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CSCI203

# Algorithms and Data Structures

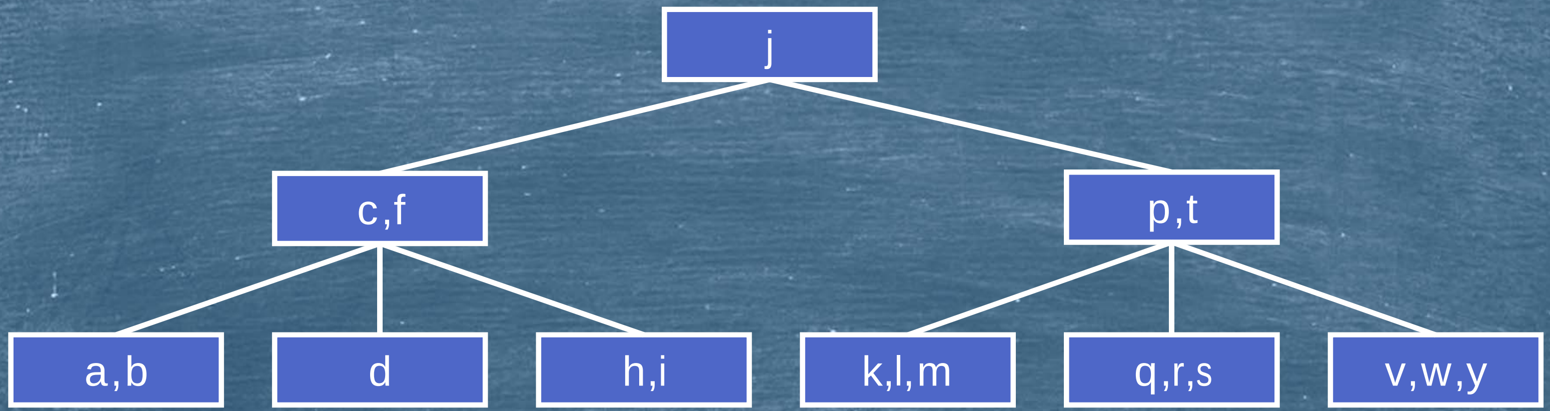
Week 5 – Lecture A

# Beyond Binary Trees

- Another balanced tree, the 2-4 tree maintains its balance in a different way.
- It has the following properties:
  1. Every internal node (except possibly the root) has between two and four children.
  2. The keys within each node are ordered from smallest to largest.
  3. Internal nodes have one less key than they have children. Each such key is conceptually positioned between two consecutive children. Its value is larger than the largest key in the subtree to its left and smaller than the smallest key in the subtree to its right.
  4. All leaves of the tree are at the same depth.

# An Example2-4 tree

- The following is an example of a 2-4 tree.




- Note: only the keys are shown here. Each node also contains associated data.
- Note: a *k*-node contains *k*-1 keys and, if it is not a leaf, has *k* children.

# Searching a 2-4 tree

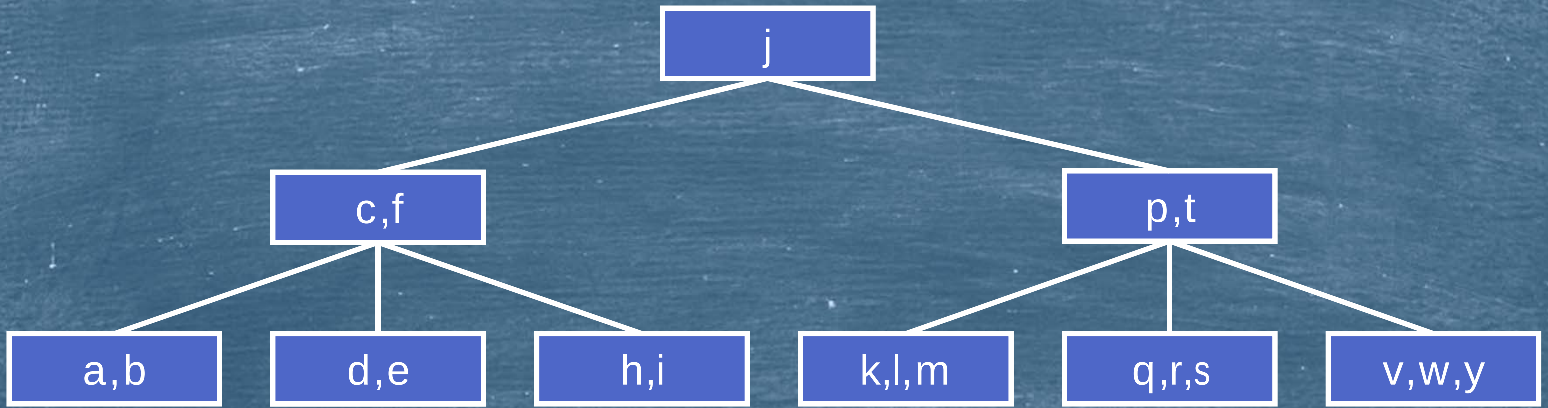
- This process is similar to searching a binary tree
- If searching for value, compare value with each key
  - If value == key return the associated data
  - If value < key recursively search the subtree left of key
  - If value > key, repeat the process using the next key, if there is no next key, search the last subtree.

