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On an improved local convergence analysis for the Secant method

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Abstract We provide a local convergence analysis for the Secant method in a Banach space setting under Hölder continuous conditions. Using more precise estimates, and under the same computational cost, we enlarge the radius of convergence obtained in Ren and Wu (J Comput Appl Math 194:284–293, 2006).

Keywords Secant method · Banach space · Fréchet-derivative · Hölder continuity · Local convergence · Radius of convergence · Newton's method

Mathematics Subject Classifications (2000) 65H10 · 65G99 · 47H17 · 49M15

1 Introduction

In this study, we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

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where F is a Fréchet-differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$ for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We use the Secant method

$$x_{n+1} = x_n - \left[x_{n-1}, x_n; F\right]^{-1} F(x_n) \quad (x_{-1},\ x_0\ \in\ D),\ (n \geq 0), \tag{1.2}$$

to generate a sequence $\{x_n\}$ approximating $x^\star$. Here, $[x, y; F] \in L(X, Y)$, the space of bounded linear operators from X into Y, denotes the divided difference of order one for the operator F at the point x, $y (x \neq y)$ satisfying

$$\left[x, y; F\right](x - y) = F(x) - F(y). \tag{1.3}$$

Sergeev [18] and Schmidt [17] generalized the Secant method in Banach spaces. Ever since, there has been an extensive literature on the local as well as the semilocal convergence of the Secant method under various Lipschitz-type conditions [1–5, 7–9, 11–15].

In the case of the local convergence the idea is to find a ball $B(x^\star, r) \subseteq D$ such that convergence to $x^\star$ can be achieved for initial guesses chosen from the convergence ball $B(x^\star, r)$. Obviously, we would like $B(x^\star, r)$ to be as large as possible, so that we can increase the number of initial guesses [2, 3].

Under the Lipschitz condition

$$\| F'(x^\star)^{-1} \left[F'(x) - F'(y)\right] \| \leq L \|x - y\| \tag{1.4}$$

for all $x, y \in D$ and $L > 0$, Rheinboldt in [16] provided the convergence radius

$$r_R = \frac{2}{3L} \tag{1.5}$$

for Newton's method. In view of (1.4) there exists $L_0 \in (0, L]$ such that

$$\| F'(x^\star)^{-1} \left[F'(x) - F'\left(x^\star\right)\right] \| \leq L_0 \|x - x^\star\| \tag{1.6}$$

for all $x \in D$. Using a combination of (1.4) and (1.6), Argyros in [2, 3] provided the convergence radius

$$r_A = \frac{2}{2L_0 + L}. \tag{1.7}$$

Note that

$$L_0 \leq L \tag{1.8}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [2, 3]. In case $L_0 < L$ it follows from (1.5) and (1.7) that

$$r_R < r_A. \tag{1.9}$$

Huang in [10] generalized the result for Newton's method to the Hölder case:

$$\| F'(x^\star)^{-1} \left(F'(x) - F'(x_\tau) \right) \| \leq L(1-\tau)^p \|x - x_\tau\|^p \tag{1.10}$$

for all $x \in B(x^\star, r_H)$, where $x_\tau = x^\star + \tau(x - x^\star)$, $0 < p \leq 1$, and

$$r_H = \left(\frac{1+p}{(1+2p)L} \right)^{\frac{1}{p}}. \tag{1.11}$$

In view of (1.10) there exists $L_0 \in (0, L]$ satisfying (1.6). Using a combination of (1.6) and (1.10), Argyros in [6] provided the convergence radius

$$r_{AA} = \left(\frac{1+p}{(1+p)L_0 + L} \right)^{\frac{1}{p}}. \tag{1.12}$$

In case $L_0 < L$ it follows from (1.11) and (1.12) that

$$r_H < r_{AA}. \tag{1.13}$$

Note also that in [2, 6] the error bounds on the distances $\|x_n - x^\star\|$ are finer than in [16] and [10] respectively.

Argyros in [1] provided a convergence analysis for the Secant method (1.2) under the hypothesis

$$\|[x_{-1}, x_0; F]^{-1}([y, u; F] - [x, y; F])\| \leq M_1\|x-u\|^p + M_2\|x-y\|^p + M_3\|u-y\|^p \tag{1.14}$$

for all $x, y, u \in D$ and some $M_1 > 0$, $M_2 > 0$, $M_3 > 0$. Pavaloiu provided some refinements of the Argyros' analysis in [11, 12]. Ren and Wu in [13] used the special condition of (1.14) namely:

$$\| F'(x^\star)^{-1} \left([x, y; F] - F'(z) \right) \| \leq K \left(\|x - z\|^p + \|y - z\|^p \right) \tag{1.15}$$

