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Matrix decomposition algorithms for the finite element Galerkin method with piecewise Hermite cubics

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Abstract Matrix decomposition algorithms (MDAs) employing fast Fourier transforms are developed for the solution of the systems of linear algebraic equations arising when the finite element Galerkin method with piecewise Hermite bicubics is used to solve Poisson's equation on the unit square. Like their orthogonal spline collocation counterparts, these MDAs, which require $O(N^2 \log N)$ operations on an $N \times N$ uniform partition, are based

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on knowledge of the solution of a generalized eigenvalue problem associated with the corresponding discretization of a two-point boundary value problem. The eigenvalues and eigenfunctions are determined for various choices of boundary conditions, and numerical results are presented to demonstrate the efficacy of the MDAs.

Keywords Elliptic boundary value problems · Finite element Galerkin method · Piecewise Hermite cubics · Generalized eigenvalue problem · Eigenvalues and eigenfunctions · Matrix decomposition algorithm

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1 Introduction

Over the past few decades, several matrix decomposition algorithms (MDAs) have been developed for the solution of the linear systems arising from discretizations of certain separable elliptic boundary value problems on rectangular domains. These algorithms are commonly discussed in terms of solving the linear systems associated with the five-point finite difference approximation to Poisson's equation with Dirichlet boundary conditions, although they are applicable to other finite difference approximations and to more general boundary value problems. While most attention has been devoted to finite difference methods, MDAs have also been developed for solving the linear systems arising in finite element Galerkin (FEG) methods, nodal and orthogonal spline collocation (OSC) methods, and spectral methods; see, for example, [1, 3, 4, 6–8, 12].

The focus in this paper is on the development of MDAs for solving the linear systems arising when tensor products of piecewise Hermite (C^1) cubics are used in the FEG method for Poisson's equation in the unit square,

$$-\Delta u = f(x, y), \qquad (x, y) \in \Omega = (0, 1) \times (0, 1), \tag{1.1}$$

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with u satisfying on the horizontal sides of $\partial\Omega$, the homogeneous Dirichlet boundary conditions

$$u(x,0) = u(x,1) = 0, \quad x \in (0,1), \tag{1.2}$$

and on the vertical sides of $\partial\Omega$, homogeneous Dirichlet boundary conditions

$$u(0,y) = u(1,y) = 0, \quad y \in [0,1], \tag{1.3}$$

or homogeneous Neumann boundary conditions

$$u_x(0,y) = u_x(1,y) = 0, \quad y \in [0,1], \tag{1.4}$$

or homogeneous Dirichlet-Neumann boundary conditions

$$u(0,y) = u_x(1,y) = 0, \quad y \in [0,1], \tag{1.5}$$

or homogeneous Neumann-Dirichlet boundary conditions

$$u_x(0,y) = u(1,y) = 0, \quad y \in [0,1], \tag{1.6}$$

or the periodic boundary conditions

$$u(0,y) = u(1,y), \quad u_x(0,y) = u_x(1,y), \quad y \in [0,1]. \tag{1.7}$$

Throughout, we consider an $N \times N$ partition of Ω with the same uniform partition of $[0,1]$ in the x and y directions.

The FEG method with piecewise Hermite cubics applied to (1.1), (1.2), and any one of the above boundary conditions on the vertical sides of $\partial\Omega$, gives rise to a linear system of the form

$$(A_1 \otimes B_2 + B_1 \otimes A_2)\mathbf{u} = \mathbf{f}, \tag{1.8}$$

where A_i and B_i are square matrices of order $M = 2N$, $\otimes$ denotes the matrix tensor product, and

$$\mathbf{u} = [u_{1,1}, \ldots, u_{1,M}, \ldots, u_{M,1}, \ldots, u_{M,M}]^T,$$

$$\mathbf{f} = [f_{1,1}, \ldots, f_{1,M}, \ldots, f_{M,1}, \ldots, f_{M,M}]^T. \tag{1.9}$$

Assume that a real diagonal matrix Λ and a real nonsingular matrix Z are known such that

$$A_1 Z = B_1 Z \Lambda, \tag{1.10}$$

$$Z^T B_1 Z = I, \tag{1.11}$$

where I is the identity matrix. Premultiplying (1.10) by Z^T and using (1.11), we obtain

$$Z^T A_1 Z = \Lambda. \tag{1.12}$$

With $\mathbf{v} = (Z^{-1} \otimes I)\mathbf{u}$, the system (1.8) can then be written in the form

$$(Z^T \otimes I)(A_1 \otimes B_2 + B_1 \otimes A_2)(Z \otimes I)\mathbf{v} = (Z^T \otimes I)\mathbf{f},$$

which becomes, on using (1.11) and (1.12),

$$(\Lambda \otimes B_2 + I \otimes A_2)\mathbf{v} = (Z^T \otimes I)\mathbf{f}.$$

Thus, from the preceding, we obtain the following MDA for solving (1.8):

MDA

1. Compute $\mathbf{g} = (Z^T \otimes I)\mathbf{f}$.
2. Solve $(\Lambda \otimes B_2 + I \otimes A_2)\mathbf{v} = \mathbf{g}$.
3. Compute $\mathbf{u} = (Z \otimes I)\mathbf{v}$.

