convergence of sequences.

- Take $a \in \mathbb{R}^n$ ... lim $x(k) \to a$ ... $\{x(k)\}^{\infty}$ ...

Take $p \in \mathbb{R}^n$. $\lim_{k\to\infty} x(k) = p$ / $x(k) \to p$

- $\{x(k)\}_k$ in $\mathbb{R}^n$ converges to p if

  $\forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t. $\forall k \in \mathbb{N}, k \geq K \Rightarrow \|x(k) - p\| < \varepsilon$.

> - $\{x(k)\}_k$ converges if there exist $p \in \mathbb{R}^n$ s.t. $\lim_{k\to\infty} x(k) = p$.
> - Otherwise, $\{x(k)\}$ diverges. ($\lim_{k\to\infty} x(k) = DNE$)

- every subsequence of $\{x(k)\}$ also converge to p.

$\{x(k)\}_{k=1}$ converges to $(2,0)$.

$x(k) = \left(2 + \frac{1}{k}, \frac{\sin k}{k}\right)$

Proof.

Let $\varepsilon > 0$.

Take $K = \left\lceil \frac{\sqrt{2}}{\varepsilon} \right\rceil + 1$

Assume $k \in \mathbb{N}^+$ and $k \geq K$.

Then $\|x(k) - p\|$

$= \left\|\left(2 + \frac{1}{k}, \frac{\sin k}{k}\right) - (2, 0)\right\|$

$= \sqrt{\left(2 + \frac{1}{k} - 2\right)^2 + \left(\frac{\sin k}{k}\right)^2}$

$= \sqrt{\frac{1 + \sin^2 k}{k^2}}$

$\leq \sqrt{\frac{2}{k^2}}$ (as $0 \leq \sin^2 k \leq 1$)

$< \varepsilon$ (as $k \geq \left\lceil \frac{\sqrt{2}}{\varepsilon} \right\rceil + 1 > \sqrt{\frac{2}{k^2}}$)

$\therefore \|x(k) - p\| < \varepsilon. \Rightarrow \lim_{k\to\infty} x(k) = p$

Theorem:

1. Constant: $x(k) = p$ for all except finitely many $k \in \mathbb{N}^+ \Rightarrow x(k) \to p$
2. Linearity: $x(k) \to p$ and $y(k) \to q \Rightarrow x(k) + \lambda y(k) \to p + \lambda q$
3. Dot Product: $x(k) \to p$ and $y(k) \to q \Rightarrow x(k) \cdot y(k) \to p \cdot q$

Theorem 2.2.16 Convergence component-wise

- Let $\{x(k)\}_k$ be sequence in $\mathbb{R}^n$ with $x(k) = (x_1(k), x_2(k), \dots, x_n(k))$
- Fix $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ (components of "p")
- $\{x(k)\}_k$ converges to p iff $\{x_i(k)\}_k$ converges to p_i, for ALL $i \in \{1, 2, \dots, n\}$

Theorem 2.2.17 Interior point, boundary point, limit point

Interior point of A: every sequence $\{x(k)\}_k$ of points converging to p, exist $K \in \mathbb{N}^+$ s.t. $\{x(k)\}_{k=K}^{\infty} \subseteq A$ (zoom [illegible])

Boundary point of A: exist sequences of points in A and A^c both converging to p.

Limit point of A: exist points in $A \setminus \{p\}$ converge to p.

2.3 Open sets and closed sets.

2.31 Open sets

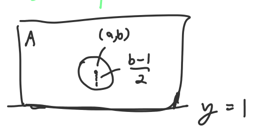
(None of its boundary points are included (every point has wiggle room)

Definition 2.3.1 Open sets.

- A set $A \subseteq \mathbb{R}^n$ is open if every point of A is an interior point of A.
  - empty set
  - $\mathbb{R}^n$
  - (a, b)
  - Open ball $\{x \in \mathbb{R}^n : \|x - p\| < r\}$ in $\mathbb{R}^n$
- Interior of a set $A \subseteq \mathbb{R}^n$ is open

  (the set contains all boundary points, $\overset{\text{actual}}{\text{limit}}$ stays in C)

- $\{(x, y) \in \mathbb{R}^2 : y > 1\}$ is open



- Proof.
  - $(a,b) \in A$
  - Fix $r = \frac{b-1}{2}$

  $\xrightarrow{\text{wts}} B_r((a,b)) \subseteq A$

  - For $(x,y) \in B_r((a,b))$,

  $|y - b| \leq \|(x,y) - (a,b)\| < r$

  $\therefore |y - b| < \frac{b-1}{2}$

  $b - \frac{b-1}{2} < y < b + \frac{b-1}{2}$

  $y > \frac{b+1}{2} > 1$ as $b > 1$.

