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CONTROL IN STOCHASTIC SYSTEMS AND UNDER UNCERTAINTY CONDITIONS

Analytical Synthesis of Robust Controllers of the Given Accuracy under External Perturbations

N. I. Sel'vesyuk

Research Institute of Aviation Systems, State Scientific Centre of the Russian Federation, Federal State Unitary Enterprise, ul. Viktorenko 7, Moscow, 125319 Russia

e-mail: niselvesuk@2100.gosniias.ru

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Abstract—An approach that allows finding a set of robust statistic feedback controllers that ensure the given control accuracy for the controlled parameters in a multiconnected control system under bounded parametric and external perturbations of various classes is proposed. To describe the set of controllers of the given accuracy, the results of analytically solving the matrix equations based on matrix canonization and parametrization of the Lyapunov equation are used. The solvability conditions of the synthesis problem ensure the robustness and optimal suppression of the perturbations. They are homogeneously represented as linear matrix inequalities obtained based on the Lyapunov functions. A methodological example is given.

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INTRODUCTION

Methods to solve the perturbation suppression problem depend on the class of the system and the type of perturbations acting upon it. Recently, various statements of the external perturbation suppression problem have widely employed the methods based on the concept of invariant ellipsoids that are some approximations of the attainability sets of a dynamic system [1]. With this approach applied to linear systems, the control synthesis methods are described in detail in [2] for the perturbations bounded in the L_2-norm and in the series of works [3, 4] for the perturbations bounded in the L_∞-norm. In all cases, solving the control synthesis problem is reduced to the semidefinite programming optimization problem with the set of conditions in the form of linear matrix inequalities (LMIs). The almost exhaustive generalization of the approach involved based on the results of international and Russian authors for various variants of the problem statement, including the robust one, is given in [5]. The robustness problems are stated via the Lyapunov function, which also allows stating its solution via LMI.

This approach has a number of advantages, such as the flexibility of the problem statement, obtaining a stabilizing solution, and the ease of solving the problem numerically using the MATLAB packages. However, from the practical point of view, there are some drawbacks here; for instance, it uses integral optimization criteria that make it more difficult to state the requirements for the control quality for particular parameters of the system in the form of clear engineering quality indices; and it yields a unique solution to the synthesis problem, which restricts the choice of the structure for the control law and the possibility of satisfying the additional synthesis requirements.

In this work, we propose an approach that allows synthesizing the set of robust controllers for a multiconnected linear system that ensure the given control accuracy with respect to particular coordinates of the state vector of the system and optimal suppression of the uncontrolled bounded perturbations with respect to the rest of the coordinates. The controllers are robust in the sense of the inaccuracy of description of the system model.

The proposed approach is based on the results we obtained in [6]. It assumes the analytical construction of the set of controllers that ensure the given requirements on the control accuracy are met. We state the requirements for each element of the controlled system's output and formalize them via the elements of the matrices that are the solution of the Lyapunov equations that describe invariant ellipsoids. To find the realizable matrix of the invariant ellipsoid, we use the numerical methods with LMI used to formalize the conditions of suppressing perturbations and ensuring robustness. The approach involved is considered earlier in [7] for the perturbations bounded in the L_2-norm and in [8] for the perturbations bounded in the

L_∞-norm. In this work, we consider the robust problem statement for various classes of bounded perturbations.

1. GENERAL PROBLEM STATEMENT

In the general case, we consider the linear multiconnected system

$$\dot{\mathbf{x}}(\mathrm{t}) = \mathbf{A}\mathbf{x}(\mathrm{t}) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\mathbf{w}(\mathrm{t}) + \mathbf{h}(\mathbf{x}(t), t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{z}(\mathrm{t}) = \mathbf{D}\mathbf{x}(\mathrm{t}), \tag{1.1}$$

where $\mathbf{x} \in \Re^{n_x}$ is the state vector with the initial state $\mathbf{x}_0$; $\mathbf{w} \in \Re^{n_w}$ is the vector of uncontrolled external perturbations; $\mathbf{z} \in \Re^{n_z}$ is the vector of controlled parameters; $\mathbf{u} \in \Re^{n_u}$ is the control vector; and $\mathbf{h}(\mathbf{x}, t) \in \Re^{n_x}$ is the vector function of both arguments that characterizes uncertainty of the model description with the nominal matrix $\mathbf{A}$. The pair of matrices $(\mathbf{A}, \mathbf{G})$ is completely controllable, the pair $(\mathbf{A}, \mathbf{B})$ is stabilizable, and matrix $\mathbf{D}$ has a full rank. Matrix $\mathbf{A}$ is not supposed to be Hurwitz.

The nonlinearity $\mathbf{h}(\mathbf{x}, t)$ is an arbitrary piecewise-continuous function that satisfies the inequality in the quadratic form

$$\mathbf{h}^{\mathrm{T}}(\mathbf{x}, t)\mathbf{h}(\mathbf{x}, t) \le \gamma^2 \mathbf{x}^{\mathrm{T}} \mathbf{H}_{\mathrm{A}}^{\mathrm{T}}(t) \mathbf{H}_{\mathrm{A}}(t) \mathbf{x}, \tag{1.2}$$

where $\gamma > 0$ is the parameter that characterizes the uncertainty boundary and $\mathbf{H}_{\mathrm{A}}(t)$ is the matrix that describes the uncertainty structure.

We call system (1.1) robustly stabilizable with degree γ if its state is asymptotically stable for any $\mathbf{h}(\mathbf{x}, t)$ satisfying (1.2).

In what follows, for the sake of convenience, we consider the structured parametric uncertainty

$$\mathbf{h}(\mathbf{x}, t) = \mathbf{H}_{\mathrm{A}}(t)\mathbf{x}(t), \quad \mathbf{H}_{\mathrm{A}}(t) = \mathbf{F}\Delta(t)\mathbf{H}, \tag{1.3}$$

where $\Delta(t) \in \Re^{p \times q}$, $p \le n_x$, $q \le n_x$ is the perturbing matrix of an arbitrary structure (in the general case, it is rectangular and depends on time) that satisfies the condition

$$\|\Delta(t)\|_2 \le \gamma, \quad \gamma = \mathrm{const}, \tag{1.4}$$

where $\|\cdot\|_2$ is the spectral norm and $\mathbf{F} \in \Re^{n_x \times p}$, $\mathbf{H} \in \Re^{q \times n_x}$ are the constant matrices that form the structure of the perturbation of the nominal matrix of system $\mathbf{A}$.

