1 Defining the Problem

Curve and surface modeling play a major role in Reverse Engineering during object reconstruction through CAD software [9, 18] or for CFD simulations involving moving interfaces such as flame propagation problems [11]. Usually, one has access only to a discrete (finite) data set and a smooth curve (surface) is needed in order to perform operations such as numerical integration or differentiation. Although not unique, the solution to this problem is generally sought as a spline curve (surface) that either interpolates the data or provides an approximation which is *closest* to it. An approximation *fit* is desirable for experimental measurements or computational data which may be slightly inaccurate or noisy whereas an interpolating approach [15, 10] is more suitable when the data (or part of it) is known to be exact. This paper focuses on approximation techniques since the algorithm presented here was developed for curve reconstruction for numerical solutions on aircraft icing accretion [].

Splines are determined by the approximation degree, control polygon (net) and knot sequence. The ideal spline *fit* is an approximation that minimizes the error (under certain metric) employing a minimum number of knots (control points). Unfortunately, this represents a great challenge since both the "optimal" number of control points and knot location are unknown and in general, it is not possible to derive explicit formulas [6]. Thus, the solution is found by casting a suitable optimization problem, usually based on the least-squares minimization [14, 2, 19]. The methodology proposed here attempts to solve the *minimal* spline problem using a heuristic approach where knots are strategically introduced as part of an iterative scheme that drives the solution towards a desired accuracy.

The simplest solution to the spline least-squares optimization problem is to fix the knot sequence (number and distribution) alongside a suitable curve parametrization and solve a linear system for the control points [14, Ch. 9.4], [3]. The usual parametrization choices are based on the chord-length or centripetal models which have proven to improve the quality of the approximation, especially for detection of sharp features such as "cusps" [5, 21, 7, 10]. However, a fixed knot sequence may be too oversimplified and shape manipulation (knot insertion/removal) [13, 23, 4] can be introduced as part of an error controlled iterative scheme. For example, continuously inserting knots and applying least-squares minimization (*i.e.* finding new control points) until a specified tolerance is attained [16, 12]. Alternatively, one can starting with a large number of control points and remove knots successively as long as the quality of the approximation is not destroyed [8, 22]. Ultimately, the problem can be formulated as a non-linear optimization, allowing "free knots" as part of the least-squares solution [20, 1]. The free-knots problem generally yields to better results but this occurs at the expense of computational efficiency [17] and in addition, it can result in many stationary points [6].

This paper presents a new algorithm from the *knot insertion* category; starting with a minimum number of knots, we look for the span with greatest weighted error, insert a new knot and find the control points using a linear least-squares solver. If the new approximation is above the tolerance step, the knot sequence is updated. Otherwise, the step is rejected and a new knot is inserted at the next "high-error" span. When no new knots overcome the step control, the scheme proceeds with the "best so far", tags the knot location and when a valid step is found, attempts to remove any potential redundant knot. The algorithm stops if either the global error hits certain tolerance (1), it has reached the maximum number of iterations (2) or no new knots can improve the current approximation (3). Although we cannot prove that this strategy is optimal, our results suggest that this knot distribution leads to accurate results remaining at relatively low computational costs, thus becoming suitable for engineering applications.

In the following sections we discuss the different aspects of our method. Section **??** defines the algorithm building blocks: Bspline functions, least squares approximation and the "best knot" sequence problem. In section **??** implementation details are given and section **??** shows the results of applying this tool on several data types including curve fitting on noisy noisy data as well as ice-layer reconstruction. Finally, in section **??** we conclude and highlight the areas for improvement as well as future applications.

2 Background

We begin by introducing B-spline functions, briefly discuss their components and provide formulas for their computation. We assume that the reader is familiar with these functions and full details can be found in [14, 2, 19].