# Insertion into a 2-4 tree

- Find the leaf where the item is to be inserted
- Insert the item
- Update the node
  1. Insertion into a 2-node $\Rightarrow$ 3-node
  2. Insertion into a 3-node $\Rightarrow$ 4-node
  3. Insertion into a 4-node $\Rightarrow$ 5-node
     - If an immediately adjacent sibling is not full, send a key from the parent down to this sibling, and a key up from the 5-node to the parent.
     - If all such siblings are full, split the node into two by passing the median key up to the parent, and relink subtrees if needed.
       - Repeat with the parent if necessary.
       - Create a new root layer if necessary.

# An Example of Insertion

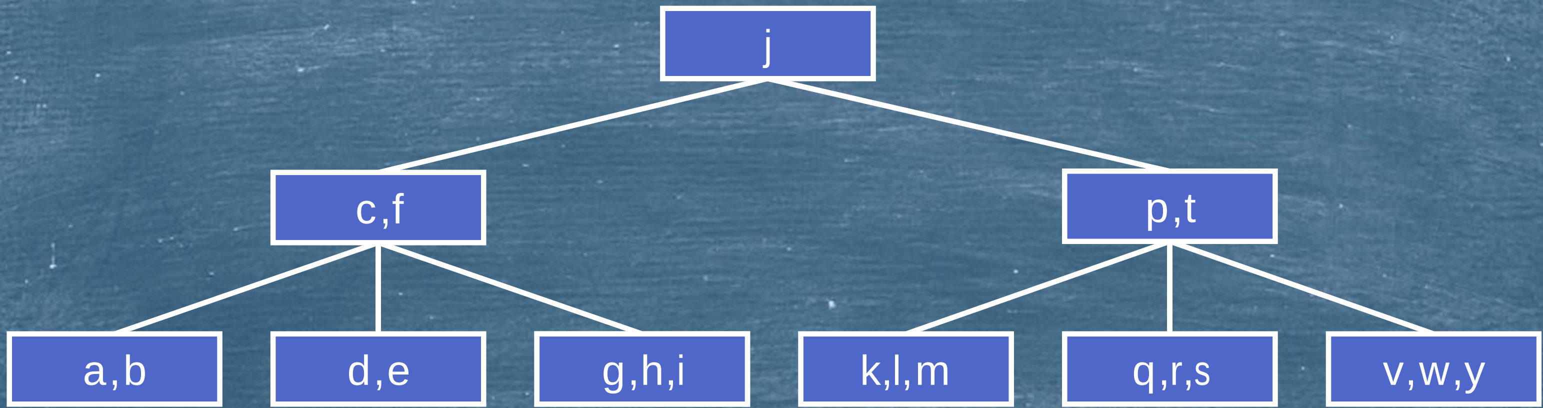
- Insert ‘e’ into the following tree.




- The 2-node becomes a 3-node.

# An Example of Insertion, continued

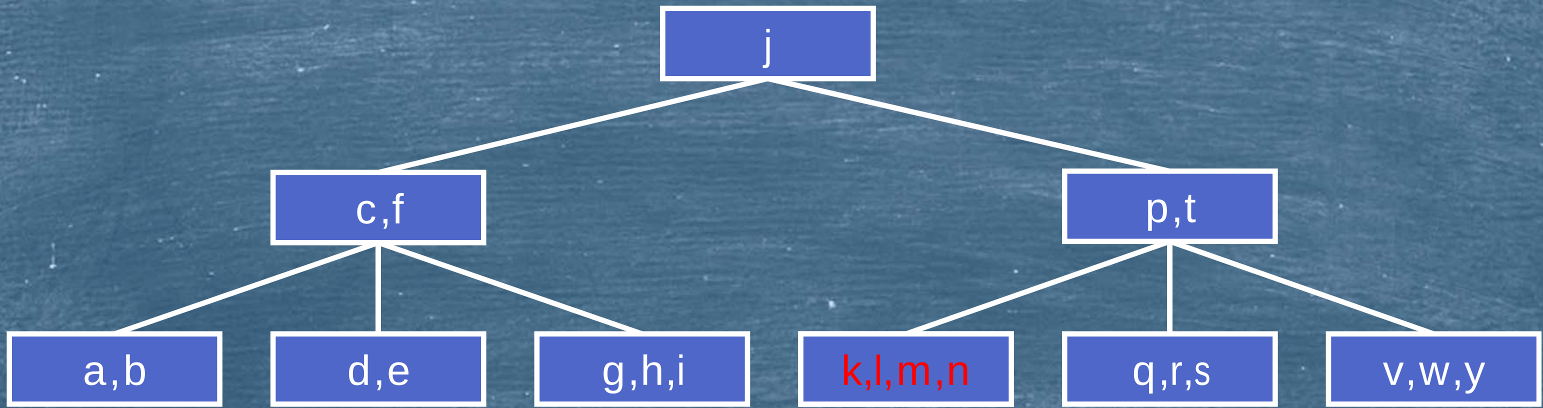
- Now, insert ‘g’.




- The 3-node becomes a 4-node.

# An Example of Insertion, continued

- Finally, insert ‘n’.




- The 4-node becomes a 5-node.
- This must be remedied.

