Ch3

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Vector Subspace

Definition 1.

Contains the zero vector

A Subspace V of R^n is any non-empty subset of R^n;

1. V is closed under vector addition; ($\vec{u}, \vec{v} \in V$, we have $\vec{u} + \vec{v} \in V$)
2. V is closed under scalar multiplication ($\vec{u} \in V$, $k \in \mathbb{R}$, we have $k\vec{u} \in V$)

Example 2:

$$S = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

- NOT vector space
- $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓ $2\vec{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ ✗

Lemma 3.3.

If S is a set of m linearly independent vectors in R^n, Then $m \leq n$.

Theorem 3.4.

A subset V is a vector subspace of R^n iff it exists $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ s.t. $V = \text{Span}(\vec{v_1}, \vec{v_2}, \dots, \vec{v_n})$

Definition 3.5.

- If V is a vector space with $V = \text{Span}(v_1, \dots, v_n)$, then $\{v_1, \dots, v_n\}$ is a spanning/generating set for V.

Activity: Show $V = \left\{ \begin{pmatrix} x-y \\ x+y \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$

(1) Using Definition 3.1:
- Can take any kx, ky.
- Can take any addition

(2) Using finding a generating set for V.

Rouche Capelli

- Unique: - last rref(A) no pivot
  - every rref(C) have pivot
- Inf. many: - last rref(A) no pivot
  - last rref(C) no pivot
- Inconsistent: - last rref(A) has pivot

$$\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}\vec{x}, \begin{pmatrix}-1\\1\\1\end{pmatrix}\vec{y} \right\}$$

3.2 Bases.

Definition 3.6.

Let V be a vector subspace of R^n.

A subset $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \dots, \vec{b}_m\}$ of R^n is called a Basis.

- if it is a linearly independent generating set.
  > $\vec{b}_1, \dots, \vec{b}_m$ linearly independent
  > $V = \text{Span}(\vec{b}_1, \dots, \vec{b}_m)$
  - Bases are not unique.

Theorem 3.8. Let V be a vector subspace of R^n.
Then size of any basis for V is unique.

Definition 3.9
Let V be a vector subspace of R^n.
Dimension of V as $\dim V$, is equal to the size of any basis for V.
- $\dim\{\vec{0}\} = 0$.

Activity

Let $V = \text{Span}(v_1, v_2, v_3, v_4)$ be a vector subspace of R^n

$A = (v_1, v_2, v_3, v_4)$

- rref(A) has pivot at column 1, 3, 4.

(1) $\vec{v}_2 \in \text{Span}(\vec{v}_1, \vec{v}_3, \vec{v}_4)$

(2) $\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$ is a linearly independent set.

(3) $\dim V = 3$

Lemma. 3.10.

$A = (\vec{v_1} \quad \vec{v_2} \quad \dots \quad \vec{v_m})$

$\text{rref}(A) = (\vec{x_1} \quad \vec{x_2} \quad \dots \quad \vec{x_n})$

rref(A) column $\vec{x_m}$ has no pivot (NOT pivot column)

Then $\text{Span}(v_1, v_2, \dots, v_n) = \text{Span}(v_1, v_2, \dots v_{n-1})$

REMOVE $\vec{v_m}$.

Theorem 3.11 (Finding Bases)

Let V be the vector subspace of $\mathbb{R}^n$

$V = \text{Span}(\vec{v_1}, \dots, \vec{v_m})$

→ If A is matrix with column vectors $\vec{v_1}, \dots, \vec{v_m}$

→ pivot columns of A will form basis for V.

→ RREF of A has k pivots → $\dim(V) = k$.

Problem set 3. P.3.1, P.3.3, P.3.6, P.3.7, P.3.10

P.3.1: Prove if S is a set of m linearly independent vectors in $\mathbb{R}^n$, then $m \le n$.

Let $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}, \dots, \vec{v_m}\}$

Since $\mathbb{R}^n$ has dimension of n,

any basis of $\mathbb{R}^n$ consists exactly n linearly independent vectors.

... f linearly independent vectors

∴ The maximum no. of linearly independent vectors in $\mathbb{R}^n$ is n.

∴ $m \le n$, as the set S cannot have more linearly independent vectors than the dimension of $\mathbb{R}^n$.

P.3.3 (Theorem 3.11)

- Show that any set of n linearly independent vectors in $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a set with n linearly independent vectors

Let $C = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{pmatrix}$.

Since all vectors in S are linearly independent, all columns in C are pivot columns.

Thus, n columns (linearly independent vectors) form the basis for $\mathbb{R}^n$.

P.3.6. True / False:

If W and V are subspaces of $\mathbb{R}^n$, then $W \cup V := \{\vec{x} \in \mathbb{R}^n : \vec{x} \in W \text{ or } \vec{x} \in V\}$ is a subspace of $\mathbb{R}^n$.

- True.

$W \cap V$ satisfies the 3 subspace properties:

1. $\vec{0} \in W$ and $\vec{0} \in V$, $\therefore \vec{0} \in \vec{W} \cap \vec{V}$

2. Let $\vec{x}, \vec{y} \in W \cap V$.

   Since W is a subspace, $\vec{x}+\vec{y} \in W$.

   Since V is a subspace, $\vec{x}+\vec{y} \in V$.

   $\therefore$ $\vec{x}+\vec{y} \in W \cap V$, thus closure under addition.

3. Let $\vec{x} \in W \cap V$. Let k be any scalar.

   Since W is a subspace, $k\vec{x} \in W$.

   Since V is a subspace, $k\vec{x} \in V$.

   $\therefore$ $k\vec{x} \in W \cap V$, thus closure under multiplication.

$\therefore$ $W \cap V$ is a subspace of $\mathbb{R}^n$

P.3.7 True or False;

if W and V are subspaces of $\mathbb{R}^n$, then

$$W \cup V := \{\vec{x} \in \mathbb{R}^n : \vec{x} \in W \text{ or } \vec{x} \in V\}$$

is a subspace of $\mathbb{R}^n$.

False.

Let $n=2$. i.e. dimension $\mathbb{R}^2$.

Take $W = \{ \begin{pmatrix} t \\ 0 \end{pmatrix} \mid t \in \mathbb{R} \}$

Take $V = \{ \begin{pmatrix} 0 \\ s \end{pmatrix} \mid s \in \mathbb{R} \}$

Take $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ from $W \cup V$,

addition of both gives $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

However, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in $W \cup V$.

$\therefore$ Closure under addition does not apply here.

$\therefore$ $W \cup V$ is not a subspace of $\mathbb{R}^2$.

$\therefore$ We have disproved the required statement.

P. 3.10 Let V and W be vector subspaces of $\mathbb{R}^5$ with $V \cap W = \{\vec{0}\}$.

Suppose that V has basis $\{\vec{v_1}, \vec{v_2}\}$,

W has basis $\{\vec{w_1}, \vec{w_2}\}$.

Find a basis for the vector space $V+W$, and justify how you know this is a basis.

Since $V \cap W = \{\vec{0}\}$, we know that

$\vec{v_1}, \vec{v_2}, \vec{w_1}, \vec{w_2}$ are linearly independent

vectors of $V+W$, as none of the 4 vectors

overlap except the zero vector from V and W.

$\therefore$ Since $\vec{v_1}, \vec{v_2}, \vec{w_1}, \vec{w_2}$ are all in $V+W$

and are linearly independent, $\therefore$

$\therefore \{\vec{v_1}, \vec{v_2}, \vec{w_1}, \vec{w_2}\}$ form the basis for $V+W$.