Ch9 Coordinate Systems and Change of Basis

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※ Coordinates in new Basis:

$$[\vec{v}]_B = B^{-1}\vec{v}$$

B-coordinate to standard coordinates:

$$\vec{v} = B\cdot[\vec{v}]_B$$

9.1 Coordinate Systems.

$\boxed{\text{Definition 9.1}}$

Let $\beta = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ be an ordered basis for vector V.

Recall: Every $\vec{v} = x_1\vec{b_1} + \dots + x_n\vec{b_n}$

new basis vectors

The β-COORDINATES of $\vec{x}$ is

$$[\vec{x}]_\beta := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Theorem 9.2.

Let V be a vector subspace of $\mathbb{R}^n$ and B a basis for V.

Then every vector V has a unique representation in terms basis B

i.e. If $B = \{b_1, b_2, \dots, b_n\}$, then every $\vec{v} \in V$, there are unique real number $x_1, \dots, x_m$ so that

$$\vec{v} = x_1\vec{b_1} + x_2\vec{b_2} + \dots + x_m\vec{b_m}.$$

$$(\vec{b_1} \ \dots \ \vec{b_m} \mid \vec{v}) \text{ must be consistent}$$

∴ There is only exactly one solution to

$$\vec{v_1} = x_1\vec{b_1} + \dots + x_m\vec{b_m}$$

9.2 Change of Basis Matrices

Definition 9.4

- Let C and B be bases for a vector space V.
- Change of BASIS MATRIX $M_{C\leftarrow B}$ is matrix satisfying

$$M_{C\leftarrow B}[\vec{x}]_B = [\vec{x}]_C$$

Theorem 9.5

- $B = \{\vec{b_1}, \dots, \vec{b_n}\}$ is a basis for $\mathbb{R}^n$.
- For any $\vec{x} \in \mathbb{R}^n$,

$$M_{\varepsilon\leftarrow B} = \left([\vec{b_1}]_\varepsilon \ \dots \ [\vec{b_n}]_\varepsilon \right)$$

※ $M_{\varepsilon\leftarrow B}$ is invertible and we have $M^{-1}_{\varepsilon\leftarrow B} = M_{B\leftarrow\varepsilon}$

Lemma 9.6. Let C be a basis for a vector space V.

Then for any $\vec{x}, \vec{y} \in V$ and scalar $k \in \mathbb{R}$,

$$[\vec{x}+\vec{y}]_C = [\vec{x}]_C + [\vec{y}]_C \quad \text{and} \quad [k\vec{x}]_C = k[\vec{x}]_C.$$

Lemma 9.7 Let V be a vector space with basis $B = \{\vec{b_1}, \dots, \vec{b_n}\}$

Activity 9.5.

Let $B = \{b_1, b_2, b_3\}$ be the ordered basis for $\mathbb{R}^3$

$$b_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad b_2 = \begin{pmatrix}2\\1\\-1\end{pmatrix} \quad b_3 = \begin{pmatrix}0\\1\\3\end{pmatrix}$$

Find B-coordinates of $\vec{v} = \begin{pmatrix}1\\3\\4\end{pmatrix}$.

$$[\vec{v}]_B = \begin{pmatrix}1 & 2 & 0\\ 1 & 1 & 1\\ 0 & -1 & 3\end{pmatrix}^{-1}\begin{pmatrix}1\\3\\4\end{pmatrix}$$

※ Practice!

※ $[x]_B = \begin{pmatrix}a\\b\end{pmatrix}$

$\vec{x} = ab_1 + b\vec{b_2}$

Given: B-Basis to Standard.

- Basis: $b_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $b_2 = \begin{pmatrix}3\\-1\end{pmatrix}$ → $\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix}$ ≡ $M_{E\leftarrow B}$.
- $[x]_B = \begin{pmatrix}2\\-1\end{pmatrix}$. ∴ $\vec{x} = x_1 b_1 + x_2 b_2$.

