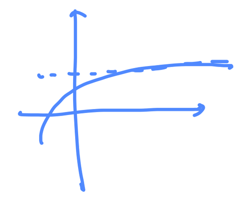
Ch1 - Ch4

Saturday, 5 October 2024 7:51 pm

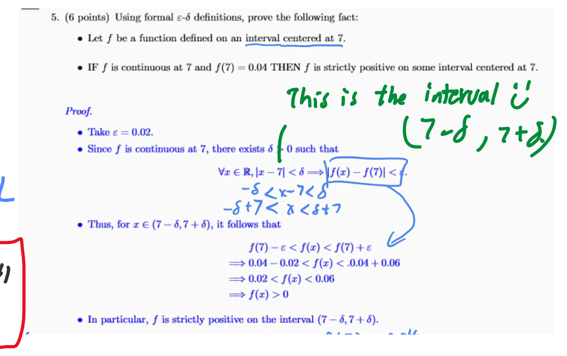
- Always add 's.t.' after $\exists$

- Is negation. • Is addition on known definition. (✪ Add $\forall x \in \mathbb{R}$ for all definitions)

	5a. $\lim_{x\to a} f(x) = L$	$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < \lvert x-a \rvert < \delta \Rightarrow \lvert f(x) - L \rvert < \varepsilon$.
5a: $\exists L \in \mathbb{R}$	5b. $\lim_{x\to a} f(x)$ exist	$\boxed{\exists L \in \mathbb{R}}$ s.t. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < \lvert x-a \rvert < \delta \Rightarrow \lvert f(x) - L \rvert < \varepsilon$.
5a: Neg	5c. $\lim_{x\to a} f(x) \neq L$	$\exists \varepsilon > 0$ s.t. , $\forall \delta > 0, \exists x \in \mathbb{R}$ s.t $0 < \lvert x-a \rvert < \delta$ AND $\lvert f(x) - L \rvert \geq \varepsilon$
5a: Neg($\exists L \in \mathbb{R}$ + 5a)	5d. $\lim_{x\to a} f(x)$ DNE	$\forall L \in \mathbb{R}\ \exists \varepsilon > 0$ s.t. $\forall \delta > 0, \exists x \in \mathbb{R}$. s.t. $0 < \lvert x-a \rvert < \delta$ AND $\lvert f(x) - L \rvert \geq \varepsilon$
5a: $a < x < a + \delta$	5e. $\lim_{x\to a^+} f(x) = L$	$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\boxed{a < x < a + \delta} \Rightarrow \lvert f(x) - L \rvert < \varepsilon$.
✪ $f(x) = \infty$ use M.	5f. $\lim_{x\to a} f(x) = \infty$	$\boxed{\forall M \in \mathbb{R}}, \exists \delta > 0$ s.t. $0 < \lvert x-a \rvert < \delta \Rightarrow \boxed{f(x) > M}$ (This means the smaller the δ, the larger the M)
	5g. $\lim_{x\to a^-} f(x) = -\infty$	$\boxed{\forall M \in \mathbb{R}}, \exists \delta > 0$ s.t. $\boxed{a - \delta < x < a} \Rightarrow f(x) < M$
✪ $x \to \infty$ use K.	5h. $\lim_{x\to\infty} f(x) = L$.	$\forall \varepsilon > 0, \boxed{\exists K \in \mathbb{R}}$ s.t. $\boxed{x > K} \Rightarrow \lvert f(x) - L \rvert < \varepsilon$. (The larger the x, the closer to L.)
	5i. $\lim_{x\to -\infty} f(x) = \infty$	$\forall M \in \mathbb{R}, \boxed{\exists K \in \mathbb{R}}$ s.t. $x < K \Rightarrow f(x) > M$
	✪ $\lim_{x\to a} f(x) = f(a)$	$\forall a \in \mathbb{R}$ (if for all x.), $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < \lvert x-a \rvert < \delta \Rightarrow \lvert f(x) - \boxed{f(a)} \rvert < \varepsilon$.
	Removable continuity.	$\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < \lvert x-a \rvert < \delta \Rightarrow \lvert f(x)$ $< \varepsilon$, but $f(a)$ is undefined / $f(a) \neq L$
	Unremovable continuity	See DNE.



$f(a) = L$.



5. (6 points) Using formal ε-δ definitions, prove the following fact:

- Let f be a function defined on an interval centered at 7.
- IF f is continuous at 7 and $f(7) = 0.04$ THEN f is strictly positive on some interval centered at 7.

