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Relational semantics for full linear logic

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ABSTRACT

Relational semantics, given by Kripke frames, play an essential role in the study of modal and intuitionistic logic. In [4] it is shown that the theory of relational semantics is also available in the more general setting of substructural logic, at least in an algebraic guise. Building on these ideas, in [5] a type of frames is described which generalise Kripke frames and provide semantics for substructural logics in a purely relational form.

In this paper we study full linear logic from an algebraic point of view. The main additional hurdle is the exponential. We analyse this operation algebraically and use canonical extensions to obtain relational semantics. Thus, we extend the work in [4,5] and use their approach to obtain relational semantics for full linear logic. Hereby we illustrate the strength of using canonical extension to retrieve relational semantics: it allows a modular and uniform treatment of additional operations and axioms.

Traditionally, so-called phase semantics are used as models for (provability in) linear logic [8]. These have the drawback that, contrary to our approach, they do not allow a modular treatment of additional axioms. However, the two approaches are related, as we will explain.

1. Introduction

Relational semantics, given by Kripke frames, play an essential role in the study of modal and intuitionistic logic [3]. They provide an intuitive interpretation of the logic and a means to obtain information about it. The possibility of applying semantical techniques to obtain information about a logic motivates the search for relational semantics in a more general setting.

Many logics are closely related to corresponding classes of algebraic structures which provide *algebraic semantics* for the logics. The algebras associated to classical modal logic are Boolean algebras with an additional operator (BAOs). Kripke frames arise naturally from the duality theory for these structures in the following way. Boolean algebras are dually equivalent to Stone spaces [11]. A modal operator on Boolean algebras translates to a binary relation with certain topological properties on the corresponding dual spaces, hence giving rise to so-called descriptive general frames. Forgetting the topology yields Kripke frames, which are in a discrete duality with perfect modal algebras, *i.e.*, modal algebras whose underlying Boolean algebra is a powerset algebra and whose operator is complete. This may be depicted as follows:

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$$\begin{tikzcd}
\begin{array}{c}\text{syntactic}\\\text{specification}\end{array} \arrow[d, leftrightsquigarrow] & & \\
\begin{array}{c}\text{modal}\\\text{algebras}\end{array} \arrow[rr, shift left, "\text{topological duality}"] \arrow[d, "\text{canonical extension}"'] & & \begin{array}{c}\text{descriptive}\\\text{general frames}\end{array} \arrow[ll, shift left] \arrow[d, "\text{forget topology}"] \\
\begin{array}{c}\text{perfect}\\\text{modal algebras}\end{array} \arrow[rr, shift left, "\text{discrete duality}"] & & \begin{array}{c}\text{Kripke}\\\text{frames}\end{array} \arrow[ll, shift left] \arrow[d, leftrightsquigarrow] \\
 & & \begin{array}{c}\text{relational}\\\text{semantics}\end{array}
\end{tikzcd}$$

Hence, one may retrieve relational semantics for modal logic by first moving horizontally using the duality and thereafter going down by forgetting the topology.

Many other interesting logics, including substructural logics, however, have algebraic semantics which are not based on distributive lattices and for these duality theory is vastly more complicated or even non-existent. Luckily, the picture above also indicates an alternative route to obtain relational semantics: going down first and thereafter going right. The (left) downward mapping is given by taking the *canonical extension* of a BAO. Canonical extensions were introduced in the 1950s by Jónsson and Tarski exactly for BAOs [9,10]. Thereafter their ideas have been developed further, which has led to a smooth theory of canonical extensions applicable in a broad setting [6,7]. In [4] canonical extensions of partially ordered algebras are defined to obtain relational semantics for the implication–fusion fragment of various substructural logics. Their approach is purely algebraic. In [5] this work is translated to the setting of possible world semantics. A class of frames (RS-frames) is described which generalise Kripke frames and provide semantics for substructural logics in a purely relational form. This is summarised in the following picture:

$$\begin{tikzcd}
\begin{array}{c}\text{syntactic}\\\text{specification}\end{array} \arrow[d, leftrightsquigarrow] & \\
\begin{array}{c}\text{partially ordered}\\\text{algebras}\end{array} \arrow[d, "\text{canonical extension}"'] & \\
\begin{array}{c}\text{perfect lattices}\\\text{with additional operations}\end{array} \arrow[r, shift left, "\text{discrete duality}"] & \begin{array}{c}\text{RS-frames}\\\text{with additional relations}\end{array} \arrow[l, shift left] \arrow[d, leftrightsquigarrow] \\
 & \begin{array}{c}\text{relational}\\\text{semantics}\end{array}
\end{tikzcd} \tag{1}$$

A well-known substructural logic that extends the basic implication–fusion fragment is linear logic. Linear logic was introduced by Jean-Yves Girard [8]. Formulas in linear logic represent resources that may be used exactly once. Proof-theoretically this is witnessed by the fact that the structural rules contraction and weakening are not admissible in general. However, these structural rules are allowed in a controlled way by means of a new modality, the exponential !, which expresses the case of unlimited availability of a specific resource. Traditionally, phase spaces are used as semantics for linear logic. These have the drawback that, contrary to the approach described above, they do not allow a modular treatment of additional operations and axioms.

In [4] relational semantics for the basic implication–fusion fragment of linear logic was obtained. In this paper we extend this approach to derive relational semantics for full linear logic. We show that the axioms of the logic in question satisfy canonicity, and we identify the corresponding relational structures. We show that this method of canonicity and correspondence allows a modular and uniform treatment of the additional operations and axioms of linear logic. The modularity distinguishes our work from earlier derivations of Kripke-style semantics for linear logic [1]. Furthermore, we translate our results to one-sorted frames in order to compare these to phase semantics.

The paper is structured as follows: first, we discuss the general method of obtaining relational semantics for substructural logics using canonical extension, essentially by explaining how to move 'down-right' in the picture above (Section 2) and by indicating how to show that this indeed yields complete relational semantics (Section 3). We focus on the parts of this general theory that are important for the remainder of our paper and refer the reader to [4,5] for more details. In Section 4 this method is applied to obtain relational semantics for the multiplicative additive fragment of linear logic (MALL). The modular set-up allows us to augment this result by deriving relational semantics for the exponential, as we work out in Section 5. This gives relational semantics for full linear logic. In Section 6, which serves as an intermezzo, we look at the exponentials from the algebraic perspective. Finally, in Section 7 we discuss how our results relate to phase semantics.

2. Duality between perfect lattices and RS-polarities

Algebraic semantics for substructural logics are given by partially ordered sets (posets) with additional operations on them (*partially ordered algebras*). Hence, the first step in obtaining relational semantics for substructural logics, using the method depicted in (1), is to define canonical extensions for posets. This is worked out in Section 2 of [4] where one can find a careful and clear explanation of this theory. We quickly recap the relevant definitions below.

Definition 1. Let $e : P \to Q$ be an embedding of partially ordered sets. We identify the elements of P with their image in Q. An element of Q is called a *filter element* if it is the infimum in Q of some filter in P. We write $F(Q)$ for the set of filter elements of Q. Dually, an element of Q is called an *ideal element* if it is the supremum in Q of some ideal in P. We denote the set of ideal elements of Q by $I(Q)$. In [4], these elements are called closed and open, respectively.

Definition 2. A *canonical extension* of a poset P is an order embedding $e : P \hookrightarrow C$ of P in a complete lattice C, satisfying

- every element of C is both the supremum of all filter elements below it and the infimum of all ideal elements above it (*denseness*);
- for F a filter of P, I an ideal of P, $\bigwedge e[F] \leqslant \bigvee e[I]$ implies $F \cap I \neq \emptyset$ (*compactness*).

Every poset P has a canonical extension which is unique up to an isomorphism fixing P. We denote this extension by P^δ.

Note that, by denseness, every element of the canonical extension may be written as a join of elements of $F(P^\delta)$ and as a meet of elements of $I(P^\delta)$. This yields two ways to extend an order-preserving map f between posets to a map between their canonical extensions, namely:

Definition 3. Let P and Q be posets, and $f : P \to Q$ an order-preserving map. Define maps $f^\sigma, f^\pi : P^\delta \to Q^\delta$ by setting, for $u \in P^\delta$,

$$f^\sigma(u) = \bigvee\Big\{\bigwedge\big\{f(p)\colon\ x \leqslant p \in P\big\}\colon\ u \geqslant x \in F\big(P^\delta\big)\Big\},$$
$$f^\pi(u) = \bigwedge\Big\{\bigvee\big\{f(p)\colon\ y \geqslant p \in P\big\}\colon\ u \leqslant y \in I\big(P^\delta\big)\Big\}.$$

It can be shown that f^σ and f^π are order-preserving extensions of f, that send filter elements to filter elements and ideal elements to ideal elements. We will use the sigma-extension to describe canonical extension of (additional operations on) the algebras corresponding to linear logic. At the end of Section 5.1 we explain, for the experts on canonical extension, why working with the sigma-extension is most natural in this setting.

In case P and Q are lattices and $f : P \to Q$ preserves joins (or meets), f^π and f^σ coincide and we denote this unique extension by f^δ. The structures arising as canonical extensions of posets are perfect lattices.

Definition 4. A *perfect lattice* L is a complete lattice that is both join-generated by its completely join-irreducible elements $\mathcal{J}^\infty(L)$ and meet-generated by its completely meet-irreducible elements $\mathcal{M}^\infty(L)$.

To move horizontally in (1) one should identify relational structures that are in a duality with perfect lattices. In [5] a class of (two-sorted) frames fulfilling this requirement is described. These frames generalise the traditional notion of a Kripke frame. We introduce these structures here and briefly discuss this duality.

Definition 5. A (*two-sorted*) *frame* is a triple $F = (X, Y, \preccurlyeq)$ where X and Y are sets and $\preccurlyeq \subseteq X \times Y$ is a relation from X to Y.