Then, we can rewrite the dynamics equation (1.1) as

$$\dot{\mathbf{x}}(\mathrm{t}) = (\mathbf{A} + \mathbf{F}\Delta(t)\mathbf{H})\mathbf{x}(\mathrm{t}) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\mathbf{w}(\mathrm{t}). \tag{1.5}$$

As external perturbations, we consider three classes of signals

(1) arbitrary perturbations of bounded intensity (bounded in the L_∞-norm)

$$\|\mathbf{w}_1(t)\|_\infty \le 1, \ \forall t \ge 0 \tag{1.6}$$

(the unit function $\mathrm{w}(\mathrm{t}) \equiv 1$ or polyharmonical signals with bounded amplitudes of the harmonics can serve as an example of such a function);

(2) arbitrary perturbations decreasing in time (bounded in L_2-norm)

$$\|\mathbf{w}_2(t)\|_2^2 = \int_0^\infty \mathbf{w}_2^{\mathrm{T}}(t)\mathbf{w}_2(t)dt \le 1 \tag{1.7}$$

(the pulse functions that characterize shock actions on the system or polyharmonical signals with damped amplitudes of harmonics can serve as an example of such a function);

(3) a stationary Gaussian random process with the zero mean $M[\mathbf{w}_3(t)] = 0$ and the given intensity matrix $M[\mathbf{w}_3(t)\mathbf{w}_3^{\mathrm{T}}(t)] = \mathbf{Q}$ as a particular case of (2).

In what follows, we show that for all three types of perturbations, $\mathbf{w}(t)$ one can give a unified statement and solution to the control synthesis problem.

We state the requirements on the control accuracy as restrictions on each element of vector $\mathbf{z}(t)$ in the form

$$|z_i| \le c_i, \quad i = \overline{1, n_z}, \tag{1.8}$$

where $c_i > 0$ are real numbers that characterize the accuracy of the control. Such a statement of requirements on the accuracy of the control is the most characteristic one for engineering applications.

The synthesis problem is to analytically describe the set of transmission matrices K in the form of statistic linear feedback with respect to the state

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) \tag{1.9}$$

that robustly stabilizes the closed-loop system

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{F}\Delta(t)\mathbf{H} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{G}\mathbf{w}(t) \tag{1.10}$$

and ensures the given control accuracy with respect to the controlled parameters $\mathbf{z}(t)$ in the form of restrictions (1.8).

2. ANALYZING THE APPROACHES TO SOLVING THE SYNTHESIS PROBLEM

2.1. Using Invariant Ellipsoids

We consider system (1.5) with the nominal matrix **A** without control and with external perturbations $\mathbf{w}(t)$ that belong to any of the three classes considered above

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{G}\mathbf{w}(t). \tag{2.1}$$

We assume matrix **A** to be Hurwitz. Then, the limiting attainability set of system (2.1) is the totality of the ends of the trajectories of the system on the segment $t \in [0, \infty)$ subjected to any of the perturbations $\mathbf{w}(t)$

$$\aleph = \{\mathbf{x} \in \Re^{n_x}: \quad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{G}\mathbf{w}(t), \quad \|\mathbf{w}(t)\| \le 1\}. \tag{2.2}$$

It is shown [1, 2] that the set $\aleph$ for the stable matrix **A** is an invariant, closed, and bounded convex set of an arbitrary form in the general case. In [3, 4, 9, 10], for discrete and continuous linear systems, invariant ellipsoids of the system are proposed to be used as the approximation of attainability set (2.2) that are convenient in terms of computations. The ellipsoid

$$\Im_x = \{\mathbf{x} \in \Re^{n_x} : \mathbf{x}^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{x} \le 1\}, \quad \mathbf{P} = \mathbf{P}^{\mathrm{T}} > 0 \tag{2.3}$$

is called an invariant for system (2.1) if the condition $\mathbf{x}(0) \in \Im_x$ yields $\mathbf{x}(t) \in \Im_x$ for all instants $t \ge 0$. The positive definite matrix **P** is called the matrix of the ellipsoid $\Im_x$. The way to find matrix **P** depends on the perturbation class.

When system (2.1) is subjected to the perturbations $\mathbf{w}_1(t)$ bounded in the L_∞-norm and satisfying condition (1.6), ellipsoid (2.2) is a conservative approximation of the attainability set. The matrix of the minimal ellipsoid $\mathbf{P} = \mathbf{P}(\alpha)$ for $\mathbf{x}(0) = 0$ satisfies the Lyapunov equation

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathrm{T}} + \alpha\mathbf{P} + \alpha^{-1}\mathbf{G}\mathbf{G}^{\mathrm{T}} = 0, \ \mathbf{P} > 0, \tag{2.4}$$

where $0 < \alpha < -2\max \mathrm{Re}\,\lambda_{\mathrm{i}}(\mathbf{A})$ and $\lambda_{\mathrm{i}}(\mathbf{A})$ are the eigenvalues of matrix **A**.

When system (2.1) is subjected to the perturbations $\mathbf{w}_2(t)$ bounded in the L_2-norm and satisfying condition (1.7), the attainability set exactly coincides with the invariant ellipsoid (2.3). Matrix **P** is a controllability Gramian of the system and can be found from the Lyapunov equation

$$\mathbf{P}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{G}\mathbf{G}^{\mathrm{T}} = 0, \quad \mathbf{P} > 0. \tag{2.5}$$

If the perturbation vector $\mathbf{w}_3(t)$ is a stationary Gaussian random process with zero mean and the intensity matrix **Q**, the attainability set also exactly coincides with the invariant ellipsoid (2.3) and matrix **P** is a covariance matrix of the system and can be found from the Lyapunov equation symmetric to (2.5)

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathrm{T}} + \mathbf{G}\mathbf{Q}\mathbf{G}^{\mathrm{T}} = 0, \quad \mathbf{P} > 0. \tag{2.6}$$

Note that unlike the case of perturbations bounded in the L_2-norm, where the attainability set of the system is exactly described by the invariant ellipsoid [5], the ellipsoid is only the approximation of the attainability set for the perturbations bounded in the L_∞-norm. The properties of such an approximation are studied in [5], where its conservative nature is shown. However, from the point of perturbation compensation, this is not a principal drawback.