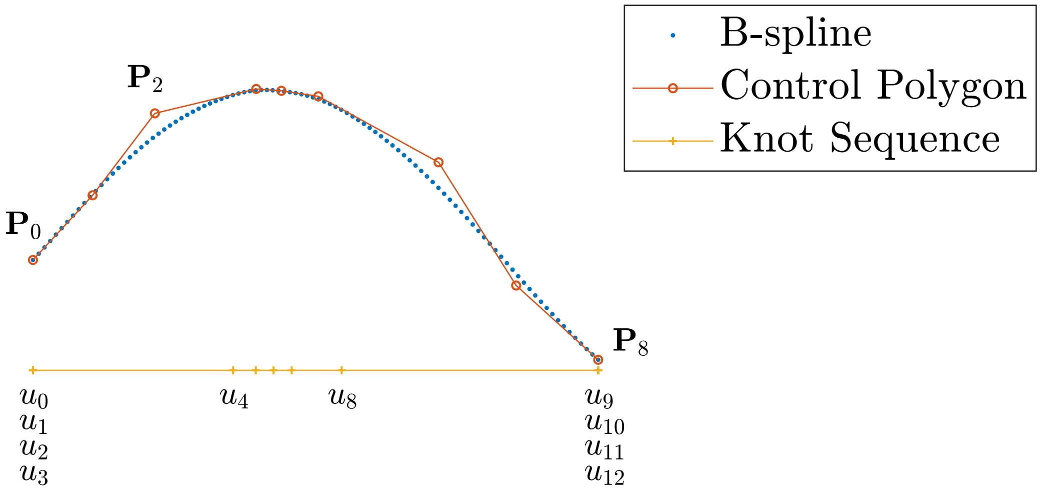


Figure 1: A cubic B-spline curve approximation to a piece of a sine wave employing nine control poinst ant the knot sequence $U = \{u_0, \ldots, u_{12}\}$.

B-splines and Least Squares Approximations

Given a knot sequence $U = \{u_0, \ldots, u_r\}$ such that $u_i \leq u_{i+1},\ i = 0, \ldots r-1$, we define the i^{th} B-spline of degree p through the recurrence formula:

$$N_{i,0} = \begin{cases} 1 & \text{if } u_i \leq t \leq u_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$N_{i,p} = \frac{t - u_i}{u_{i+p} - u_i} N_{i,p-1}(t) + \frac{u_{i+p+1} - t}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(t). \tag{2}$$

B-spline curves are a combination of these functions driven by a control polygon:

$$C(t) = \sum_{i=0}^{n} N_{i,p}(t)\mathbf{P}_i, \quad \mathbf{P}_i = \text{control point}. \tag{3}$$

From this definition and as shown in Figure 1, it is clear that the approximation degree, knot sequence and control polygon fully determine a B-spline curve.

Assume we are given a data set $\{X_i\}_{i=1}^{m}$, and a fixed knot sequence $U = \{u_0, \ldots, u_r\}$. Then, we look for the B-spline curve

$$C(t) = \sum_{i=0}^{n=r-p-1} N_{i,p}(t)\mathbf{P}_i, \quad t \in [0, 1] \tag{4}$$

that approximates the data in the least squares sense:

$$C(0) = X_0, \quad C(1) = X_m \quad \text{and} \tag{5}$$

$$\sum_{k=1}^{m-1} |X_k - C(t_k)|^2 \quad \text{is a minimum with respect } \mathbf{P}_i. \tag{6}$$

Here the t_k are associated to each data point (X_k) by a previously chosen parametrization which in our case corresponds to the centripetal method [7]:

$$t_0 = 0, \quad t_m = 1 \tag{7}$$

$$t_k = t_{k-1} + \frac{\sum_{i=1}^{k} \sqrt{|X_i - X_{i-1}|}}{\sum_{i=1}^{m} \sqrt{|X_i - X_{i-1}|}}, \quad k = 1, \ldots, m-1. \tag{8}$$

Other parametrization models such as uniform [] or chord-length [] and more recently [] may be used. However, our results suggested that the centripetal choice leads consistently to a better approximation.

The solution to equation (5) can be found as follows [14, Ch. 9.4.1]. Let

$$R_k = Q_k - N_{0,p}(t_k) \cdot Q_0 - N_{m,p}(t_k) \cdot Q_m, \quad k = 1, \ldots, m-1. \tag{9}$$

Since the B-spline knots are fixed, minimizing this equation gives:

$$\sum_{i=1}^{n-1} \left(N_{\ell,p}(t_k) N_{i,p}(t_k)\right) P_i = \sum_{k=1}^{m-1} N_{\ell,p}(t_k) R_k, \quad \ell = 1 \ldots, n-1 \tag{10}$$

Note 2.1 *The parametrization should be reflected in the knot distribution. If it satisfies the Schoenberg-Whitney condition*

$$u_i < t_k < u_{i+k}, \quad i = 1, \ldots, m, \tag{11}$$

then the matrix in (**??**) *has full rank and can be solved applying any linear matrix solver. Hence, each span must contain at least one* t_k.

