Linear Algebra Assignment 1

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Problem 1

Mandy wants to build a computer, and he found a magic seller that can help her customize all the parts to the exact performance, including CPU, memory, disk storage, graphics card and power supply. We just need to give the seller our needs.

But Mandy has a tight budget, so she wants to build the cheapest computer that meets all her requirements

Here are her requirements:

She wants to run Google Chrome on the computer, and the minimum requirement of Google Chrome is:

- 4 CPU cores
- 10GB of disk space.

Chrome also uses 2GB of memory per process (Chrome will launch as many processes as CPU cores)

As everybody knows, Chrome is a RAM monster, and here's a fun fact:
If Chrome sees that it is using under 50% of total memory, it will enable **MONSTER MODE**, which will eat up all the remaining memory and causing OOM, crashing the computer.

Mandy also wants to play GTA6 on her computer. To play GTA6, the computer needs:

- at least 8 CPU cores
- 500 GFlops of graphics computing power
- 300GB of disk space
- 12 GB of memory.

Note that Mandy only opens an application at one time

And we also have to choose a power supply that can handle our build, and here are the power consumption of each components:

- CPU: 15W per core
- RAM: 2W per GB
- Storage: 5W per 50GB
- GPU: 100W per 250GFlops

For the price:

- CPU: 1500$ per core
- RAM: 300$ per GB
- Storage: 10$ per GB
- GPU: 40$ per GFlops of performance
- Power: 10$ per Watt of capacity

What's the cheapest computer she can get?

Problem 2

The problem can be expressed as a linear programming problem:

Make a build of:

- a CPU cores
- b GB of RAM
- c GB of storage
- d GFlops of GPU performance
- e Watts of power supply capacity

Minimize: $G = 1500a + 300b + 10c + 40d + 10e$

With constraints:

- $a \geq 4$
- $b \geq 2a$
- $2a \geq \frac{b}{2}$
- $a \geq 8$
- $d \geq 500$
- $c \geq 300 + 10$
- $b \geq 12$
- $e \geq 15a + 2b + 5\frac{c}{50} + 100\frac{d}{250}$

After minimizing:

- $a \geq 8$
- $b \geq 12$
- $-2a + b \geq 0$
- $4a - b \geq 0$
- $c \geq 310$
- $d \geq 500$
- $-150a - 20b - c - 4d + 10e \geq 0$

Result

When

- $a = 8$
- $b = 16$
- $c = 310$
- $d = 500$
- $e = 383$

G has the minimal: 43730

That means:
Mandy can build the cheapest computer with 8 CPU cores, 16GB of memory, 310GB of disk storage, 500 GFlops of GPU performance, and a 383 Watt power supply for 43740$ that meets all her requirements

Problem 4

4.1

$$A' = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & & \ddots & \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,m} \\ b_1 & b_2 & \dots & b_m \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & & \ddots & \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,m} \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 0 \\ -a_{n,1} + b_1 & -a_{n,2} + b_2 & \dots & -a_{n,m} + b_m \end{bmatrix}$$

$$= A + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -a_{n,1} + b_1 & -a_{n,2} + b_2 & \dots & -a_{n,m} + b_m \end{bmatrix}$$

$$= A + uv^T$$

4.2

By

$$
\begin{aligned}
&\left(A^{-1}-\frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}\right)\times A' \\
&=\left(A^{-1}-\frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}\right)\times(A+uv^T) \\
&=A^{-1}A+A^{-1}uv^T-\frac{A^{-1}uv^TA^{-1}A}{1+v^TA^{-1}u}-\frac{A^{-1}uv^TA^{-1}uv^T}{1+v^TA^{-1}u} \\
&=I+A^{-1}uv^T-\frac{A^{-1}uv^T+A^{-1}uv^TA^{-1}uv^T}{1+v^TA^{-1}u} \\
&=I+A^{-1}uv^T-\frac{A^{-1}u(v^T+v^TA^{-1}uv^T)}{1+v^TA^{-1}u} \\
&=I+A^{-1}uv^T-\frac{A^{-1}u(1+v^TA^{-1}u)v^T}{1+v^TA^{-1}u} \\
&=I+A^{-1}uv^T-A^{-1}uv^T \\
&=I
\end{aligned}
$$

And

$$
\begin{aligned}
&A'\times\left(A^{-1}-\frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}\right) \\
&=(A+uv^T)\times\left(A^{-1}-\frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}\right) \\
&=I+uv^TA^{-1}-\frac{AA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}-\frac{uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \\
&=I+uv^TA^{-1}-\frac{uv^TA^{-1}+uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \\
&=I+uv^TA^{-1}-\frac{u(v^TA^{-1}+v^TA^{-1}uv^TA^{-1})}{1+v^TA^{-1}u} \\
&=I+uv^TA^{-1}-\frac{u(1+v^TA^{-1}u)v^TA^{-1}}{1+v^TA^{-1}u} \\
&=I+uv^TA^{-1}-uv^TA^{-1} \\
&=I
\end{aligned}
$$