The attainability set in the form of the invariant ellipsoid for the controlled output has the form

$$\Im_z = \{\mathbf{z} \in \Re^{n_z} : \mathbf{z}^{\mathrm{T}}(\mathbf{DPD}^{\mathrm{T}})^{-1}\mathbf{z} \le 1\}, \quad \mathbf{P} > 0. \tag{2.7}$$

We can treat ellipsoid (2.7) as the characteristic of influence of external perturbations on the system's trajectory and use it to estimate the degree of influence of perturbations $\mathbf{w}(t)$ on the controlled parameters $\mathbf{z}(t)$.

From the computational and engineering points of view, it is convenient to consider the trace function $f(\mathbf{P}) = \mathrm{trace}[\mathbf{DPD}^{\mathrm{T}}]$ as such an estimate. Then, for the particular element of the controlled output $z_i = \mathbf{d}_i\mathbf{x}$, $i = \overline{1, n_z}$, where $\mathbf{d}_i$ is the ith row of the matrix $\mathbf{D}$, invariant ellipsoid (2.7) degenerates into the segment

$$|z_i| \le \left(\mathbf{d}_i\mathbf{P}\mathbf{d}_i^{\mathrm{T}}\right)^{1/2}. \tag{2.8}$$

Expression (2.8) allows us to state clear engineering requirements on the control quality

(1) the restrictions on the maximal values (with a certain degree of conservatism) of the controlled parameters

$$\max_{t\ge 0}\max_{\|\mathbf{w}\|\le 1}|z_i(t)|$$

for $\mathbf{w}_1$(t);

(2) the restrictions on the values of the steady-state error

$$e_{z_i} = \lim_{t\to\infty} z_i(t)$$

for $\mathbf{w}_2(t)$;

(3) the restrictions on the value of the root-mean-square deviation σ_z for $\mathbf{w}_3(t)$.

2.2. Robust Stabilization Based on the Common Lyapunov Function

We consider system (1.5) without control and external perturbations and take $\Delta(t) = \mathrm{const}$ for the sake of simplicity

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{F}\Delta\mathbf{H})\mathbf{x}(t),\ \mathbf{x}(0) = 0. \tag{2.9}$$

To ensure robust stability in the quadratic sense, we need to construct the common quadratic Lyapunov function $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{P}_\Delta\mathbf{x}$, $\mathbf{P}_\Delta = \mathbf{P}_\Delta^{\mathrm{T}} > 0$ for family (2.9). The simplest way to find the matrix $\mathbf{P}_\Delta$ is to use the Lyapunov inequality

$$(\mathbf{A} + \mathbf{F}\Delta\mathbf{H})\mathbf{P}_\Delta + \mathbf{P}_\Delta(\mathbf{A} + \mathbf{F}\Delta\mathbf{H})^{\mathrm{T}} < 0.$$

Taking into account condition (1.4), we can rewrite this inequality as

$$\mathbf{AP}_\Delta + \mathbf{P}_\Delta\mathbf{A}^{\mathrm{T}} + \gamma(\mathbf{F}\Delta\mathbf{HP}_\Delta + \mathbf{P}_\Delta\mathbf{H}^{\mathrm{T}}\Delta^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}) < 0 \tag{2.10}$$

for all uncertainties $\|\Delta\| \le 1$.

Using Petersen's lemma [5], we can represent expression (2.10) as the linear matrix inequality

$$\begin{bmatrix} \mathbf{AP}_\Delta + \mathbf{P}_\Delta\mathbf{A}^{\mathrm{T}} + \gamma^2\mathbf{FF}^{\mathrm{T}} & \mathbf{P}_\Delta\mathbf{H}^{\mathrm{T}} \\ \mathbf{HP}_\Delta & -\mathbf{I} \end{bmatrix} \le 0, \tag{2.11}$$

whose solution allows finding matrix $\mathbf{P}_\Delta$ of the Lyapunov function for the parametrically perturbed family (2.9). Here, we can treat parameter γ as the radius of robust stability.

2.3. Methodological Basis for Synthesis

As the methodological basis for synthesis, we use the methods for solving analytically linear matrix equations based on canonization of matrices that are fully described in [11] and the results of parametriz-

Table

Equation		Solvability Condition	Formula Representation
Name	Formula		
Left-hand side	$\mathbf{AX} = \mathbf{B}$	$\bar{\mathbf{A}}^{\mathrm{L}}\mathbf{B} = 0$	$\{\mathring{\mathbf{X}}\}_{\boldsymbol{\nu}} = \tilde{\mathbf{A}}\mathbf{B} + \bar{\mathbf{A}}^{\mathrm{R}}\boldsymbol{\nu}$
Left-hand side skew-symmetric	$\mathbf{AX} = \mathbf{B}$ $\mathbf{X} = -\mathbf{X}^{\mathrm{T}}$	$\bar{\mathbf{A}}^{\mathrm{L}}\mathbf{B} = 0$ $\mathbf{BA}^{\mathrm{T}} = -\mathbf{AB}^{\mathrm{T}}$	$\{\mathring{\mathbf{X}}\}_{\boldsymbol{\mu}} = \tilde{\mathbf{A}}\mathbf{B} - \mathbf{B}^{\mathrm{T}}\tilde{\mathbf{A}}^{\mathrm{T}}$ $+\tilde{\mathbf{A}}\mathbf{A}\mathbf{B}^{\mathrm{T}}\tilde{\mathbf{A}}^{\mathrm{T}} + \bar{\mathbf{A}}^{\mathrm{R}}\boldsymbol{\mu}(\bar{\mathbf{A}}^{\mathrm{R}})^{\mathrm{T}}$
Right-hand side	$\mathbf{XC} = \mathbf{B}$	$\mathbf{B}\bar{\mathbf{C}}^{\mathrm{R}} = 0$	$\{\mathring{\mathbf{X}}\}_{\boldsymbol{\lambda}} = \mathbf{B}\tilde{\mathbf{C}} + \boldsymbol{\lambda}\bar{\mathbf{C}}^{\mathrm{L}}$
Two-sided	$\mathbf{AXC} = \mathbf{B}$	$\bar{\mathbf{A}}^{\mathrm{L}}\mathbf{B} = 0, \mathbf{B}\bar{\mathbf{C}}^{\mathrm{R}} = 0$	$\{\mathring{\mathbf{X}}\}_{\boldsymbol{\nu},\boldsymbol{\lambda}} = \tilde{\mathbf{A}}\mathbf{B}\tilde{\mathbf{C}} + \bar{\mathbf{A}}^{\mathrm{R}}\boldsymbol{\nu} + \boldsymbol{\lambda}\bar{\mathbf{C}}^{\mathrm{L}}$