The motivation for introducing iterative knot insertion is that equation (5) requires *a priori* knowledge on the amount of control points needed which is often unavailable. Furthermore, inserting new knots locally, at regions with high error, can lead to a better approximation as shown in our results section (*e.g.* Figure **??**).

2.0.1 Knot Placement

Given an approximation degree (p), the knot insertion technique presented here starts with a minimum knot sequence $U = \{0, \underbrace{\ldots}_{p-1}, 0, 1, \underbrace{\ldots}_{p-1}, 1\}$. Then, each iteration consists of finding the span with greatest weighted error:

$$e_k = \frac{1}{N_k} \sum_{i=1}^{N_k} (C(t_i) - X_i)^2, \quad N_k = \# t_i \in [u_k, u_{k+1}),\ i = 1, \ldots, m-1,$$

where the distance function is computed as follows:

$$C(t_i) - X_i = \min_{t \in [0,1]} |C(t) - X_i|, \tag{12}$$

i.e., the *actual* distance to the data. The new knot position is determined by the first t_i such that $t_i \geq \frac{e_k}{2}$ and takes the value $\tilde{u} = \frac{t_i + t_{i+1}}{2}$. This knot choice ensures that the split $[u_i, \tilde{u})$, $[\tilde{u}, u_{i+1})$ are non empty spans (see Note 2.1).

In [14, Ch.], it was propsed to find all the spans that are above a targeted tolerance and introduce a knot at the highest error. However, this strategy does not account for the fact that each knot has impact in more than one control point and it may lead to an excessive number of control points. It is well known that spline functions develop wiggles when employing a high number of control points; if high error regions have few data support, iterative approximation may lead to "jammed knots" which might lead to local oscillations when each span is only supported by one data point. Such artifacts may not be visible to the Least-Squares solver which is evaluating only at the parametrization values. This is shown in Figure 3 where the resulting curve is shown at the values used during minimization (centripetal) and over an uniform sampling where we can see that wiggles have developped. This is because in that region, almost all the knot spans contain only one $t-$value forcing the curve to interpolate through the data and producing a zig-zagging control polygon. Finally, notice that for each data point X_k, the error function from equation (12) finds a parameter along the curve taht doesn't necessarily coincide with t_k. Hence, when placing the knot at the maximum t_k, this doesn't reflect the actual position of this maximum. Figure 2 shows an example of applying knot insertion based on half-way weighted error (see), at the maximum error and at the middle of the span with highest error. We can see that the weighted error produces the most accurate curve.