for all $x, y, z \in D$ and some $K > 0$ to provide the convergence radius for the Secant method (1.2) given by

$$r_{RW} = \left(\frac{1+p}{2(1+p+2^p)K} \right)^{\frac{1}{p}}. \tag{1.16}$$

In view of (1.15) there exists $K_0, K_1 \in (0, K]$ such that

$$\|F'(x^\star)^{-1}\left([x, y; F] - F'(x^\star)\right)\| \leq K_0\left(\|x - x^\star\|^p + \|y - x^\star\|^p\right) \tag{1.17}$$

for all $x, y \in D$ and

$$\|F'(x^\star)^{-1}\left(\left[x, x^\star; F\right] - F'(x^\star)\right)\| \leq K_1\|x - x^\star\|^p \tag{1.18}$$

for all $x \in D$. Note also that

$$K_1 \leq K_0 \leq K \tag{1.19}$$

holds in general and $\frac{K}{K_0}$, $\frac{K_0}{K_1}$ can be arbitrarily large [2, 3].

In this study we show how to enlarge radius r_{RW}. In particular, let $c \in [0, 2]$ and define r_0, r_1 and r_2 by

$$r_0 = \begin{cases} \left(\dfrac{1+p}{2(1+p)K_0 + (2+p)K}\right)^{\frac{1}{p}}, & \text{if } c = 1, \\ \left(\dfrac{(c-1)(1+p)}{2(c-1)(1+p)K_0 + (c^{1+p} + c - 2)K}\right)^{\frac{1}{p}}, & \text{if } c \neq 1, \end{cases}$$

$$r_1 = \left(\frac{1}{2K_0 + 2c^pK}\right)^{\frac{1}{p}},$$

$$r_2 = \left(\frac{1}{6K_0}\right)^{\frac{1}{p}}.$$

We denote r_0 as $\bar{r}_0$ for the special case $c = 2$, that is

$$\bar{r}_0 = \left(\frac{1+p}{2(1+p)K_0 + 2^{1+p}K}\right)^{\frac{1}{p}}. \tag{1.20}$$

Further, define r_{RWA} by

$$r = r_{RWA} = \begin{cases} r_0, & \text{if } \max(r_1, r_2) > r_0, \\ \max(\bar{r}_0, r_1, r_2), & \text{if } \max(r_1, r_2) \leq r_0. \end{cases} \tag{1.21}$$

It follows from (1.16) and (1.20) that $r_{RW} \leq \bar{r}_0$. Moreover in case $K_0 < K$, we have

$$r_{RW} < \bar{r}_0. \tag{1.22}$$

For any fixed parameter $p \in (0, 1]$, we define a function $h(c)$ as

$$h(c) = \begin{cases} 1 + p, & \text{if } c = 1, \\ \dfrac{c^{1+p} - 1}{c - 1}, & \text{if } c \neq 1. \end{cases} \tag{1.23}$$

It is easy to verify that $h(c)$ increases monotonically, and thus $h(c) \leq h(2)$ is true for any $c \in [0, 2]$. Therefore,

$$\bar{r}_0 \leq r_0 \tag{1.24}$$

is also true for any $c \in [0, 2]$. Using a combination of (1.21), (1.22) and (1.24), we deduce $r_{RW} \leq r$ holds for any case. Especially in case $K_0 < K$, we have

$$r_{RW} < r_{RWA}. \tag{1.25}$$

We shall show that r_{RWA} is the convergence radius of the Secant method (1.2). It follows from (1.25) that we have enlarged the radius of convergence given by [13]. Note that if $c = 2$ and $K_0 = K$, we easily deduce $r_0 = \bar{r}_0 = r_{RW} \geq r_1 \geq r_2$, that is, (1.21) reduces to (1.16). Note also that the improvements in this study are done under the same computational cost as in [13], since in practice the computation of constant K requires that of K_0 and K_1.

2 Local convergence analysis of Secant method (1.2)

We can show the main local convergence theorem for the Secant method (1.2).

Theorem 2.1 *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume there exists $x^\star \in D$ such that $F(x^\star) = 0$, and $F'(x^\star)^{-1} \in L(Y, X)$; Condition (1.15) holds for all*

$$x, y, z \in B(x^\star, r),$$

and

$$B(x^\star, r) \subseteq D. \tag{2.1}$$

Then sequence $\{x_n\}(n \geq -1)$ generated by the Secant method (1.2) is well defined, remains in $B(x^\star, r)$ for all $n \geq -1$, and converges to the unique solution $x^\star$ in $B(x^\star, (\frac{1}{K_1})^{\frac{1}{p}})$ which contains $B(x^\star, r)$, provided that x_{-1}, x_0 are chosen so that $x_{-1}, x_0 \in B(x^\star, r)$, and

$$max \left\{ \| \left[x_{-1}, x_0; F\right]^{-1} F(x_0) \|, \|x_0 - x_{-1}\| \right\} < cr. \tag{2.2}$$