  $\therefore (x,y) \in A \Rightarrow B_r((a,b))$ is contained in A.

  (Intuition: in open ball of the point,

2.32 Closed sets.

like closed addition of vectors.

Definition 2.3.7

A set $A \subseteq \mathbb{R}^n$ is closed if every limit point of A belongs to A.

- empty set
- $\mathbb{R}^n$
- $a, b \in \mathbb{R}$ with a, b, $[a,b]$.
- $p \in \mathbb{R}^n$ and $r > 0$, $\{x \in \mathbb{R}^n : \|x - p\| < r\}$

- Closure of set A is closed

as it is smaller than radius

We can find the point inside A too.

Example. $\{(x,y) \in \mathbb{R}^2 : y \geq 1\}$ is closed

- Let (a,b) be limit point of A.
- $\{(x(k), y(k))\}_k$ be sequence in $A \setminus \{(a,b)\}$ converging to (a,b)

$\begin{cases} \to \cdot\, x(k) \to a \\ \to \cdot\, y(k) \to b \end{cases}$

As $y(k) \geq 1$ for $\forall k \in \mathbb{N}$,
and $y(k) \to b$,

$\therefore b = \lim_{k \to \infty} y(k) \geq 1$

$\therefore (a,b) \in A$.

2.33 Set Operations

Theorem 2.3.14

- $A \subseteq \mathbb{R}^n$ is open

$\iff$

$A^C = \mathbb{R}^n \setminus A$ is closed

- $\emptyset$, $\mathbb{R}^n$ are both open and closed

- Neither opened / closed:
  - $[137, 237)$
  - $\{\frac{1}{n} : n \in \mathbb{N}\}$
  - $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1, y \geq 0\}$
  - $\mathbb{Q}$: $\mathbb{Q}^\circ = \emptyset$ and $\bar{\mathbb{Q}} = \mathbb{R}$.

2.4.1 Compact Sets.

Closed + bounded

Definition 2.4.2 Compact

- $A \subseteq \mathbb{R}^n$ is **compact** if
- every sequence of A has subsequence which converges to a point in A

Bolzano-Weierstrass;

- A set in $\mathbb{R}^n$ is compact if and only if
- both closed and bounded.

- cannot make sequence run to infinity (Boundness)
- cannot converge to a missing boundary point (Closedness)
- force to settle down somewhere inside.

- Compact:
  - empty set (vacuously)
  - any finite set A
  - $[a,b]$ in $\mathbb{R}$.

- NOT compact:
  - $A = (0, 1]$ as $\{\frac{1}{k}\}_{k=1}^{\infty}$ converge 0
  - $B = \{\frac{1}{n} : n \in \mathbb{N}\}$
  - $\mathbb{R}^n$, $x(k) = (k, k, \ldots, k)$

$(1, 1, 1, \ldots)$
$(2, \ldots, 2)$
$(\ldots)$
$(\infty, \ldots, \infty)$

Bounded;

$A \subseteq \mathbb{R}^n$ is bounded if ...

- $\exists R > 0$ s.t. $A \subseteq \{x \in \mathbb{R}^n : \|x\| < R\}$

· If set is not bounded, then it is unbounded

2.4.2 Set operations and subsets

- Finite union of compact sets ⇒ compact
- Finite or infinite intersection of compact sets is compact
- Finite Cartesian product of compact sets is compact

2.5 Limits (2.5 ~ 2.5.3)

· Definition 2.5.1 Limit

Let $f: A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$.
Let $a \in \mathbb{R}^n$ be a limit point of A, let $b \in \mathbb{R}^m$
Define b to be limit of f at a provided