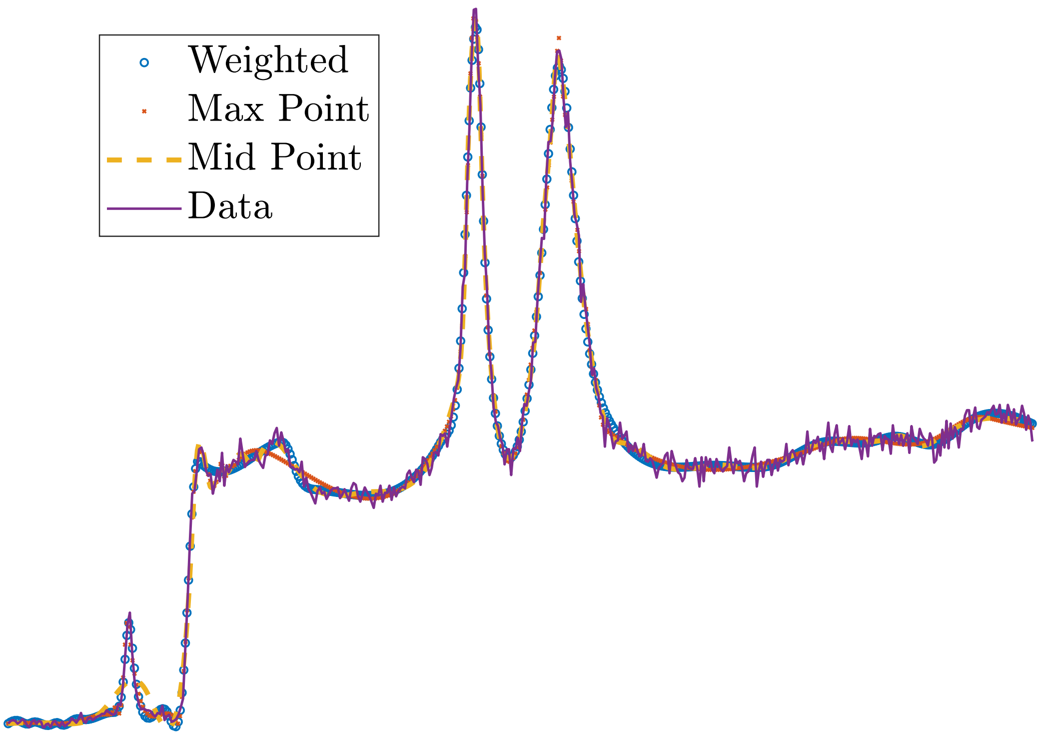


Figure 2: Bspline curves employing a weighted error knot insertion, a maximum error position and at the middle of the span with greatest error. The experimental data corresponds to a Ramana Spectroscopy.

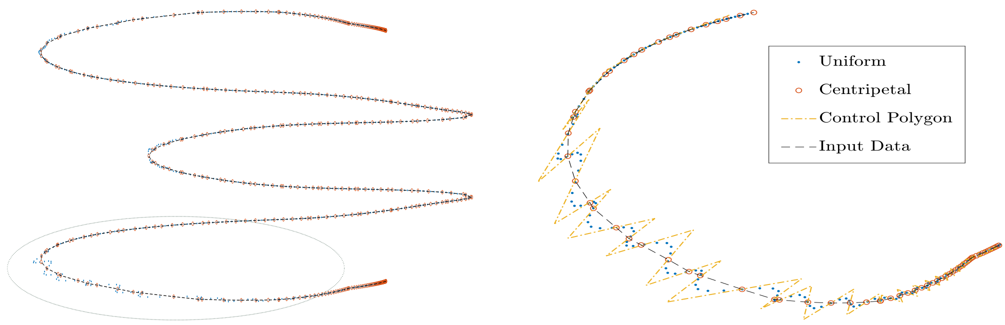


Figure 3: Curve reconstruction of a level set solution sampling the B-spline at the centripetal nodes and at a uniform sample. The rigth plot is a zoom of the highlighted region revealing the "hidden" wiggles.

2.0.2 The Algorithm

Now that the least squares approach and knot placement have been discussed, we introduce the main algorithm and discuss implementation details. Once the desired tolerance is selected, we use the centripetal method in order to obtain a curve parametrization:

$$d = \sum_{k=1}^{n} \sqrt{|X_k - X_{k-1}|}, \tag{13}$$

$$t_0 = 0,\ t_m = 1, \quad t_k = t_{k-1} + \frac{\sqrt{|X_k - X_{k-1}|}}{d}, \quad k = 1, \ldots, m-1. \tag{14}$$

This method has proven to be suitable for detecting curve features such as "cusps" [].

Note 2.2 *If the initial number of control points is set $n > p$, we define the internal knots sequence proposed by proposed by []:*

$$d = \frac{m+1}{n-p+1} \Rightarrow i = int(j \cdot d), \quad \alpha = jd - i \tag{15}$$

$$u_{p+j} = (1-\alpha)u_{i-1} + \alpha u_i, \quad j = 1, \ldots, n-p. \tag{16}$$

At each iteration, we solve (**??**) and compute the global error. If we haven't reached the desired tolerance, we insert a knot as discussed in section 2.0.1 and find the new control points. If the new configuration has not significantly improved the previous iteration, we reject the knot and find the next worst error. If all knots were found to be "unsuitable" we proceed with the best configuration, insert a new knot and tag its value. Once a significant improvement has been found, we try to remove all the "redundant" points computed between the tagged knot and the knot at valid iteration and when possible, the knots in between are removed. The algorithm stops when either the tolerance has reached, the number of iterations have exceeded the maximum or when the approximation has stagnated: no new knot produces a better approximation. The implementation steps are shown in Algorithm **??**.