Moreover, if $r \leq \bar{r}_0$, the following estimate holds for all $n \geq 1$

$$\|x_n - x^\star\| \leq r \left(\frac{1}{r} \left(max \left(\|x_{-1} - x^\star\|, \|x_0 - x^\star\| \right) \right) \right)^{(p+1)^{\lfloor \frac{n+1}{2} \rfloor}}, \tag{2.3}$$

where $\lfloor s \rfloor$ denotes the largest integer that is not larger than s.

Note that points x_{-1} and x_0 can always be chosen close enough so that (2.2) holds true.

Proof We will prove this theorem by induction. First, using hypotheses (1.17) and $x_{-1}, x_0 \in B(x^\star, r)$, we have

$$\begin{aligned} \|I - F'(x^\star)^{-1} \left[x_{-1}, x_0; F\right] \| &= \|F'(x^\star)^{-1} \left(F'(x^\star) - \left[x_{-1}, x_0; F\right]\right) \| \\ &\leq K_0 \left(\|x_{-1} - x^\star\|^p + \|x_0 - x^\star\|^p\right) \\ &< 2K_0 r^p. \end{aligned} \tag{2.4}$$

From the definitions of $r_0, \bar{r}_0, r_1$ and r_2, it follows that

$$2K_0 r_0^p = \begin{cases} \dfrac{2(1+p)K_0}{2(1+p)K_0 + (2+p)K}, & \text{if } c = 1, \\ \dfrac{2(1+p)K_0}{2(1+p)K_0 + \dfrac{c^{1+p}-1+c-1}{c-1}K}, & \text{if } c \neq 1, \end{cases}$$

$$2K_0 \bar{r}_0^p = \frac{2(1+p)K_0}{2(1+p)K_0 + 2^{1+p}K}, \quad 2K_0 r_1^p = \frac{2K_0}{2K_0 + 2c^p K}, \quad 2K_0 r_2^p = \frac{1}{3}.$$

Using the definition of r gives

$$2K_0 r^p < 1. \tag{2.5}$$

By the Banach lemma, it follows from (2.4) and (2.5) that $F'(x^\star)^{-1}[x_{-1}, x_0; F]$ is invertible, and thus x_1 is well defined. Moreover, we have

$$\begin{aligned} \left\| \left(F'(x^\star)^{-1}\left[x_{-1}, x_0; F\right]\right)^{-1} \right\| &\leq \frac{1}{1 - K_0\left(\|x_{-1} - x^\star\|^p + \|x_0 - x^\star\|^p\right)} \\ &< \frac{1}{1 - 2K_0 r^p}. \end{aligned} \tag{2.6}$$

Now we estimate $\|x_1 - x^\star\|$. From (1.3), $F(x^\star) = 0$ and the convexity of D, it follows that

$$\begin{aligned} \|x_1 - x^\star\| &= \|x_0 - x^\star - [x_{-1}, x_0; F]^{-1}(F(x_0) - F(x^\star))\| \\ &= \|(F'(x^\star)^{-1}[x_{-1}, x_0; F])^{-1} F'(x^\star)^{-1}([x_{-1}, x_0; F] \\ &\quad - \int_0^1 F'(tx_0 + (1-t)x^\star)dt)(x_0 - x^\star)\|. \end{aligned} \tag{2.7}$$

By (2.6), (2.7) and hypothesis (1.17), it yields

$$\begin{aligned} \|x_1 - x^\star\| &\leq \frac{\|x_0 - x^\star\|}{1 - K_0(\|x_{-1} - x^\star\|^p + \|x_0 - x^\star\|^p)} \\ &\quad \times \left\| \int_0^1 F'(x^\star)^{-1}([x_{-1}, x_0; F] - F'(tx_0 + (1-t)x_\star))dt \right\| \\ &\leq \frac{\|x_0 - x^\star\|}{1 - K_0(\|x_{-1} - x^\star\|^p + \|x_0 - x^\star\|^p)} \\ &\quad \times \left(\int_0^1 K(\|x_{-1} - tx_0 - (1-t)x^\star\|^p + \|x_0 - tx_0 - (1-t)x^\star\|^p)dt \right) \\ &\leq \frac{K\|x_0 - x^\star\|}{1 - K_0(\|x_{-1} - x^\star\|^p + \|x_0 - x^\star\|^p)} \\ &\quad \times \left(\int_0^1 (t\|x_{-1} - x_0\| + (1-t)\|x_{-1} - x^\star\|)^p dt + \int_0^1 (1-t)^p \|x_0 - x^\star\|^p dt \right). \end{aligned} \tag{2.8}$$

