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A simple proof of logarithmic convexity of extended mean values

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Abstract In the note, a simple proof is provided for the logarithmic convexity of extended mean values.

Keywords Simple proof · Logarithmic convexity · Extended mean values

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1 Introduction

Let x, y be two positive numbers and r, s be two real variables. Then extended mean values $E(r, s; x, y)$ were defined in [4, 11] as

$$E(r, s; x, y) = \left(\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r}\right)^{1/(s-r)}, rs(r-s)(x-y) \neq 0;$$

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$$
\begin{aligned}
&E(r,0;x,y)=\left(\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right)^{1/r}, && r(x-y)\neq 0;\\
&E(r,r;x,y)=\frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, && r(x-y)\neq 0;\\
&E(0,0;x,y)=\sqrt{xy}, && x\neq y;\\
&E(r,s;x,x)=x, && x=y.
\end{aligned} \tag{1}
$$

There are many papers on extended mean values $E(r,s;x,y)$, see [2] and related references therein.

By very complex and intricate calculations on more than four printed pages, the logarithmic convexity of extended mean values $E(r,s;x,y)$ were presented in [5, 6] as follows.

Theorem 1 *For given positive numbers x and y,*

1. *if $(r,s)\in(0,\infty)\times(0,\infty)$, then extended mean values $E(r,s;x,y)$ are logarithmically concave with respect to either r or s;*
2. *if $(r,s)\in(-\infty,0)\times(-\infty,0)$, then extended mean values $E(r,s;x,y)$ are logarithmically convex with respect to either r or s.*

The aim of this note is to provide a simple and elegant proof of Theorem 1, which will spend not more than one and a half printed page.

2 A simple proof of Theorem 1

For real numbers a and b with $b>a$, let

$$
F_{a,b}(t)=\begin{cases}\dfrac{t}{e^{bt}-e^{at}}, & t\neq 0;\\[2mm] \dfrac{1}{b-a}, & t=0.\end{cases} \tag{2}
$$

Straightforward computation gives

$$
\begin{aligned}
\ln F_{a,b}(t)&=\ln|t|-\ln\left|e^{bt}-e^{at}\right|,\\
[\ln F_{a,b}(t)]'&=\frac{1}{t}-\frac{be^{bt}-ae^{at}}{e^{bt}-e^{at}},\\
[\ln F_{a,b}(t)]''&=\frac{(a-b)^2e^{(a+b)t}}{(e^{at}-e^{bt})^2}-\frac{1}{t^2}
\end{aligned}
$$

and

$$
\begin{aligned}
&[\ln F_{a,b}(t)]^{(3)} \\
&\quad= \frac{2}{t^3} - \frac{(a-b)^3 e^{(a+b)t}(e^{at}+e^{bt})}{(e^{at}-e^{bt})^3} \\
&\quad= \frac{2}{t^3}\left(\frac{at-bt}{e^{at}-e^{bt}}\right)^3\left[\left(\frac{e^{at}-e^{bt}}{at-bt}\right)^3 - \frac{e^{(a+b)t}(e^{at}+e^{bt})}{2}\right] \\
&\quad= \frac{2e^{3(a+b)t/2}}{t^3}\left(\frac{at-bt}{e^{at}-e^{bt}}\right)^3\left\{\left[\frac{e^{(a-b)t/2}-e^{(b-a)t/2}}{(a-b)t}\right]^3 - \frac{e^{(a-b)t/2}+e^{(b-a)t/2}}{2}\right\} \\
&\quad\triangleq \frac{2e^{3(a+b)t/2}}{t^3}\left(\frac{at-bt}{e^{at}-e^{bt}}\right)^3 Q\left(\frac{a-b}{2}t\right).
\end{aligned}
$$

The Lazarević's inequality (see [1, p. 131] and [3, p. 300]) reads as follows

$$
\frac{\sinh t}{t} < \cosh t < \left(\frac{\sinh t}{t}\right)^3, \quad t \neq 0. \tag{3}
$$

This shows that

$$
Q(t) = \left(\frac{e^t - e^{-t}}{2t}\right)^3 - \frac{e^{-t}+e^t}{2} = \left(\frac{\sinh t}{t}\right)^3 - \cosh t > 0
$$

for $t \in \mathbb{R}$ with $t \neq 0$. Hence $[\ln F_{a,b}(t)]^{(3)} > 0$ on $(0, \infty)$ and $[\ln F_{a,b}(t)]^{(3)} < 0$ on $(-\infty, 0)$ for all real numbers a and b with $a \neq b$.

It is easy to see that extended mean values $E(r, s; x, y)$ can be expressed in terms of $F_{a,b}(t)$ as

$$
E(r, s; x, y) = \begin{cases} \left[\dfrac{F_{\ln x, \ln y}(r)}{F_{\ln x, \ln y}(s)}\right]^{1/(s-r)}, & (r-s)(x-y) \neq 0; \\ \exp\left(-\dfrac{F'_{\ln x, \ln y}(r)}{F_{\ln x, \ln y}(r)}\right), & r = s, \ x - y \neq 0 \end{cases} \tag{4}
$$

and

$$
\ln E(r, s; x, y) = \begin{cases} \dfrac{-1}{s-r}\displaystyle\int_r^s \dfrac{F'_{\ln x, \ln y}(u)}{F_{\ln x, \ln y}(u)}\,\mathrm{d}u, & (r-s)(x-y) \neq 0; \\ -\dfrac{F'_{\ln x, \ln y}(r)}{F_{\ln x, \ln y}(r)}, & r = s, \ x - y \neq 0. \end{cases} \tag{5}
$$

It is known [5, p. 1788, Lemma 1] that if f is a twice-differentiable convex function on an interval I, then the arithmetic mean

$$
\phi(r, s) = \begin{cases} \dfrac{1}{s-r}\displaystyle\int_r^s f(t)\,\mathrm{d}t, & r \neq s \\ f(r), & r = s \end{cases} \tag{6}
$$

of $f(t)$ is also convex with respect to both r and s on I. Consequently, using the symmetric properties $E(r,s;x,y)=E(s,r;x,y)=E(r,s;y,x)$ and the fact that

$$[\ln F_{a,b}(t)]^{(3)}=\left[\frac{F'_{a,b}(t)}{F_{a,b}(t)}\right]''\begin{cases}>0 & \text{for } t\in(0,\infty)\\ <0 & \text{for } t\in(-\infty,0)\end{cases} \tag{7}$$

for all real numbers a and b with $a\neq b$, Theorem 1 follows directly.

Remark 1 Some properties and applications of the reciprocal of the function $F_{a,b}(t)$ have been investigated in [7–10] and related references therein.

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