As explained in [5], the set X can be thought of as a set of worlds and the set Y as a set of 'information quanta' or 'co-worlds'. If $x \preccurlyeq y$, then y is said to be a part of x. Interpretants in these models consist of both a set of worlds and a set of information quanta, and we want either of these to completely determine the interpretant. This allows us to describe the interpretant in either of the two ways, whichever is most convenient given a particular situation. This requirement is fulfilled if the interpretants are Galois-closed subsets of the following Galois connection between $\wp(X)$ and $\wp(Y)$, associated with the frame F:

$$\begin{aligned} (\,)^u\ :\ \wp(X) &\to \wp(Y)\\ A &\mapsto \{y \in Y \mid \forall x.\ x \in A \Rightarrow x \preccurlyeq y\},\\ (\,)^l\ :\ \wp(Y) &\to \wp(X)\\ B &\mapsto \{x \in X \mid \forall y.\ y \in B \Rightarrow x \preccurlyeq y\}. \end{aligned}$$

The complete lattice of Galois-closed subsets of X is given by $\mathcal{G}(F) = \{A \subseteq X \mid (A^u)^l = A\}$, which is a perfect lattice.

Conversely, for every perfect lattice L, we define a frame $\mathcal{F}(L)$ by $X = \mathcal{J}^\infty(L)$, $Y = \mathcal{M}^\infty(L)$ and, for all $x \in X$, $y \in Y$,

$$x \preccurlyeq y \quad \Leftrightarrow \quad x \leqslant_L y.$$

This frame is *separating*, *i.e.*, the following two conditions hold:

1. $\forall x_1, x_2 \in X \ (x_1 \neq x_2 \Rightarrow \{x_1\}^u \neq \{x_2\}^u)$;
2. $\forall y_1, y_2 \in Y \ (y_1 \neq y_2 \Rightarrow \{y_1\}^l \neq \{y_2\}^l)$.

Furthermore it is *reduced*, *i.e.*, the following two conditions hold:

1. $\forall x \in X \ \exists y \in Y \ (x \not\preccurlyeq y$ and $\forall x' \in X \ [\{x'\}^u \supset \{x\}^u \Rightarrow x' \preccurlyeq y])$,
2. $\forall y \in Y \ \exists x \in X (y \not\succcurlyeq x$ and $\forall y' \in Y \ [\{y'\}^l \subset \{y\}^l \Rightarrow y' \succcurlyeq x])$.

A frame that is both separating and reduced is called an *RS-frame*. The separating property implies that the maps

$$\begin{array}{rcl} X & \to & \mathcal{G}(F) \\ x & \mapsto & \left(\{x\}^u\right)^l \end{array} \qquad \begin{array}{rcl} Y & \to & \mathcal{G}(F) \\ y & \mapsto & \{y\}^l \end{array}$$

are injective. Therefore we may think of X and Y as subsets of $\mathcal{G}(F)$ and we will write x both for the element of X and for the corresponding element $\{x\}^{ul}$ of $\mathcal{G}(F)$ (and similarly for elements of Y). For a separating frame, being reduced exactly means that the elements of X are completely join-irreducible in $\mathcal{G}(F)$ and the elements of Y are completely meet-irreducible in $\mathcal{G}(F)$.

An RS-frame morphism $F_1 = (X_1, Y_1, \preccurlyeq) \to (X_2, Y_2, \preccurlyeq) = F_2$ is a pair of relations $S_1 \subseteq Y_1 \times X_2$, $S_2 \subseteq X_1 \times Y_2$ satisfying some conditions. These conditions ensure that the pair of relations gives rise to a complete lattice homomorphism $\mathcal{G}(S_1, S_2) : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$. Conversely, for each complete lattice homomorphism $f : L_1 \to L_2$ between perfect lattices, one may define an RS-frame morphism $\mathcal{F}(f) : \mathcal{F}(L_2) \to \mathcal{F}(L_1)$.

Proposition 6. *The mappings $\mathcal{F}$ and $\mathcal{G}$ form a duality between the category of perfect lattices and the category of RS-frames.*

For further details and a proof of the above proposition, the reader is referred to [5].

3. Relational semantics via canonical extension

We will now extend and apply the basic theory of the previous section to describe the general method for obtaining relational semantics for substructural logics.

The basic substructural logic we consider is non-associative Lambek calculus (NLC). Its signature consists of three binary operations $\otimes$, $\to$, $\leftarrow$. The axioms of NLC state that the implications $\to$ and $\leftarrow$ are residuals of the fusion $\otimes$. Algebraic semantics for this logic is given by residuated algebras.

Definition 7. A *residuated algebra* is a structure $(P, \otimes, \to, \leftarrow)$, where P is a partially ordered set and, for all $x, y, z \in P$,

$$x \otimes y \leqslant z \quad \Leftrightarrow \quad y \leqslant x \to z \quad \Leftrightarrow \quad x \leqslant z \leftarrow y.$$

A residuated algebra is called *perfect* if and only if its underlying poset is a perfect lattice.

For a perfect residuated algebra, the underlying perfect lattice L corresponds dually to the RS-frame $\mathcal{F}(L) = (\mathcal{J}^\infty(L), \mathcal{M}^\infty(L), \leqslant_L)$, as explained in Section 2. The action of the fusion (and thereby of its residuals) may be encoded on this dual frame as follows. First note that, as the fusion is residuated, it is completely join-preserving in both co-ordinates. Therefore, its action is completely determined by its action on pairs from $\mathcal{J}^\infty(L) \times \mathcal{J}^\infty(L)$. Define a relation $R_\otimes \subseteq \mathcal{J}^\infty(L) \times \mathcal{J}^\infty(L) \times \mathcal{M}^\infty(L)$ by

$$R_\otimes(x_1, x_2, y) \quad \Leftrightarrow \quad x_1 \otimes x_2 \leqslant y.$$

The relation $R_\otimes$ is *compatible*, that is, for all $x_1, x_2 \in \mathcal{J}^\infty(L)$, $y \in M^\infty(L)$, the sets

$$R_\otimes[x_1, x_2, \_] \qquad R_\otimes[x_1, \_, y] \qquad R_\otimes[\_, x_2, y]$$

are Galois-closed.[1]

[1] We may also witness the fusion $\otimes$ dually by the relation $R_\downarrow \subseteq (\mathcal{J}^\infty(L))^3$ defined by $R_\downarrow(x_1, x_2, x_3) \Leftrightarrow x_3 \leqslant x_1 \otimes x_2$. In that case, however, the conditions stating that R arises from a fusion are less natural.

Definition 8. A structure $F = (X, Y, \preccurlyeq, R)$, where $(X, Y, \preccurlyeq)$ is an RS-frame and $R \subseteq X \times X \times Y$ is a compatible relation, is called a *relational RS-frame*.

Conversely, for an RS-frame $F = (X, Y, \preccurlyeq)$, a relation $R \subseteq X \times X \times Y$ gives rise to a fusion $\otimes_R$ on $\mathcal{G}(F)$, by defining

$$\begin{aligned} x_1 \otimes_R x_2 &= \bigwedge \{ y \in Y \mid R(x_1, x_2, y) \} && \text{for all } x_1, x_2 \in X, \\ u_1 \otimes_R u_2 &= \bigvee \{ x_1 \otimes_R x_2 \mid x_1, x_2 \in X,\ x_1 \leqslant u_1,\ x_2 \leqslant u_2 \} && \text{for all } u_1, u_2 \in \mathcal{G}(F). \end{aligned}$$

This operation is completely join-preserving in both coordinates and therefore it is residuated, with residuals $\to_R$ and $\leftarrow_R$. For any residuated fusion operation $\otimes$ on a perfect lattice, $\otimes_{R_\otimes} = \otimes$ and, for any compatible relation R on an RS-frame, $R_{\otimes_R} = R$.

Proposition 9. *(See Proposition 6.6 in [4].) The above defined maps $(L, \otimes, \to, \leftarrow) \mapsto (\mathcal{F}(L), R_\otimes)$ and $(X, Y, \preccurlyeq, R) \mapsto (\mathcal{G}(X, Y, \preccurlyeq), \otimes_R, \to_R, \leftarrow_R)$ yield a duality between perfect residuated algebras and relational RS-frames.*[2]

We denote the extended mappings of the above proposition by $\mathcal{F}_{ra}$ and $\mathcal{G}_{ra}$. For a residuated algebra P, the σ-extension of its fusion, $\otimes^\sigma : P^\delta \times P^\delta \to P^\delta$, is a residuated operator on the canonical extension P^δ (Corollary 3.7 of [4]). This completes the description of the walk through (1) for NLC: We start with a residuated algebra P, its canonical extension is a perfect residuated algebra P^δ which yields a relational frame via the mapping $\mathcal{F}_{ra}$.

We will now describe how to interpret the logic NLC on relational RS-frames.

Definition 10. Let S be a set of propositional letters. An *interpretation of S in a frame* $F = (X, Y, \preccurlyeq, R)$ is a map $V : S \to \mathcal{G}_{ra}(F)$. This yields a satisfaction relation defined by, for $x \in X$, $s \in S$,

$$(F, V), x \Vdash s \quad \Leftrightarrow \quad x \leqslant V(s).$$

In case $(F, V), x \Vdash s$ holds, we say *s holds at x in (F, V)*. We also obtain an information content relation defined by, for $y \in Y$,

$$(F, V), y \succ s \quad \Leftrightarrow \quad y \geqslant V(s).$$

In case $(F, V), y \succ s$ holds, we say *y is part of s in (F, V)*.