# Fixing a 5-node

- Consider the 5 node on the last slide:

k,l,m,n

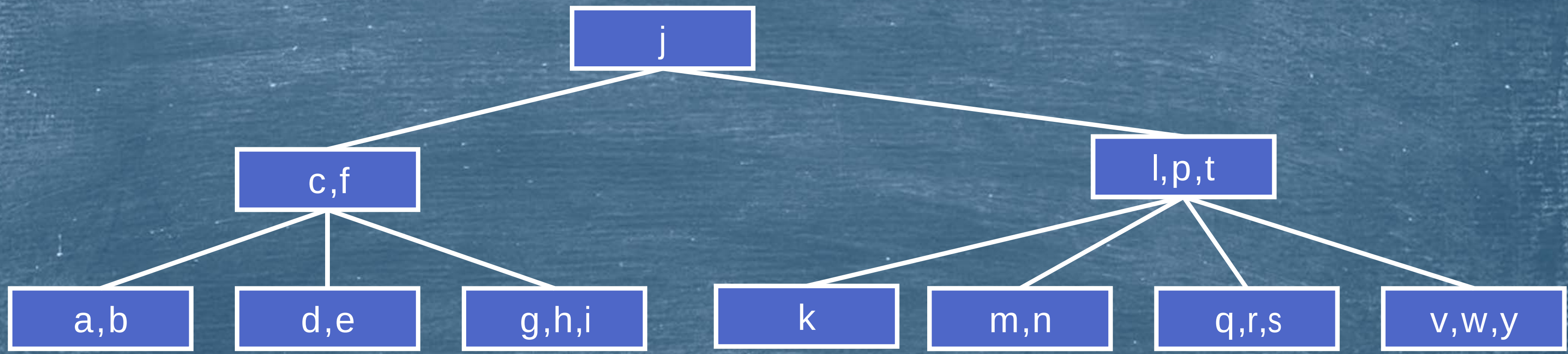
- To repair this, we must reduce the number of keys in the node.
- We could possibly do this by moving keys to its sibling. q,r,s
- The sibling is full so we split the node, instead.

k　　m,n

- The ‘l’ moves into the node above.

# Splitting a 5-node

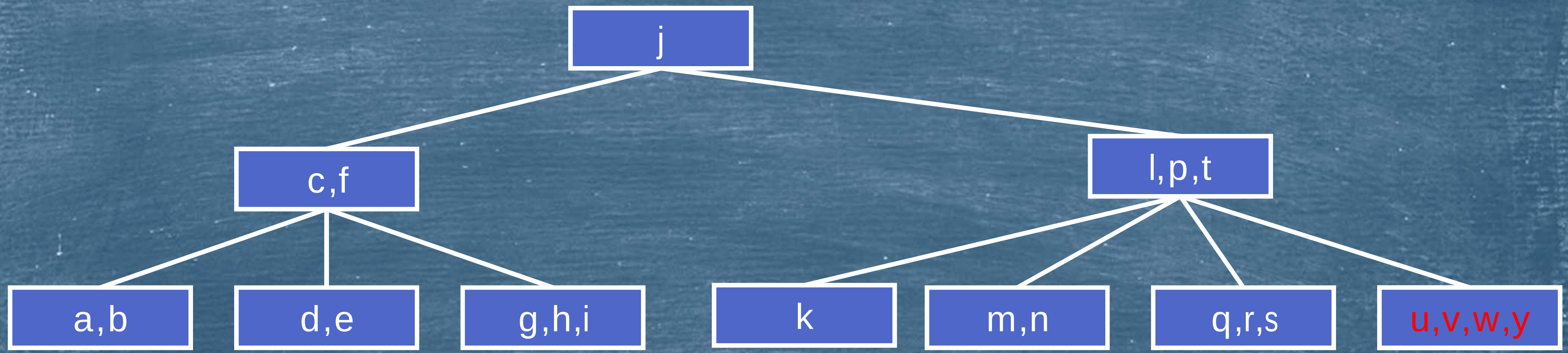
- So, the broken tree…




- …becomes this repaired tree.

# Another Insertion

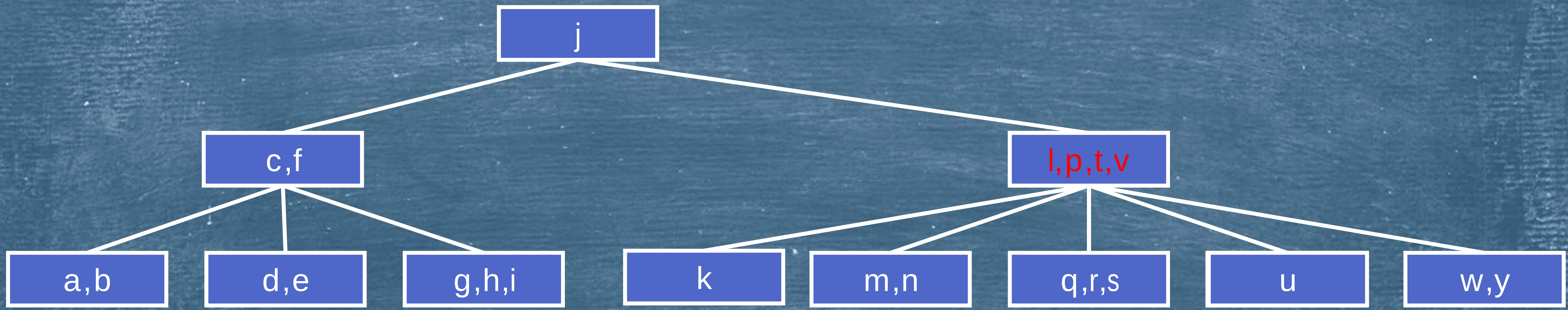
- Now insert ‘u’.




- We have another 5-node.
- Again, the sibling is full.

