Ch11

Sunday, 30 March 2025 7:13 PM

11.1 Dot Product

- Property:
  1. $u \cdot v = v \cdot u$
  2. $(u+v) \cdot w = u \cdot w + v \cdot w$
  3. $(cu) \cdot v = c(u \cdot v)$

- Norm = $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$ = distance
- Distance = $\|\vec{u} - \vec{v}\|$

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right)$$

- $\vec{u} \cdot \vec{v} = 0$ (orthogonal)

11.2 Orthonormal Bases and Orthogonal Matrices

Orthogonal:

- $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ every dot product $= 0$ with another

Orthonormal:

- orthogonal
- $\|v_i\| = 1$ for all vectors
- $[\vec{x}]_B \cdot [\vec{y}]_B = \vec{x} \cdot \vec{y}$
  $\Rightarrow \|\vec{x}\| = \|[\vec{x}]_B\|$ } if B is orthonormal

- $Q^{-1} = Q^T \Leftrightarrow B$ is orthonormal for $B = \{v_1, \dots, v_n\}$ $Q = (v_1 \dots v_n)$
  $(Q^T Q = I)$

- Matrix Q Orthogonal if $Q^{-1} = Q^T$, if its column vectors form orthonormal basis
  (don't know why not called orthonormal)

P. 11.5. True or False:

For any vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$,
we have $\|\vec{x} + \vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$.

False!

Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then $x + y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\|\vec{x}\| = 1$.

$\|\vec{y}\| = 1$.

$\|\vec{x} + \vec{y}\| = \sqrt{1^2 + 1^2} = \sqrt{2} \neq 2$.

P. 11.8. Show that if $\beta = \{\vec{v_1}, \dots, \vec{v_n}\}$ is a set of nonzero orthogonal vectors in $\mathbb{R}^n$ then β forms a basis for $\mathbb{R}^n$.

Assume $c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_n\vec{v_n} = 0$

Assume $v_i \cdot v_k = 0$ for $i \neq k$, $i, k \in \{1, 2, \dots, n\}$

$(c_1\vec{v_1} + \dots + c_n\vec{v_n}) \cdot \vec{v_k} = \vec{0} \cdot \vec{v_k}$

$c_1(\vec{v_1} \cdot \vec{v_k}) + \dots + c_k(\vec{v_k} \cdot \vec{v_k}) + \dots + c_n(\vec{v_n} \cdot \vec{v_k}) = 0$

$c_k \|\vec{v_k}\| = 0$

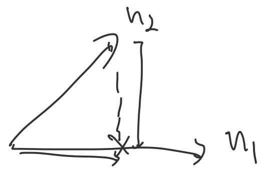
Since $\vec{v_k}$ is nonzero vector, $c_k = 0$.

$\therefore$ Since $k \in \{1, 2, \dots, n\}$, $c_1 = c_2 = \dots = c_n = 0$.

$\Rightarrow$ The set is linearly independent.

$\Rightarrow$ The set forms a basis.

- $Q\vec{v} \cdot Q\vec{w} = v \cdot w$.
- $\|Q\vec{v}\| = \|\vec{v}\|$
- $\theta = \arccos\left(\frac{u \cdot v}{\|u\|\|v\|}\right) = \arccos\left(\frac{Qv \cdot Qu}{\|Qv\|\|Qu\|}\right)$



11.3 The Gram-Schmidt Process

- Orthogonal Projection of $\vec{x}$ onto $\vec{y}$

$$\text{proj}_{\vec{y}}\vec{x} = \frac{x \cdot y}{y \cdot y} y$$

- The Gram-Schmidt Process (Find the orthonormal basis from basis)

If V is a vector subspace of $\mathbb{R}^n$ with basis $\{\vec{v_1}, \dots, \vec{v_n}\}$

and let $\vec{u_1} = \vec{v_1}$

$$\vec{u_2} = \vec{v_2} - \text{proj}_{\vec{u_1}}\vec{v_2}$$

$$\vec{u_3} = \vec{v_3} - \text{proj}_{\vec{u_1}}\vec{v_3} - \text{proj}_{\vec{u_2}}\vec{v_3}$$

$$\vdots$$

$$\vec{u_m} = \vec{v_m} - \text{proj}_{\vec{u_1}}\vec{v_m} - \text{proj}_{\vec{u_2}}\vec{v_m} - \dots - \text{proj}_{\vec{u_{m-1}}}\vec{v_m}$$

Then $\{\vec{u_1}, \dots, \vec{u_m}\}$ is an orthogonal basis for V.

11.4. The Spectral Theorem

- Orthogonally diagonalizable A if
  - $Q^T A Q = D$ (if exist orthogonal matrix Q, diagonal matrix D)

    $A = QDQ^T$
  - orthogonally diagonalizable $\iff$ Any symmetric ($A = A^T$) $\to A = QDQ^T$
    - 1. A has at least one real eigenvalue
    - 2. If λ, μ are distinct eigenvalue of A, any $\vec{x} \in E_\lambda$, $\vec{y} \in E_\mu$, $\vec{x}$ and $\vec{y}$ are orthogonal.

11.5 The Singular Value Decomposition

- Any linear transformation
  - ⇒ Decomposed into a 3 transformation
    1. Rotation / Reflection
    2. Dilation
    3. Rotation / Reflection

- A : $m \times n$ matrix
  - exist an orthonormal basis $\mathbb{R}^n$ of eigenvectors of A^TA $\{v_1, \dots, v_n\}$ (nxn matrix; unit vectors)
  - $\{A\vec{v_1}, \dots, A\vec{v_n}\}$ is an orthogonal subset of $\mathbb{R}^m$.
  - $\{A\vec{v_1}, \dots, A\vec{v_r}\}$ forms orthogonal basis for $Col(A)$. (for $A\vec{v}_{r+1} = \dots = A\vec{v_n} = 0$)

- 1. Find eigenvector of $A^TA \Rightarrow \{v_1, \dots, v_n\}$
  2. $\{Av_1, \dots, Av_r\}$ is orthogonal basis for $Col(A)$

- Singular Value Decomposition
  - $A = U \Sigma V^T$
    - U: $m \times m$ orthogonal
    - V: $n \times n$ orthogonal

tells you how much it stretches the vector on certain direction

Singular values

, $\sigma_i = \|A\vec{v_i}\|$ unit eigenvector of A^TA

$= \sqrt{\lambda_i}$ ← eigenvalues of A^TA

$(\lambda_i > 0)$