Proof.

- Take $\varepsilon = 0.02$.
- Since f is continuous at 7, there exists $\delta > 0$ such that

$$\forall x \in \mathbb{R}, |x - 7| < \delta \Longrightarrow \boxed{|f(x) - f(7)| < \varepsilon}.$$

$-\delta < x - 7 < \delta$

$-\delta + 7 < x < \delta + 7$

This is the interval :) $(7-\delta, 7+\delta)$

- Thus, for $x \in (7 - \delta, 7 + \delta)$, it follows that

$$f(7) - \varepsilon < f(x) < f(7) + \varepsilon$$
$$\Longrightarrow 0.04 - 0.02 < f(x) < .0.04 + 0.06$$
$$\Longrightarrow 0.02 < f(x) < 0.06$$
$$\Longrightarrow f(x) > 0$$

- In particular, f is strictly positive on the interval $(7 - \delta, 7 + \delta)$.

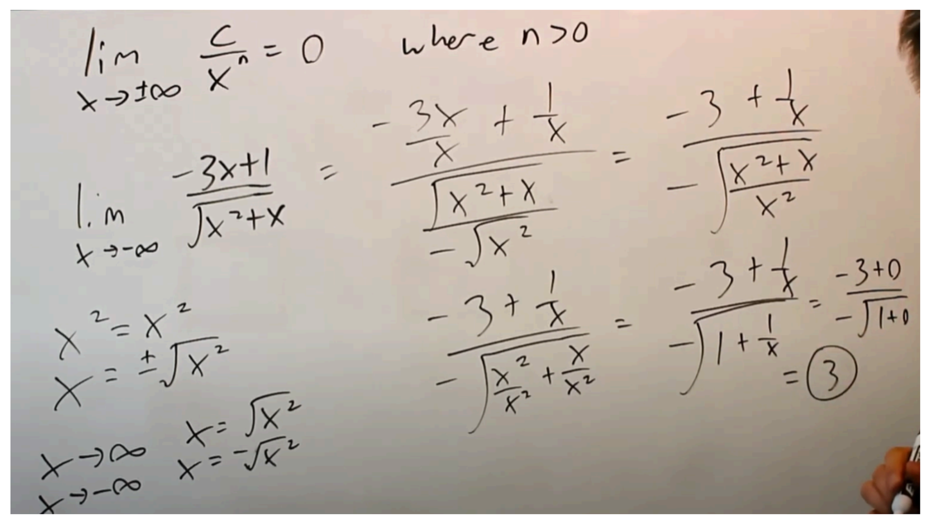
Notes:
- Add $\exists L \in \mathbb{R}$ for 'exist'
- Use negation for $\neq$ / exist $\to$ DNE.
- For $x \to a^{+/-}$, use $a - \delta < x < a$ / $a < x < a + \delta$
- $A \to B$ is NOT A or B!
- $\sim(A \to B)$ is A AND NOT B!

Tackle limit + composite... "$4|f(3x+1) - 2|$"

1. Let $\varepsilon > 0$ be fixed.
2. Use $\frac{\varepsilon}{4}$ as epsilon, $\exists M_1 > 0$, s.t. $x_1 > M_1 \Rightarrow |f(x_1) - 2| < \frac{\varepsilon}{4}$
3. Take $M = \frac{M_1 - 1}{3}$
4. Let $x \in \mathbb{R}$. Assume $x > M$. $x > M = x > \frac{M_1 - 1}{3} = 3x + 1 > M_1 \Rightarrow |f(3x+1) - 2| < \frac{\varepsilon}{4}$
5. $|4f(3x+1) - 8|$
$= 4|f(3x+1) - 2|$
$< 4 \cdot \frac{\varepsilon}{4}$
$= \varepsilon$

Infinities:
- Use $\forall M \in \mathbb{R}$ for $\lim_{x\to} = \infty$
- Use $\exists K \in \mathbb{R}$ for $\lim_{x\to\infty}$
- Do $f(x) < M$ / $x < K$ for $-\infty$!

For Computation, becareful of negative inf.



2.19/20 Computation

✪ $1 - \cos x \Rightarrow (1 - \cos x)(1 + \cos x) = \sin^2 x$.

✪ Use $x = -\sqrt{x^2}$ for $x \to -\infty$!!

e.g. $x + \sqrt{x^2 + x}$

$= x\left(1 - \sqrt{1 + \frac{1}{x}}\right)$

Because x is negative and we add a negative to make value of sqrt. positive.