3 The Algorithm

The work presented here provides a solution to data approximation in the least-squares sense through an iterative scheme that aims to attain a user specified accuracy. In order to avoid solving the non-linear "free knots" problem, we propose the following approach: start with few (or minimal) number of control points and perform a curve fit solving the linear least-squares problem in order to find the points location. If the error is unacceptable, locate the knot span with highest squared error and insert a new knot such that the error is split in halves. The process iterates until the desired tolerance is reached or it is not possible to introduce new suitable knots. We will discuss later what unsuitable knots mean.

are not visible to the Hence, we believe that a "knot increasing" iterative scheme is more stable than approximating the curve starting with a high number of control points and iterative remove the redundant ones. The usual approach for solving the least squares problem consists of a pre-stage that finds a suitable curve parametrization (*e.g.*, centripetal or chord-length) ${t_j}_{j=1}^m$ that targets the m data points and then compute the knots location followed by finding the control points location that minimizes the error in the least squares sense. This curve discretization does observe the curve behavior between the evaluations $B(t_j)$ and $B(t_{j+1})$, where a "parasitic" loop may develop. If the local error is smaller than the tolerance, that curve zone will not be altered (*i.e.*, candidate for knot-removal in a knot decreasing iterative scheme) thus, producing undesirable results. Furthermore, knot spans that contain few t-values can end up having a dense control point area producing wiggles. Again, from the least-squares perspective, such wiggles do not exist as illustrated in Figure **??**.

3.0.1 Computational Costs

The methodology presented here attempts to avoid the "free knot problem" which involves solving a non-linear least squares solver. By sequentially increasing the number of control points together with a well chosen knot position, we can obtain a suitable approximation which relies on solving a linear system thus, avoiding excessive computational costs. Although we cannot prove that our knot position is optimal, it is designed to increase the accuracy at where the maximum error in the least squares sense occur, thus it is in accordance with the minimization solver. Furthermore as shown in the next section, our results reveal that this routine is suitable for noisy data and does not "destroy" sharp features.

4 Results

We begin by studying the performance of the solver on "basic shapes" as well as smooth data. Then we apply this routine on a level-set solution employed for studying ice formation and finally, we study its behavior on nosy data.

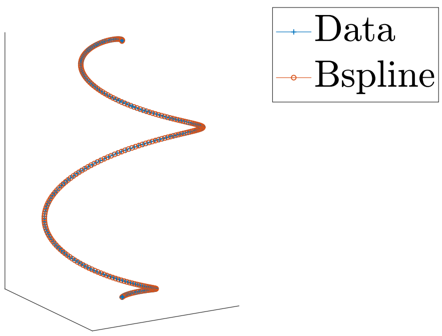
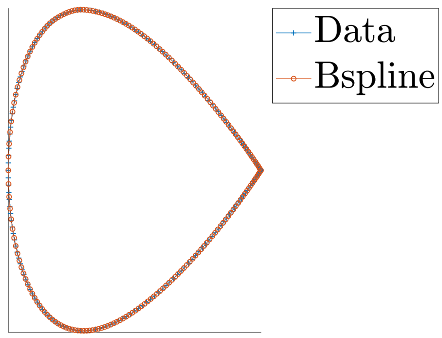
R.M.S.	Control Points			
	Sphere Curve		**NACA**	
TOL	ECILS	DLS	ECILS	DLS
1.0e-05	30	22	10	19
1.0e-06	52	36	19	35
1.0e-07	84	60	42	54
				

Table 1: Results when approximating a curve moving along a sphere (see Figure ??) using 200 points.

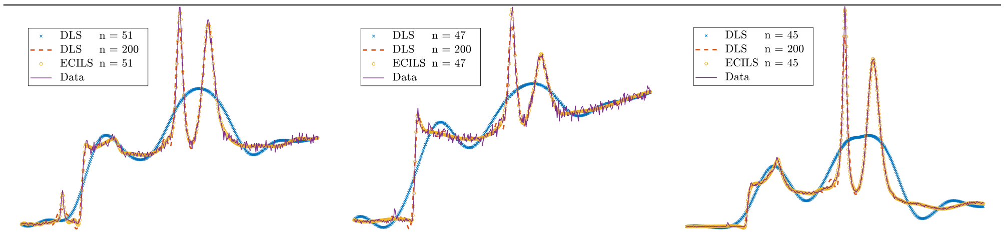


Figure 4: Three data samples from Raman Spectroscopies (*) and the resulting approximation using the new algorithm versus a direct fit for different number of control points.

Figure ?? shows a helix obtained from

4.1 Basic Shapes Approximation

Consider

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