Let $Fm(S)$ be the collection of all formulas in the language $(\otimes, \to, \leftarrow)$ over S. Quotienting $Fm(S)$ by provable equivalence (in NLC) and defining the operations $\otimes$, $\to$ and $\leftarrow$ on equivalence classes yields a residuated algebra $\widetilde{Fm}(S)$. An interpretation $V : S \to G_{ra}(F)$ uniquely extends to a homomorphism $\overline{V} : \widetilde{Fm}(S) \to \mathcal{G}_{ra}(F)$. We may inductively extend the relations $\Vdash$ and $\succ$ to the collection $Fm(S)$, in such a way that, for $\phi \in Fm(S)$,

$$(F, V), x \Vdash \phi \quad \Leftrightarrow \quad x \leqslant \overline{V}([\phi]),$$

$$(F, V), y \succ \phi \quad \Leftrightarrow \quad y \geqslant \overline{V}([\phi]),$$

where $[\phi]$ denotes the equivalence class of ϕ in $\widetilde{Fm}(S)$. A concrete description of these relations is given in Section 4 of [5]. For example, for $\phi, \psi \in Fm(S)$, $x \in X$, $y \in Y$,

$$y \succ \phi \otimes \psi \quad \Leftrightarrow \quad \forall x_1, x_2 \in X.\ (x_1 \Vdash \phi \text{ and } x_2 \Vdash \psi) \quad \Rightarrow \quad R(x_1, x_2, y);$$

$$x \Vdash \phi \otimes \psi \quad \Leftrightarrow \quad \forall y \in Y.\ y \succ \phi \otimes \psi \quad \Rightarrow \quad x \preccurlyeq y.$$

For formulas ϕ and ψ in $Fm(S)$, we say the sequent $\phi \vdash \psi$ is valid in the frame F if and only if, for every valuation V in F, the following equivalent conditions hold:

1. $\forall x \in X.\ (F, V), x \Vdash \phi \Rightarrow (F, V), x \Vdash \psi$;
2. $\forall y \in Y.\ (F, V), y \succ \phi \Rightarrow (F, V), y \succ \psi$;
3. $\overline{V}([\phi]) \leqslant \overline{V}([\psi])$.

It immediately follows that,

$$\phi \Vdash \psi \quad \text{is valid in } F \quad \Leftrightarrow \quad \phi \leqslant \psi \quad \text{holds in } \mathcal{G}_{ra}(F). \tag{2}$$

We are now ready to describe our method for obtaining relational semantics for a substructural logic. Let $\mathcal{E}$ be a collection of inequalities axiomatising a logic $\mathcal{L}_\mathcal{E}$ in the connectives $\otimes, \to, \leftarrow$, extending NLC. The collection $\mathcal{A}lg_\mathcal{E}$ of residuated algebras

[2] Note that we have not spelled out which morphisms we consider in both categories. The reader interested in more details is referred to [4].

satisfying the inequalities in $\mathcal{E}$ provides complete algebraic semantics for $\mathcal{L}_{\mathcal{E}}$, in the sense that, for all formulas ϕ, ψ,

$$\phi \vdash \psi \text{ is derivable in } \mathcal{L}_{\mathcal{E}} \quad \text{iff} \quad \phi \leqslant \psi \text{ holds in all residuated algebras in } \mathcal{A}lg_{\mathcal{E}}.$$

Our aim is to describe a collection of relational frames $\mathcal{K}$ which provides complete relational semantics for $\mathcal{L}_{\mathcal{E}}$. We define, for a collection of relational frames $\mathcal{K}$,

$$\mathcal{K}^{+} = \{\mathcal{G}_{ra}(F) \mid F \in \mathcal{K}\}.$$

By (2), $\mathcal{K}$ provides complete relational semantics for $\mathcal{L}_{\mathcal{E}}$ if and only if $\mathcal{L}_{\mathcal{E}} = \mathcal{E}qThr(\mathcal{K}^{+})$, where $\mathcal{E}qThr(\mathcal{K}^{+})$ is the equational theory of $\mathcal{K}^{+}$, *i.e.*, the collection of inequalities that hold in all algebras in $\mathcal{K}^{+}$.

We will show that, to obtain complete relational semantics for $\mathcal{L}_{\mathcal{E}}$, it suffices to obtain:

1. Canonicity: show that $\mathcal{A}lg_{\mathcal{E}}$ is closed under canonical extension, that is, show that, for all $P \in \mathcal{A}lg_{\mathcal{E}}$, $P^{\delta} \in \mathcal{A}lg_{\mathcal{E}}$.
2. Correspondence: describe a class of frames $\mathcal{K}$ which satisfies $\mathcal{A}lg_{\mathcal{E}}^{\delta} \subseteq \mathcal{K}^{+} \subseteq \mathcal{A}lg_{\mathcal{E}}$, where $\mathcal{A}lg_{\mathcal{E}}^{\delta} = \{P^{\delta} \mid P \in \mathcal{A}lg_{\mathcal{E}}\}$.

Proposition 11. *If $\mathcal{A}lg_{\mathcal{E}}$ is closed under canonical extension, then $\mathcal{E}qThr(\mathcal{A}lg_{\mathcal{E}}) = \mathcal{E}qThr(\mathcal{A}lg_{\mathcal{E}}^{\delta})$.*

Proof. As, by assumption, $\mathcal{A}lg_{\mathcal{E}}^{\delta} \subseteq \mathcal{A}lg_{\mathcal{E}}$, clearly $\mathcal{E}qThr(\mathcal{A}lg_{\mathcal{E}}) \subseteq \mathcal{E}qThr(\mathcal{A}lg_{\mathcal{E}}^{\delta})$. For the converse, suppose $\phi \leqslant \psi$ holds in $\mathcal{A}lg_{\mathcal{E}}^{\delta}$ and $P \in \mathcal{A}lg_{\mathcal{E}}$. As P embeds in its canonical extension P^{δ} and $P^{\delta} \in \mathcal{A}lg_{\mathcal{E}}^{\delta}$, $\phi \leqslant \psi$ holds in P. $\square$

If $\mathcal{A}lg_{\mathcal{E}}$ is closed under canonical extension we say the collection of axioms $\mathcal{E}$ is *canonical*. It follows from the above proposition that in this case any collection $\mathcal{K}$ of frames satisfying $\mathcal{A}lg_{\mathcal{E}}^{\delta} \subseteq \mathcal{K}^{+} \subseteq \mathcal{A}lg_{\mathcal{E}}$ provides complete relational semantics for $\mathcal{L}_{\mathcal{E}}$. In case the axioms in $\mathcal{E}$ are 'sufficiently simple' one may obtain, in a mechanical way, first-order conditions on relational frames describing a class $\mathcal{K}$ with this property. Many well-known logics may be axiomatised by canonical and 'sufficiently simple' axioms, hence the above described procedure may be applied to these logics to obtain complete relational semantics. In [4] this is worked out for the fusion–implication fragment of the Lambek calculus, linear logic, relevance logic, BCK logic and intuitionistic logic.

We will apply an extension of the above method to obtain relational semantics for linear logic. Algebraic semantics for linear logic is given by residuated algebras equipped with additional operations (corresponding to the additional connectives of linear logic). To obtain relational semantics, one has to give a description of these additional operations on the relational frames. In the next section we will illustrate this procedure by deriving relational semantics for multiplicative additive linear logic. In the section thereafter we augment this result by describing the exponentials on the frame side, thereby obtaining relational semantics for full linear logic.

4. Relational semantics for MALL

We start by deriving relational semantics for the multiplicative additive fragment of linear logic (MALL). Its algebraic semantics are given by classical linear algebras, which are extensions of the residuated algebras studied in the previous section.

Definition 12. A *classical linear algebra* (CL-algebra) is a structure $(L, \otimes, \rightarrow, \leftarrow, 1, 0)$, where

1. $(L, \otimes, \rightarrow, \leftarrow)$ is a residuated algebra;
2. the fusion $\otimes$ is associative and commutative and has a unit 1;
3. L is a bounded lattice;
4. for all $a \in L$, $(a \rightarrow 0) \rightarrow 0 = a$.

A *perfect CL-algebra* is a CL-algebra whose underlying lattice is perfect.

In linear logic, the meet operation is denoted by & (with unit $\top$), the join by $\oplus$ (with unit 0), the implication by $\multimap$ and our constant 0 is denoted by $\perp$. However, as we will refer to the literature from lattice theory we stick to the usual lattice theoretic notation and denote meet by $\wedge$ (with unit $\top$) and join by $\vee$ (with unit $\perp$). For further details on CL-algebras the reader is referred to [12], which uses a notation similar to ours.

We denote $x \rightarrow 0$ by $x^{\perp}$ and call this operation *linear negation*. Implication sends joins in the first coordinate to meets, hence $(\_)^{\perp}$ sends joins to meets. As $(\_)^{\perp}$ is a bijection, it follows that it is a (bijective) lattice homomorphism $L \rightarrow L^{\partial}$, where L^{∂} is the lattice obtained by reversing the order in L. It follows that the σ- and π-extension of the linear negation coincide and we denote the unique extension of $(\_)^{\perp}$ to the canonical extension L^{δ} by $(\_)^{\perp^{\delta}}$.

The first step in obtaining relational semantics for MALL is checking canonicity, *i.e.*, ensuring that the class **CL** of CL-algebras is closed under canonical extension.

Proposition 13. *The class* **CL** *is closed under canonical extension.*

Proof. Let L be a CL-algebra and let L^δ be its canonical extension. In [4] it is shown that L^δ is a perfect residuated algebra. Hence, in particular, it is a bounded lattice. Furthermore, it is shown that, if $\otimes$ is associative (resp. commutative), then so is its extension $\otimes^\sigma$. The unit 1_L of the fusion in L is the unit for $\otimes^\sigma$ as, for all $w \in L^\delta$,

$$\begin{aligned} w \otimes^\sigma 1_L &= \bigvee \{x \mid w \geqslant x \in F(L^\delta)\} \otimes^\sigma 1_L \\ &= \bigvee \{x \otimes^\sigma 1_L \mid w \geqslant x \in F(L^\delta)\} \\ &= \bigvee \Big\{ \bigwedge \{p \otimes 1_L \mid x \leqslant p \in L\} \Bigm| w \geqslant x \in F(L^\delta) \Big\} \\ &= \bigvee \Big\{ \bigwedge \{p \mid x \leqslant p \in L\} \Bigm| a \geqslant x \in F(L^\delta) \Big\} \\ &= \bigvee \{x \mid w \geqslant x \in F(L^\delta)\} = w. \end{aligned}$$

It follows from the results in [2] that, for all $w \in L^\delta$, $(w^{\perp^\delta})^{\perp^\delta} = w$. This completes the proof that L^δ is a CL-algebra. □

To describe the constants 1 and 0 dually, we have to extend the relational frames with two Galois-closed subsets, $U \subseteq X$ and $Z \subseteq Y$. Starting from a perfect CL-algebra L, these sets are given by[3]

$$U_L = \{x \in \mathcal{J}^\infty(L) \mid x \leqslant 1\} \quad \text{and} \quad Z_L = \{y \in \mathcal{M}^\infty(L) \mid 0 \leqslant y\}.$$

Our next step is to characterise the collection of frames $F = (X, Y, \preccurlyeq, R, U, Z)$ such that $\mathcal{G}(F)$ is a (perfect) CL-algebra (this constitutes the correspondence result). In the remainder of this section, we assume that any element named x (resp. y) with any super- or subscript comes from X (resp. Y).