Since

$$A' \times \left(A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u}\right) = \left(A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u}\right) \times A' = I$$

so $A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1+v^T A^{-1}u}$ is the inverse of A', that is

$$A'^{-1} = (A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u}$$

Problem 5

To find the optimal solution without enumerating all the intersection points, we have to find a way to walk along the edges and find the best path to get to the optimal intersection point. And I found an algorithm called Simplex Method:

To utilize matrix row operations, we first convert the constraint inequalities ($\leq$) to equalities($=$) by adding a slack variable:
$x + 2y \leq 16 \Rightarrow x + 2y + s_1 = 16$
$x + y \leq 9 \Rightarrow x + y + s_2 = 9$
$3x + 2y \leq 24 \Rightarrow 3x + 2y + s_3 = 24$

Note that the constraints have to be in this form.
Every inequality can be converted into this form using some tricks

And we move the variables of our objective function G to LHS:
$G = 40x + 30y \Rightarrow -40x - 30y + G = 0$

By this way, we can create a matrix:

$$\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & G & \\ 1 & 2 & 1 & 0 & 0 & 0 & 16 \\ 1 & 1 & 0 & 1 & 0 & 0 & 9 \\ 3 & 2 & 0 & 0 & 1 & 0 & 24 \\ \hline -40 & -30 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Where the top rows are constraints, and the last row is the objective function

Now we are at $(x, y) = (0, 0)$, where $G = 0$

The last row is the objective function,and we wish to maximize it.
If a variable's coefficient is large, then increasing it would lead to quicker grow of the objective function
Since the variables are moved to LHS, so we should choose the most negative number in the last row as the pivot column to increase, and that will be the optimal way to move.
In this case, it's x
We can see that we can increase 16 to x for the first row, and 9 for the second row, and 8 for the third row.
But if we move x over any constraints, it will fall out of the feasible area
So we could only increase 8 to x

Now we do the row operations, making x in the third row to 1, and other rows to 0, thus moving to $(8, 0)$

Dividing the third row by 3

$$\begin{array}{cccccc|c}
x & y & s_1 & s_2 & s_3 & G & \\
1 & 2 & 1 & 0 & 0 & 0 & 16 \\
1 & 1 & 0 & 1 & 0 & 0 & 9 \\
1 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 8 \\
\hline
-40 & -30 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Making the x column 0 for other rows

$$\begin{array}{cccccc|c}
x & y & s_1 & s_2 & s_3 & G & \\
0 & \frac{4}{3} & 1 & 0 & -\frac{1}{3} & 0 & 8 \\
0 & \frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 1 \\
1 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 8 \\
\hline
0 & -\frac{10}{3} & 0 & 0 & \frac{40}{3} & 1 & 320
\end{array}$$

Now we are at $(8, 0)$, and we increased the objective G to 320

We keep finding the most negative number in the last row for the pivot column
In this case, it is y

We can only move y by 3 because of the second row
Again, make y on the second row 1 and other rows 0

Dividing the second row by $\frac{1}{3}$

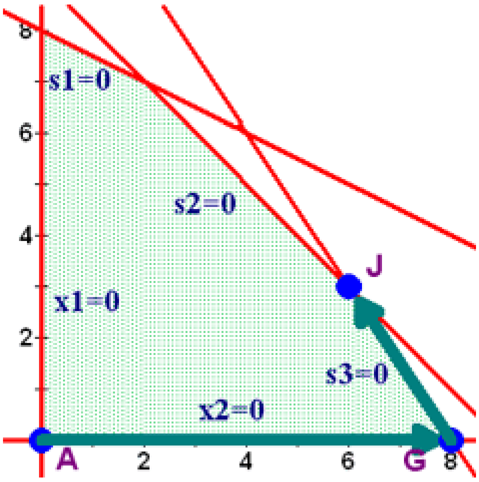
$$\begin{array}{cccccc|c}
x & y & s_1 & s_2 & s_3 & G & \\
0 & \frac{4}{3} & 1 & 0 & -\frac{1}{3} & 0 & 8 \\
0 & 1 & 0 & 3 & -1 & 0 & 3 \\
1 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 8 \\
\hline
0 & -\frac{10}{3} & 0 & 0 & \frac{40}{3} & 1 & 320
\end{array}$$

Making the y column 0 for other rows

$$\begin{array}{cccccc|c}
x & y & s_1 & s_2 & s_3 & G & \\
0 & 0 & 1 & -4 & 1 & 0 & 4 \\
0 & 1 & 0 & 3 & -1 & 0 & 3 \\
1 & 0 & 0 & -2 & 1 & 0 & 6 \\
\hline
0 & 0 & 0 & 10 & 10 & 1 & 330
\end{array}$$

Now we are at $(6, 3)$, and we increased the objective to 330!

Now we look at the last row. There are no more negative numbers.
That means increasing any column will only decrease the objective (because other rows would subtract it)
So we are already at the optimal point!



We move from the basic solution $(x, y) = (0, 0)$ to the optimal solution $(x, y) = (6, 3)$ without enumerating all the intersection points!

This amazing article explains the Simplex Method very well!
Now I totally understand it now!
https://people.richland.edu/james/ictcm/2006/simplex.html