# Another Repair

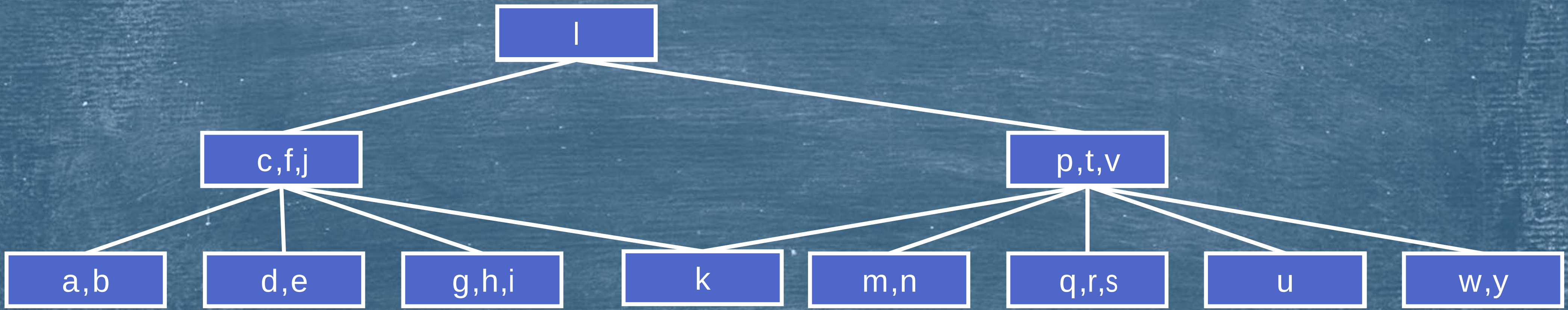
- So we split the node into two...




- ...but the parent is now a 5-node.

# Another Repair

- This time, the sibling has spare room…




- …send key from parent down to sibling and key from node up to parent.
- Note: the child node k moves across.

# The Worst-Case Scenario

- If the root node becomes a 5-node, split the root node and create a new root containing the median key of the old root.
- This increases the height of the tree by one.

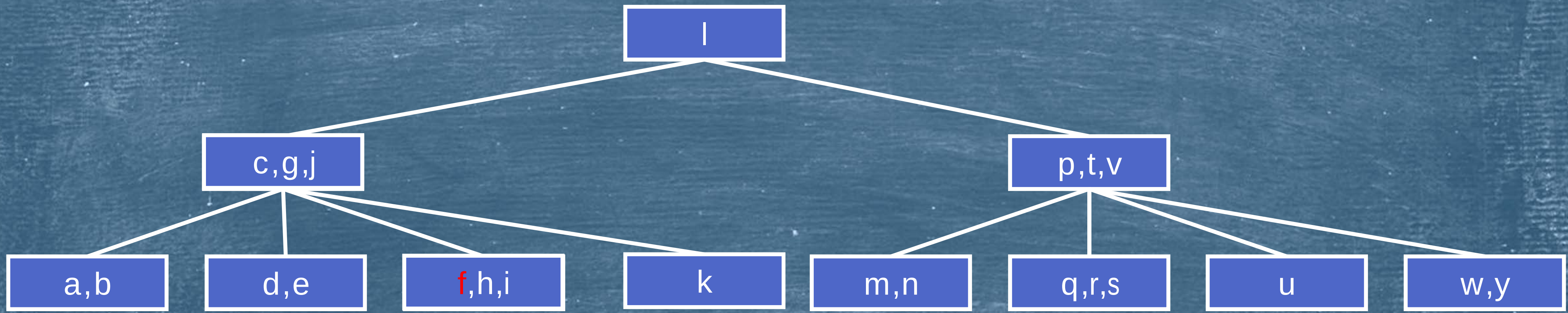
# Deletion

- Find where the item to be deleted is located.
- If this is an internal node, swap the item with its immediate in-order successor.
  - Repeat this until the item to be deleted is in a leaf node.
- Delete the item.
- Update the node.

- Update the node:
  - Deletion from a 4-node $\Rightarrow$ 3-node.
  - Deletion from a 3-node $\Rightarrow$ 2-node.
  - Deletion from a 2-node $\Rightarrow$ 1-node.
- If the 1-node has an immediate sibling with more than one key, send a key from the sibling up to the parent, and a key from the parent down to the node and relink if necessary
  - If not, remove the node and send the key down from the parent into a sibling node
  - This may cause the parent to become a 1-node
  - Fix this recursively
  - If the root becomes a 1-node, remove it.

# Deletion: Some Examples

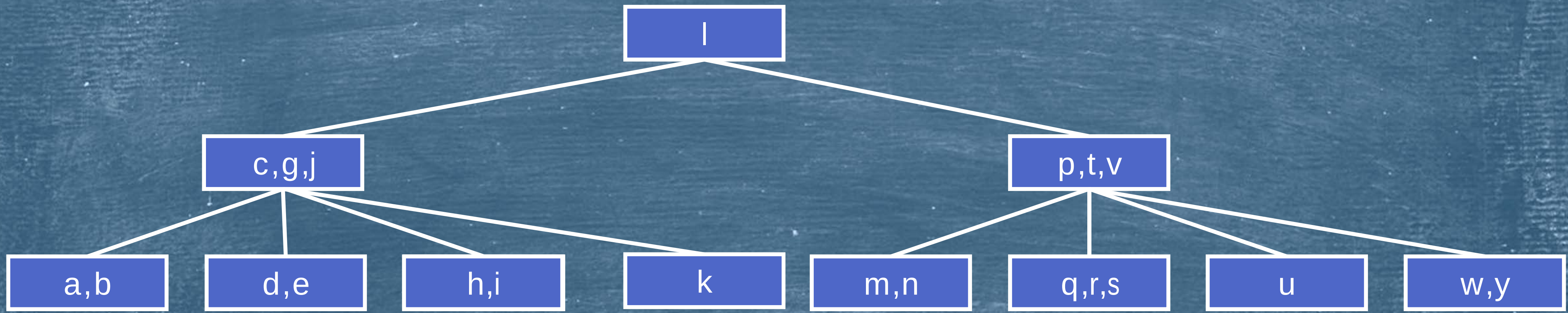
- Delete ‘f’ from the following tree.