ing the Lyapunov equation [12]. Here, we briefly give the identities and relations used when solving the synthesis problem.

Canonization of the arbitrary matrix $\mathbf{A}$ of dimension $m \times n$ and rank r matches it with a set of five matrices that is not unique in the general case

$$\mathbf{A}_{m\times n} \to (\bar{\mathbf{A}}^{L}_{(m-r)\times m}, \tilde{\mathbf{A}}^{L}_{r\times m}, \tilde{\mathbf{A}}_{n\times m}, \tilde{\mathbf{A}}^{R}_{n\times r}, \bar{\mathbf{A}}^{R}_{n\times(n-r)}),$$

where $\bar{\mathbf{A}}^{L}$ and $\bar{\mathbf{A}}^{R}$ are the left and right divisors of zero of the maximal rank; $\tilde{\mathbf{A}}^{L}$, $\tilde{\mathbf{A}}^{R}$, and $\tilde{\mathbf{A}}$ are the left, right, and aggregated canonizers. These matrices satisfy the identities

$$\begin{bmatrix} \tilde{\mathbf{A}}^{L} \\ \bar{\mathbf{A}}^{L} \end{bmatrix} \mathbf{A} \begin{bmatrix} \tilde{\mathbf{A}}^{R} & \bar{\mathbf{A}}^{R} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{\mathrm{r}} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix}, \quad \tilde{\mathbf{A}}^{R}\tilde{\mathbf{A}}^{L} = \tilde{\mathbf{A}}, \quad \mathbf{A}\tilde{\mathbf{A}}\mathbf{A} = \mathbf{A}. \tag{2.12}$$

The repeated canonization of the divisors of zero of the maximal rank yields the following result

$$\bar{\mathbf{A}}^{L} \to (\mathbf{0}, \mathbf{0}, (\bar{\mathbf{A}}^{L})^{\mathrm{R}}, (\bar{\mathbf{A}}^{L})^{R}, \mathbf{A}\tilde{\mathbf{A}}^{R}), \ \bar{\mathbf{A}}^{R} \to (\tilde{\mathbf{A}}^{L}\mathbf{A}, (\bar{\mathbf{A}}^{R})^{L}, (\bar{\mathbf{A}}^{R})^{L}, \mathbf{0}, \mathbf{0}). \tag{2.13}$$

Here, the aggregate $(\bar{\mathbf{A}}^{L})^{R}$ means the right divisor of the unity of the left divisor of zero of the maximal rank for matrix $\mathbf{A}$.

If the matrix $\mathbf{B}$ is invertible in the product of the matrices $\mathbf{AB}$, the formula holds for canonization of the product of these matrices

$$\mathbf{AB} \to (\overline{\mathbf{AB}}^{L}, (\mathbf{AB})^{\sim}, \overline{\mathbf{AB}}^{R}) = (\bar{\mathbf{A}}^{L}, \mathbf{B}^{-1}\tilde{\mathbf{A}}, \mathbf{B}^{-1}\bar{\mathbf{A}}^{R}). \tag{2.14}$$

We put the formulas for analytically solving the equations used in this work together in a table.

Here, $\boldsymbol{\nu}$ and $\boldsymbol{\lambda}$ are arbitrary matrices of the appropriate dimensions and $\boldsymbol{\mu} = -\boldsymbol{\mu}^{\mathrm{T}}$ is an arbitrary skew-symmetric matrix of the appropriate dimension. The curly brackets are used to designate sets, while the subscripts are used for the parameters that generate them.

We see parameterization of an equation as a construction of the entire set of its matrix coefficients that do not affect its solution. Here, we use parameterization of the algebraic Lyapunov equation of the form

$$\mathbf{AP} + \mathbf{PA}^{\mathrm{T}} + \mathbf{Q} = 0, \quad \mathbf{P} = \mathbf{P}^{\mathrm{T}} > 0. \tag{2.15}$$

The entire set of numerical matrices $\mathbf{A}$ such that (2.15) possesses one and the same fixed solution $\mathbf{P}$ is given by the formula

$$\{\mathbf{A}\}_{\boldsymbol{\eta}} = -\frac{1}{2}\mathbf{QP}^{-1} - \mathbf{P}\boldsymbol{\eta}, \tag{2.16}$$

where $\boldsymbol{\eta} = -\boldsymbol{\eta}^{\mathrm{T}}$ is an arbitrary skew-symmetric matrix.

3. SOLVING THE SYNTHESIS PROBLEM

We consider the process of solving the synthesis problem by the example of the perturbation $w_1(t)$. To satisfy the given restrictions on the accuracy of the controlled output, closed-loop system (1.10) with the nominal matrix $\mathbf{A}$ and subjected to the perturbations $w_1(t)$ should satisfy the Lyapunov equation

$$(\mathbf{A}-\mathbf{BK})\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{BK})^{\mathrm{T}}+\alpha\mathbf{P}+\alpha^{-1}\mathbf{GG}^{\mathrm{T}}=0,\ \mathbf{P}>0, \tag{3.1}$$

with matrix $\mathbf{P}$, with its respective diagonal elements specified by the given restrictions (2.8).