2.21 Extreme Value Theorem

If f is a continuous function on $[a,b]$

Then f has a max and a min on $[a,b]$.

($\forall x \in I, \exists c \in I$ s.t. $f(x) \leq f(c)$)

2.22 The Intermediate Value Theorem.

Let f defined $[a,b]$. Let $M \in \mathbb{R}$.

If
- $f(a) < M < f(b)$
- f is continuous on $[a,b]$.

Then ...

Then
$\exists c \in (a,b)$ such that $f(c) = M$.
(i.e. f takes all value between $f(a)$ to $f(b)$)

☆ Proof of ε-δ : Strategies.

- Intervals I : relate it to $-\delta < x < \delta$
- Try to add two consequents from assumption :)
- Take epsilon to be $\frac{\varepsilon}{4}$ if its $4f(x)$ we trying to prove. Do it inverse :)
- For ∞ or $-\infty$, you can use equal or greater / smaller.
- To prove a limit DNE.
  $\Rightarrow 0 < |x-a| < \delta$ AND $|f(x) - L| \geq \varepsilon$ — outside the bounded region, There is an x such that the function is outside ε.
  Case 1 : $L \geq 0$; choose x s.t. $f(x) = -1$, $|\underbrace{f(x) - L}_{-1 - \square}| > 1 > \varepsilon$ — for $\varepsilon = \frac{1}{2}$. $f(x) = \cos(\pi/x)$
  Case 2 : $L < 0$: Choose $f(x) = 1$, $|\underbrace{f(x) - L}_{1 + \square}| > 1 > \varepsilon$
- Negation : remember $\forall x \in \mathbb{R}$!

Ch 3. Derivatives.

Product rule : d前留後 + 留前d後

Quotient rule : $\dfrac{vu' - v'u}{v^2}$

Differentiable : $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists.

$\Downarrow$ $\Downarrow$

Continuous : $\lim\limits_{x \to c} f(x) = f(c)$

If f is differentiable at c.
Then f is continuous at c.

Proof. Assume f is differentiable at c.
Then

$\lim\limits_{x \to c} [f(x) - f(c)]$

$= \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}(x - c)$

$= \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} \cdot \lim\limits_{x \to c} (x - c)$

$= f'(c) \cdot 0$

$= 0$ $\therefore \lim\limits_{x \to c} f(x) - f(c) = 0$

$\lim\limits_{x \to c} f(x) = f(c)$

Chain Rule

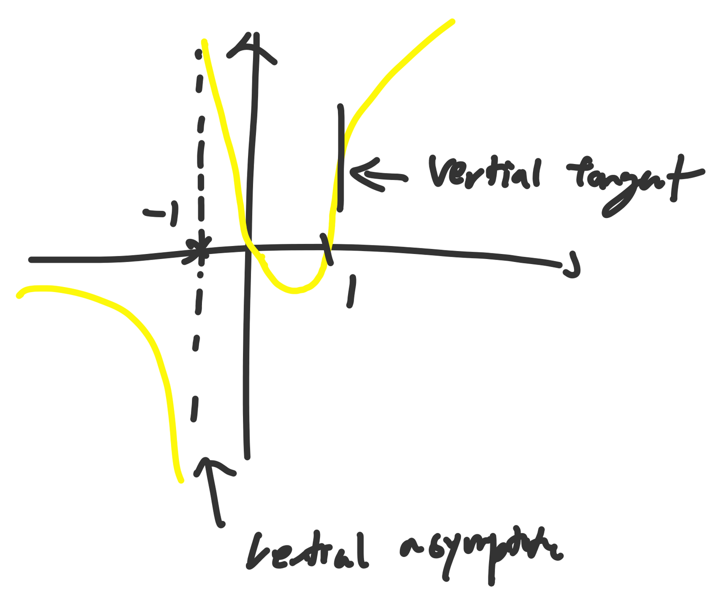
- If g is differentiable at a
- If f is differentiable at g(a)
- Then $f \circ g$ is differentiable at a.