· $\lim_{x \to a} f(x) = b$

⇒ $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$,

$0 < \|x - a\| < \delta \Rightarrow \|f(x) - b\| < \varepsilon$

· Definition 2.5.6. Isolated point

- $a \in A$
- a is not a limit point of A

(limit at isolated point of f's domain is undefined) (Not 'not exist' is NOT defined)

e.g. $A = [1,3) \cup \{7\}$
- defined: $[1,3)$ (may / may not exist)
- undefined: 7 (not defined since isolated)

Theorem 2.5.9 Sequential definition of limits (Main tool proving non-existence of limits) (Similar to side limits, all function approaching $(0,0)$ should give the same value)

- $A \subseteq \mathbb{R}^n$, $f: A \to \mathbb{R}^m$ be a function.
- Let $a \in \mathbb{R}^n$ be limit point of A,
- Let $b \in \mathbb{R}^m$
- $\lim_{x \to a} f(x) = b$

⟺ · every sequence of points $\{x(k)\}_k$ in $A \setminus \{a\}$ with $x(k) \to a$, the sequence of points $\{f(x(k))\}_k$ in $\mathbb{R}^m$ converges to b (i.e. $f(x(k)) \to b$)

Theorem 2.5.11 Basic properties

$\lim_{x \to a} f(x) = b \iff \lim_{x \to a} f_i(x) = b_i$

iff for $\forall i \in \{1, \dots, m\}$

($f: A \to \mathbb{R}^m$,
a is limit point of A, b is $(b_1, \dots, b_m) \in \mathbb{R}^m$,
$f_1, \dots, f_m$ be component functions of f, $f = (f_1, \dots, f_m)$)

$f(x,y) = (x+y, xy)$
$f_1(x,y) = x+y$, $f_2(x,y) = xy$
$\lim_{(x,y) \to (2,3)} f_1(x,y) = 5$ and $\lim_{(x,y) \to (2,3)} f_2(x,y) = 6$
$\therefore \lim_{(x,y) \to (2,3)} f(x,y)$
$= \left(\lim_{(x,y) \to (2,3)} f_1(x,y), \lim_{(x,y) \to (2,3)} f_2(x,y) \right)$
$= (5, 6)$

Lemma 2.5.13 Uniqueness of limits.

· $f: A \to \mathbb{R}^m$ and fix $b_1, b_2 \in \mathbb{R}^m$
$\lim_{x \to a} f(x) = b_1$, and $\lim_{x \to a} f(x) = b_2$,
$\boxed{b_1 = b_2}$

Theorem 2.5.14 More properties.

- Let $A \subseteq \mathbb{R}^n$ be a set.
- Let a be limit point of A.

Proof. $\lim_{(x,y) \to (2,3)} (x+y) = 5$

Proof.
WTS: $\forall \varepsilon > 0, \exists \delta > 0$ s.t.
$0 < \|(x,y) - (2,3)\| < \delta$
$\Rightarrow \|(x+y) - 5\| < \varepsilon$.

Let $\varepsilon > 0$.
Take $\delta =$
Assume $0 < \|(x,y) - (2,3)\| < \delta$.
$\therefore$ $|x-2| \le \sqrt{(x-2)^2 + (y-3)^2} < \delta$
$|y-3| \le \sqrt{(x-2)^2 + (y-3)^2} < \delta$
Then $\|(x+y) - 5\| = |x-2 + y-3|$
$\le |x-2| + |y-3|$ (by Δ inequality)
$< 2\delta$ (as $|x-2| < \delta$, $|y-3| < \delta$)
$= \varepsilon$

Proof. $\lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2}$ does not exist

· Define $\mathbb{R}^2 \setminus \{(0,0)\}$ converge to
$x(k) = (0, \frac{1}{k}) \to (0,0)$
$y(k) = (\frac{1}{k}, \frac{1}{k}) \to (0,0)$

· $f(x(k)) = \frac{0}{0^2 + (\frac{1}{k})^2} = 0$

· $f(y(k)) = \frac{\frac{1}{k^2}}{\frac{1}{k^2} + \frac{1}{k^2}} = \frac{1}{2}$ (For all $k \in \mathbb{N}^+$)

$\therefore \lim_{k \to \infty} f(x(k)) = 0$
$\ne \lim_{k \to \infty} f(y(k)) = \frac{1}{2}$,
but $\{x(k)\}_k$ and $\{y(k)\}_k$ both converge to $(0,0)$
$\therefore$ Limit DNE.

