

# 1 Introduction

## 2 Defining the Problem

Curve and surface modelling play a major role in Reverse Engineering during object reconstruction through CAD software [10] or for CFD simulations involving moving interfaces such as flame propagation problems [11]. Usually, one has access only to a discrete (finite) data set and a smooth curve (surface) is needed in order to perform operations such as numerical integration or differentiation. Although not unique, the solution to this problem is generally sought as a spline curve (surface) that either interpolates or provides an approximation which is *closest* to the data. The latter one is desirable for experimental measurements or computational data which may be slightly inaccurate or noisy whereas cases for which data (or part of it) is known to be exact, an interpolation approach [] is more suitable. Since the work presented here was originally developed for curve reconstruction during aircraft icing simulations [], we focus our attention on approximation techniques and the challenges associated to them

Defining a spline implies finding its control points (net) and knot sequence and an ideal *fit* would be a spline that minimizes the error (under certain metric) employing a minimum number of knots (control points). Unfortunately, this represents a great challenge since both the “optimal” number of control points and knot location are unknown and in general, it is not possible to derive explicit formulas [6]. Thus, the solution is found by casting a suitable optimization problem, usually based on the least-squares minimization [15, 2, 16]. The methodology proposed here attempts to solve the *minimal* spline curve from a heuristic approach, where knots are strategically introduced as part of an iterative scheme that drives the error towards a desired tolerance.

The simplest solution to the spline least-squares optimization problem is to fix the number of control points, prescribe a knot sequence alongside with a suitable curve parametrization and solve a linear system for the control points [15, Ch. 5.4], [3]. It has been shown that chord-length and centripetal based parameterizations improve the accuracy of the approximation [5, 18, 7, 9], especially for detection of sharp features. However, a fixed knot sequence may be too oversimplified and shape manipulation (knot insertion/removal) [13, 19, 4] can be introduced as part of an error controlled iterative scheme; one can continuously insert new knots and find a new least-squares solution, driving the curve towards a desired tolerance [14, 12] or alternatively, begin with a large number of control points and remove knots successively until the

quality of the approximation is affected [8? ]. Ultimately, the problem can be formulated as a non-linear optimization, allowing “free knots” as part of the least-squares solution [17, 1].

The algorithm presented here is based on a heuristic approach and combines a least squares solver together with an error controlled knot insertion. The scheme starts with the minimum number of control points and at each iteration, attempts to insert a new knot at the span with highest weighted error. The insertion step is error controlled and the knot is rejected if the new approximation does not improve according to a tolerance. When none of the new knots are capable to overcome tolerance step, the scheme proceeds with the *best* candidate, tags it and iterate until a valid step is found and it attempts to remove any possible *redundant* knots, starting from the tagged location. Although we cannot prove that this number is optimal, our results suggest that this knot distribution as well as an increasing knot sequence approach leads to satisfactory results. In the following section we discuss the building blocks of our algorithm: B-spline functions, least squares approximation and the “best knot” sequence problem and in section ?? we provide implementation details. Section ?? shows several experiments including approximation to noisy data and ice layer reconstruction. Finally, in section ?? we conclude and highlight the areas for improvement as well as future applications.

## 3 background

B-splines

Curve Reconstruction

Least Squares Solvers Linear versus Non Linear

## 4 The Algorithm

The work presented here provides a solution to data approximation in the least-squares sense through an iterative scheme that aims to attain a user specified accuracy. In order to avoid solving the non-linear “free knots” problem, we propose the following approach: start with few (or minimal) number of control points and perform a curve fit solving the linear least-squares problem in order to find the points location. If the error is unacceptable, locate the knot span with highest squared error and insert a new knot such that the error is split in halves. The process iterates until the desired tolerance is reached or it is not possible to introduce new suitable knots. We will discuss