- Swap the ‘f’ with its in-order successor, ‘g’.

# Deletion: Some Examples

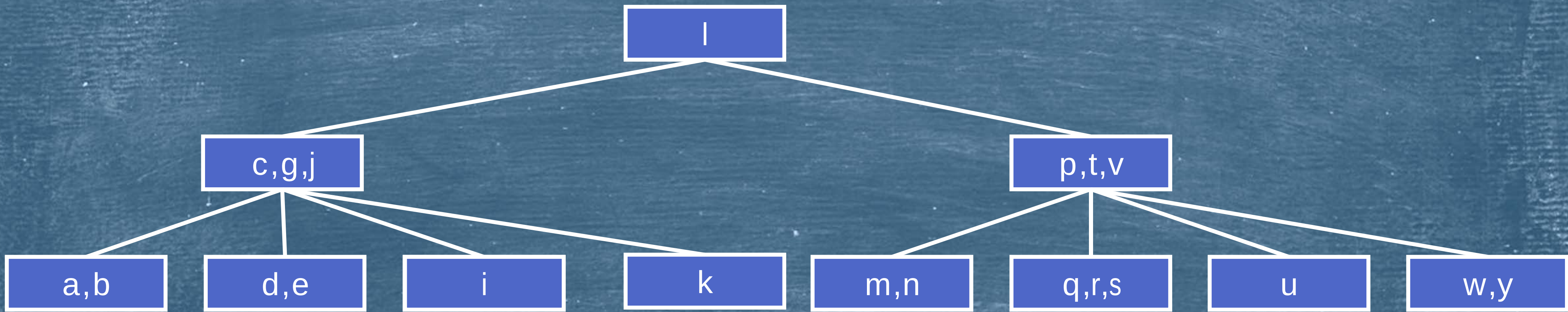
- The ‘f’ is now in a leaf node.




- So we delete it, leaving a 3-node.

# Deletion: Some Examples

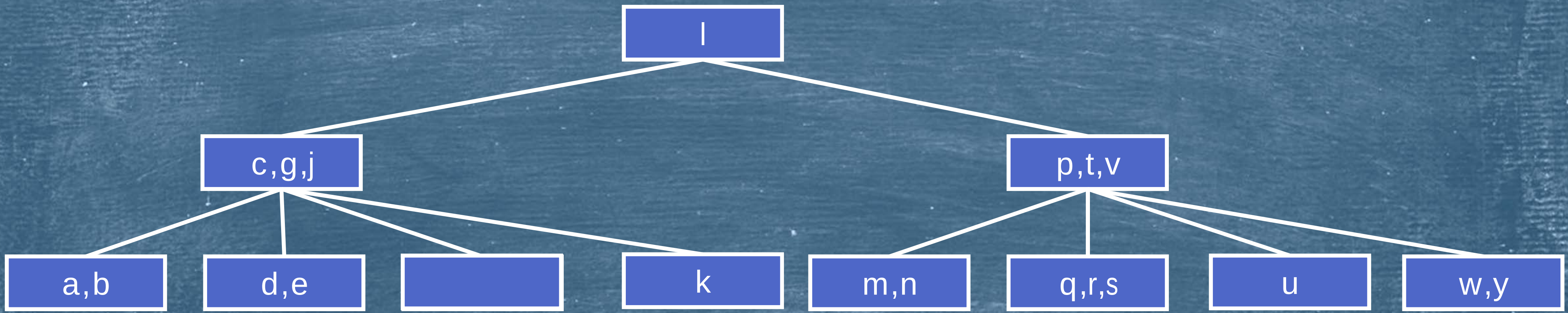
- Now delete the ‘h’ key.




- It is already in a leaf so we just delete it.
- The 3-node becomes a 2-node.

# Deletion: Some Examples

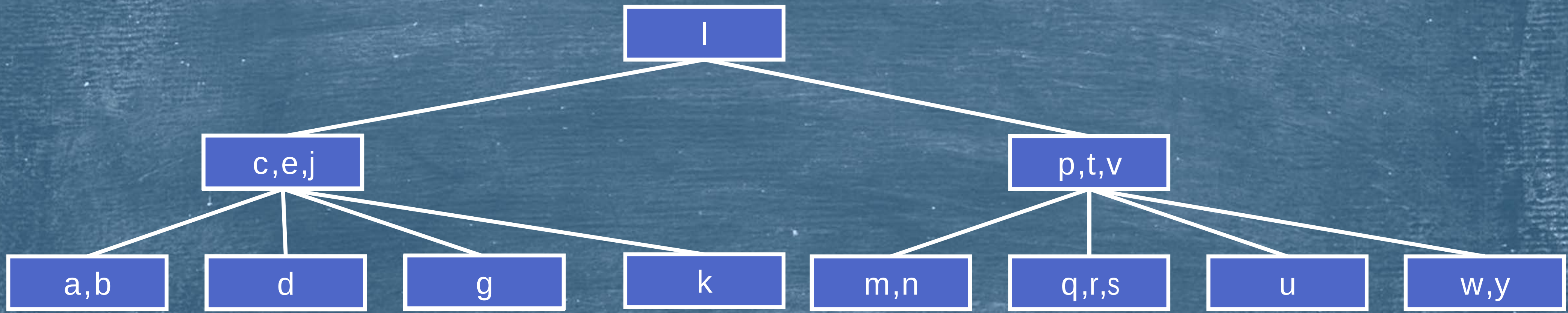
- Now delete the 'i' key.