Proof: $\lim_{(x,y) \to (2,3)} xy = 6$ More to cover...

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.
$0 < \|(x,y) - (2,3)\| < \delta$.
$\Rightarrow \|xy - 6\| < \varepsilon$.

Let $\varepsilon > 0$.
Take $\delta =$
Assume $0 < \|(x,y) - (2,3)\| < \delta$.
Then $|x-2| \le \sqrt{(x-2)^2 + (y-3)^2} < \delta$
$|y-3| \le \sqrt{(x-2)^2 + (y-3)^2} < \delta$
$\|xy - 6\|$
$= |xy - 2y + 2y - 6|$
$\le |xy - 2y| + |2y - 6|$ (Δ ineq.)
$\le \boxed{|y|}\, |x-2| + 2|y-3|$
$\le 4|x-2| + 2|y-3|$
$< 6\delta$
$\le \varepsilon$

Then $\delta \le 1$
$\|y-3\| < 1$
$-1 < y-3 < 1$
$2 < y < 4$

- Let f and g be $\mathbb{R}^n$-valued function defined on A.
- Let ϕ be real-valued function defined on A.
- Let $\lambda \in \mathbb{R}$ and $b \in \mathbb{R}^m$, constants.

1. Constants $\lim_{x\to a} b = b$ and $\lim_{x\to a} x = a$
2. Linearity $\boxed{\text{If } \lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x) \text{ exists}}$

$\to \lim_{x\to a}(f(x) + \lambda g(x))$ exists

$= \lim_{x\to a} f(x) + \lambda \lim_{x\to a} g(x)$

3. Scalar multiple If $\lim_{x\to a}\phi(x)$ and $\lim_{x\to a} f(x)$ exists,

$\to \lim_{x\to a}(\phi(x) f(x))$

$= \left(\lim_{x\to a}\phi(x)\right)\left(\lim_{x\to a} f(x)\right)$

scalar vector

4. Dot product If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists

$\to \lim_{x\to a}(f(x)\cdot g(x))$ exists

$= \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right)$

Theorem 2.5.15 Squeeze Theorem

Let $A \subseteq \mathbb{R}^n$.

Let a be limit point of A.

Let f, g, h be real-valued function with domain A

Assume $\delta > 0$ s.t.

$\forall x \in A,\ 0 < \|x - a\| < \delta$

$\Rightarrow f(x) \le g(x) \le h(x)$

If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = b$ for some $b \in \mathbb{R}$.

Then $\lim_{x\to a} g(x) = b$.

consider norm $\to \infty$.

2.5.3 Limits with Infinity.

Definition 2.5.16 limit of $f(x)$ as $\|x\| \to \infty$ (i.e. $x \to \infty$)

- Let $A \subseteq \mathbb{R}^n$ be unbounded.
- Let $f: A \to \mathbb{R}^m$, let $b \in \mathbb{R}^m$ to be limit of $f(x)$ as $\boxed{\|x\| \to \infty}$
- $\forall \varepsilon > 0,\ \exists M > 0,$ s.t. $\forall x \in A,$

$\|x\| > M \Rightarrow \|f(x) - b\| < \varepsilon$

- ✓ $\lim_{\|x\|\to\infty} f(x) = b$
- ✗ $\lim_{\|x\|\to\infty} f(x)$ DNE.

Definition 2.5.18 limit of f at a diverges to $+\infty$ (i.e. $f(x) \to \infty$)

Let $A \subseteq \mathbb{R}^n$ be a set.

Let a be a limit point of A.

Let $f: A \to \mathbb{R}$ be a real-valued function.