Construct a Vertical Tangent line $\sqrt[3]{x}$ at $x = 1$: $\sqrt[3]{(x-1)}$

Vertical Asymptote line $\frac{1}{x}$ at $x = -1$: $\frac{1}{x+1}$

Construct a function that has both ;

$f(x) = \dfrac{1}{x+1} + \sqrt[3]{x-1}$



A B

Ch 4 Transcendental functions

4n. A function f consists of

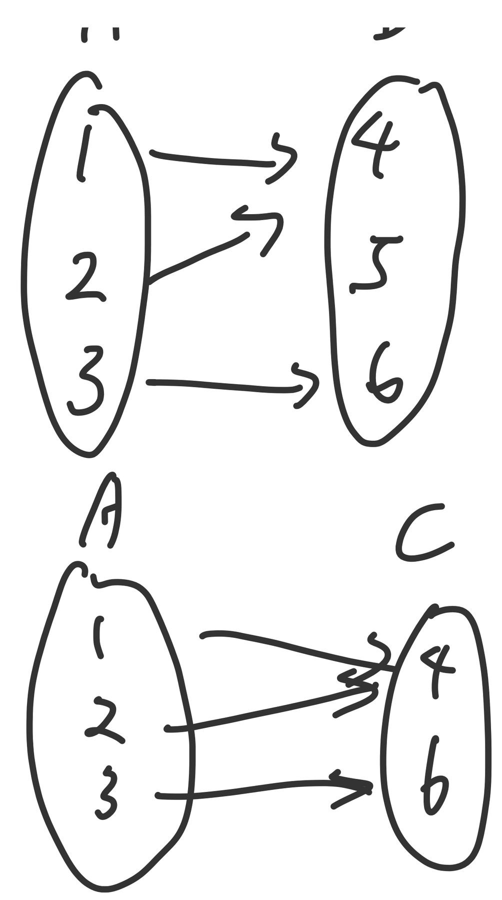
1. A domain

For CS, (codomain = range) 2. A codomain (codomain: potential output; Range: actual outputs)

(range = image) 3. A rule that matches each input to exactly one output

$f : A \to B$

↑ Name ↑ Domain ↑ Codomain



Codomain = $\{4, 5, 6\}$

Range = $\{4, 6\}$

They are not the same.

In Single-Variable Calculus...

- Domain = Largest subset of $\mathbb{R}$ possible
- Codomain = always $\mathbb{R}$.

$g(x) = \frac{1}{x^3}$

Domain $g = (-\infty, 0) \cup (0, \infty)$

Codomain $g = \mathbb{R}$.

☆ Range = real output

☆ Codomain = possible output

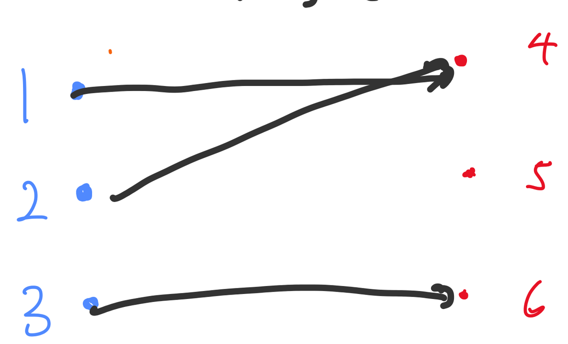
Injective function: each function input → one distinct output, no repeat.

Inverse: if and only if

- f is injective (Calculus)
- f is injective and surjective (Linear Alg)

Swap inputs and outputs

- A function f (reverse the arrow!)



- Inverse:

$f : A \to B$ be a one-to-one function.

The inverse of f is another function

- $f^{-1} : B \to A$

$$\boxed{\forall x \in A, \forall y \in B, \quad y = f(x) \iff x = f^{-1}(y)}$$

	f	f^{-1}
Domain	A	C
Range	C	A

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & C \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

$\forall x \in A, \ f^{-1}(f(x)) = x$

$\forall y \in C, \ f(f^{-1}(y)) = y$

- **Surjective : (RANGE = Co-domain)**

  Let $f : A \to B$ be a function

  - f is surjective or onto when $\boxed{\text{Range } f = B}$ (i.e. all possible outputs are actual outputs)

