Linear Algebra Assignment 2

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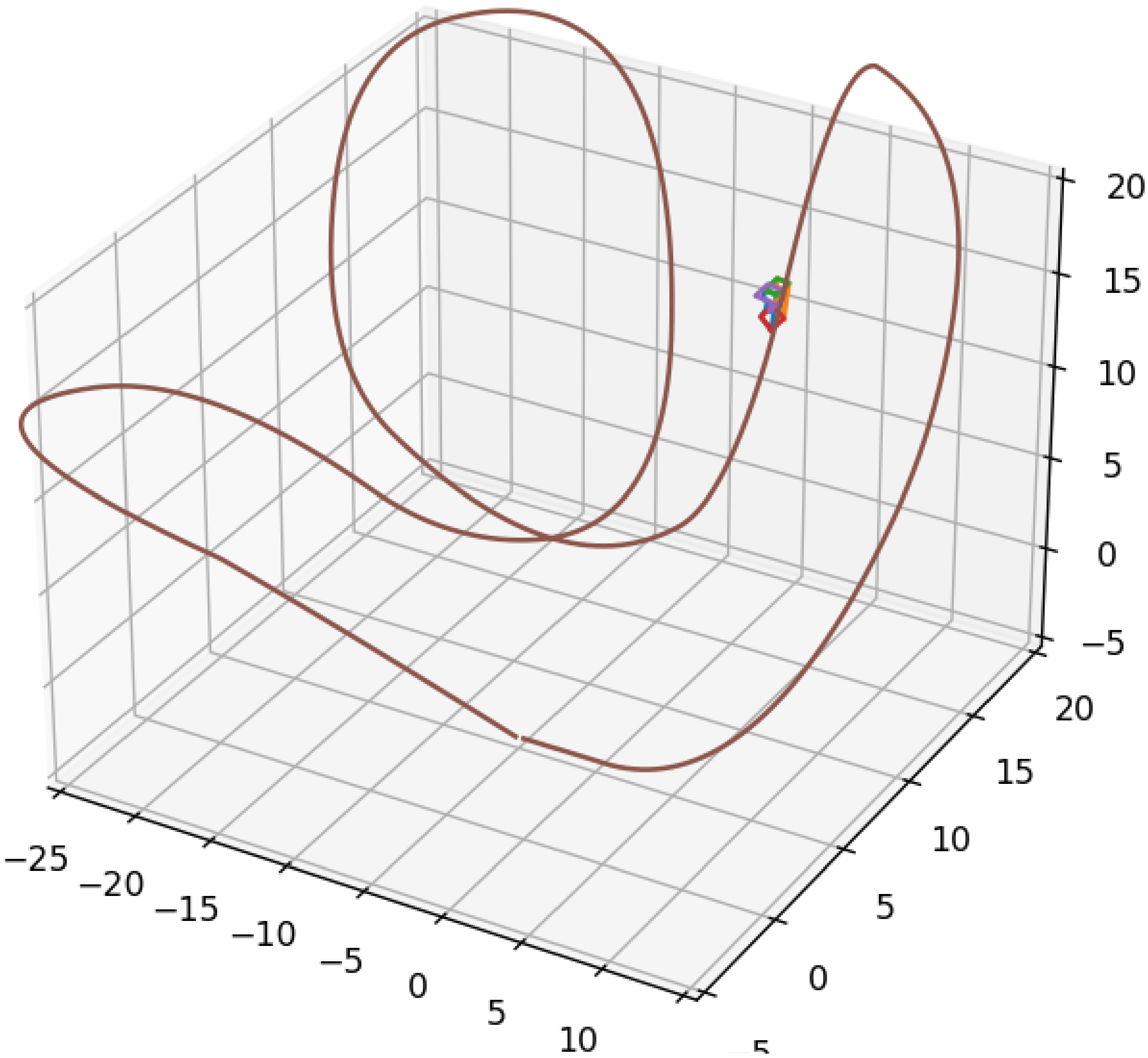
Python versions