From $x_{-1}, x_0 \in B(x^\star, r)$, hypothesis (2.2) and the definition of r, it follows that

$$
\begin{aligned}
\|x_1 - x^\star\| &< \frac{Kr}{1-2K_0r^p}\int_0^1 (ct + (1-t))^p r^p + (1-t)^p r^p)dt \\
&= \frac{Kr^{p+1}}{1-2K_0r^p}\int_0^1 ((1+(c-1)t)^p + (1-t)^p)dt \\
&= \begin{cases} \dfrac{Kr^{p+1}}{1-2K_0r^p}\left(1+\dfrac{1}{1+p}\right), & \text{if } c = 1, \\[2ex] \dfrac{Kr^{p+1}}{1-2K_0r^p}\left(\dfrac{c^{1+p}-1}{(c-1)(1+p)}+\dfrac{1}{1+p}\right), & \text{if } c \neq 1. \end{cases}
\end{aligned}
\tag{2.9}
$$

With the definition of r_0, in case $r = r_0$, we easily deduce

$$
r = \begin{cases} \dfrac{Kr^{p+1}}{1-2K_0r^p}\left(1+\dfrac{1}{1+p}\right), & \text{if } c = 1, \\[2ex] \dfrac{Kr^{p+1}}{1-2K_0r^p}\left(\dfrac{c^{1+p}-1}{(c-1)(1+p)}+\dfrac{1}{1+p}\right), & \text{if } c \neq 1. \end{cases}
\tag{2.10}
$$

Since $\bar{r}_0$ is a special case r_0 for $c = 2$, in case $r = \bar{r}_0$, the relation (2.10) is also true. In case $r = r_1$, with the definition of r, the inequality $r_1 \le r_0$ is true, and thus it is easy to verify that

$$
r \ge \begin{cases} \dfrac{Kr^{p+1}}{1-2K_0r^p}\left(1+\dfrac{1}{1+p}\right), & \text{if } c = 1, \\[2ex] \dfrac{Kr^{p+1}}{1-2K_0r^p}\left(\dfrac{c^{1+p}-1}{(c-1)(1+p)}+\dfrac{1}{1+p}\right), & \text{if } c \neq 1. \end{cases}
\tag{2.11}
$$

Using the similar technique, we can verify (2.11) is also true in case $r = r_2$. Therefore, in any case of r, we can deduce that $x_1 \in B(x^\star, r)$.

Now we suppose $\{x_k\}(1 \le k \le n)$ is well defined, $x_k \in B(x^\star, r)(1 \le k \le n)$ and $\|x_k - x_{k-1}\| < cr(1 \le k \le n)$. Similarly to the argumentation about x_{-1} and x_0, we have

$$
\begin{aligned}
\|I - F'(x^\star)^{-1}\left[x_{n-1}, x_n; F\right]\| &\le K_0\left(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right) \\
&< 2K_0r^p < 1.
\end{aligned}
\tag{2.12}
$$

By the Banach lemma, $[x_{n-1}, x_n; F]$ is invertible, and thus x_{n+1} is well defined. Moreover we have

$$
\|(F'(x^\star)^{-1}[x_{n-1}, x_n; F])^{-1}\| \le \frac{1}{1 - K_0(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p)}.
\tag{2.13}
$$

Similarly to the estimate of $\|x_1 - x^\star\|$ and by the induction hypotheses, we obtain

$$\begin{aligned}
\|x_{n+1} - x^\star\| &\le \frac{K\|x_n - x^\star\|}{1 - K_0(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p)} \\
&\quad \times \left(\int_0^1 (t\|x_{n-1} - x_n\| + (1-t)\|x_{n-1} - x^\star\|)^p dt \right. \\
&\qquad \left. + \int_0^1 (1-t)^p \|x_n - x^\star\|^p dt\right) \\
&< \frac{Kr}{1 - 2K_0 r^p} \int_0^1 ((ct + (1-t))^p r^p + (1-t)^p r^p) dt \\
&\le r. \qquad (2.14)
\end{aligned}$$

That is, $x_{n+1} \in B(x^\star, r)$.