Limit of f at a diverges to $+\infty$

- $\forall M > 0,\ \exists \delta > 0$ s.t. $\forall x \in A,$

$0 < \|x - a\| < \delta \Rightarrow f(x) > M$

- ✓ $\lim_{x\to a} f(x) = +\infty$

2.6.1 Continuity

- $\lim_{x\to a} f(x) = f(a)$ (2.6.1)
- Let $f: A \to \mathbb{R}^n$ be a function with domain $A \subseteq \mathbb{R}^n$.
- Let $a \in A$ be a point.
- The function f is continuous at a provided
  - $\forall \varepsilon > 0,\ \exists \delta > 0.$ s.t. $\forall x \in A,$

limit point: zoom into a,
any neighborhood has points in A (a not must in A)

Example: $f(x,y) = x+y$
$g(x,y) = xy$

- $\lim_{(x,y)\to(2,3)} f(x,y) = 5 = f(2,3)$ } f is continuous at $(2,3)$
- $\lim_{(x,y)\to(2,3)} g(x,y) = 6 = g(2,3)$ } f is continuous at $(2,3)$

$||x-a|| < \delta \implies ||f(x) - f(a)|| < \varepsilon$ (includes isolated points)

Lemma 2.6.2.

- a is isolated point of $A \implies f$ is continuous (vacuously)
- a is a limit point of $A \implies \left(f \text{ is continuous} \iff \lim_{x\to a} f(x) = f(a)\right)$

Theorem 2.6.5 Sequential Definition of Continuity

- Let $f: A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$.
  Let $a \in A$.
  Then f is continuous at a if and only if
- for every sequence $\{x(k)\}_k$ in A converging to a
- $\{f(x(k))\}_k$ in $\mathbb{R}^m$ converges to $f(a)$

Definition 2.6.6

Let $f: A \to \mathbb{R}^m$ be a function with domain $A \subseteq \mathbb{R}^n$.
For $S \subseteq A$,
- f is continuous on S if f is continuous for $\forall a \in S$.
(. f is continuous on its domain A.)

Example 2.6.7 (checking continuity on subsets)

Example 2.6.11

$$F(x,y) = \begin{cases} x+y & \text{if } (x,y) \neq (2,3) \\ 237 & \text{otherwise} \end{cases}$$

- F continuous on $\mathbb{R}^2 \setminus \{(2,3)\}$
- F not continuous at $(2,3)$ since $\lim_{(x,y)\to(2,3)} F(x,y) = 5 \neq 237 = F(2,3)$

2.6.2 Basic Properties

Theorem 2.6.14 Continuity of components

- $f = (f_1, \dots, f_m): A \to \mathbb{R}^m$ is continuous at $a \in A$
  if and only if
  f_i is continuous at a for each $i \in \{1, \dots, m\}$

Lemma 2.6.16 Linear Transformation continuity

- every linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ = continuous.
- identity map
- Coordinate projection map $\pi_i : \mathbb{R}^n \to \mathbb{R}$ by $\pi_i(a) = x_i$
- $f(x) = Ax$ for $m\times n$ matrix A, $f: \mathbb{R}^n \to \mathbb{R}^m$

Theorem 2.6.18 Basic Properties

Let $A \subseteq \mathbb{R}^n$, $a \in A$.
Let f, g be $\mathbb{R}^m$-valued functions defined on A.
Let ϕ be a real-valued function of A. $\mathbb{R}^n \to \mathbb{R}$
Let $\lambda \in \mathbb{R}$.
- If f and g are continuous at a:
  - $f + \lambda g$ is continuous at a
  - ϕf (scalar) is continuous at a
  - f dot g is continuous at a

Example 2.6.19
- $f: \mathbb{R}^n \to \mathbb{R}$ is $||x||^2 = x_1^2 + \dots + x_n^2$
- f is dot product of identity map.
  $id(x) \cdot id(x) = x \cdot x = ||x||^2 = f(x)$

Corollary 2.6.20

Let $f: A \to B$ where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$.
If
- f is continuous at a
- g is continuous at $f(a)$

Then $g \circ f$ is continuous at a.

Definition 2.6.24

- monomial in n variables $x_1, \dots, x_n$ is a function of the form $x_1^{r_1}, \dots, x_n^{r_n}$, for some $a_1, \dots, a_n \in \mathbb{N}$.

- polynomial in n variables $x_1, \dots, x_n$ is a linear combination of monomials in n variables with real coefficient.

Example 2.6.25

$p(x,y,z) = xy + 3z^4$: polynomial

xy, z^4 : monomials

Lemma 2.6.26

- all polynomials in n variables are continuous on $\mathbb{R}^n$.