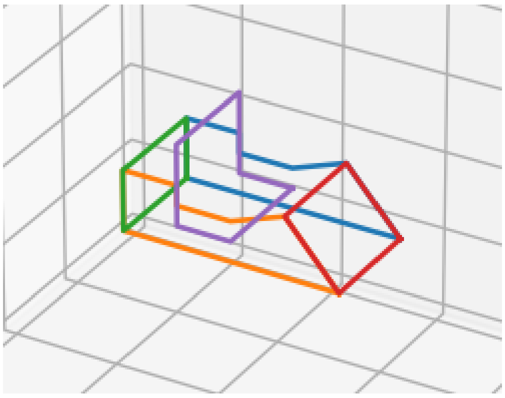
- Python 3.9.7 on Arch Linux
- matplotlib 3.4.3

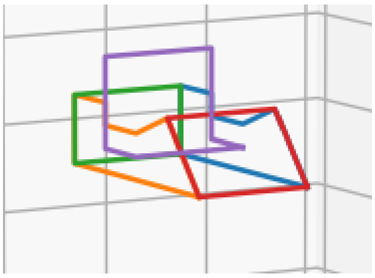
P1

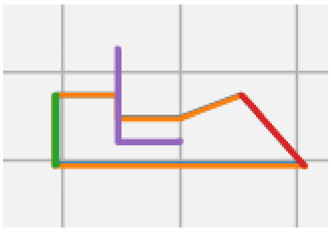
I’ve designed an animation of a roller coaster:



The roller coaster model looks like this:



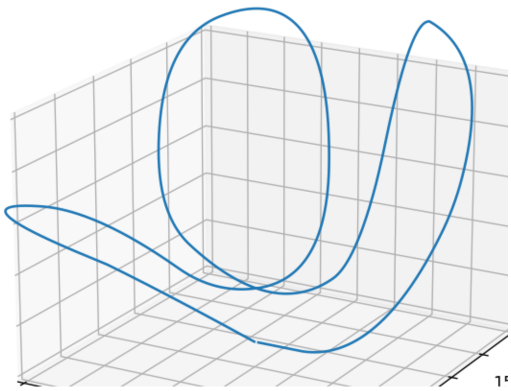




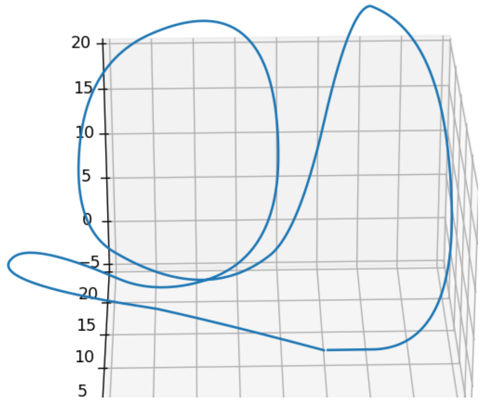
It contains 5 polygons:

- Right panel (Orange)
- Left panel (Blue)
- Back panel (Green)
- Front panel (Red)
- Seat (Purple)

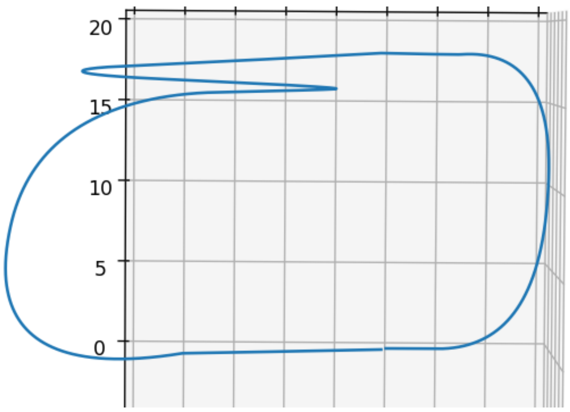
And there is also the roller coaster rails (path):