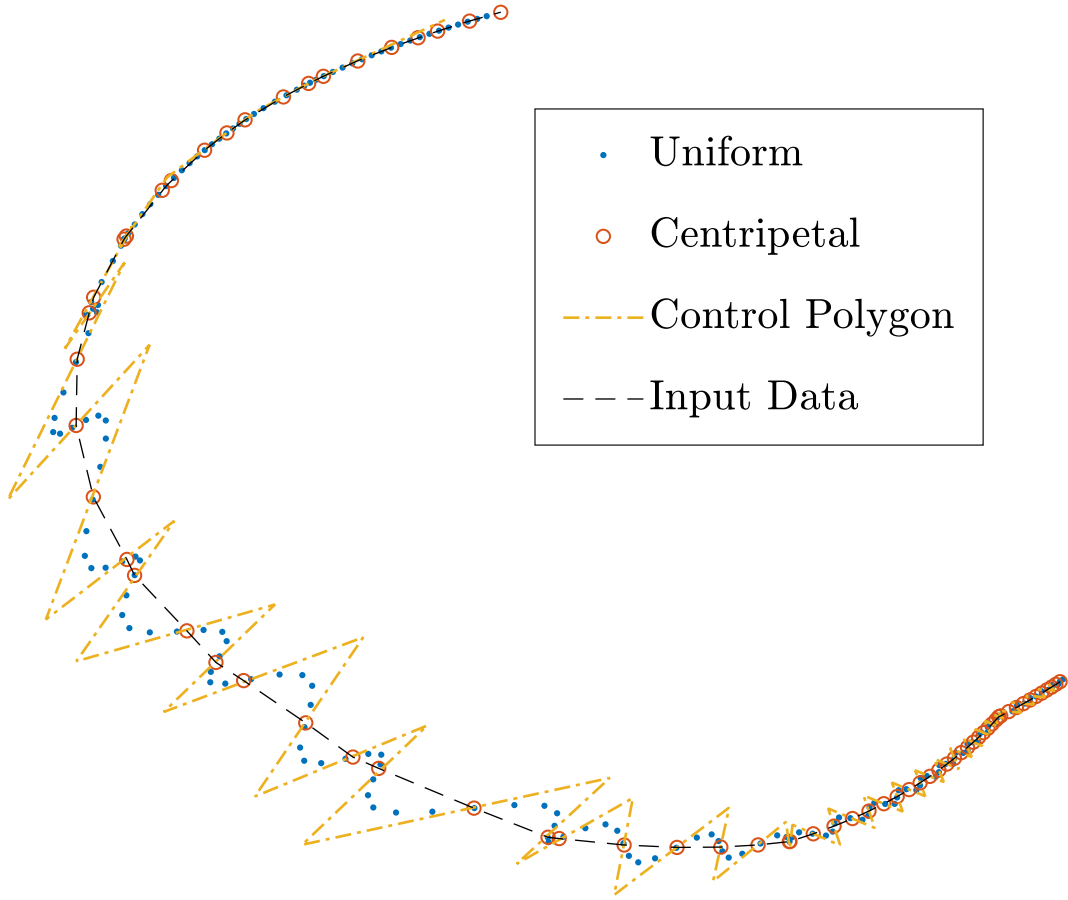


Figure 1: B-spline approximation sampled at two different parametrizations (centripetal and uniform) revealing the “hidden” wiggles as a result excessive number of control points.

later what unsuitable knots mean.

It is well known that as the number of control points approach the number of data points, the resulting approximating B-spline curve may develop undesirable wiggles or artificial loops (self-intersections). However, such artifacts may not be visible to the Least-Squares solver since curve has is smooth along the chosen parametrization . Figure 1 illustrates this phenomena: the curve values that are fed to the solver correspond to the centripetal parametrization (smooth) and the uniform values correspond to sampling the curve uniformly around the resulting curve. Observe how the B-spline wiggles in between the centripetal knots. This is because in that region, almost all the knot spans contain only one  $t$  - value resulting in a zig-zagging control polygon.

are not visible to the Hence, we believe that a “knot increasing” iterative

scheme is more stable than approximating the curve starting with a high number of control points and iterative remove the redundant ones. The usual approach for solving the least squares problem consists of a pre-stage that finds a suitable curve parametrization (*e.g.*, centripetal or chord-length)  $t_{j=1}^m$  that targets the  $m$  data points and then compute the knots location followed by finding the control points location that minimizes the error in the least squares sense. This curve discretization does observe the curve behavior between the evaluations  $B(t_j)$  and  $B(t_{j+1})$ , where a “parasitic” loop may develop. If the local error is smaller than the tolerance, that curve zone will not be altered (*i.e.*, candidate for knot-removal in a knot decreasing iterative scheme) thus, producing undesirable results. Furthermore, knot spans that contain few  $t$ -values can end up having a dense control point area producing wiggles. Again, from the least-squares perspective, such wiggles do not exist as illustrated in Figure ??.