$M_{E\leftarrow B}\cdot[x]_B = \vec{x}$

2個 b_1 -1個 b_2

$$\text{Vector } [v]_E = (2)\cdot\begin{pmatrix}1\\2\end{pmatrix} + (-1)\begin{pmatrix}3\\-1\end{pmatrix}$$

$$= \begin{bmatrix}2\\4\end{bmatrix} + \begin{bmatrix}-3\\1\end{bmatrix}$$

$$= \begin{pmatrix}-1\\5\end{pmatrix}$$

Standard to B-Basis

Given Basis: $b_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $b_2 = \begin{pmatrix}3\\-1\end{pmatrix}$

$\vec{x} = \begin{pmatrix}-1\\5\end{pmatrix}$

- Solve

$$\vec{x} = a\cdot\vec{b_1} + b\cdot\vec{b_2}$$

$$\begin{pmatrix}-1\\5\end{pmatrix} = a\begin{pmatrix}1\\2\end{pmatrix} + b\begin{pmatrix}3\\-1\end{pmatrix}$$

$$\begin{pmatrix}-1\\5\end{pmatrix} = \begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 1 & 3 & -1\\ 2 & -1 & 5\end{array}\right)$$

For any basis C of V, $\{[\vec{b_1}]_C, \dots, [\vec{b_n}]_C\}$ is linearly independent.

✰ Example Question 1. (WW9 : P4)

$B = \left(\begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)$

$C = \left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right)$

✰

Find a matrix P s.t. $[\vec{x}]_C = P[\vec{x}]_B$ for all $x \in \mathbb{R}^2$.

✰ Given: $P = M_{C \leftarrow B}$.

i.e. Want to find Change of Basis from B to C's matrix.

Step 1: From B! Start from basis of B!

$b_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = a \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} + b \cdot \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

$\therefore \begin{cases} -2a + 3b = 3 \\ a - 2b = -1 \end{cases} \rightarrow [b_1]_C = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

$b_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ -

$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = a \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{cases} -2a + 3b = 2 \\ a + 2b = 3 \end{cases} \rightarrow [b_2]_C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

WW9 : Problem 5.

(a) Find $x = \begin{bmatrix} -1 \\ 4 \\ -5 \end{bmatrix}$ in $E = \left\{ \underset{b_1}{\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}}, \underset{b_2}{\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}, \underset{b_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$

✰ $\begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix} = a \cdot b_1 + b \cdot b_2 + c \cdot b_3$

$\begin{bmatrix} -1 \\ 4 \\ -5 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 7 & -2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 14 \end{pmatrix}$

$[\vec{x}]_E = \begin{pmatrix} -1 \\ 6 \\ 14 \end{pmatrix}$

(b), Let $F_1 = \left\{ \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

$F_2 = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$

Find transition matrix $P_{F_2 \leftarrow F_1}$ s.t. $[x]_{F_2} = P_{F_2 \leftarrow F_1}[x]_{F_1}$ for $x \in \mathbb{R}^2$.

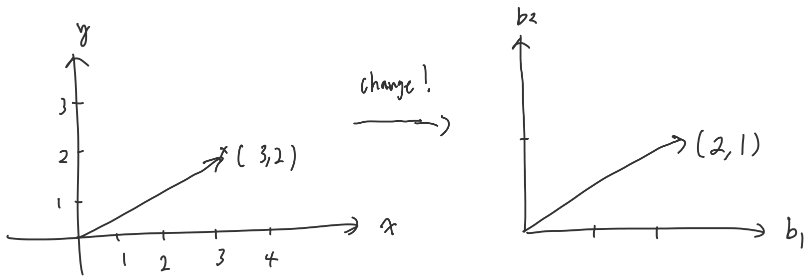
✰ i.e. Want to find Change of Basis matrix from F_1 to F_2.

Start from basis of F_1!

$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = a \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 3 \\ -2 \end{bmatrix} \rightarrow \begin{matrix} -2a + 3b = 4 \\ a - 2b = 2 \end{matrix} \quad \begin{pmatrix} -14 \\ -8 \end{pmatrix}$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = a \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \begin{matrix} -2a + 3b = 1 \\ a - 2b = 3 \end{matrix} \quad \begin{pmatrix} -11 \\ -7 \end{pmatrix}$

✰ Change of Coordinate System and Basis!



✰ $(b_1, b_2) = (1.5, 2)$

Exercise.

P.9.1 True or False:

If $B = \{\vec{b_1}, \dots, \vec{b_m}\}$ is a generating set for a vector space V, then every vector $\vec{v}$ in V has a unique representation in terms of B.

- False.

Let $V = \mathbb{R}^2$.