In our case, Z is a matrix whose elements are sines and/or cosines. Consequently the matrix-vector multiplications involving Z^T and Z in steps 1 and 3, respectively, are performed using fast Fourier transforms (FFTs). Since Λ is diagonal, the coefficient matrix in step 2 is block diagonal and the system reduces to M independent systems of order M. Moreover, since A_2 and B_2 are banded matrices, step 2 is carried out using a banded Gauss elimination solver. Hence the total cost of the MDA is $O(N^2 \log N)$.

For a uniform partition of $[0, \pi]$, Strang and Fix [13, page 227] present a solution of the generalized eigenvalue problem

$$A_1\mathbf{z} = \lambda B_1\mathbf{z} \tag{1.13}$$

resulting from the piecewise Hermite cubic FEG discretization of the continuous eigenvalue problem

$$-u'' = \lambda u, \quad x \in (0, \pi), \quad u(0) = u'(\pi) = 0.$$

This information could perhaps be used to develop an MDA for the solution of (1.8) resulting from the Hermite bicubic FEG approximation of (1.1) subject to the Dirichlet-Neumann boundary condition in the x-direction; cf., Section 5. However, no indication is given in [13] of how one could handle other boundary conditions.

For the solution of (1.1) with homogeneous Dirichlet boundary conditions, Bank [2] formulated matrix decomposition-like algorithms for solving the FEG linear systems (1.8) arising from tensor products of piecewise Hermite cubics. In this approach, the matrices A_i, B_i, $i = 1, 2$, corresponding to Bank's choice of basis functions are first reordered before the application of an orthogonal transformation with the block matrix

$$Q = \sqrt{\frac{2}{N}} \begin{bmatrix} \mathcal{S}_{N-1} & 0 \\ 0 & \mathcal{C}_{N+1}D \end{bmatrix},$$

where D is the diagonal matrix of order $N + 1$ given by

$$D = \text{diag}(1/2, 1, \ldots, 1, 1/2), \tag{1.14}$$

and $\mathcal{S}_{N-1}$ and $\mathcal{C}_{N+1}$ are the matrices given by

$$\mathcal{S}_{N-1} = \left(\sin\frac{ik\pi}{N}\right)_{i,k=1}^{N-1}, \quad \mathcal{C}_{N+1} = \left(\cos\frac{(i-1)(k-1)\pi}{N}\right)_{i,k=1}^{N+1}, \tag{1.15}$$

with i and k denoting the row and column indices, respectively. The resulting transformed matrices are reordered to become block diagonal with all of the blocks being 2×2 except the first and the last which are 1×1. With these transformed matrices, the original system can be written as one in which the coefficient matrix, after reordering, is block diagonal with $(N-1)^2$ blocks of order 4, $4(N-1)$ blocks of order 2, and four 1×1 blocks. While the total computational cost of Bank's method is $O(N^2 \log N)$, it requires twice as much work as the corresponding method in Section 3 of the present paper because it requires twice as many FFTs. An approach in which the orthogonal transformation is only applied to A_1 and B_1, reordered as before, is very briefly mentioned in [2]. No details are provided, but it would appear that the systems of equations corresponding to step 2 of the MDA algorithm would be much more involved than those arising in our approach. Moreover, it is not clear how either of Bank's approaches would extend to other boundary conditions. No numerical results are presented in [2].

In [11], explicit formulas are given for the eigenvalues of (1.13) resulting from the piecewise Hermite cubic FEG discretization of the continuous eigenvalue problem

$$-u'' = \lambda u, \quad x \in (0, 1), \quad u(0) = u(1) = 0,$$

and a method for solving (1.8) corresponding to (1.1)–(1.3) is developed. This method, whose cost is claimed to be half of that in [2], is based on the preliminary elimination of half of the unknowns and the use of FACR(l) method with $l = O(\log\log N)$, or the marching algorithm. No numerical evidence is provided to demonstrate the efficacy of this approach. Moreover, there is no mention of how this approach could be extended to other boundary conditions.

The costs of the Galerkin MDAs of this paper are approximately the same as the costs of the corresponding orthogonal spline collocation MDAs of [6]. For example, for the Dirichlet boundary conditions, the constant multiplier of the cost term $N^2 \log N$ is the same for the Galerkin MDA of this paper and for Algorithm II of [5] since the same number of the same FFTs is performed in each algorithm.