By Corollary 6.14 in [4], the fusion in $\mathcal{G}(F)$ is associative if and only if F satisfies (Φ_a):

$$\begin{aligned} &\forall x_1, x_2, x_3\, \forall y. \\ &\big[\forall x_2' \,\big(\forall y' \,[R(x_2, x_3, y') \Rightarrow x_2' \preccurlyeq y'] \Rightarrow R(x_1, x_2', y)\big)\big] \\ &\quad \Leftrightarrow \quad \big[\forall x_1' \,\big(\forall y'' \,[(R(x_1, x_2, y'') \Rightarrow x_1' \preccurlyeq y'')] \Rightarrow R(x_1', x_3, y)\big)\big]. \end{aligned} \tag{Φ_a}$$

Furthermore, by Corollary 6.17 in [4], the fusion in $\mathcal{G}(F)$ is commutative if and only if F satisfies (Φ_c):

$$\forall x_1, x_2\, \forall y. \quad R(x_1, x_2, y) \quad \Leftrightarrow \quad R(x_2, x_1, y). \tag{Φ_c}$$

For U to be the unit of the fusion in $\mathcal{G}(F)$ we have to ensure that $W \otimes U = W$ for all $W \in \mathcal{G}(F)$. As the fusion on $\mathcal{G}(F)$ is completely join-preserving, it suffices to ensure $x \otimes U = x$ for all $x \in X (= \mathcal{J}^\infty(\mathcal{G}(F)))$. Note that,

$$\begin{aligned} x \otimes U \leqslant y \quad &\Leftrightarrow \quad \bigvee \{x \otimes x' \mid x' \leqslant U\} \leqslant y \\ &\Leftrightarrow \quad \forall x' \in U. x \otimes x' \leqslant y \\ &\Leftrightarrow \quad U \subseteq R[x, \_, y]. \end{aligned}$$

Hence, U is the unit of the fusion in $\mathcal{G}(F)$ if and only if F satisfies (Φ_u):

$$\forall x\, \forall y. \quad x \preccurlyeq y \quad \Leftrightarrow \quad U \subseteq R[x, \_, y]. \tag{Φ_u}$$

Now we have come to the last axiom: $(a \to 0) \to 0 = a$. First note that, by the adjunction property,

$$a \leqslant (a \to 0) \to 0 \quad \Leftrightarrow \quad (a \to 0) \otimes a \leqslant 0 \quad \Leftrightarrow \quad a \to 0 \leqslant a \to 0.$$

So in any case $a \leqslant (a \to 0) \to 0$. Furthermore, the mapping $a \mapsto (a \to 0) \to 0$ is completely join-preserving and therefore it again suffices to consider completely join-irreducible elements. Note that, for $x' \in \mathcal{J}^\infty(\mathcal{G}(F))$,

$$\begin{aligned} x' \leqslant (x \to 0) \to 0 \quad &\Leftrightarrow \quad (x \to 0) \otimes x' \leqslant 0 \\ &\Leftrightarrow \quad x \to 0 \leqslant x' \to 0 \\ &\Leftrightarrow \quad \forall x''.\, x'' \leqslant x \to 0 \quad \Rightarrow \quad x'' \leqslant x' \to 0 \\ &\Leftrightarrow \quad \forall x''.\, x \otimes x'' \leqslant 0 \quad \Rightarrow \quad x' \otimes x'' \leqslant 0 \\ &\Leftrightarrow \quad \forall x''.\, Z \subseteq R[x, x'', \_] \quad \Rightarrow \quad Z \subseteq R[x', x'', \_]. \end{aligned}$$

[3] We could also have described Z as a subset of X, however as it occurs in the axiom $(a \to 0) \to 0 = a$ and the implication is meet-preserving in the second coordinate, it is more convenient to describe it by the collection of meet-irreducibles above it, *i.e.*, by a subset of Y.

Hence, the equation $(a \to 0) \to 0 = a$ holds in $\mathcal{G}(F)$ if and only if F satisfies (Φ_{dd}):

$$\forall x, x'. \left(\forall x''. Z \subseteq R\left[x, x'', \_\right] \quad \Rightarrow \quad Z \subseteq R\left[x', x'', \_\right]\right) \quad \Rightarrow \quad x' \leqslant x.^4 \qquad (\Phi_{dd})$$

The above calculations lead to the following duality result.

Definition 14. A *CL-frame* is an extended relational RS-frame $F = (X, Y, \preccurlyeq, R, U, Z)$, where U is a Galois-closed subset of X, Z is a Galois-closed subset of Y and the extended frame F satisfies Φ_a, Φ_c, Φ_u and Φ_{dd}.

Theorem 15. *The mappings $L \mapsto (\mathcal{F}_{ra}(L), U_L, Z_L) = \mathcal{F}_{cl}(L)$ and $F \mapsto (\mathcal{G}_{ra}(F), U, Z) = \mathcal{G}_{cl}(F)$ yield a duality between perfect CL-algebras and CL-frames.*

Combining (an extension of) Proposition 11 and Proposition 13, it follows that the perfect CL-algebras provide complete semantics for MALL. Using the duality result obtained above we may now conclude the following.

Corollary 16. *The class of CL-frames gives complete semantics for MALL.*

5. Relational semantics for full linear logic

We are now ready to consider full linear logic. This is an extension of the previously defined multiplicative additive fragment with a unary operation which is called exponential. A special feature of this exponential is the fact that on formulas that are in its image, the structural rules contraction and weakening are allowed. We start by describing the algebraic semantics of the exponential. The following definition is equivalent to the one given in [12].

Definition 17. Let L be a CL-algebra. An *exponential* on L is a mapping $! : L \to L$ such that, for all $a, b \in L$,

1. $!!a = !a \leqslant a$;
2. $a \leqslant b \Rightarrow !a \leqslant !b$;
3. $!\top = 1$;
4. $!a \otimes !b = !(a \wedge b)$.

In this case, $(L, !)$ is called a *CLS-algebra*. We write **CLS** for the class of CLS-algebras.

To obtain relational semantics for full linear logic, we have to prove that the properties of the exponential satisfy canonicity, and we have to describe the exponentials dually.

5.1. Canonicity of the exponential

In this subsection we prove that the class of CLS-algebras is closed under canonical extension. In Proposition 13 of Section 4, we already showed that the canonical extension of a CL-algebra is again a CL-algebra. Using this result, it is only left to prove that the extended version of ! is an exponential on L^δ.

Theorem 18. *The class* **CLS** *is closed under canonical extension.*

Proof. We will check that the map $!^\sigma$ on L^δ satisfies the four properties of Definition 17.

1. Using that $!p \leqslant p$ for all $p \in L$, we have, for all $x \in F(L^\delta)$,

$$\begin{aligned} !^\sigma x &= \bigwedge \{!p \mid x \leqslant p \in L\} \\ &\leqslant \bigwedge \{p \mid x \leqslant p \in L\} \\ &= x. \end{aligned}$$

This result for filter elements implies that for all $a \in L^\delta$,

[4] Note that the statement $x' \leqslant x$ uses the ordering of $\mathcal{G}(F)$. We may also write this in the language of the frame as: $\forall y.\ x \preccurlyeq y \Rightarrow x' \preccurlyeq y$. For readability we use the shorthand $x' \leqslant x$.

$$\begin{aligned} !^\sigma a &= \bigvee\big\{!^\sigma x \,\big|\, a \geqslant x \in F\big(L^\delta\big)\big\} \\ &\leqslant \bigvee\big\{x \,\big|\, a \geqslant x \in F\big(L^\delta\big)\big\} \\ &= a. \end{aligned}$$

It follows that $!^\sigma !^\sigma a \leqslant !^\sigma a$, so it is left to show that $!^\sigma a \leqslant !^\sigma !^\sigma a$. Lemma 3.4 from [4] implies that $!^\sigma$ sends filter elements to filter elements, hence for $x \in F(L^\delta)$, we have that $!^\sigma x \in F(L^\delta)$, and thus $!^\sigma !^\sigma x = \bigwedge\{!p \mid !^\sigma x \leqslant p \in L\}$. For $x \in F(L^\delta)$, we will prove that

$$\begin{aligned} !^\sigma !^\sigma x &= \bigwedge\big\{!p \,\big|\, !^\sigma x \leqslant p \in L\big\} \\ &\geqslant \bigwedge\{!q \mid x \leqslant q \in L\} \\ &= !^\sigma x, \end{aligned}$$

by showing that for every meetand of the first meet, there exists a meetand in the second meet that is below it. Let $p \in L$ be such that $!^\sigma x \leqslant p$. Then

$$!^\sigma x = \bigwedge\{!q \mid x \leqslant q \in L\} \leqslant p.$$

By compactness, there exist $q_1, \ldots, q_n \in L$ such that $x \leqslant q_i$ for $i \in \{1, \ldots, n\}$, and $!q_1 \wedge \cdots \wedge !q_n \leqslant p$. Then $x \leqslant q_1 \wedge \cdots \wedge q_n$, and since ! is order-preserving we have that $!(q_1 \wedge \cdots \wedge q_n) \leqslant !q_1 \wedge \cdots \wedge !q_n \leqslant p$. We denote $q_1 \wedge \cdots \wedge q_n =: q$. By assumption, $!q \leqslant !!q$. This implies, together with $!q \leqslant p$ and the fact that ! is order-preserving, that $!q \leqslant !!q \leqslant !p$. Furthermore, $!q$ is a meetand of the second meet since $x \leqslant q$ and $q \in L$. Thus, we have now proved, for all filter elements x, $!^\sigma x \leqslant !^\sigma !^\sigma x$.
For $a \in L^\delta$, we want to show that

$$\begin{aligned} !^\sigma a &= \bigvee\big\{!^\sigma x \,\big|\, a \geqslant x \in F\big(L^\delta\big)\big\} \\ &\leqslant \bigvee\big\{!^\sigma x \,\big|\, !^\sigma a \geqslant x \in F\big(L^\delta\big)\big\} \\ &= !^\sigma !^\sigma a. \end{aligned}$$

Let x' be such that $!^\sigma x'$ is a joinand of the first join, *i.e.*, such that $a \geqslant x' \in F(L^\delta)$. Lemma 3.4 from [4] states that $!^\sigma$ is order-preserving, hence $!^\sigma a \geqslant !^\sigma x'$. Since $!^\sigma$ sends filter elements to filter elements we have that $!^\sigma x' \in F(L^\delta)$. Thus,

$$!^\sigma !^\sigma x' \in \big\{!^\sigma x \,\big|\, !^\sigma a \geqslant x \in F\big(L^\delta\big)\big\}.$$

Since $!^\sigma !^\sigma x' \geqslant !^\sigma x'$ by the previous result for filter elements, this means that there is a joinand in the second join that is above the joinand of the first join that we started with.