- **Injective (One-to-one)**

  - f is injective / one-to-one when

    $$\forall x_1, x_2 \in A,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

Theorem

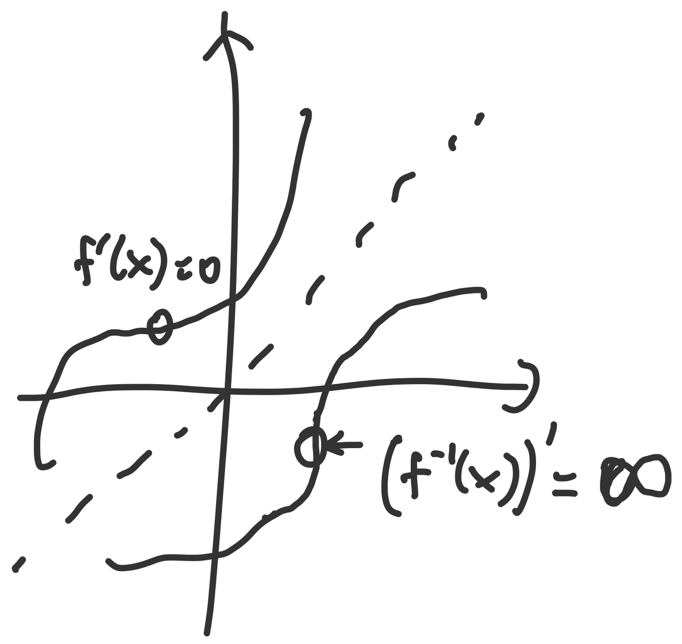
$$\boxed{f \text{ has an inverse} \iff \begin{cases} f \text{ is injective} \\ f \text{ is surjective} \end{cases}}$$

- **Derivative of INVERSE function**

  If 1. f has inverse
  2. f is differentiable
  3. For all $x \in I$, $f'(x) \neq 0$

  Then f^{-1} is differentiable.

  $$\boxed{(f^{-1})'(y) = \frac{1}{f'(x)}}$$



$$f'(f^{-1}(y)) = y$$

$$\underbrace{f'(f^{-1}(y))}_{x} \cdot (f^{-1})'(y) = 1$$

$$(f^{-1})'y = \frac{1}{f'(x)}$$

To make it one-to-one

ARCSIN : inverse function of restriction of sin to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$x = \arcsin y \iff y = \sin x$$

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

e.g.

$g(x) = \ln(f^{-1}(x))$

$g'(3) = \frac{1}{f^{-1}(3)} \cdot \frac{1}{f'(f^{-1}(3))}$

$= \frac{1}{2} \cdot \frac{1}{f'(2)}$

$= \frac{1}{2} \cdot \frac{1}{-0.8}$

$= -0.625$

$h(x) = f(x)^{g(x)}$

$\ln h(x) = g(x) \ln f(x)$

$\frac{1}{h(x)} \cdot h'(x) = g'(x) \ln f(x) + g(x) \frac{1}{f(x)} \cdot f'(x)$

$h'(-3) = \left(g'(-3) \ln f(-3) + \frac{g(-3)}{f(-3)} f'(-3) \right) h(-3)$

$= \left(\underset{-2}{g'(-3)} \ln(2.5) + \frac{-1}{2.5} \underset{\frac{1}{2}}{f'(-3)} \right) \underset{2.5}{f(-3)}^{\overset{-1}{g(-3)}}$