In case $r = r_1$, from (1.15), (2.13), the induction hypotheses and the identity

$$F(x_n) = \left(\left[x_{n-1}, x_n; F\right] - \left[x_{n-2}, x_{n-1}; F\right]\right)(x_n - x_{n-1}), \qquad (2.15)$$

it follows that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\left[x_{n-1}, x_n; F\right]^{-1} F'(x^\star)F'(x^\star)^{-1}F(x_n)\| \\
&\le \frac{\|F'(x^\star)^{-1}\left(\left[x_{n-1}, x_n; F\right] - F'(x_{n-1}) + F'(x_{n-1}) - \left[x_{n-2}, x_{n-1}; F\right]\right)\|\,\|x_n - x_{n-1}\|}{1 - K_0\left(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right)} \\
&\le \frac{K\left(\|x_n - x_{n-1}\|^p + \|x_{n-2} - x_{n-1}\|^p\right)\|x_n - x_{n-1}\|}{1 - K_0\left(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right)} \\
&\le \frac{2Kc^p r^p}{1 - 2K_0 r^p}\|x_n - x_{n-1}\| \\
&= 1 \bullet \|x_n - x_{n-1}\| < cr \qquad (2.16)
\end{aligned}$$

by the choice of r_1.

In case $r = r_2$, from (1.15), (2.13), the induction hypotheses and the identity (2.15), it follows that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\left[x_{n-1}, x_n; F\right]^{-1} F'(x^\star)F'(x^\star)^{-1}F(x_n)\| \\
&\le \frac{\|F'(x^\star)^{-1}\left(\left[x_{n-1}, x_n; F\right] - F'(x^\star) + F'(x^\star) - \left[x_{n-2}, x_{n-1}; F\right]\right)\|\,\|x_n - x_{n-1}\|}{1 - K_0\left(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right)} \\
&\le \frac{K_0\left(\|x_{n-2} - x^\star\|^p + 2\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right)\|x_n - x_{n-1}\|}{1 - K_0\left(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right)} \\
&\le \frac{4K_0 r^p}{1 - 2K_0 r^p}\|x_n - x_{n-1}\| \\
&= 1 \bullet \|x_n - x_{n-1}\| < cr \qquad (2.17)
\end{aligned}$$

by the choice of r_2.

In case $r = r_0$, from the definition of r, this case happens as $\max(r_1, r_2) > r_0$. If $r_1 > r_0$, we can take the similar argumentation for the case $r = r_1$. In fact, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \| \left[x_{n-1}, x_n; F\right]^{-1} F'(x^\star) F'(x^\star)^{-1} F(x_n)\| \\
&\leq \frac{\|F'(x^\star)^{-1}\left(\left[x_{n-1}, x_n; F\right] - F'(x_{n-1}) + F'(x_{n-1}) - \left[x_{n-2}, x_{n-1}; F\right]\right)\| \|x_n - x_{n-1}\|}{1 - K_0\left(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right)} \\
&\leq \frac{K\left(\|x_n - x_{n-1}\|^p + \|x_{n-2} - x_{n-1}\|^p\right)\|x_n - x_{n-1}\|}{1 - K_0\left(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right)} \\
&\leq \frac{2Kc^p r_0^p}{1 - 2K_0 r_0^p}\|x_n - x_{n-1}\| \\
&\leq \frac{2Kc^p r_1^p}{1 - 2K_0 r_1^p}\|x_n - x_{n-1}\| \\
&= 1 \bullet \|x_n - x_{n-1}\| < cr.
\end{aligned} \tag{2.18}$$

On the other hand, if $r_2 > r_0$, we can take the similar argumentation for the case $r = r_2$. In fact, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \| \left[x_{n-1}, x_n; F\right]^{-1} F'(x^\star) F'(x^\star)^{-1} F(x_n)\| \\
&\leq \frac{\|F'(x^\star)^{-1}\left(\left[x_{n-1}, x_n; F\right] - F'(x^\star) + F'(x^\star) - \left[x_{n-2}, x_{n-1}; F\right]\right)\| \|x_n - x_{n-1}\|}{1 - K_0\left(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right)} \\
&\leq \frac{K_0\left(\|x_{n-2} - x^\star\|^p + 2\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right)\|x_n - x_{n-1}\|}{1 - K_0\left(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p\right)} \\
&\leq \frac{4K_0 r_0^p}{1 - 2K_0 r_0^p}\|x_n - x_{n-1}\| \\
&\leq \frac{4K_0 r_2^p}{1 - 2K_0 r_2^p}\|x_n - x_{n-1}\| \\
&= 1 \bullet \|x_n - x_{n-1}\| < cr.
\end{aligned} \tag{2.19}$$

In case $r = \bar{r}_0$, $\|x_{n+1} - x_n\| \leq cr$ is also true since $\bar{r}_0$ is a special case of r_0 for $c = 2$.

Hence, by induction, the sequence $\{x_n\}$ generated by the Secant method (1.2) is well defined, $x_n \in B(x^\star, r)(n \geq -1)$.