- It is already in a leaf so we just delete it.
- The 2-node becomes a 1-node and we have a broken tree.

# Deletion: Some Examples

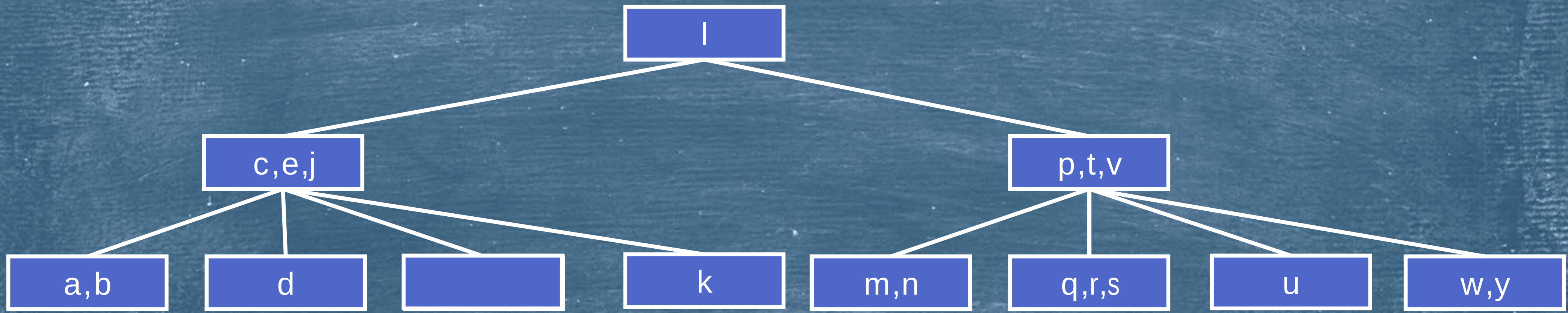
- The empty node has a sibling with more than one key.




- Send a key from the sibling up to the root and a key from the root down to the (empty) 1-node.

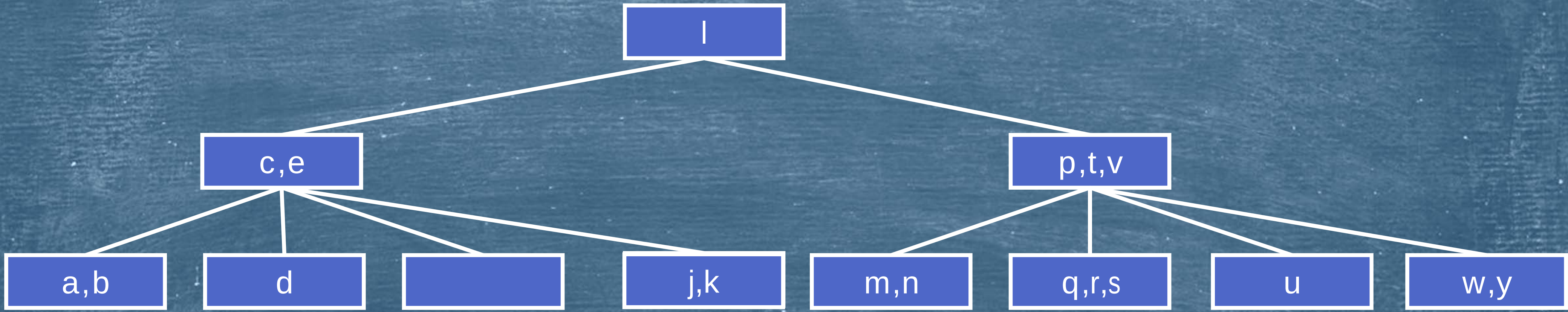
# Deletion: Some Examples

- Delete the ‘g’ key.




- This time no sibling has spare keys.