2. The exponential ! is order-preserving, hence, by Lemma 3.4 from [4], $!^\sigma$ is order-preserving.
3. As $!^\sigma \top_{L^\delta} = !^\sigma \top_L = !\top_L = 1_L$ and 1_L is the unit of the fusion $\otimes^\sigma$ on L^δ, it follows that $!^\sigma \top_{L^\delta} = 1_{L^\delta}$.
4. Define $[!, !] : (a, b) \mapsto (!a, !b)$. For all $a, b \in L^\delta$,

$$\begin{aligned} !^\sigma a \otimes^\sigma !^\sigma b &= \big(\otimes^\sigma \circ [!, !]^\sigma\big)(a, b) \\ &\overset{(1)}{=} \big(\otimes \circ [!, !]\big)^\sigma (a, b) \\ &\overset{(2)}{=} (! \circ \wedge_L)^\sigma (a, b) \\ &\overset{(3)}{=} \big(!^\sigma \circ \wedge_L^\sigma\big)(a, b) \\ &= \big(!^\sigma \circ \wedge_{L^\delta}\big)(a, b) \\ &= !^\sigma (a \wedge_{L^\delta} b). \end{aligned}$$

The σ-extension does not preserve composition in general. In [6] it is analysed in which case it does. The equalities $^{(1)}$ and $^{(3)}$ rely on special instances of this general theory, $^{(2)}$ follows from the assumption for ! on L. □

In our definition of canonical extensions of CL-algebras and CLS-algebras we have used the sigma-extensions of the additional operations and not their pi-extensions (cf. Definition 3). For canonical extension experts, it should come as no surprise that we use the sigma-extension of the fusion operation: In a residuated algebra fusion is a lower adjoint. This is precisely the property preserved by sigma-extension (whereas pi-extension preserves the property of being an upper adjoint in any one coordinate).

However, the choice in the case of the exponential is a bit more subtle. To explain it, note first that properties (1) and (2) of exponentials (Definition 17) are equivalent to saying that !, viewed as a map from L to Im(!), is the *upper* adjoint of the inclusion map $e : \mathrm{Im}(!) \to L$ (Lemma 19). This indicates that the pi-extension might be the more natural extension. However, since ! and e are unary and are, as maps between Im(!) and L, meet-preserving and join-preserving, respectively, they are in fact both smooth and their pi- and sigma-extensions agree.

The map !, viewed as a map from L to L, is in this light actually the composition $e \circ !$, and the question arises how the sigma- and pi-extension of this composition relate to the composition of the (unique) extensions of the two maps individually. Using the fact that the outer map is an operator (or that the inner is meet-preserving), we see that $(e \circ !)^\sigma = e^\sigma \circ (!)^\sigma = e^\delta \circ (!)^\delta$ and thus the sigma-extension of !, viewed as a map from L to L, captures the 'right' extension of !. On the other hand, since e is not a dual operator and ! is not join-preserving, there is no reason to believe that $(e \circ !)^\pi$ agrees with these other extensions of $e \circ !$. In fact, we expect that there are plenty of examples where ! is non-smooth as an operation on L.

5.2. Correspondence for the exponential

For the correspondence result for full linear logic, we have to extend the relational structures from Section 4, which give complete semantics for MALL, such that they also account for the exponential. We start by observing some properties of the exponential.

Lemma 19. *Let L be a CL-algebra, $! : L \to L$ an operation, and $I = \mathrm{Im}(!)$. Then the following are equivalent:*

i. *! satisfies 17.1 and 17.2,*
ii. *! is the upper adjoint of the embedding $I \hookrightarrow L$.*

And, in this case, I is closed under all (possibly infinitary) joins that exist in L. If, in addition, ! satisfies 17.3, then I is a meet-semilattice in the induced order and $! : L \to I$ is meet-preserving. Finally, under the additional assumption of 17.4, the meet of I is given by the fusion of L.

Proof. Proving the equivalence statement is left to the reader. Now let $S \subseteq I$ and suppose $\alpha := \bigvee_L S$ exists. For all $s \in S$, $s \leqslant \alpha$, hence by 17.2, $!s \leqslant !\alpha$. Since $s \in I$, we have that $!s = s$. Thus $!\alpha$ is an upper bound for S and therefore $!\alpha \geqslant \bigvee_L S = \alpha$. By 17.1, $!\alpha \leqslant \alpha$, hence $\alpha = !\alpha \in I$. Thus I is closed under all joins that exist in L.

To show that I is a meet-semilattice, let $i, j \in I$. Using 17.2, it follows that $!(i \wedge_L j)$ is a common lower bound for $!i = i$ and $!j = j$, and $!(i \wedge_L j) \in I$. For all $k \in I$ such that k is a common lower bound for i and j, $k \leqslant i \wedge_L j$, hence by the adjunction property $k \leqslant !(i \wedge_L j)$. Thus $i \wedge_I j = !(i \wedge_L j)$. Using 17.3 we see that, for all $i \in I$, $i = !i \leqslant !\top_L = 1_L$. Hence, 1_L is the top element of I and this is the unit of the meet. Since $! : L \to I$ is the upper adjoint of the embedding, it is meet-preserving, *i.e.*, for all $a, b \in L$, $!(a \wedge_L b) = !a \wedge_I !b$.

Finally, in case ! satisfies 17.4 as well, the meet on I is given by the fusion on L: $i \wedge_I j = !(i \wedge_L j) = !i \otimes !j = i \otimes j$. □

The following proposition shows that exponentials on L can be characterised by certain subsets of L. Later on we will use this result to describe exponentials dually. In Theorem 8.18 in [12], it was already shown that certain subsets give rise to exponentials. However, contrary to [12], we restrict to subsets that are closed under certain joins. In this way, we obtain a one-to-one correspondence between specific subsets and exponentials.

Proposition 20. *There is a bijective correspondence between exponentials on L and collections $I \subseteq L$ such that*

(I1) *for all $a \in L$, $\bigvee\{i \in I \mid i \leqslant a\}$ exists and is an element of I;*
(I2) *I is closed under $\otimes$;*
(I3) *for all $i \in I$, $i \otimes i = i$;*
(I4) *$1 \in I$ and for all $i \in I$, $i \leqslant 1$.*

Proof. Let $I \subseteq L$ satisfy (I1)–(I4). Define, for $a \in L$, $!_I(a) = \bigvee\{i \in I \mid i \leqslant a\}$. Using (I1), it follows that the image of $!_I$ is exactly I. It is readily checked that $!_I$ is the upper adjoint of the inclusion map $I \hookrightarrow L$, whence, by Lemma 19, the first two properties of Definition 17 are satisfied. Furthermore, since $!_I$ is an upper adjoint, it sends $\top$ to the top of I, which is 1 according to (I4).

It remains to check that, for all $a, b \in L$, $!_I(a \wedge b) = !_I(a) \otimes !_I(b)$. As, by Lemma 19, $!_I : L \to I$ is meet-preserving, $!_I(a \wedge_L b) = !_I(a) \wedge_I !_I(b)$. Note that $!_I(a) \otimes !_I(b) \leqslant !_I(a) \otimes 1 = !_I(a)$, and similarly for b. This implies $!_I(a) \otimes !_I(b) \leqslant !_I(a) \wedge_I !_I(b) = !_I(a \wedge_L b)$. The other inequality is shown as follows,

$$
\begin{aligned}
!_I(a \wedge_L b) &= \bigvee \{i \in I \mid i \leqslant a \wedge_L b\} \\
&= \bigvee \{i \in I \mid i \leqslant a, i \leqslant b\} \\
&= \bigvee \{i \otimes i \mid i \in I, i \leqslant a, i \leqslant b\} \quad (\text{by (I3)}) \\
&\leqslant \bigvee \{i \otimes i' \mid i, i' \in I,\ i \leqslant a,\ i' \leqslant b\} \\
&= \bigvee \{i \in I \mid i \leqslant a\} \otimes \bigvee \{i \in I \mid i \leqslant b\} \\
&= !_I(a) \otimes !_I(b).
\end{aligned}
$$

Now, let $!$ be an exponential on L. Define $I_! = \{a \in L \mid !a = a\}$. Note that $I_! = \mathrm{Im}(!)$, since for all $a \in I_!$, $a = !a \in \mathrm{Im}(!)$, and for all $!b \in \mathrm{Im}(!)$, $!!b = !b$, hence $!b \in I_!$. We leave it to the reader to check that $I_!$ satisfies (I1)–(I4).

The bijectivity of this correspondence follows from the fact that $I = I_{!_I}$ and $! = !_{I_!}$. $\square$

Remark 21. In the setting of complete lattices, the first condition of Proposition 20 is equivalent to the condition that I is closed under all joins.

Our next goal is to describe the exponentials dually, *i.e.*, to describe a class of extended frames $\mathcal{K}$ such that the class $\mathcal{K}^+ = \{\mathcal{G}(F) \mid F \in \mathcal{K}\}$ is contained in **CLS** and contains at least all CLS-algebras of the form L^δ (where $L \in$ **CLS**) (cf. Section 3).