Next we shall show $\{x_n\}$ converges to $x^\star$. Since $r > 0$, we must have $r' > 0$ such that $r > r'$, and conditions (1.15), (2.1), (2.2) and $x_0, x_{-1} \in B(x^\star, r)$ are true by replacing r with r'. Then we can take similar analysis above and deduce that $x_n \in B(x^\star, r')(n \geq -1)$. Moveover, it follows from (2.14) that

$$\|x_{n+1} - x^\star\| \leq \frac{K\|x_n - x^\star\|}{1 - K_0(\|x_{n-1} - x^\star\|^p + \|x_n - x^\star\|^p)}$$

$$\times \left(\int_0^1 (t\|x_{n-1} - x_n\| + (1-t)\|x_{n-1} - x^\star\|)^p dt + \int_0^1 (1-t)^p \|x_n - x^\star\|^p dt \right) \leq M\|x_n - x^\star\|, \quad (n \geq 0), \tag{2.20}$$

where

$$M = \frac{K}{1 - 2K_0(r')^p} \int_0^1 \left((ct + (1-t))^p (r')^p + (1-t)^p (r')^p \right) dt < \frac{K}{1 - 2K_0 r^p} \int_0^1 \left((ct + (1-t))^p r^p + (1-t)^p r^p \right) dt \leq 1. \tag{2.21}$$

So, sequence $\{x_n\}$ converges to $x^\star$ linearly at least. Furthermore, it follows from (2.20) that

$$\|x_{n+1} - x^\star\| \leq \frac{K\|x_n - x^\star\|}{1 - 2K_0 r^p} \left(\int_0^1 (\|x_{n-1} - x^\star\| + t\|x_n - x^\star\|)^p dt + \int_0^1 (1-t)^p \|x_n - x^\star\|^p dt \right)$$
$$= \frac{K\left((\|x_{n-1} - x^\star\| + \|x_n - x^\star\|)^{p+1} - \|x_{n-1} - x^\star\|^{p+1} + \|x_n - x^\star\|^{p+1} \right)}{(1+p)(1 - 2K_0 r^p)}, \quad (n \geq 0). \tag{2.22}$$

Denote

$$\omega = \left(\frac{K}{(1+p)(1 - 2K_0 r^p)} \right)^{\frac{1}{p}}, \tag{2.23}$$

$$\theta_n = \omega \|x_n - x^\star\|, \quad (n \geq -1), \tag{2.24}$$

then from (2.22)–(2.24), we have

$$\theta_{n+1} \leq (\theta_n + \theta_{n-1})^{1+p} - \theta_{n-1}^{1+p} + \theta_n^{1+p}, \quad (n \geq 0). \tag{2.25}$$

Define a function $g(u, v)$ as follows

$$g(u, v) = (u + v)^{1+p} - v^{1+p} + u^{1+p}, \quad u \geq 0, \; v \geq 0. \tag{2.26}$$

It is easy to verify that $g(u, v)$ increases monotonically about $u \geq 0$ and $v \geq 0$, and thus we have

$$\theta_{n+1} \leq (2max(\theta_n, \theta_{n-1}))^{1+p}, \quad (n \geq 0). \tag{2.27}$$

On the other hand, it follows from (2.20), (2.21) and (2.24) that

$$\theta_{n+1} \leq \theta_n, \quad (n \geq 0). \tag{2.28}$$

Denote

$$\theta = max(\theta_0, \theta_{-1}), \tag{2.29}$$

then we will prove the following relations hold

$$\theta_{2n-1} \leq 2^{(1+p)+(1+p)^2+\cdots+(1+p)^n} \theta^{(1+p)^n}, \quad (n \geq 1), \tag{2.30}$$

$$\theta_{2n} \leq 2^{(1+p)+(1+p)^2+\cdots+(1+p)^n} \theta^{(1+p)^n}, \quad (n \geq 1). \tag{2.31}$$

For n=1, from (2.27) we have

$$\theta_1 \leq (2\theta)^{1+p} = 2^{1+p}\theta^{1+p}, \tag{2.32}$$

and from (2.27), (2.28) and (2.29) we have

$$\theta_2 \leq (2max\,(\theta_1, \theta_0))^{1+p} \leq (2\theta_0)^{1+p} \leq (2\theta)^{1+p} = 2^{1+p}\theta^{1+p}. \tag{2.33}$$

Hence (2.30) and (2.31) hold for $n = 1$.