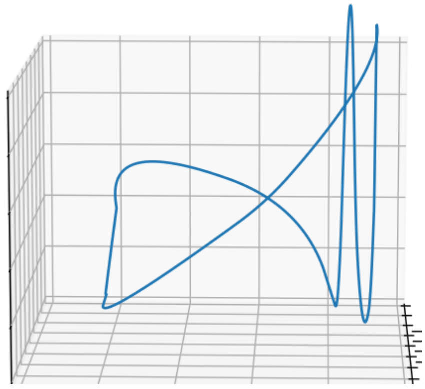
Front:



Top:



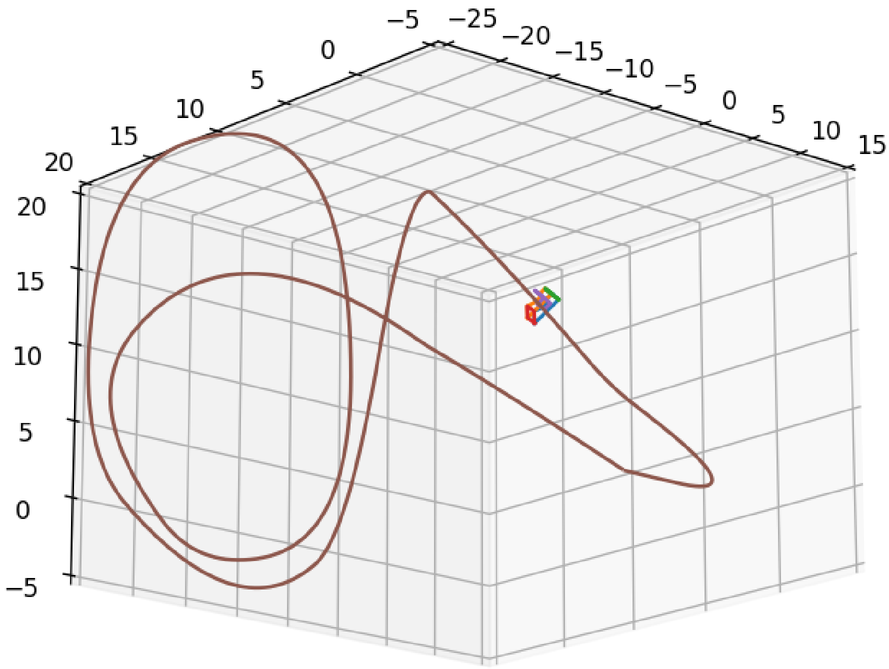
Side:



P3

If we change the order of rotation matrices from
Yaw -> Pitch -> Roll
to
Pitch -> Yaw -> Roll

It has a different effect on rotation
It’s especially obvious when the roller coaster is climbing:



Originally, when the roller coaster is climbing, it first yaws left (on the x-y plane), and then pitches up

After changing the order of rotation, it now first pitches up, and then yaws left.
But when it yaws left, it is no longer turning on the x-y plane, because the pitch changes first, it is now yawing on the pitched plane, thus scrambling the behavior of turning

This shows the effect of applying the rotation matrices in different order

P4

1.

R_{roll} is orthogonal:

$$
\begin{gathered}
{R_{roll}}^T \times R_{roll} \\
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & cos(\theta) \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & cos(\theta) \end{bmatrix} \\
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta)^2 + sin(\theta)^2 & -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) \\ 0 & -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) & \sin(\theta)^2 + cos(\theta)^2 \end{bmatrix} \\
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \\
R_{roll} \times {R_{roll}}^T \\
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & cos(\theta) \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & cos(\theta) \end{bmatrix} \\
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta)^2 + sin(\theta)^2 & \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) \\ 0 & \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) & \sin(\theta)^2 + cos(\theta)^2 \end{bmatrix} \\
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
\end{gathered}
$$