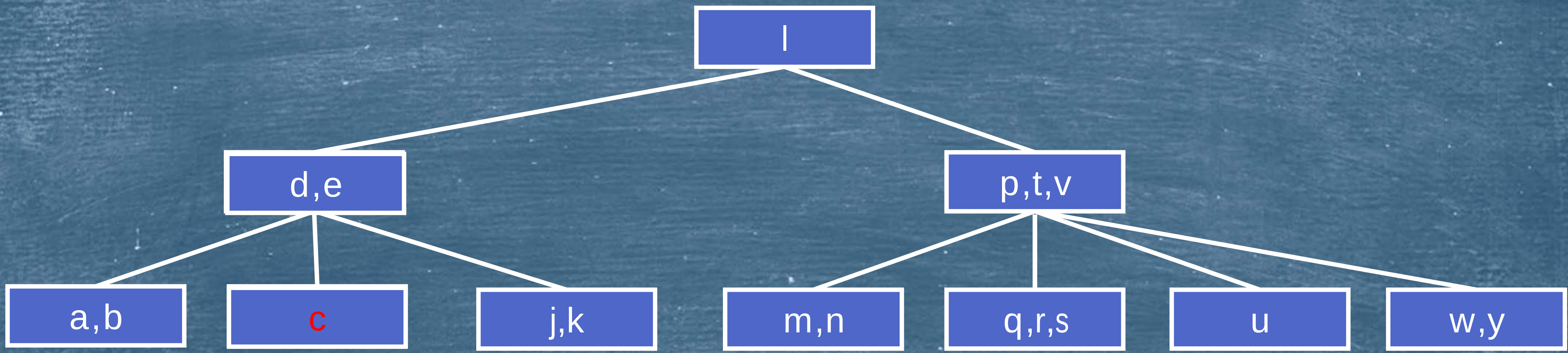
# Deletion: Some Examples




- Remove the node and send a key from the parent down to a sibling.
- And we are ok again.

# Deletion: Some Examples

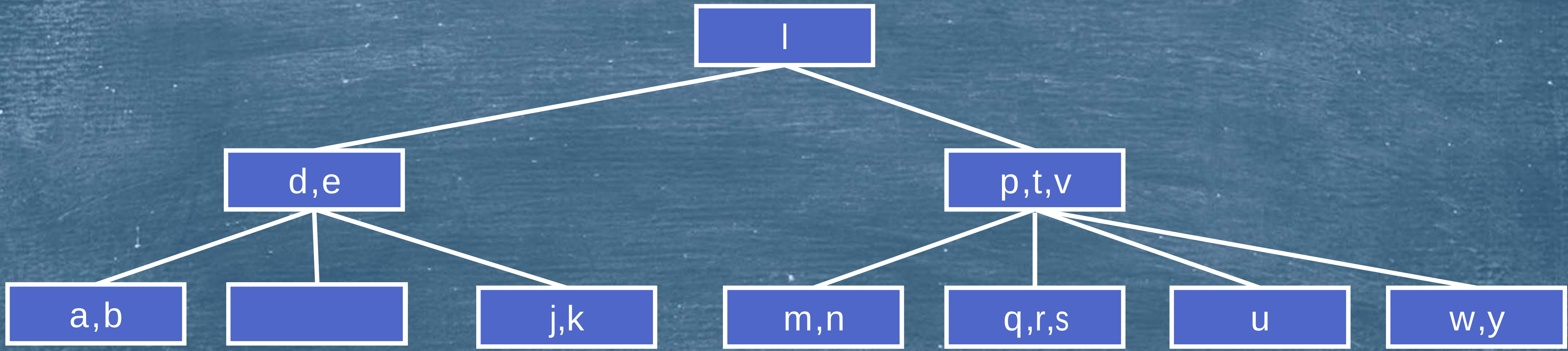
- Delete the ‘c’ key.




- It is internal, swap with ‘d’.

# Deletion: Some Examples

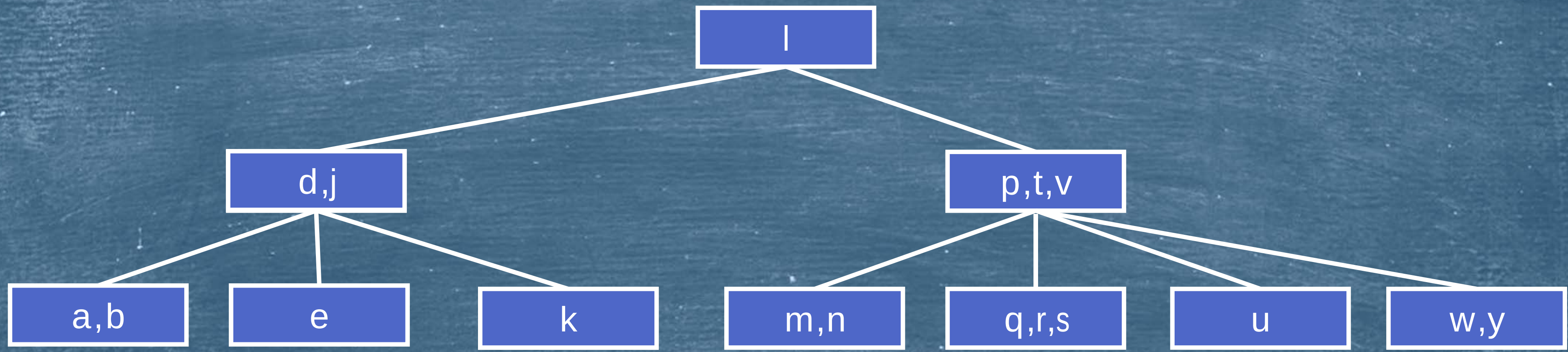
- Delete the ‘c’ key.




- We have a 1-node.
- A sibling has more than one key.

# Deletion: Some Examples

- A sibling has more than one key.




- Move a sibling key up and a parent key down.
- And we are done.