Let $B = \{ \vec{b_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{b_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \vec{b_3} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \}$

B is the generating set of V as

$\mathrm{Span}(b_1, b_2, b_3) = \mathrm{Span}(b_1, b_2) = \mathbb{R}^2$.

Take $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

$$\left.\begin{aligned} \vec{v} &= 1 \cdot b_1 + 2 \cdot b_2 + 0 \cdot b_3 \\ \vec{v} &= 0 \cdot b_1 + 1 \cdot b_2 + 1 \cdot b_3. \end{aligned}\right\} \text{2 different representation}$$

Let $B = \{b_1, b_2, b_3\}$

$v = b_1 + 2b_2$

$v = b_2 + b_3$.

9.2. Prove

Let C be a basis for a vector space V.

Then for any $\vec{x}, \vec{y} \in V$ and scalar $k \in \mathbb{R}$,

1. $[\vec{x}+\vec{y}]_C = [\vec{x}]_C + [\vec{y}]_C$

and

2. $[k\vec{x}]_C = k[\vec{x}]_C$

☆ Let $x = a_1\vec{c_1} + \dots + a_n\vec{c_n}$

$y = b_1\vec{c_1} + \dots + b_n\vec{c_n}$

↳ coefficient of $\vec{c}$ vectors are coordinates in basis C.

Find $\vec{x}+\vec{y}$, $[\vec{x}+\vec{y}]_C$ ... and $k\vec{x}$, $[k\vec{x}]_C$ ...

Proof. 1. Let

$\vec{x} = a_1\vec{c_1} + a_2\vec{c_2} + \dots + a_n\vec{c_n}$. $\Rightarrow [\vec{x}]_C = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$\vec{y} = b_1\vec{c_1} + b_2\vec{c_2} + \dots + b_2\vec{c_2}$. $\Rightarrow [\vec{y}]_C = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$\therefore \vec{x}+\vec{y} = (a_1+b_1)\vec{c_1} + \dots + (a_n+b_n)\vec{c_n}$

$\therefore [\vec{x}+\vec{y}]_C = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [\vec{x}]_C + [\vec{y}]_C$

2. Let $k \in \mathbb{R}$.

$k\vec{x} = ka_1\vec{c_1} + ka_2\vec{c_2} + \dots + ka_n\vec{c_n}$

$[k\vec{x}]_C = \begin{bmatrix} ka_1 \\ \vdots \\ ka_n \end{bmatrix} = k\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = k[\vec{x}]_C$

9.1. False.

Let $B = \{ \vec{b_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{b_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \vec{b_3} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \}$

$\mathrm{Span}\, B = \mathrm{span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \mathrm{span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \mathbb{R}^2$

Let $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in V$.

$\vec{v} = 1 \cdot b_1 + 2b_2$

$\vec{v} = 1 \cdot b_1 + 1 \cdot b_3$

$\therefore$ 2 representation $\therefore$ False

9.2 Let $\vec{x} = a_1\vec{c_1} + \dots + a_n\vec{c_n}$

$\vec{y} = b_1\vec{c_1} + \dots + b_n\vec{c_n}$

$k \in \mathbb{R}$.

$\vec{x}+\vec{y} = (a_1+b_1)\vec{c_1} + \dots + (a_n+b_n)\vec{c_n}$

$[\vec{x}+\vec{y}]_C = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [\vec{x}]_C + [\vec{y}]_C$

$k\vec{x} = ka_1\vec{c_1} + \dots + ka_n\vec{c_n}$

$[k\vec{x}]_C = \begin{bmatrix} ka_1 \\ \vdots \\ ka_n \end{bmatrix} = k\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = k[\vec{x}]_C$

9.3

9.3 Prove.

Let V be a vector space with basis $B = \{\vec{b_1}, \dots, \vec{b_n}\}$

Then, for any basis C of V, $\{[\vec{b_1}]_C, \dots, [\vec{b_n}]_C\}$ is linearly independent.

WTS: If $a_1[\vec{b_1}]_C + \dots + a_n[\vec{b_n}]_C = 0$, $a_1 = \dots = a_n = 0$.

Proof. Since $[\vec{x}+\vec{y}]_C = [\vec{x}]_C + [\vec{y}]_C$, $[k\vec{x}]_C = k[\vec{x}]_C$.