#### 4.1 Linear Least Squares Approximation

For a given a data set  $\{X_i\}_{i=1}^m$  and  $n > m$  control points, we look for the B-spline curve

$$C(u) = \sum_{i=0}^n N_{i,p} P_i, \quad u \in [0, 1] \quad (1)$$

that approximates the data in the least squares sense:

$$\sum_{k=1}^{m-1} |X_k - C(t_k)|^2, \quad C(0) = X_0 \text{ and } C(1) = X_m.$$

Let

$$R_k = Q_k - N_{0,p}(t_k)Q_0 - N_{m,p}(t_k)Q_m, \quad k = 1, \dots, m-1,$$

since the B-spline knots are fixed, minimizing equation (4.1) gives:

$$\sum_{i=1}^{n-1} (N_{\ell,p}(t_k) N_{i,p}(t_k)) P_i = \sum_{k=1}^{m-1} N_{\ell,[}(t_k) R_k, \quad \ell = 1 \dots, n-1 \quad (2)$$

which produces a determined system of equations that can be solved applying any linear matrix solver. For more details, see [15]. The limitations of this solver is that a suitable approximation assumes prior knowledge of the “correct” number of control points as well as the knot location. The algorithm presented next consists of inserting a new knot at each iteration followed by solving the new least squares problem applying equation (4.1).

#### 4.1.1 Knot Placement

?? A suitable knot sequence must ensure that each span contains at least one  $t_k$  value so that (2) is well posed. We start with the minimum number of control points (  $n = p$  ) and the following knot sequence  $U = \left\{ 0, \underbrace{\dots}_{p-1}, 0, 1, \underbrace{\dots}_{p-1}, 1 \right\}$ .

At each iteration, we find the span with greatest squared error:

$$e_k = \frac{1}{N_k} \sum_{i=1}^{N_k} (C(t_i) - X_i)^2, \quad N_k = \#t_i \in [u_k, u_{k+1}), i = 1, \dots, m-1.$$

At  $e_k$ , we look for the first  $t_i$  such that  $t_i \geq e_k/2$  and we insert a knot  $\tilde{u} = 0.5(t_i + t_{i+1})$ .

**Remark 4.1** *This knot choice ensures that every new span is non empty.*

In order to avoid concentrating too many control points at a particular zone, we introduce two additional constraints in our knot choice:

1. The new knot has to be at a *minimum* distance from the its neighboring knots which we have set to  $1.e - 06$ .
2. The user can specify a minimum  $t_k$ 's per span, ( $> 1$  ).

#### 4.1.2 The Algorithm

Now that the least squares approach and knot placement have been discussed, we introduce the main algorithm and discuss implementation details. Once the desired tolerance is selected, we use the centripetal method in order to obtain a curve parametrization:

$$d = \sum_{k=1}^n \sqrt{|X_k - X_{k-1}|}, \quad (3)$$

$$t_0 = 0, \quad t_m = 1, \quad t_k = t_{k-1} + \frac{\sqrt{|X_k - X_{k-1}|}}{d}, \quad k = 1, \dots, m-1. \quad (4)$$

This method has proven to be suitable for detecting curve features such as "cusps" [].