# 2-4 Tree Efficiency

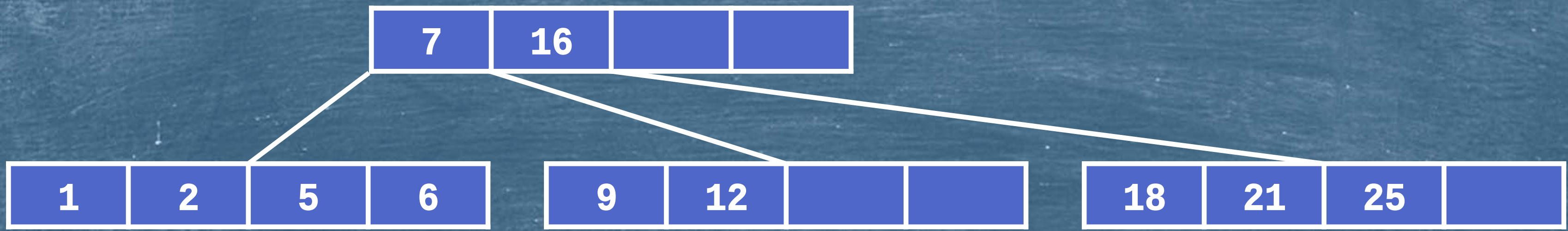
- Height of tree is O(log *n*)
- Searching:
  - Each node checked takes O(1)
  - Search takes O(log *n*)
- Insertion:
  - Split takes O(1) at each level
  - At most log *n* splits (1 per level)
  - Insertion takes O(log *n*)
- Deletion
  - Merge takes O(1)
  - At most log *n* merges
  - Deletion takes O(log *n*)

# B-Trees; Generalizing 2-4 Trees

- Created by Rudolph Bayer and Ed McGreight
- An *m*-ary search tree with the following additional properties
  - The root is either a leaf or has at least 2 children
  - All other internal nodes have at between $\lceil m/2 \rceil$ and $m$ subtrees.
  - All non-root nodes have between $\lceil (m/2) \rceil$-1 and $m$-*1* keys inclusive.
  - All leaves are at the same level

# An Example B-Tree

- This is a B-Tree with order 5 (m=5)
- Again, only the keys are shown.




- Nodes have between 2 and 4 keys.
- Internal nodes have between 3 and 5 children.

# Searching B-Trees

- Similar to searching a binary tree
- If searching for value, compare value with each key
  - If value == key return the associated data
  - If value < key recursively search the subtree left of key
  - If value > key, repeat the process using the next key, if there is no next key, search the last subtree.

# Inserting Into B-Trees

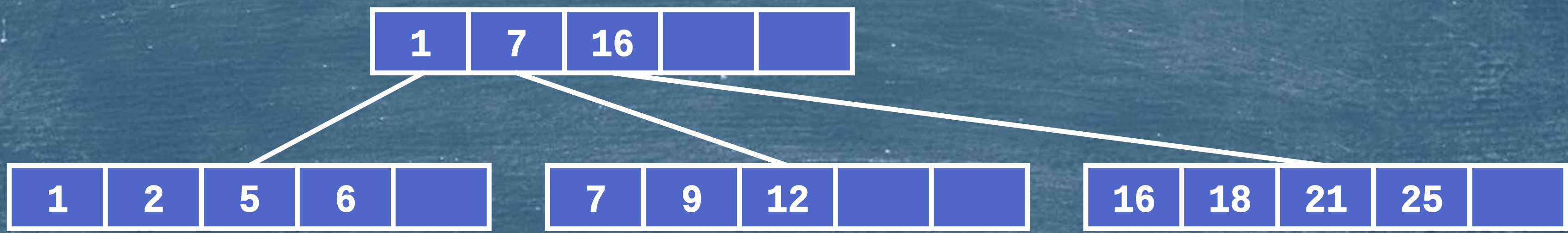
- This process is analogous to insertion into 2-4 trees.
  - Find the leaf where the item is to be inserted
  - Insert the item
  - If the node overflows (now has *m* keys)
    - Split the node into two by passing the median key up to the parent
      1. Repeat with the parent if necessary.
      2. Create a new root layer if necessary.

# Deletion From B-Trees

- This process is analogous to deletion from 2-4 Trees.
  - Find the key to be deleted.
  - Swap it down to a leaf.
  - Delete the key from the leaf.
  - If the node underflows (now has $(m/2)-1$ keys):
    - If a sibling has spare keys, borrow from the sibling and adjust the parent node.
    - If not, merge with a sibling and reduce the number of parent keys by one.
      - This may result in the parent being underfilled.
      - In this case repeat the procedure.

# Variations on B-Trees

- B+ Tree:
  - Data associated with keys are only stored in the leaf nodes.
  - Each internal node contains copies of the first key of each child .
  - Our previous example now looks like this:




  - Note: the nodes now contain between 3 and 5 keys.

# Variations on B-Trees

- B* Tree:
  - Nodes are kept at least 2/3 full
  - When a node fills, try to share its contents with a sibling.
    - Only split a node if siblings are full as well.
  - This keeps the nodes more nearly full, and the tree less deep.

# Real-Life B-Trees

- Although small ($m$=5) B-Trees are useful examples, in real applications $m$ can be 100 or even larger.
- If $m$=100 we can store 1,000,000 records in a tree that has only 3 levels!
- Thus, we need only examine at most 3 nodes to find a given record.
- Compare this with 20 nodes using a BST.

# B-Trees and Databases

- Databases frequently contain millions, if not billions, of records.
- Often, the records are stored on disk, so the access time is significant.
- B-Trees, because they allow access with a small number of steps (disk reads), are frequently used as the storage mechanism for such databases.

# B-Trees and Databases

- The Following table shows the number of records for each level of B-Tree.
- We assume *m*=100.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 100 | 10,000 | 1,000,000 | 100,000,000 | 10,000,000,000 |

- As you can see, we can search a database of 10 billion records with only 5 disk accesses!