R_{yaw} is orthogonal:

$$
{R_{yaw}}^T \times R_{yaw}
$$

$$
= \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\theta)^2 + sin(\theta)^2 & -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) & 0 \\ -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) & \sin(\theta)^2 + cos(\theta)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
$$

$$
R_{yaw} \times {R_{yaw}}^T
$$

$$
= \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\theta)^2 + sin(\theta)^2 & \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) & 0 \\ \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) & \sin(\theta)^2 + cos(\theta)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
$$

R_{pitch} is orthogonal:

$$
\begin{aligned}
&{R_{pitch}}^T \times R_{pitch} \\
&= \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & cos(\theta) \end{bmatrix} \times \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta)^2 + sin(\theta)^2 & 0 & -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) & 0 & \sin(\theta)^2 + cos(\theta)^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
\end{aligned}
$$

$$
\begin{aligned}
&R_{pitch} \times {R_{pitch}}^T \\
&= \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & cos(\theta) \end{bmatrix} \times \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta)^2 + sin(\theta)^2 & 0 & \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) \\ 0 & 1 & 0 \\ \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) & 0 & \sin(\theta)^2 + cos(\theta)^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
\end{aligned}
$$

2.

According to QR decomposition, any real square matrix A may be decomposed as $A = QR$, where $Q$is an orthogonal matrix and R is an upper triangular matrix

We first show that $R_{pitch}R_{yaw}R_{roll}$ is an orthogonal matrix:

$$
\begin{gathered}
(R_{pitch}R_{yaw}R_{roll}) \times (R_{pitch}R_{yaw}R_{roll})^T \\
= R_{pitch}R_{yaw}R_{roll} \times {R_{roll}}^T {R_{yaw}}^T {R_{pitch}}^T \\
= R_{pitch}R_{yaw}(R_{roll}{R_{roll}}^T){R_{yaw}}^T {R_{pitch}}^T \\
= R_{pitch}R_{yaw}I{R_{yaw}}^T {R_{pitch}}^T \\
= R_{pitch}(R_{yaw}{R_{yaw}}^T){R_{pitch}}^T \\
= R_{pitch}I{R_{pitch}}^T \\
= R_{pitch}{R_{pitch}}^T \\
= I \\
(R_{pitch}R_{yaw}R_{roll})^T \times (R_{pitch}R_{yaw}R_{roll}) \\
= {R_{roll}}^T {R_{yaw}}^T {R_{pitch}}^T \times R_{pitch}R_{yaw}R_{roll} \\
= {R_{roll}}^T {R_{yaw}}^T ({R_{pitch}}^T R_{pitch})R_{yaw}R_{roll} \\
= {R_{roll}}^T {R_{yaw}}^T I R_{yaw}R_{roll} \\
= {R_{roll}}^T ({R_{yaw}}^T R_{yaw})R_{roll} \\
= {R_{roll}}^T I R_{roll} \\
= {R_{roll}}^T R_{roll} \\
= I
\end{gathered}
$$

Since

$$(R_{pitch}R_{yaw}R_{roll}) \times (R_{pitch}R_{yaw}R_{roll})^T = (R_{pitch}R_{yaw}R_{roll})^T \times (R_{pitch}R_{yaw}R_{roll}) = I$$

, $R_{pitch}R_{yaw}R_{roll}$ is an orthogonal matrix

Then according to QR decomposition, any 3×3 matrix A may be expressed as $A = QR$, where $Q$is an orthogonal matrix and R is an upper triangular matrix

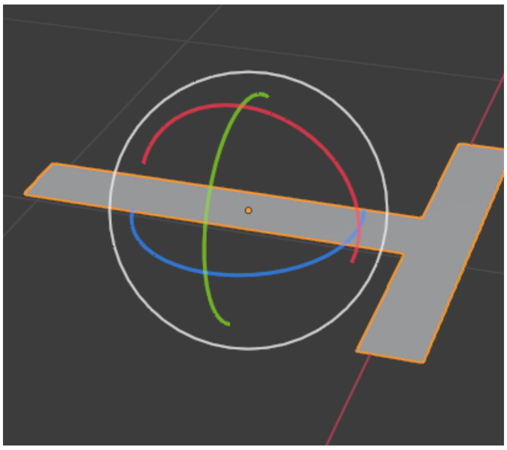
Since $R_{pitch}R_{yaw}R_{roll}$ is an orthogonal matrix, and U is an upper triangular matrix,
any 3×3 matrix A could be expressed as