We rewrite (3.1) in the form

$$(\mathbf{A}+\frac{1}{2}\alpha\mathbf{I}-\mathbf{BK})\mathbf{P}+\mathbf{P}(\mathbf{A}+\frac{1}{2}\alpha\mathbf{I}-\mathbf{BK})^{\mathrm{T}}+\alpha^{-1}\mathbf{GG}^{\mathrm{T}}=0. \tag{3.2}$$

Taking into account parameterization formula (2.16), we can use expression (3.2) to find the equation for synthesis of controllers that ensure the given control accuracy for the nominal system

$$\mathbf{BK}=\mathbf{A}+\frac{1}{2}\alpha\mathbf{I}+\frac{1}{2\alpha}\mathbf{GG}^{\mathrm{T}}\mathbf{P}^{-1}+\mathbf{P}\boldsymbol{\eta}. \tag{3.3}$$

We can treat (3.3) as a left-hand side matrix equation for the unknown matrix $\mathbf{K}$. According to the table, the necessary and sufficient solvability condition for (3.3) takes the form of equality

$$\bar{\mathbf{B}}^{L}\left(\mathbf{A}+\frac{1}{2}\alpha\mathbf{I}+\frac{1}{2\alpha}\mathbf{GG}^{\mathrm{T}}\mathbf{P}^{-1}+\mathbf{P}\boldsymbol{\eta}\right)=0 \tag{3.4}$$

with some skew-symmetric matrix $\boldsymbol{\eta}=-\boldsymbol{\eta}^{\mathrm{T}}$.

Thus, existence of the matrix $\boldsymbol{\eta}$ of the special structure such that equality (3.4) becomes an identity is necessary and sufficient for the given matrix $\mathbf{P}$ to be realizable in a closed-loop system (1.10) with controller $\mathbf{K}$.

To take into account the restrictions on the diagonal elements of matrix $\mathbf{P}$ that formalize the requirements on the control accuracy, we need to introduce additional conditions in the form of inequalities

$$\mathbf{d}_i\mathbf{P}\mathbf{d}_i^T\le c_i^2,\quad i=\overline{1,n_z}, \tag{3.5}$$

where c_i are real numbers that characterize the given control accuracy.

By the properties of the solution of the Lyapunov equation (3.1) and if the requirements on the matrices of system (1.10) stated in the problem statement are met, the set of controllers found from (3.3) is stabilizing for a system with the nominal matrix $\mathbf{A}$. However, to take into account the parametric perturbations (1.3), i.e., to ensure robust stability, we need to single out a subset that specifies the common Lyapunov function for the perturbed system (2.9), i.e., satisfies the matrix inequality (2.10), from the set of realizable matrices $\mathbf{P}$ given by expression (3.4) and restrictions (3.5). We call the matrices from such set $\mathbf{S}\in\mathbf{P}\cap\mathbf{P}_\Delta$ *attainable matrices* for system (1.10).

Since it is difficult to find the attainable matrix $\mathbf{S}$ analytically, we use numerical methods based on LMIs to solve this problem. The Matlab LMI Control Toolbox, as well as SeDuMi, YALMIP, and cvx, provides powerful numerical procedures for solving these inequalities. To use this approach, we solve the linear matrix inequalities of the form $\mathrm{F}(\mathbf{X})\le 0$, $\mathbf{X}=\mathbf{X}^{\mathrm{T}}>0$, the first of these resulting from the requirement for the system to be stable, rather than the matrix equations of the form $\mathrm{F}(\mathbf{X})=0$ under condition $\mathbf{X}=\mathbf{X}^{\mathrm{T}}>0$ and with the matrix $\mathbf{X}$ linearly included in it. To make the numerical solving process efficient, we state the general attainability problem as an optimization problem from the class of semidefinite programming (SDP) problems.

We state the theorem based on the previous results.

Theorem. Suppose $\mathbf{S}$ is the solution to the semidefinite programming problem

$$\mathrm{tr}(\mathbf{S})\to\min$$

under the restrictions

$$\begin{bmatrix}\bar{\mathbf{B}}^{L}(\mathbf{AS}+\mathbf{SA}^{\mathrm{T}}+\alpha\mathbf{S}+\gamma\mathbf{FF}^{\mathrm{T}})(\bar{\mathbf{B}}^{L})^{\mathrm{T}} & \bar{\mathbf{B}}^{L}\mathbf{SH}^{\mathrm{T}} & \bar{\mathbf{B}}^{L}\mathbf{G}\\ \mathbf{HP}(\bar{\mathbf{B}}^{L})^{\mathrm{T}} & -\mathbf{I} & 0\\ \mathbf{G}^{\mathrm{T}}(\bar{\mathbf{B}}^{L})^{\mathrm{T}} & 0 & -\alpha\mathbf{I}\end{bmatrix}\le 0, \tag{3.6}$$

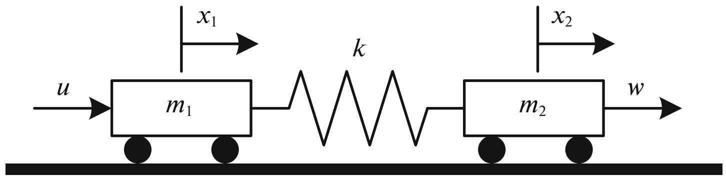

Fig. 1. Scheme of the two-mass system.

$$\mathbf{d}_i\mathbf{S}\mathbf{d}_i^T \le c_i^2, \quad i = \overline{1, n_z} \tag{3.7}$$

with respect to the matrix variable $\mathbf{S} = \mathbf{S}^{\mathrm{T}} > 0$ and scalar variables $\alpha > 0$ and $\gamma > 0$.

Then, matrix $\mathbf{S}$ is an attainable matrix in the closed-loop system (1.10) and corresponds to the given restrictions on the output subjected to external perturbations with the uncertainty radius α and on the robust quadratic stability of the system with the stability radius γ.

