Richardson Extrapolation

Jonah Miller
jonah.maxwell.miller@gmail.com

January 17, 2014

Suppose that we are numerically solving a differential equation by finite differences. The true answer is a function

$$f^*(x) \tag{1}$$

which we will often denote f^* for brevity. A numerical solution to the problem is modified by the lattice spacing h and a factor $\alpha(x)$ which is a function of the problem itself:

$$f(h) = f^* + \alpha h^n, \tag{2}$$

where n is the order convergence of the method.

Now suppose that we know the numerical solution for three values of h, h_1, h_2, and h_3, with corresponding numerical solutions f_1, f_2, and f_3. From this information, can we extract n and pointwise solutions for f^* and α? We begin by devising a numerical way to extract n.

$$\begin{aligned} \frac{f_1 - f_3}{f_2 - f_3} &= \frac{(f^* + \alpha h_1^n) - (f^* + \alpha h_3^n)}{(f^* - \alpha h_2^n) - (f^* + \alpha h_3^n)} \\ \Rightarrow \frac{f_1 - f_3}{f_2 - f_3} &= \frac{h_1^n - h_3^n}{h_2^n - h_3^n}. \end{aligned} \tag{3}$$

Although in general it is not analytically solvable, one can use one-dimensional root-finding to solve equation (3) for n. Theoretically, one should perform this test at every lattice point or use curve-fitting methods like least-squares to solve for the value of n. However, it is probably sufficient to solve for n at a single well-chosen grid point.

Once n is known, we can solve for α:

$$\begin{aligned} f_1 - f_3 &= \alpha(h_1^n - h_3^n) \\ \Rightarrow \alpha &= \frac{f_2 - f_3}{h_2^n - h_3^n}. \end{aligned} \tag{4}$$

α will very likely depend on x.

Finally, we can extract the true solution using n, α and any one of the numerical solutions:

$$f^* = f_i - \alpha h_i^n \ \forall \ i \in \{1, 2, 3\}. \tag{5}$$

It's probably best to use the most fine-grained solution one has to extract the true solution.