2.6.3 Topological Properties (open, closed, bounded, compact)

Theorem 2.6.27 preimage of continuous map preserves open/closed. (does not preserve boundness, compactness)

Let $f: \mathbb{R}^n \to \mathbb{R}^m$. They are equivalent:

(a) f is continuous on $\mathbb{R}^n$

(b) $f^{-1}(U)$ is open on every open set $U \subseteq \mathbb{R}^m$

(c) $f^{-1}(V)$ is closed on every closed set $V \subseteq \mathbb{R}^m$.

EG 2.6.30

- $f(x,y) = 237$
- $x^2+y^2 < 1$
- $f(x,y) = 0$

$\Rightarrow f^{-1}(\{237\}) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$
$= B_1((0,0))$

- Open: $f^{-1}(\{237\})$
- Closed: $\{237\}$

$\}$ f cannot be continuous!

Theorem 2.6.35 Image of continuous map preserves compactness

If
- A is compact subset of $\mathbb{R}^n$
- f is an $\mathbb{R}^m$-valued function continuous on A.

Then $f(A)$ is a compact subset of $\mathbb{R}^m$.

Example 2.6.36

- $f(x,y,z) = \{(xy, yz, xz) : 0 \le x,y,z \le 1\}$
(Continuous by Thm 2.6.14, Lemma 2.6.26, polynomial in each component)

- $[0,1]^3$ is compact

$\Rightarrow B = f([0,1]^3)$ is ALSO COMPACT by Theorem 2.6.35.

Example 2.6.37

- $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

- Set $A = [\varepsilon, 1] \xrightarrow{\text{image}}$ set $f(A) = [1, \frac{1}{\varepsilon}]$ $\boxed{\text{COMPACT}}$
- Set $B = [0,1] \longrightarrow$ set $f(A) = \{0\} \cup [1, \infty)$ $\boxed{\text{NOT COMPACT}}$

$\Rightarrow$ Not continuous!

2.7 Path-connected sets

Definition 2.7.1 Path-connected (C^k path-connected, if k-times differentiable)

- A set $S \subseteq \mathbb{R}^n$ is path-connected

If every pair of points $p, q \in S$,
there exit continuous function $\gamma : [a,b] \to \mathbb{R}^n$ s.t.

$\gamma(a) = p$, $\gamma(b) = q$, $\mathrm{im}(\gamma) \subseteq S$

Definition 2.7.4 Convex

- line segment between any 2 points $p, q \in S$ lies in S.

Convex

Theorem 2.7.8 Image of continuous map preserves path-connectedness (cont. + path-con $\to$ $f(s)$ path con)

· f: $\mathbb{R}^n$-valued function

× not for preimage!

· If f is continuous on $S \subseteq \mathbb{R}^n$, a path-connected set

· Then f(S): path-connected

Corollary 2.7.10 (Intermediate Value Theorem)

real-valued function defined on [a,b]

· If f is continuous on [a,b]

· Then f([a,b]) is path-connected

2.8 Global Extrema

Definition 2.8.1 Global max point / value. not using norm anymore! evaluate the output $\mathbb{R}$ value

Let $A \subseteq \mathbb{R}^n$. f be real-valued function defined on A

· Global max point of f on A: point p if $f(p) \geq f(x)$ for $\forall x \in A$

· Global maximum value f(p).

· Such point exist = f attains global max on A.

Theorem 2.8.7 Extreme value theorem (Single variable)

· If $f: [a,b] \to \mathbb{R}$ is $\boxed{\text{continuous}}$

· Then f $\boxed{\text{attains a maximum and minimum on } [a,b]}$

Extreme Value Theorem (Multi-variable)

If · $A \subseteq \mathbb{R}^n$ is non-empty compact set

· $f: A \to \mathbb{R}$ is continuous

Then · f attains max and min at points of A

Theorem 2.8.8 Least upper bound principle (LUB) We can assume :)

upper bound b

If · $S \subseteq \mathbb{R}$ has upper bound ($\forall x \in S$, $x \leq b$, $b \in \mathbb{R}$)

Then · Supremum of S exist. (Sup DNE = sup(S) = ∞)

Lemma 2.8.14

· $A \subseteq \mathbb{R}^n$ closed and unbounded

· f is continuous real-valued function on A

If · $f(x) \to -\infty$ as $\|x\| \to \infty$

Then · f attains max on A