Suppose (2.30) and (2.31) hold for $n = k$, i.e.,

$$\theta_{2k-1} \leq 2^{(1+p)+(1+p)^2+\cdots+(1+p)^k} \theta^{(1+p)^k}, \tag{2.34}$$

$$\theta_{2k} \leq 2^{(1+p)+(1+p)^2+\cdots+(1+p)^k} \theta^{(1+p)^k}. \tag{2.35}$$

Then, we have

$$\begin{aligned} \theta_{2k+1} &\leq (2max\,(\theta_{2k}, \theta_{2k-1}))^{1+p} \leq (2\theta_{2k-1})^{1+p} \\ &\leq 2^{1+p}\left(2^{(1+p)+(1+p)^2+\cdots+(1+p)^k} \theta^{(1+p)^k}\right)^{1+p} \\ &= 2^{(1+p)+(1+p)^2+\cdots+(1+p)^{k+1}} \theta^{(1+p)^{k+1}}, \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} \theta_{2k+2} &\leq (2max\,(\theta_{2k+1}, \theta_{2k}))^{1+p} \leq (2\theta_{2k})^{1+p} \\ &\leq 2^{1+p}\left(2^{(1+p)+(1+p)^2+\cdots+(1+p)^k} \theta^{(1+p)^k}\right)^{1+p} \\ &= 2^{(1+p)+(1+p)^2+\cdots+(1+p)^{k+1}} \theta^{(1+p)^{k+1}}. \end{aligned} \tag{2.37}$$

That is, (2.30) and (2.31) hold for $n = k + 1$. By induction, (2.30) and (2.31) hold for all $n \geq 1$.

On the other hand, it is easy to see

$$\begin{aligned} 2^{(1+p)+(1+p)^2+\cdots+(1+p)^n} \theta^{(1+p)^n} &= 2^{\frac{(1+p)^{n+1}-(1+p)}{p}} \theta^{(1+p)^n} \\ &= \left(2^{1+p}\theta^p\right)^{\frac{(1+p)^n}{p}} \left(\frac{1}{2}\right)^{1+\frac{1}{p}}, \end{aligned} \tag{2.38}$$

then from (2.24), (2.29), (2.30) and (2.38), we have

$$\begin{aligned}\|x_{2n-1}-x^{\star}\| &\le \frac{1}{\omega}\left(2^{1+p}\theta^{p}\right)^{\frac{(1+p)^n}{p}}\left(\frac{1}{2}\right)^{1+\frac{1}{p}}\\ &= \frac{1}{\omega}\left(2^{1+p}\omega^{p}\left(max\left(\|x_{-1}-x^{\star}\|,\|x_0-x^{\star}\|\right)\right)^{p}\right)^{\frac{(1+p)^n}{p}}\left(\frac{1}{2}\right)^{1+\frac{1}{p}}\\ &= \left(2^{1+p}\omega^{p}\right)^{\frac{(1+p)^n-1}{p}}\left(max\left(\|x_{-1}-x^{\star}\|,\|x_0-x^{\star}\|\right)\right)^{(1+p)^n}.\end{aligned} \tag{2.39}$$

Using the definitions of $\bar{r}_0$ and ω, we can verify that the condition $r \le \bar{r}_0$ is equivalent to $2^{1+p}\omega^p \le \frac{1}{r^p}$, and thus from the condition $r \le \bar{r}_0$ and (2.39), we have

$$\begin{aligned}\|x_{2n-1}-x^{\star}\| &\le \left(\frac{1}{r}\right)^{(1+p)^n-1}\left(max\left(\|x_{-1}-x^{\star}\|,\|x_0-x^{\star}\|\right)\right)^{(1+p)^n}\\ &= r\left(\frac{max\left(\|x_{-1}-x^{\star}\|,\|x_0-x^{\star}\|\right)}{r}\right)^{(1+p)^n}.\end{aligned} \tag{2.40}$$

Similarly, from the condition $r \le \bar{r}_0$, we can deduce

$$\|x_{2n}-x^{\star}\| \le r\left(\frac{max\left(\|x_{-1}-x^{\star}\|,\|x_0-x^{\star}\|\right)}{r}\right)^{(1+p)^n}. \tag{2.41}$$

Merging (2.40) and (2.41), we obtain (2.3) at once.

To show the uniqueness, we assume that there exists a second solution $y^{\star} \in B(x^{\star}, (\frac{1}{K_1})^{\frac{1}{p}})$ and consider operator $A = [y^{\star}, x^{\star}; F]$. Since $A(y^{\star}-x^{\star}) = F(y^{\star}) - F(x^{\star})$, if operator A is invertible then $y^{\star} = x^{\star}$. Indeed, from (1.18), we have

$$\|I - F'(x^{\star})^{-1}A\| = \|F'(x^{\star})^{-1}(F'(x^{\star}) - A)\| \le K_1\|y^{\star}-x^{\star}\|^{p} < 1, \tag{2.42}$$

then by the Banach lemma, we can see operator A is invertible. From the definition of r, it is easy to verify that the ball $B(x^{\star}, (\frac{1}{K_1})^{\frac{1}{p}})$ is bigger than $B(x^{\star}, r)$.