$(-4, 1)$ $\quad \frac{1+1}{-4+3} = \frac{2}{-1} = -2 = g'(-3)$

$f'(-3) = \frac{1}{2}$

$(-3, -1)$ $\quad (-2, 3)$ $(-4, 2)$ $\quad \frac{1}{-2+4} = \frac{1}{2}$

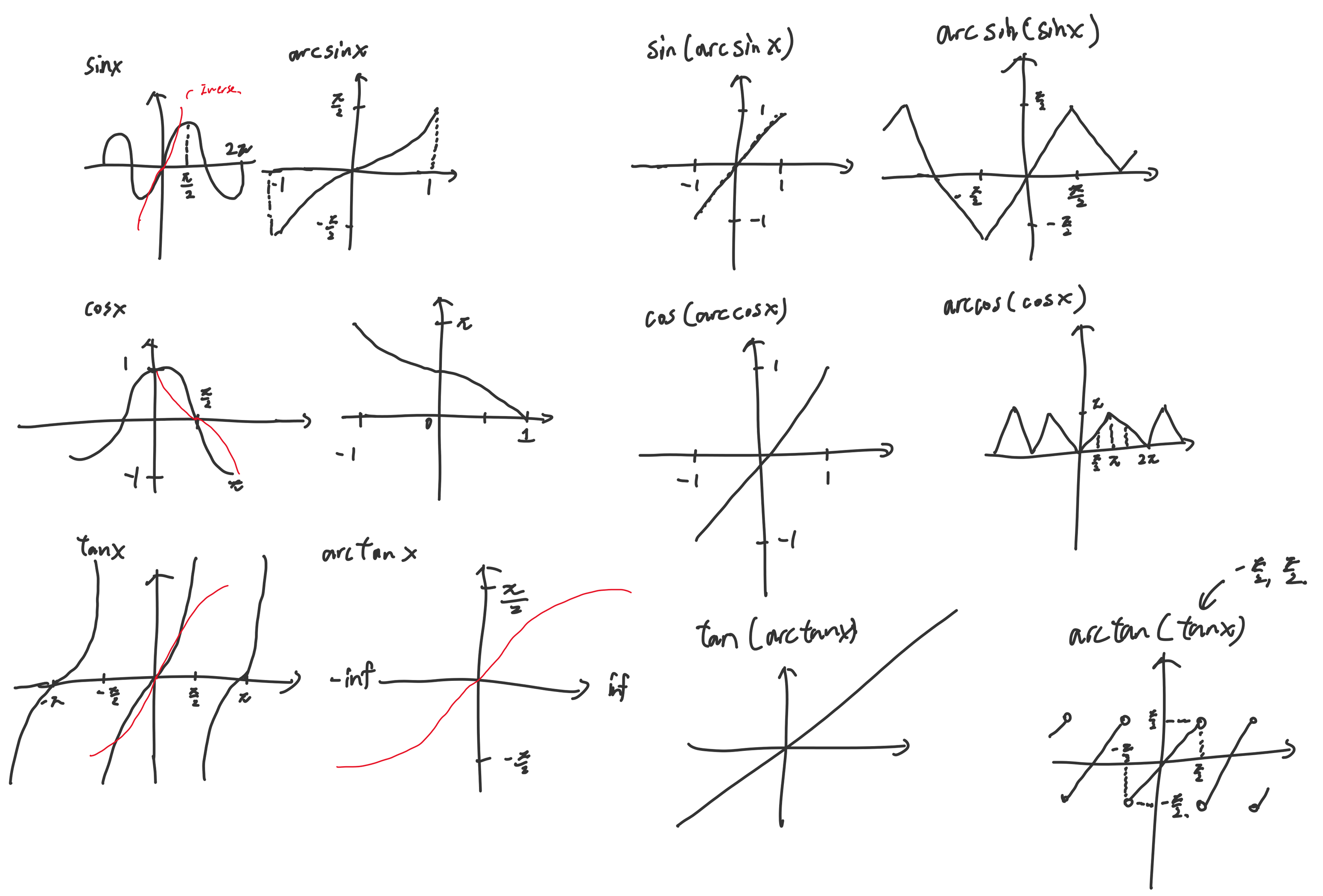
$2^{f(x)} \ln 2 \cdot f'(x)$

ARCTAN : inverse function of restriction of tan to $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$\boxed{\begin{array}{ccc} x = \arctan y & \Longleftrightarrow & y = \tan x \\ x \in (-\frac{\pi}{2}, \frac{\pi}{2}) & & y \in \mathbb{R}. \end{array}}$$

ARCCOS : inverse function of restriction of cos to $[0, \frac{\pi}{2}]$

$$\boxed{\begin{array}{ccc} x = \arccos y & \Longleftrightarrow & y = \cos x \\ x \in [0, \frac{\pi}{2}] & & y \in [-1, 1] \end{array}}$$



$$\boxed{\begin{array}{l} \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}} \\ \\ \frac{d}{dx} \arctan x = \frac{1}{1+x^2} \end{array}}$$

$= 2^{f(1)} \ln 2 \cdot f'(x)$

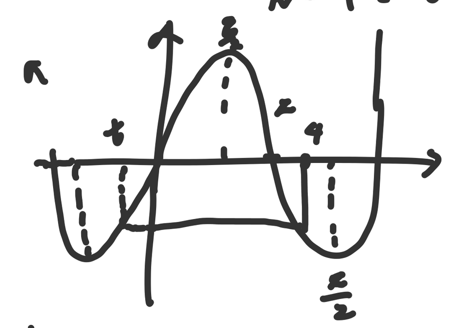
$= 2^2 \ln 2 \cdot \cancel{f'(x)}\ 2.5$

$= 4 \ln 2$

$= 2.77$

$\frac{\pi}{2} - t = 4 - \frac{\pi}{2}$

$\pi - 4 = t$



$\frac{d}{dx} \arctan(f(x) + \pi x)$

$= \frac{1}{1 + (f(x) + \pi x)^2} \cdot f'(x) + \pi$

$= \frac{1}{1 + (\pi)^2} \cdot (-0.3 + \pi) = 0.261$