It follows from Proposition 20 that an exponential on a CL-algebra is completely determined by its image. Hence, we could describe an exponential dually by extending the frame with the collection of Galois-closed sets corresponding to the elements in the image of the exponential. But in fact, we can do with a much smaller set. As the canonical extension L^δ, of a CL-algebra L, is a perfect lattice, by Remark 21, the image $\mathrm{Im}(!^\sigma)$ of the extended exponential $!^\sigma$ is a complete join-sublattice of L^δ. Furthermore, $\mathrm{Im}(!^\sigma)$ is isomorphic to $(\mathrm{Im}(!))^\delta$, whence, it is a perfect lattice itself. These observations allow us to describe the exponential dually by remembering only the completely join-irreducible elements of $\mathrm{Im}(!^\sigma)$. The remainder of this section is devoted to making these ideas precise.

Definition 22. A *perfect CLS-algebra* is a CLS-algebra $(L, !)$ such that L is a perfect CL-algebra and, in addition, the image of the exponential $\mathrm{Im}(!)$ is a complete join-sublattice of L, which is generated by its completely join-irreducible elements.

Recall from Theorem 15 that the class of perfect CL-algebras is dually equivalent to the class of CL-frames. To obtain relational semantics for full linear logic, we extend such frames with a collection J of Galois-closed sets. For a perfect CLS-algebra $(L, !)$, $J_!$ is the collection of completely join-irreducible elements of the image of the exponential, *i.e.*,

$$J_! = \{v \in L \mid !v = v \text{ and } v \text{ is completely join-irreducible in } \mathrm{Im}(!)\}.$$

We now have to determine which frames $F = (X, Y, \preccurlyeq, R, U, Z, J)$ give rise to a perfect CLS-algebra. Let I_J be the $\bigvee$-closure of J in $\mathcal{G}(F)$. We have to ensure that I_J satisfies the four conditions of Proposition 20 and that the Galois-closed sets in J are the completely join-irreducible elements of I_J. As before, we assume that any element named x (resp. y) with any super- or subscript comes from X (resp. Y).

(I1) This condition is automatically satisfied, since I is the $\bigvee$-closure of J in $\mathcal{G}(F)$.

(I2) As the fusion $\otimes$ in $\mathcal{G}(F)$ is completely join-preserving in both coordinates, I_J is closed under $\otimes$ if and only if, for all $w, w' \in J$, $w \otimes w' \in I_J$. This is the case if and only if, for all $w, w' \in J$, $w \otimes w' = \bigvee\{v \in J \mid v \leqslant w \otimes w'\}$. Since $w \otimes w' \geqslant \bigvee\{v \in J \mid v \leqslant w \otimes w'\}$ is always true, the property is satisfied if and only if the converse inequality holds, for all $w, w' \in J$. This can be rewritten as:

$$
\begin{aligned}
& w \otimes w' \leqslant \bigvee \{v \in J \mid v \leqslant w \otimes w'\} \\
\Leftrightarrow \quad & \forall y.\ \bigvee \{v \in J \mid v \leqslant w \otimes w'\} \leqslant y \quad \Rightarrow \quad w \otimes w' \leqslant y \\
\Leftrightarrow \quad & \forall y.\ (\forall v \in J.\ v \leqslant w \otimes w' \Rightarrow v \leqslant y) \quad \Rightarrow \quad w \otimes w' \leqslant y.
\end{aligned}
$$

Note that, as the collection J is a collection of Galois-closed sets, this statement is intrinsically second order: we cannot get around quantifying over J. However, all the other parts of the statement can be rewritten in the language of the frame, quantifying only over X and Y. For example,

$$
\begin{aligned}
w \otimes w' \leqslant y \quad & \Leftrightarrow \quad \forall x, x'.\ x \leqslant w \text{ and } x' \leqslant w \quad \Rightarrow \quad x \otimes x' \leqslant y \\
& \Leftrightarrow \quad \forall x, x'.\ x \in w \text{ and } x' \in w \quad \Rightarrow \quad R(x, x', y)
\end{aligned}
$$

For the sake of readability we choose not to write down all the derivations to statements in the language of the frame, but use some operations of the corresponding CLS-algebra as a short-hand. We conclude that I_J is closed under $\otimes$ if and only if F satisfies (Φ_{e2}):

$$\forall w, w' \in J\ \forall y.\ (\forall v \in J.\ v \leqslant w \otimes w' \Rightarrow v \leqslant y) \quad \Rightarrow \quad w \otimes w' \leqslant y. \qquad (\Phi_{e2})$$

(I3) Again using the fact that the fusion is completely join-preserving, it follows that the fusion is idempotent on I_J if and only if it is idempotent on J. Hence, $\otimes$ is idempotent on I_J if and only if F satisfies (Φ_{e3}):

$$\forall w \in J.\, w = w \otimes w. \tag{Φ_{e3}}$$

As above, the reader should view $w = w \otimes w$ as an abbreviation of a statement in the language of the frame.

(I4) We have to ensure $1 \in I_J$ and, for all $W \in I_J$, $W \leqslant 1$. Note that this is equivalent to $1 = \bigvee_{\mathcal{G}(F)}\{w \in J\}$, and

$$\begin{aligned} 1 = \bigvee\{w \in J\} \quad &\Leftrightarrow \quad \forall y.\, 1 \leqslant y \quad \Leftrightarrow \quad \bigvee\{w \in J\} \leqslant y \\ &\Leftrightarrow \quad \forall y.\, (\forall x.\, x \leqslant 1 \Rightarrow x \leqslant y) \quad \Leftrightarrow \quad (\forall w \in J.\, w \leqslant y) \\ &\Leftrightarrow \quad \forall y.\, (\forall x.\, x \in U \Rightarrow x \preccurlyeq y) \quad \Leftrightarrow \quad (\forall w \in J.\, w \leqslant y). \end{aligned}$$

Thus, the fourth property is satisfied for I_J if and only if F satisfies (Φ_{e4}):

$$\forall y.\, (\forall x.\, x \in U \Rightarrow x \preccurlyeq y) \quad \Leftrightarrow \quad (\forall w \in J.\, w \leqslant y). \tag{Φ_{e4}}$$

Finally, we want J to be the collection of completely join-irreducible elements of the lattice I_J. Note that $v \in J$ is completely join-irreducible in I_J if and only if $\bigvee\{w \in J \mid w < v\} < v$. This can be rewritten in the following way:

$$\begin{aligned} \bigvee\{w \in J \mid w < v\} < v \quad &\Leftrightarrow \quad \exists y.\, \bigvee\{w \in J \mid w < v\} \leqslant y \text{ and } v \nleqslant y \\ &\Leftrightarrow \quad \exists y.\, (\forall w \in J.\, w < v \Rightarrow w \leqslant y) \text{ and } v \nleqslant y. \end{aligned}$$

So the elements of J are completely join-irreducible in I_J if and only if

$$\forall v \in J\, \exists y.\, (\forall w \in J.\, w < v \Rightarrow w \leqslant y) \text{ and } v \nleqslant y. \tag{Φ_{cji}}$$

These calculations yield the following duality result.

Definition 23. A *CLS-frame* is an extended relational RS-frame $F = (X, Y, \preccurlyeq, R, U, Z, J)$, where $(X, Y, \preccurlyeq, R, U, Z)$ is a CL-frame and J is a collection of Galois-closed subsets of X, such that F satisfies the above conditions (Φ_{e2}), (Φ_{e3}), (Φ_{e4}), (Φ_{cji}).

A CLS-frame F gives rise to a perfect CLS-algebra $(L, !_J)$, where L is the perfect CL-algebra corresponding to the frame F as in Theorem 15 and, for all $W \in \mathcal{G}(F)$,

$$!_J(W) = \bigvee\{V \in J \mid V \leqslant W\}.$$

Theorem 24. *The mappings $(L, !) \mapsto (\mathcal{F}_{cl}(L), J_!)$ and $(F, J) \mapsto (\mathcal{G}_{cl}(F), !_J)$ yield a duality between perfect CLS-algebras and CLS-frames.*

Combining this duality theorem with the fact that, for every CLS-algebra L, L^δ is a perfect CLS-algebra yields the following.

Theorem 25. *The class of CLS-frames provides complete relational semantics for full linear logic.*

Up to now we have computed the conditions on the relational frames corresponding to the axioms in a mechanical way, not worrying about getting the simplest possible formulation. For the multiplicative additive fragment, the axioms could all be reduced to statements concerning only join-irreducible elements, hence these mechanical translations yield first-order statements on the dual. To witness the exponential dually, second order structure is needed.

This mechanical approach illustrates the strength of using duality theory in the search for relational semantics: it allows a modular and uniform treatment of additional operations and axioms. In Section 7 we will see that we may rewrite the conditions to get a cleaner representation and we will show that the semantics are closely related to phase semantics which are traditionally used as semantics for linear logic.

6. Properties of exponentials

It is well known that, on a given CL-algebra, there is not a unique admissible !. In this intermezzo section, we will look at the family of admissible exponentials from an algebraic perspective.

We say that an exponential ! is *larger than* an exponential !′, if, for all $a \in L$, $!'a \leqslant !a$ or, equivalently, if $I_{!'} \subseteq I_!$, where $I_! = \{a \in L \mid !a = a\}$. It is clear that every CL-algebra L has a smallest exponential, namely the exponential corresponding to the subset $\{\perp, 1\}$. Furthermore, every idempotent element of L below 1 gives rise to an exponential in the following way.

Lemma 26. *Let L be a CL-algebra. For all $a \in L$ such that $a \otimes a = a$ and $a \leqslant 1$, the subset $I_a := \{\perp, a, 1\}$ corresponds to an exponential.*

Proof. It is left to the reader to check that, for $a \in L$ with $a \leqslant 1$, I_a satisfies the four properties of Proposition 20. $\square$

Not every CL-algebra admits a largest exponential. We characterise the CL-algebras which do.

Lemma 27. *A CL-algebra L has a largest exponential if and only if the collection of idempotents of $\otimes$ below 1, i.e., $\{a \in L \mid a \otimes a = a \leqslant 1\}$, defines an exponential (which then is the largest exponential on L).*

Proof. It follows immediately from Proposition 20 that, for an exponential !, $I_!$ is contained in $\{a \in L \mid a \otimes a = a \leqslant 1\}$. Hence, if this set defines an exponential, then it is the largest exponential. Conversely, by Lemma 26, for all $a \in L$ with $a \otimes a = a \leqslant 1$, the set $I_a := \{\perp, a, 1\}$ defines an exponential. Hence, if L has a largest exponential, the corresponding subset has to contain $\{a \in L \mid a \otimes a = a \leqslant 1\}$ and is, by the first remark, in fact equal to it. $\square$

Proposition 28. *Every complete CL-algebra L has a largest exponential.*

Proof. By Lemma 27, it suffices to prove that the subset $I := \{a \in L \mid a \otimes a = a \leqslant 1\}$ satisfies the properties from Proposition 20.