WTS: If $a_1[\vec{b_1}]_C + a_2[\vec{b_2}]_C + \dots + a_n[\vec{b_n}]_C = 0$

Then $a_1 = a_2 = \dots = a_n = 0$.

Proof: make $a_1[\vec{b_1}]_C + \dots + a_n[\vec{b_n}]_C = 0$ to $[a_1\vec{b_1} + \dots + a_n\vec{b_n}]_C = 0$ (linearity)

Since coordinate mapping is bijective,

$[\vec{v}]_C = \vec{V}_C = 0$.

WTS: If $a_1[b_1]_C + a_2[b_2]_C + \dots + a_n[b_n]_C = 0$

$a_1 = a_2 = \dots = a_n = 0$.

$\Rightarrow$ linearity: $= [a_1\vec{b_1} + a_2\vec{b_2} + \dots + a_n\vec{b_n}] = 0$

$[\vec{v}]_C = 0 \Rightarrow \vec{v} = 0$.

$a_1\vec{b_1} + a_2\vec{b_2} + \dots + a_n\vec{b_n} = 0$

$\therefore \quad [a_1\vec{b_1} + a_2\vec{b_2} + \cdots + a_n\vec{b_n}]_C = 0$

Since the coordinate map is bijective,

$\therefore \quad [\vec{v}]_C = 0 \Rightarrow \vec{v} = 0.$

$\therefore [a_1\vec{b_1} + \cdots + a_n\vec{b_n}]_C = 0$

$a_1\vec{b_1} + \cdots + a_n\vec{b_n} = 0$ (bijective map)

$\in 0$

Since $\{b_1, \ldots, b_n\}$ is basis, it is linearly independent,

$\therefore a_1 = \cdots = a_n = 0$

$\therefore [\vec{b_1}]_C \ldots [\vec{b_n}]_C$ are linearly independent.

$\therefore \quad a_1\vec{b_1} + \cdots + a_n\vec{b_n} = 0$

Since $b_1, \ldots, b_n$ are basis $\rightarrow$ linearly independent

$\therefore \quad a_1 = \cdots = a_n = 0$

$a_1 = a_2 = \cdots = a_n = 0$

9.4. Let V be a vector subspace of $\mathbb{R}^5$ of dimension 3,

Let B and C be bases for V.

What is the size of the change of basis matrix $M_{C\leftarrow B}$?

$(3\times 1) \quad (3\times 3) \quad (3\times 1)$

- Note : $[\vec{v}]_C = M_{C\leftarrow B}[\vec{v}]_B$
- $M_{C\leftarrow B}$ is a matrix with each column representing basis vectors from B using basis C.
- Since both B and C contains only vectors of 3 dimensional, and B has 3 basis vectors,

$\therefore \quad M_{C\leftarrow B}$ is 3×3.

$M_{C\leftarrow B} = [\,[b_1]_C \quad [b_2]_C \quad \cdots \quad [\vec{b_n}]_C\,] \in \mathbb{R}^{n\times n}.$

$\star \quad M_{C\leftarrow B} = [\,[\vec{b_1}]_C \quad [\vec{b_2}]_C \quad [\vec{b_3}]_C\,] \in \mathbb{R}^{n\times n}$

Both B and C are 3 dimension, basis vectors are 3×1.

$M_{C\leftarrow B}$ is the basis vectors of B in representation of C

$[\vec{v}]_C = M_{C\leftarrow B}[\vec{v}]_B$

$3\times 1 \qquad ? \times \boxed{3} \quad 3\times 1$

$[\,[b_1]_C$

9.5. Let B, C and D be bases for vector space V.

- Show $M_{C\leftarrow B}M_{B\leftarrow D} = M_{C\leftarrow D}$.

Let $\vec{v} \in V$.

- $[\vec{v}]_B = M_{B\leftarrow D}[\vec{v}]_D$
- $[\vec{v}]_C = M_{C\leftarrow B}[\vec{v}]_B$

$= M_{C\leftarrow B}(M_{B\leftarrow D}[\vec{v}]_D)$

$= (M_{C\leftarrow B}M_{B\leftarrow D})[\vec{v}]_D$

However, $[\vec{v}]_C = M_{C\leftarrow D}[\vec{v}]_D$ by definition

$\therefore \quad M_{C\leftarrow B} \cdot M_{B\leftarrow D} = M_{C\leftarrow D}$