The entire set of controllers of the static feedback with respect to state (1.9) with the transfer matrix $\mathbf{K}$ that ensure the given value of the matrix $\mathbf{S}$ is specified by the formula

$$\{\mathbf{K}\}_{\boldsymbol{\mu},\boldsymbol{\nu}} = \tilde{\mathbf{B}}(\mathbf{N} - \boldsymbol{\mu})\mathbf{S}^{-1} + \overline{\mathbf{B}}^R\boldsymbol{\nu}, \tag{3.8}$$

where the matrix $\mathbf{N}$ can be found from the expression

$$\mathbf{N} = \mathbf{A}\mathbf{S} + \frac{1}{2}\alpha\mathbf{S} + \frac{1}{2\alpha}\mathbf{G}\mathbf{G}^{\mathrm{T}}; \tag{3.9}$$

$\boldsymbol{\mu}$ is any matrix from the set of skew-symmetric matrices

$$\boldsymbol{\mu} = \mathbf{N}^{\mathrm{T}}(\overline{\overline{\mathbf{B}}}^L)^{\mathrm{T}} - \overline{\overline{\mathbf{B}}}^L\mathbf{N} + \overline{\overline{\mathbf{B}}}^L\mathbf{N}^{\mathrm{T}}(\overline{\overline{\mathbf{B}}}^L)^{\mathrm{T}} + \mathbf{B}\boldsymbol{\rho}\,\mathbf{B}^{\mathrm{T}}; \tag{3.10}$$

$\overline{\overline{\mathbf{B}}}^L = (\overline{\mathbf{B}}^L)^R\overline{\mathbf{B}}^L$ is the reduced left divisor of zero of matrix $\mathbf{B}$; $\boldsymbol{\rho}$ is an arbitrary skew-symmetric matrix of the appropriate dimension; and $\boldsymbol{\nu}$ is an arbitrary matrix of the appropriate dimension.

The theorem is proved in the Appendix.

In most practical problems, the matrix of control efficiency $\mathbf{B}$ has the full row rank. For such a matrix, $\overline{\mathbf{B}}^R = 0$, $\tilde{\mathbf{B}} = \tilde{\mathbf{B}}^L$, and expression (3.8) describing the set of controllers of the given accuracy takes the form

$$\{\mathbf{K}\}_{\boldsymbol{\mu}} = \tilde{\mathbf{B}}^L(\mathbf{N} - \boldsymbol{\mu})\mathbf{S}^{-1}. \tag{3.11}$$

As follows from formulas (3.8) and (3.10), the set of equivalent controllers is generated by the skew-symmetric matrix $\boldsymbol{\mu}$ from some intricately arranged set.

For the perturbations $w_2(t)$ and $w_3(t)$, the calculation formulas are found similarly based on Lyapunov equations (2.5) and (2.6), respectively.

4. EXAMPLE

As an example, we consider the control problem for the two-mass system used in [5] as a testing problem for various methods of synthesizing controllers. Figure 1 gives the scheme of the system; the continuous model of perturbed oscillations is described by the system of equations

$$\begin{cases} \dot{x}_1 = v_1, \\ \dot{v}_1 = -\dfrac{k}{m_1}x_1 + \dfrac{k}{m_1}x_2 + \dfrac{1}{m_1}u, \\ \dot{x}_2 = v_2, \\ \dot{v}_2 = \dfrac{k}{m_2}x_1 - \dfrac{k}{m_2}x_2 - w, \end{cases} \tag{4.1}$$

where x_1, x_2 are the coordinates of the first and second bodies; v_1, v_2 are the velocities of the first and second bodies; m_1, m_2 are the masses of the first and second bodies; u is the control applied to the first body; w is the perturbation acting on the second body; and k is the elastic coefficient of the spring.

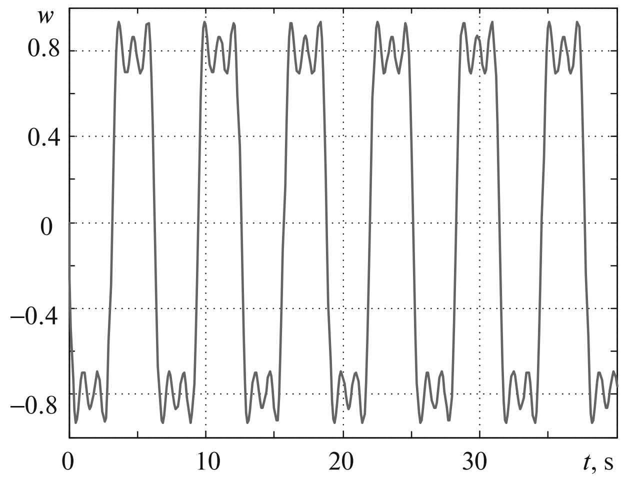

Fig. 2. Graph of perturbation.

The arbitrary perturbation $w(t)$ satisfies condition (1.6), i.e., $|w(t)| \le 1$. As the controlled parameters, we consider the coordinate and velocity of the second body subjected to the perturbation; thus, $\mathbf{z}(t) = [x_2 \;\; v_2]^{\mathrm{T}}$.

Suppose $m_1 = m_2 = 1$, and the perturbations of the nominal matrix of the dynamics are due to the uncertainty of the elastic coefficient of the spring

$$k = 1 + \delta\Delta(t), \quad |\Delta(t)| \le 1.$$

In vector-matrix form (1.5), model (4.1) has the following values of the matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 \\ -\delta \\ 0 \\ \delta \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{H} = [1 \;\; 0 \;\; -1 \;\; 0].$$

The matrix $\mathbf{A}$ of the model is unstable; the pairs $(\mathbf{A}, \mathbf{B})$ and $(\mathbf{A}, \mathbf{G})$ are completely controlled.

First, we solve the problem of minimizing the invariant ellipsoid of the closed-loop system $\mathrm{trace}(\mathbf{S}) \to \min$ with the nominal dynamics matrix, whose matrix satisfies the attainability condition (3.6). Solving this problem numerically in cvx [13], we find the following value of the matrix of the invariant ellipsoid for the controlled output (for $\alpha = 0.5$)

$$\mathbf{DSD}^{\mathrm{T}} = \begin{bmatrix} 4.47 & -1.23 \\ -1.23 & 4.18 \end{bmatrix}.$$

The respective set of controllers (3.9) has the form

$$\{\mathbf{K}\}_\mu = [4.81 - 0.27\mu \;\; 1.95 - 0.2\mu \;\; 0.11\mu - 0.46 \;\; 3.75 - 0.26\mu], \tag{4.2}$$

where μ is an arbitrary real number.