(I1) Since L is complete, $\bigvee\{i \in I \mid i \leqslant a\}$ exists for all a. We have to show that this is an element of I.

$$\begin{aligned}\bigvee\{i \in I \mid i \leqslant a\} \otimes \bigvee\{i \in I \mid i \leqslant a\} &= \bigvee\{i \otimes i' \mid i, i' \in I,\ i, i' \leqslant a\} \\ &\overset{(1)}{=} \bigvee\{i \otimes i \mid i \in I,\ i \leqslant a\} \\ &= \bigvee\{i \in I \mid i \leqslant a\},\end{aligned}$$

where $^{(1)}$ relies on the fact that $i \otimes i' \leqslant \max(i, i') \otimes \max(i, i')$. Furthermore, for all $i \in I$, $i \leqslant 1$, thus $\bigvee\{i \in I \mid i \leqslant a\} \leqslant 1$.

(I2) For all $a, b \in I$, $(a \otimes b) \otimes (a \otimes b) = (a \otimes a) \otimes (b \otimes b) = a \otimes b$. And, $a \leqslant 1, b \leqslant 1$ implies that $a \otimes b \leqslant 1 \otimes 1 = 1$, thus I is closed under $\otimes$.

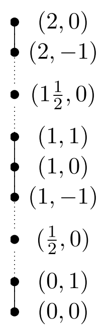
(I3) & (I4) These are clear from the definition of I. $\square$

6.1. *Example of a CL-algebra without a largest exponential*

Lemma 27 provides a tool for determining whether a largest exponential exists on a given CL-algebra. As we will see in the following, this need not always be the case. In order to construct an algebra K that does not have a largest exponential, we start by defining a CL-algebra L of which K will be a subalgebra. Consider the poset

$$L = \{(0, a) \mid a \in \mathbb{Z}^+\} \cup \{(1, b) \mid b \in \mathbb{Z}\} \cup \{(2, c) \mid c \in \mathbb{Z}^-\} \cup \{(\tfrac{1}{2}, 0), (1\tfrac{1}{2}, 0)\},$$

where the ordering is the lexicographic order on the product.



The poset L is a complete lattice with bottom $(0, 0)$ and a top $(2, 0)$. We define a binary fusion operation on L as follows,

$$(i,a)\otimes(j,b)=\begin{cases}(0,0) & \text{if } \{0\}\subseteq\{i,j\}\subseteq\{0,\frac{1}{2},1,1\frac{1}{2}\}\\ & \text{or } \{\frac{1}{2}\}\subseteq\{i,j\}\subseteq\{\frac{1}{2},1,1\frac{1}{2}\}\\ & \text{or } (i=j=1 \text{ and } a+b\leqslant 0),\\ (0,0\vee(a+b)) & \text{if } \{i,j\}=\{0,2\},\\ (\frac{1}{2},0) & \text{if } \{i,j\}=\{\frac{1}{2},2\},\\ (1,\min(a,b)) & \text{if } i=j=1 \text{ and } a+b>0,\\ (1,a) & \text{if } i=1 \text{ and } j\in\{1\frac{1}{2},2\},\\ (1,b) & \text{if } i\in\{1\frac{1}{2},2\} \text{ and } j=1,\\ (1\frac{1}{2},0) & \text{if } \{1\frac{1}{2}\}\subseteq\{i,j\}\subseteq\{1\frac{1}{2},2\},\\ (2,a+b) & \text{if } i=j=2.\end{cases}$$

Lemma 29. *The fusion on L is both associative and commutative and its unit is* $(2,0)$.

Proof. It is clear from the definition that the fusion is commutative and that $(2,0)$ is its unit. Associativity of the fusion is derived by a tedious case distinction and computation. □

Lemma 30. *The fusion on L is residuated.*

Proof. As L is a complete lattice, the fusion is residuated if and only if it is completely join-preserving in both coordinates. It is clear that the fusion is order-preserving. Hence, it suffices to check the truly infinite joins $\{(0,a)\mid a\in\mathbb{Z}^+\}$ and $\{(1,a)\mid a\in\mathbb{Z}^+\}$, which we leave to the reader. □

Lemma 31. *For all $(i,a)\in L$, $((i,a)\to(0,0))\to(0,0)=(i,a)$.*

Proof. Recall that, for $u,v\in L$, $u\to v=\bigvee\{w\mid u\otimes w\leqslant v\}$. It follows that

$$\begin{aligned}(0,a)\to(0,0)&=(2,-a) & \left(1\tfrac{1}{2},0\right)\to(0,0)&=\left(\tfrac{1}{2},0\right)\\ \left(\tfrac{1}{2},0\right)\to(0,0)&=\left(1\tfrac{1}{2},0\right) & (2,a)\to(0,0)&=(0,-a),\\ (1,a)\to(0,0)&=(1,-a)\end{aligned}$$

which proves the claim. □

Combining Lemmas 29, 30 and 31 yields that $(L,\otimes,\to,(2,0),(0,0))$ is a CL-algebra. Consider the subset $K=L-\{(\frac{1}{2},0),(1\frac{1}{2},0)\}$. One readily checks that this set is closed under fusion and linear negation. As implication is expressible in those two operations (by $u\to v=(u\otimes v^{\perp})^{\perp}$), this implies that K is the domain of a subalgebra of L, which is then a CL-algebra as well.

We claim that K does not possess a largest exponential. The collection of idempotents (below $1=\top$) of K is $\{(1,a)\mid a\in\mathbb{Z}^+\}\cup\{(0,0),(2,0)\}$. For any element of the form $(2,b)$ with $b\neq 0$, the join of the idempotents below it does not exist. So the collection of all idempotents below 1 does not yield an exponential on K and therefore, by Lemma 27, there is no largest exponential on K.

Just like any complete CL-algebra, L possesses a largest exponential corresponding to the collection of its idempotents below 1, which is

$$I=\left\{(1,a)\;\middle|\;a\in\mathbf{Z}^+\right\}\cup\left\{\left(1\tfrac{1}{2},0\right),(0,0),(2,0)\right\}.$$

The canonical extension of L may be described as

$$L^{\delta}\cong L\cup\left\{\left(\tfrac{1}{2},-1\right),\left(\tfrac{1}{2},1\right),\left(1\tfrac{1}{2},-1\right),\left(1\tfrac{1}{2},1\right)\right\},$$

in which the canonical extension of I embeds as

$$I^{\delta}\cong I\cup\left\{\left(1\tfrac{1}{2},-1\right)\right\}\subseteq L^{\delta}.$$

Note that in L^{δ}, both $(1\frac{1}{2},-1)$ and $(1\frac{1}{2},1)$ are idempotents of $\otimes^{\delta}$. The first one is an ideal element and therefore,

$$\begin{aligned}\left(1\tfrac{1}{2},-1\right)\otimes^{\delta}\left(1\tfrac{1}{2},-1\right)&=\bigvee\left\{u\otimes v\;\middle|\;u,v\in L\;\middle|\;u,v\leqslant\left(1\tfrac{1}{2},-1\right)\right\}\\ &=\bigvee\left\{(1,a)\;\middle|\;a\in\mathbf{Z}^+\right\}\\ &=\left(1\tfrac{1}{2},-1\right).\end{aligned}$$

The element $(1\frac{1}{2}, 1)$ is a filter element, whence,

$$\begin{aligned}
\left(1\tfrac{1}{2}, 1\right) \otimes^\delta \left(1\tfrac{1}{2}, 1\right) &= \bigwedge\left\{u \otimes v \;\middle|\; u, v \in L \;\middle|\; u, v \geqslant \left(1\tfrac{1}{2}, 1\right)\right\} \\
&= \bigwedge\left\{(2, a) \;\middle|\; a \in \mathbb{Z}^-\right\} \\
&= \left(1\tfrac{1}{2}, 1\right).
\end{aligned}$$

However, $(1\frac{1}{2}, 1) \notin I^\delta$. This example shows that the canonical extension of the largest exponential on L may not yield the largest exponential on the canonical extension L^δ.

7. Relational frames and phase semantics

Traditionally so-called phase semantics are used as models of (provability in) linear logic. We conclude this paper by describing the connection between these phase semantics and the relational semantics we derived in Sections 4 and 5. We first consider the multiplicative additive fragment of linear logic.

Definition 32. A *phase space* is a tuple $(M, \cdot, 1, \perp)$ where $(M, \cdot, 1)$ is a commutative monoid and $\perp \subseteq M$. One defines an operation on subsets A of M by

$$A^\perp = \{m \mid \forall n \in A.\, m \cdot n \in \perp\}. \tag{3}$$

A *fact* is a subset $F \subseteq M$ such that $(F^\perp)^\perp = F$.

MALL is interpreted in phase spaces by assigning facts to the basic propositions and interpreting the connectives as operations on facts [8]. As, for $A, B \in \wp(M)$, $B \subseteq A^\perp \Leftrightarrow A \subseteq B^\perp$, the mapping $(\_)^\perp$ yields a Galois connection on $\wp(M)$ and the Galois-closed sets are exactly the facts. The operations on facts corresponding to the connectives of MALL turn this collection of facts into a CL-algebra $\mathcal{F}ct(M)$. An inequality of MALL-formulas holds in a phase space M if and only if it holds in the corresponding CL-algebra $\mathcal{F}ct(M)$.

We now give an alternative presentation of the extended RS-frames of Theorem 25, which enables us to relate them to phase semantics.