$$A = R_{pitch}R_{yaw}R_{roll}U$$

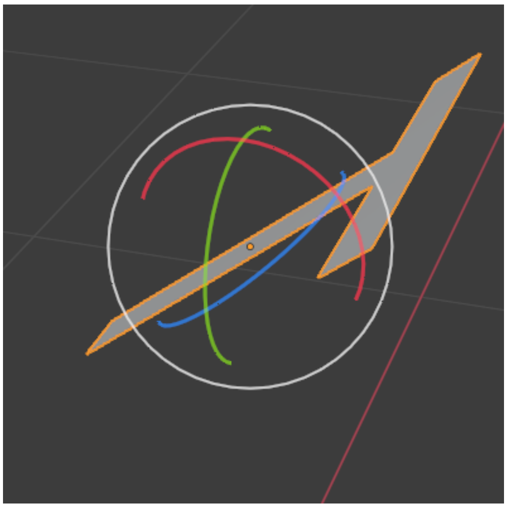
P5

Gimbal lock happens in Euler angles with gimbals, that is, the rotating axis are relative with the current state of rotation.
Just like P3, if we first pitch then yaw, then it would not yaw on the x-y plane, since the pitch changed the yaw's axis.
The rotations are relative, not absolute

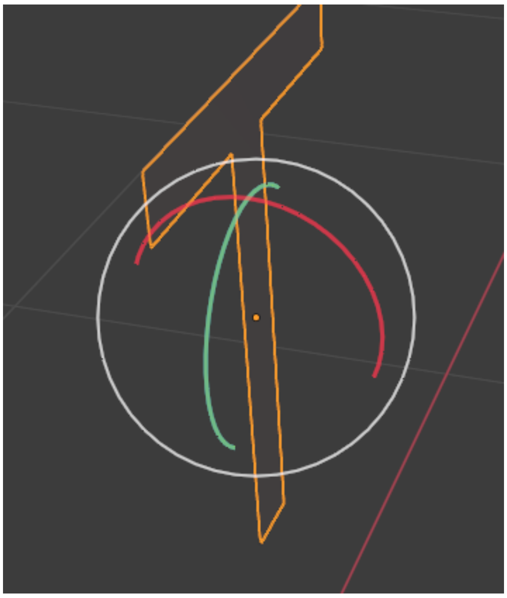
Because the rotations are ordered, it could be seen as nested, creating a gimbal



If we change the pitch (red ring), we would change the yaw's rotating axis (blue ring) too



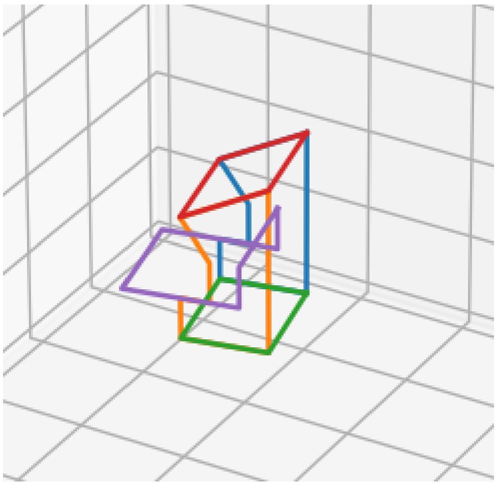
Now if we pitch to 90 degrees, bad things happens



The yaw axis and the roll axis coincide!
Now yawing and rolling has the same effect
We have now lost a degree of freedom
It is called the gimbal lock.

This could also happen on our model:

We can spin every axis when it rotates seperately
But when combining the rotations together, at some moment, there was only an axis spining.
Another spinning axis was gone!



This shows the gimbal lock, as we have lost a degree of freedom.