Suppose the system is subjected to a harmonic perturbation that satisfies restriction (1.6), the graph of which is shown in Fig. 2. Figure 3 gives the results of simulating the controlled output z(t) of the closed-loop system with controller (4.2) for $\mu = 0$ and zero initial state. The poles of the closed-loop system have the values

$$\lambda(\mathbf{A} - \mathbf{BK}_0) = \{-0.49 \pm 2.18j \;\; -0.48 \pm 0.79j\}.$$

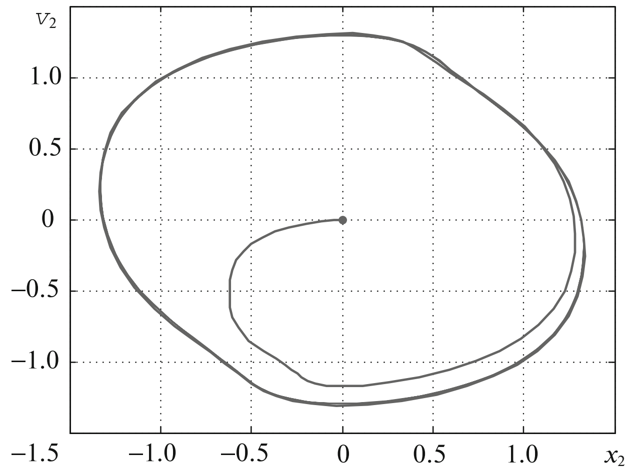

Fig. 3. Trajectory of the output of the system with controller (4.2).

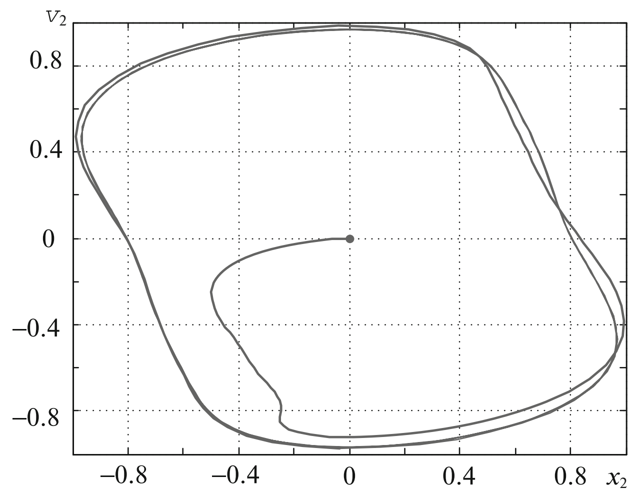

Fig. 4. Trajectory of the output of the system with controller (4.3).

Now, we calculate the transfer matrix of the static controller that ensures the influence of the perturbation on the controlled output is half that of the minimal controller (4.2), which corresponds to the values $\mathbf{z}(t) = [2.23\ \ 2.09]^{\mathrm{T}}$.

Solving the synthesis problem numerically in cvx results in the following value of the matrix of the invariant ellipsoid for the controlled output (for $\delta = 0.1$, $\alpha = 0.5$, $\gamma = 0.12$):

$$\mathbf{DSD}^{\mathrm{T}} = \begin{bmatrix} 2.23 & -0.61 \\ -0.61 & 2.09 \end{bmatrix}.$$

The respective set of controllers has the form

$$\{\mathbf{K}\}_{\mu} = \left[12.63 - 0.11\mu \ \ 2.39 - 0.05\mu \ \ 0.02\mu - 0.69 \ \ 14.29 - 0.16\mu\right], \tag{4.3}$$

where μ is an arbitrary real number.

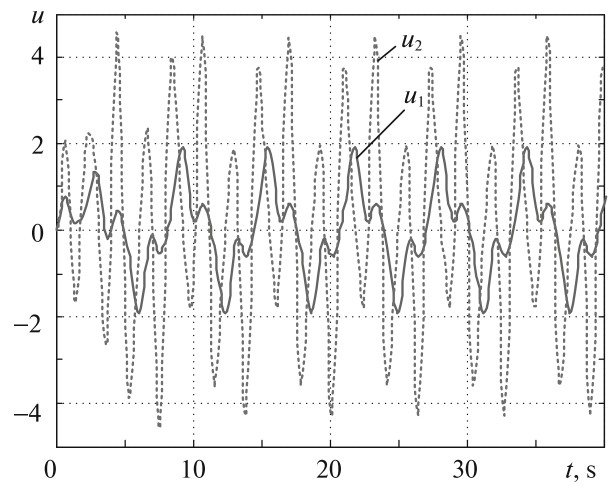

Fig. 5. Graphs of control signals.

Figure 4 shows the results of simulating the controlled output $z(t)$ of the closed-loop system with controller (4.3) for $\mu = 0$, $\Delta(t) = \sin 2t$ and zero initial state. The poles of the closed-loop system have the values

$$\lambda(\mathbf{A} - \mathbf{BK}_0) = \{-0.35 \pm 3.44j \ \ -0.58 \pm 0.78j\}.$$

The simulation results show that the requirements imposed on the control accuracy are met. Figure 5 gives the graphs of the control signals $u_1(t)$ and $u_2(t)$ for controllers (4.2) and (4.3).

CONCLUSIONS

The approach we propose in this work allows finding an analytical description of the set of robust static feedback controllers that ensure the given control accuracy for the controlled output of a linear multiconnected system subjected to various classes of perturbations. Since the set has varied parameters, we can satisfy the additional synthesis requirements or choose the structure of the controller. The solvability conditions of the synthesis problem are given as linear matrix inequalities. This allows using the well elaborated numerical MATLAB-based methods to calculate the attainable matrix of the invariant ellipsoid.