Proposition 33. *Let L be a perfect CL-algebra. The subposets $\mathcal{J}^\infty(L)$ and $\mathcal{M}^\infty(L)$ of L are dually order-isomorphic.*

Proof. We will show that $(\_)^\perp$ restricts to a map $\mathcal{J}^\infty(L) \to \mathcal{M}^\infty(L)$. The claim then follows from the fact that this operation on L is its own inverse and is order-reversing. Let $x \in \mathcal{J}^\infty(L)$ and $A \subseteq L$ such that $x^\perp = \bigwedge A$. Then

$$x = \left(x^\perp\right)^\perp = \left(\bigwedge A\right)^\perp = \bigvee\left\{a^\perp \;\middle|\; a \in A\right\}.$$

As $x \in \mathcal{J}^\infty(L)$, there exists $a \in A$ s.t. $x = a^\perp$, whence $x^\perp = (a^\perp)^\perp = a$. □

By the previous proposition, for a perfect CL-algebra L, its completely join-irreducibles and its completely meet-irreducibles are dually order-isomorphic and therefore the algebra may be described by a one-sorted frame based on the set $\mathcal{J}^\infty(L)$. Note that, for $x_1, x_2 \in \mathcal{J}^\infty(L)$,

$$x_1 \leqslant x_2^\perp \quad\Leftrightarrow\quad x_1 \leqslant x_2 \to 0 \quad\Leftrightarrow\quad x_1 \otimes x_2 \leqslant 0.$$

Hence, the order relation between $\mathcal{J}^\infty(L)$ and $\mathcal{M}^\infty(L)$ is completely determined by the fusion and the constant 0. Furthermore, in any CL-algebra, $1 = 0^\perp$, hence 1 is definable from 0 and the linear negation.

For a perfect CL-algebra L we define a (one-sorted) frame $\mathcal{F}_1(L) = (X, R_\downarrow, Z_\downarrow)$, by $X = \mathcal{J}^\infty(L)$, $Z_\downarrow = \{x \in X \mid x \leqslant 0\}$ and, for $x_1, x_2, x_3 \in X$,

$$R_\downarrow(x_1, x_2, x_3) \quad\Leftrightarrow\quad x_3 \leqslant x_1 \otimes x_2.$$

Conversely, for a one-sorted RS-frame[5] $P = (X, R_\downarrow, Z_\downarrow)$ we define a Galois connection on $\wp(X)$ by, for $A \in \wp(X)$,

$$A^\perp = \left\{x \in X \;\middle|\; \forall a \in A.\, R_\downarrow[x, a, \_] \subseteq Z_\downarrow\right\}. \tag{4}$$

We define a fusion on $\mathcal{G}_1(P)$, the Galois-closed subsets of P, by

[5] The notions 'reduced' and 'separating' are defined for one-sorted frames, as in Section 2 for two sorted frames, in such a way that they ensure that X embeds in $\mathcal{G}_1(F)$ as its completely join-irreducibles.

$$x_1 \otimes x_2 = \bigvee R[x_1, x_2, \_] \qquad \text{for all } x_1, x_2 \in X,$$
$$w_1 \otimes w_2 = \bigvee \{x_1 \otimes x_2 \mid x_1, x_2 \in X \mid x_1 \leqslant w_1,\ x_2 \leqslant w_2\} \quad \text{for all } w_1, w_2 \in \mathcal{G}_1(P).$$

For a CL-algebra L, the structures $\mathcal{F}(L) = (X, Y, \preccurlyeq, R, U, Z)$ and $\mathcal{F}_1(L) = (X, R_\downarrow, Z_\downarrow)$ are directly inter-definable. For example, for $x_1, x_2, x_3 \in X$,

$$\begin{aligned} R_\downarrow(x_1, x_2, x_3) \quad &\Leftrightarrow \quad \forall y \in Y.\ R[x_1, x_2, y] \quad \Rightarrow \quad x_3 \leqslant y \\ &\Leftrightarrow \quad x_3 \in R[x_1, x_2, \_]^l. \end{aligned}$$

This allows us to translate the conditions (Φ_a), (Φ_c), (Φ_u) and (Φ_{dd}) to statements about one-sorted frames. *E.g.*, (Φ_{dd}) becomes the statement Φ'_{dd}:

$$\forall x, x' \left(\forall x''.\ R_\downarrow\left[x, x'', \_\right] \subseteq Z_\downarrow \Rightarrow R_\downarrow\left[x', x'', \_\right] \subseteq Z_\downarrow\right) \quad \Rightarrow \quad x' \leqslant x.$$

Translation of the other statements is left to the reader. For a one-sorted RS-frame P, the algebra $\mathcal{G}_1(P)$, with constants 1 and 0 defined in the evident way, is a CL-algebra if and only if P satisfies Φ'_a, Φ'_c, Φ'_u and Φ'_{dd}.

Theorem 34. *One-sorted RS-frames $(X, R_\downarrow, Z_\downarrow)$, satisfying Φ'_a, Φ'_c, Φ'_u and Φ'_{dd} give complete semantics for MALL. We will call these structures one-sorted CL-frames.*

For a one-sorted CL-frame $P = (X, R_\downarrow, Z_\downarrow)$ we may define a phase space (only lacking a unit for the multiplication[6]) by $M_P = \wp(X)$, $\bot_P = {\downarrow} Z_\downarrow = \{A \in \wp(X) \mid A \subseteq Z_\downarrow\}$ and, for all $A, B \in \wp(X)$,

$$A \cdot_P B = \bigcup \left\{R[a, b, \_] \,\middle|\, a \in A,\ b \in B\right\}.$$

As P satisfies Φ'_c, $\cdot_P$ is commutative.

Lemma 35. *For all $\mathcal{A} \in \wp(\wp(X))$, if $\mathcal{A}$ is a fact, i.e., $(\mathcal{A}^\perp)^\perp = \mathcal{A}$, then $\mathcal{A}$ is a principal downset in $\wp(\wp(X))$. Furthermore, for all $A \in \wp(X)$, A is Galois-closed in P if and only if ${\downarrow} A$ is a fact in M_P.*

Proof. We denote both the map (3) on $\wp(M_P)$ and the map (4) on $\wp(X)$ by $(\_)^\perp$, as the reader may derive the intended meaning from the context. Note that, for $\mathcal{A} \in \wp(M_P)$,

$$\begin{aligned} \mathcal{A}^\perp &= \{B \in M_P \mid \forall A \in \mathcal{A}.\ B \cdot_P A \in \bot_P\} \\ &= \{B \in M_P \mid \forall A \in \mathcal{A}.\ B \cdot_P A \subseteq Z_\downarrow\} \\ &= \left\{B \in M_P \,\middle|\, B \cdot_P \bigcup \mathcal{A} \subseteq Z_\downarrow\right\} \\ &= \left\{B \in M_P \,\middle|\, B \subseteq \left(\bigcup \mathcal{A}\right)^\perp\right\} \\ &= {\downarrow}\left(\left(\bigcup \mathcal{A}\right)^\perp\right), \end{aligned}$$

which proves the first claim. The second claim easily follows from $(({\downarrow} A)^\perp)^\perp = {\downarrow}((A^\perp)^\perp)$. $\square$

Theorem 36. *The CL-algebras $\mathcal{G}_1(P)$ and $\mathcal{F}ct(M_P)$ are isomorphic.*

Proof. It follows from the previous lemma that the mapping $A \mapsto {\downarrow} A$ is a bijection between the two underlying sets. It is left to the reader to check that this map preserves the CL-structure. $\square$

Phase semantics for full linear logic are given by topolinear spaces. A *topolinear space* is a pair $(M, \mathcal{O})$, where M is a phase space and $\mathcal{O}$ is a set of facts of M satisfying conditions (I1)–(I4) of Proposition 20 (regarding $\mathcal{O}$ as a subset of the CL-algebra $\mathcal{F}ct(M)$). As described in that proposition, the collection $\mathcal{O}$ yields an exponential on $\mathcal{F}ct(M)$ by defining, for $U \in Fct(M)$, $!_{\mathcal{O}}(U) = \max(\{W \in \mathcal{O} \mid W \subseteq U\})$. So in fact, in a topolinear space one remembers the entire image of the exponential.

In a similar way as was done above for MALL, the extended RS-frames for full linear logic may be described in a one-sorted fashion. We denote these frames by (P, J), where P is a one-sorted CL-frame and J is a collection of Galois-closed sets of P satisfying (translations of) (Φ_{e2}), (Φ_{e3}), (Φ_{e4}) and (Φ_{cji}) (see Theorem 25). As before, the collection J yields an

[6] This is not a big issue as 1 is definable from the linear negation and 0.

exponential on $\mathcal{G}_1(P)$ by, for $U \in \mathcal{G}_1(P)$, $!_J(U) = \bigvee\{W \in J \mid W \subseteq U\}$. CLS-frames provide complete semantics for full linear logic.

A CLS-frame (P, J) gives rise to a topolinear space $(M_P, \mathcal{O}_J)$, where $\mathcal{O}_J$ is the join closure of J in $\mathcal{G}_1(P) \cong \mathcal{F}ct(M_P)$. We obtain the following extension of Theorem 36.

Theorem 37. *For a CLS-frame (P, J), the CLS-algebras $(\mathcal{G}_1(P), !_J)$ and $(\mathcal{F}ct(M_P), !_{\mathcal{O}_J})$ are isomorphic.*

Using Theorem 36 (resp. Theorem 37), completeness of the semantics of phase spaces (resp. topolinear spaces) may be derived from completeness of CL-frames (resp. CLS-frames). It is not always possible to construct, given a phase space M, a CL-frame P_M s.t. $\mathcal{G}_1(P_M) \cong \mathcal{F}ct(M)$, as the complete lattice $\mathcal{F}ct(M)$ may not be perfect.

The topolinear space describing a specific CLS-algebra, (*e.g.*, the Lindenbaum algebra used in the completeness proof) is in general much larger than the corresponding CLS-frame. This size difference is also visible in the proofs of the above two theorems: the underlying set of the topolinear space associated to a CLS-frame $(X, R_\downarrow, Z_\downarrow, J)$ is $\wp(X)$.

Obtaining relational semantics for linear logic which provide the clear intuitions of Kripke semantics in modal logic still needs further work. Neither phase semantics nor CL(S)-frames seem adequate in this regard. However, there is a great advantage of working with CL(S)-frames: they are in a duality with perfect CL(S)-algebras which enables a modular and uniform treatment of additional axioms and operations. It is remarkable that the approach illustrated in this paper derives in a mechanical fashion a semantics that is very close to phase semantics.

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