The proof is completed. □

Example 2.2 Let $X = Y = \mathbb{R}$, $x^{\star} = 2.25$, $D = [0.81, 6.25]$, and define function F on D by

$$F(x) = \frac{2}{3}x^{1.5} - x. \tag{2.43}$$

Then, $F'(x) = x^{\frac{1}{2}} - 1$, and $F'(x^{\star}) = 0.5$. Since

$$[x, y; F] = \int_0^1 F'(tx + (1-t)y)dt, \tag{2.44}$$

for any $x, y, z \in D$, we have

$$|F'(x^\star)^{-1}\left(\left[x, y; F\right] - F'(z)\right)| = 2|\int_0^1 \left(F'(tx + (1-t)y) - F'(z)\right) dt|$$
$$\leq 2\int_0^1 |\sqrt{tx + (1-t)y} - \sqrt{z}|dt$$
$$\leq 2\int_0^1 \sqrt{|tx + (1-t)y - z|}dt. \tag{2.45}$$

Here, we use a basic inequality

$$|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a-b|}, \quad \text{for any } a, b \geq 0,$$

which can be verified easily. Hence, from (2.45), we get

$$|F'(x^\star)^{-1}\left(\left[x, y; F\right] - F'(z)\right)| \leq 2\int_0^1 \sqrt{t|x-z| + (1-t)|y-z|}dt$$
$$\leq 2\int_0^1 \left(\sqrt{t}\sqrt{|x-z|} + \sqrt{(1-t)}\sqrt{|y-z|}\right) dt$$
$$= \frac{4}{3}\left(\sqrt{|x-z|} + \sqrt{|y-z|}\right). \tag{2.46}$$

Here, we use another basic inequality

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, \quad \text{for any } a, b \geq 0,$$

which holds obviously. By (2.46), we can choose constants $p = 0.5$ and $K = \frac{4}{3}$ in Condition (1.15).

Next we shall show how choose the constant K_0. In fact, for any $x, y \in D$, we have

$$|F'(x^\star)^{-1}\left(\left[x, y; F\right] - F'(x^\star)\right)| = 2|\int_0^1 \left(F'(tx+(1-t)y) - F'(x^\star)\right) dt|$$
$$\leq 2\int_0^1 |\sqrt{tx+(1-t)y} - \frac{3}{2}|dt$$
$$= 2\int_0^1 \left(|\sqrt{tx+(1-t)y} - \frac{3}{2}|^{\frac{1}{2}}\right)^2 dt. \tag{2.47}$$

Since $x, y \in D = [0.81, 6.25]$, we easily verify that for any $x, y \in D$ and any $t \in [0, 1]$, the following inequality holds:

$$2|\sqrt{tx + (1-t)y} - \frac{3}{2}|^{\frac{1}{2}} \leq \left(\sqrt{tx + (1-t)y} + \frac{3}{2}\right)^{\frac{1}{2}}. \tag{2.48}$$

Combining (2.47) and (2.48) leads to

$$|F'(x^\star)^{-1}\left([x, y; F] - F'(x^\star)\right)| \leq \int_0^1 \sqrt{|tx + (1-t)y - \tfrac{9}{4}|}dt. \tag{2.49}$$

Table 1 The results of Example 1

x_{-1}	x_0	c	r_0	$\bar{r}_0$	r_1	r_2	r	r_{RW}
2.3	2.2	2	0.067553029	0.067553029	0.038377949	0.0625	0.067553029	0.037256521
2.28	2.22	1	0.079101563	0.067553029	0.0625	0.0625	0.067553029	0.037256521
2.27	2.23	0.8	0.082266945	0.067553029	0.095174975	0.0625	0.082266945	0.037256521
2.3	2.28	0.5	0.087975919	0.067553029	0.193019485	0.0625	0.087975919	0.037256521

Now using a similar analysis as the derivation of (2.46), for any $x, y \in D$, we get

$$|F'(x^\star)^{-1}\left(\left[x, y; F\right] - F'(x^\star)\right)| \le \frac{2}{3}\left(\sqrt{|x - x^\star|} + \sqrt{|y - x^\star|}\right). \tag{2.50}$$

That means we can choose constants $p = 0.5$ and $K_0 = \frac{2}{3}$ in Condition (2.17).

In order to show the application of our main theorem, we use some different choices of initial points x_0 and x_{-1}, and list the corresponding constants c, $r_0, \bar{r}_0, r_1, r_2, r$ and r_{RW} in Table 1. From Table 1, we deduce that the radius obtained by our theorem is bigger than the radius obtained by [13] under all cases. Also note that if we can choose a smaller constant c than 2, it is possible to enlarge the radius further.

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