APPENDIX

P r o o f o f t h e t h e o r e m. Introducing designation (3.9) to reduce the writing, we rewrite condition (3.4) as the matrix left-hand side equation

$$\bar{\mathbf{B}}^L\mathbf{P}\boldsymbol{\eta} = -\bar{\mathbf{B}}^L\mathbf{N}\mathbf{P}^{-1} \qquad \text{(A.1)}$$

with respect to the skew-symmetric matrix $\boldsymbol{\eta}$. According the table, the solvability conditions of (A.1), given invertibility of the matrix $\mathbf{P}$ and formula (2.14), have the form

$$\overline{\bar{\mathbf{B}}^L}^L\bar{\mathbf{B}}^L\mathbf{N} = 0; \quad -\bar{\mathbf{B}}^L\mathbf{N}(\bar{\mathbf{B}}^L)^{\mathrm{T}} = \bar{\mathbf{B}}^L\mathbf{N}^{\mathrm{T}}(\bar{\mathbf{B}}^L)^{\mathrm{T}}.$$

The first one is met always due to formula (2.15), while the second one needs to be checked and, after the respective substitutions, takes the form of a matrix equation

$$\bar{\mathbf{B}}^L(\mathbf{AP} + \mathbf{PA}^{\mathrm{T}} + \alpha\mathbf{P} + \alpha^{-1}\mathbf{GG}^{\mathrm{T}})(\bar{\mathbf{B}}^L)^{\mathrm{T}} = 0. \qquad \text{(A.2)}$$

Taking into account the transformations by Schur's lemma [2] and based on the above reasoning, (A.2) is matched with the linear matrix inequality

$$\begin{bmatrix} \bar{\mathbf{B}}^L(\mathbf{AP} + \mathbf{PA}^{\mathrm{T}} + \alpha\mathbf{P})(\bar{\mathbf{B}}^L)^{\mathrm{T}} & \bar{\mathbf{B}}^L\mathbf{G} \\ \mathbf{G}^{\mathrm{T}}(\bar{\mathbf{B}}^L)^{\mathrm{T}} & -\alpha\mathbf{I} \end{bmatrix} \le 0. \tag{A.3}$$

By (2.4), the scalar parameter $\alpha > 0$ characterizes the uncertainty radius of external perturbations bounded in the L_∞- norm.

Then, the existence of the attainable matrix $\mathbf{S}$ for a parametrically perturbed closed-loop system (1.10) is equivalent to the existence of $\alpha(\Delta) > 0$ such that the following condition is met:

$$\begin{bmatrix} \bar{\mathbf{B}}^L\Omega(\bar{\mathbf{B}}^L)^{\mathrm{T}} & \bar{\mathbf{B}}^L\mathbf{G} \\ \mathbf{G}^{\mathrm{T}}(\bar{\mathbf{B}}^L)^{\mathrm{T}} & -\alpha(\Delta)\mathbf{I} \end{bmatrix} \le 0, \tag{A.4}$$

where $\Omega = (\mathbf{A} + \mathbf{F}\Delta\mathbf{H})\mathbf{S} + \mathbf{S}(\mathbf{A} + \mathbf{F}\Delta\mathbf{H})^{\mathrm{T}} + \alpha(\Delta)\mathbf{S}$.

Suppose there exists a number $\alpha > 0$ such that matrix inequality (A.4) holds for all admissible Δ. Then, inequality (A.4) holds if the matrix inequality holds

$$\begin{bmatrix} \bar{\mathbf{B}}^L(\mathbf{AS} + \mathbf{SA}^{\mathrm{T}} + \alpha\mathbf{S})(\bar{\mathbf{B}}^L)^{\mathrm{T}} & \bar{\mathbf{B}}^L\mathbf{G} \\ \mathbf{G}^{\mathrm{T}}(\bar{\mathbf{B}}^L)^{\mathrm{T}} & -\alpha\mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{F} \\ 0 \end{bmatrix}\Delta\begin{bmatrix} \mathbf{HP} & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{PH}^{\mathrm{T}} \\ 0 \end{bmatrix}\Delta^{\mathrm{T}}\begin{bmatrix} \mathbf{F}^{\mathrm{T}} & 0 \end{bmatrix} \le 0.$$

By Petersen's lemma and taking into account expressions (2.10) and (2.11), this relation can be transformed to condition (3.6) in the statement of the theorem and ensures the attainable matrix $\mathbf{S}$ belongs to the set of matrices of the Lyapunov function for the parametrically perturbed family (1.10). By (2.10), the scalar parameter $\gamma > 0$ characterizes the stability radius of system (1.10). Condition (3.7) in the statement of the theorem ensures that the requirements on the accuracy of each element of the output vector $\mathbf{z}(t)$ are met in the form of restrictions (2.8).

Now, we move to the solution formula. Equation (3.3), given designations (3.9), takes the form

$$\mathbf{BK} = \mathbf{NS}^{-1} + \mathbf{S}\boldsymbol{\eta},$$

and according to the table, we can write its solution as

$$\{\mathbf{K}\}_{\eta,\nu} = \tilde{\mathbf{B}}(\mathbf{NS}^{-1} + \mathbf{S}\boldsymbol{\eta}) + \bar{\mathbf{B}}^R\boldsymbol{\nu}. \tag{A.5}$$

The skew-symmetric matrix $\boldsymbol{\eta}$ must correspond to the attainable matrix $\mathbf{S}$ and be the solution of (A.1), which follows from condition (3.6). Since the matrix $\mathbf{S}$ is invertible and taking into account formulas (2.13), (2.14), we can write the solution of (A.1) as

$$\boldsymbol{\eta} = \mathbf{S}^{-1}(\mathbf{N}^{\mathrm{T}}(\bar{\bar{\mathbf{B}}}^L)^{\mathrm{T}} - \bar{\bar{\mathbf{B}}}^L\mathbf{N} - \bar{\bar{\mathbf{B}}}^L\mathbf{N}^{\mathrm{T}}(\bar{\bar{\mathbf{B}}}^L)^{\mathrm{T}} + \mathbf{B}\boldsymbol{\rho}\mathbf{B}^{\mathrm{T}})\mathbf{S}^{-1}. \tag{A.6}$$

Substituting expression (A.6) into formula (A.5), performing transformations, and introducing new designations, we arrive at the solution's formula (3.8) that is put in the theorem's statement. The theorem is proved.

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