**Note 4.1** *If the initial number of control points is set  $n > p$ , we define the internal knots sequence proposed by proposed by []:*

$$d = \frac{m+1}{n-p+1} \Rightarrow i = \text{int}(j \cdot d), \quad \alpha = jd - i \quad (5)$$

$$u_{p+j} = (1 - \alpha)u_{i-1} + \alpha u_i, \quad j = 1, \dots, n-p. \quad (6)$$

At each iteration, we solve (??) and compute the global error. If we haven't reached the desired tolerance, we insert a knot as discussed in section ?? and find the new control points. If the new configuration has not significantly improved the previous iteration, we reject the knot and find the next worst error. If all knots were found to be "unsuitable" we proceed with the best configuration, insert a new knot and tag its value. Once a significant improvement has been found, we try to remove all the "redundant" points computed between the tagged knot and the knot at valid iteration and when possible, the knots in between are removed. The algorithm stops when either the tolerance has reached, the number of iterations have exceeded the maximum or when the approximation has stagnated: no new knot produces a better approximation. The implementation steps are shown in Algorithm ??.

#### 4.1.3 Computational Costs

The methodology presented here attempts to avoid the "free knot problem" which involves solving a non-linear least squares solver. By sequentially increasing the number of control points together with a well chosen knot position, we can obtain a suitable approximation which relies on solving a linear system thus, avoiding excessive computational costs. Although we cannot prove that our knot position is optimal, it is designed to increase the accuracy at where the maximum error in the least squares sense occur, thus it is in accordance with the minimization solver. Furthermore as shown in the next section, our results reveal that this routine is suitable for noisy data and does not "destroy" sharp features.

## 5 Results

We begin by studying the performance of the solver on "basic shapes" as well as smooth data. Then we apply this routine on a level-set solution employed for studying ice formation and finally, we study its behavior on noisy data.

Figure ?? shows a helix obtained from

### 5.1 Basic Shapes Approximation

Consider

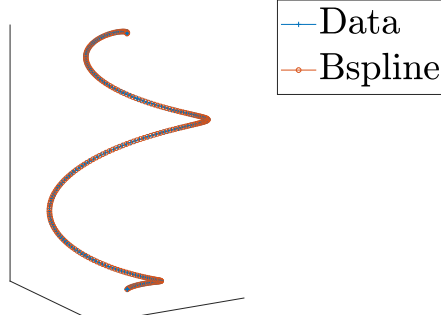
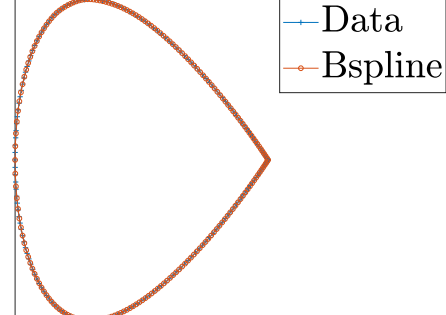
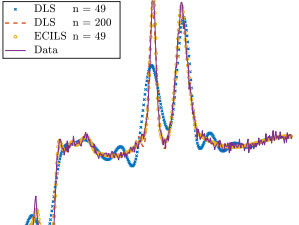
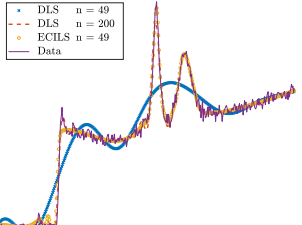
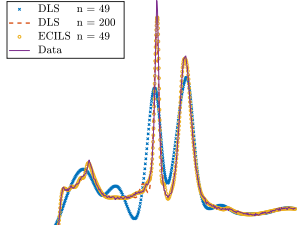
R.M.S.	Control Points			
	Sphere Curve		NACA	
	ECILS	DLS	ECILS	DLS
1.0e-05	24	20	12	19
1.0e-06	40	32	27	35
1.0e-07	73	54	40	52
				

Table 1: Results when approximating a curve moving along a sphere (see Figure ??) using 200 points.

	Noisy Data 1		Noisy Data 2		Noisy Data 3	
Solver	N°CPs	R.M.S.	N°CPs	R.M.S.	N°CPs	R.M.S.
ECILS	49	2.6e-02	45	3.0e-02	72	3.0e-02
DLS	49	9.4e-02	45	6.5e-02	72	3.6e-01
DLS	100	5.3e-02	100	4.4e-02	100	3.3e-01
DLS	200	2.7e-02	200	2.8e-02	200	2.1e-01
						

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