A brief outline of the paper is as follows. In Section 2, we introduce piecewise Hermite cubic spaces corresponding to the boundary conditions (1.3)–(1.7) and explain how (1.8) arises. Sections 3, 4, 5, 6, and 7 are devoted to the determination of the matrices Z and Λ satisfying (1.10), (1.11) for the piecewise Hermite cubic spaces for Dirichlet, Neumann, Dirichlet-Neumann, Neumann-Dirichlet, and periodic boundary conditions, respectively. It is worth noting that in contrast to the OSC approach for Hermite cubics [6], there does not appear to be a systematic approach to the determination of the FEG Hermite cubic eigensystems. However, the structure of the matrix Z is similar to that arising in the corresponding OSC problem and this observation facilitates the determination of the eigenvalues and eigenvectors of (1.13). In Section 8, we present numerical results obtained using our MDA. Some concluding remarks are presented in Section 9.

In each of the FEG methods considered in this paper, the approximate solution satisfies the corresponding boundary conditions. While this is not generally the case in standard FEG methods, it is not an uncommon approach in spectral methods [12].

2 Preliminaries

Let $\{x_i\}_{i=0}^N$ be a uniform partition of $[0,1]$ such that $x_i = ih$, $i = 0, \ldots, N$, where N is a positive integer and $h = 1/N$ is the stepsize. Let S_h be the space of piecewise Hermite cubics on $[0,1]$ defined by

$$S_h = \{v \in C^1[0,1] : v|_{[x_{i-1},x_i]} \in P_3, i = 1, \ldots, N\},$$

where P_3 is the set of polynomials of degree ≤ 3, and let

$$S_h^{\mathcal{D}} = \{v \in S_h : v(0) = v(1) = 0\}, \quad S_h^{\mathcal{N}} = \{v \in S_h : v'(0) = v'(1) = 0\},$$

$$S_h^{\mathcal{D},\mathcal{N}} = \{v \in S_h : v(0) = v'(1) = 0\}, \quad S_h^{\mathcal{N},\mathcal{D}} = \{v \in S_h : v'(0) = v(1) = 0\},$$

$$S_h^{\mathcal{P}} = \{v \in S_h : v(0) = v(1), v'(0) = v'(1)\}.$$

In the FEG method with piecewise Hermite cubics for (1.1) and (1.2), we seek $U \in V_h \otimes S_h^{\mathcal{D}}$, where $\otimes$ denotes the space tensor product, such that

$$\int_\Omega (U_x v_x + U_y v_y)\, d\Omega = \int_\Omega f(x,y) v\, d\Omega, \quad v \in V_h \otimes S_h^{\mathcal{D}}, \tag{2.1}$$

where $V_h = S_h^{\mathcal{D}}$ for (1.3), $V_h = S_h^{\mathcal{N}}$ for (1.4), $V_h = S_h^{\mathcal{D},\mathcal{N}}$ for (1.5), $V_h = S_h^{\mathcal{N},\mathcal{D}}$ for (1.6), and $V_h = S_h^{\mathcal{P}}$ for (1.7).

Let $\{\phi_i\}_{i=1}^{2N}$ be a basis for V_h, where V_h is either $S_h^{\mathcal{D}}$, or $S_h^{\mathcal{N}}$, or $S_h^{\mathcal{D},\mathcal{N}}$, or $S_h^{\mathcal{N},\mathcal{D}}$, or $S_h^{\mathcal{P}}$. Let the $2N \times 2N$ matrices A_1 and B_1 be defined by

$$A_1 = (a_{ik})_{i,k=1}^{2N}, \qquad a_{ik} = (\phi_i', \phi_k'), \quad B_1 = (b_{ik})_{i,k=1}^{2N}, \qquad b_{ik} = (\phi_i, \phi_k), \tag{2.2}$$

where $(\cdot,\cdot)$ is the L^2 inner product on $[0,1]$. Then (2.1) gives rise to (1.8)–(1.9) with A_2 and B_2 equal respectively to A_1 and B_1 for $V_h = S_h^{\mathcal{D}}$ and

$$f_{i,k} = \int_\Omega f(x,y)\phi_i(x)\phi_k(y)\, d\Omega, \quad i,k = 1, \ldots, M. \tag{2.3}$$

The basis $\{\phi_i\}_{i=1}^{2N}$ for each choice of V_h is given in terms of the *value* and *slope* functions $v_i, s_i \in S_h$, $i = 1, \ldots, N$, which are defined by

$$v_i(x_j) = \delta_{ij}, v_i'(x_j) = 0, \quad s_i(x_j) = 0, s_i'(x_j) = h^{-1}\delta_{ij} \quad i, j = 0, \ldots, N,$$

where δ_{ij} is the Kronecker delta. (The explicit formulas for v_i and s_i are given in Section 2.3 of [9].) It should be noted that the functions $\{s_i\}_{1=0}^{N}$ are the standard "slope" functions associated with the space of piecewise Hermite cubics multiplied by h^{-1}. With this normalization, the matrices $A_i,\ B_i,\ i = 1, 2$ are not only simplified but their condition numbers are improved. This normalization is not performed in [2, 13]; no details of the basis used in [11] are provided.

With

$$\begin{cases} \alpha_1 = 12/5, \quad \alpha_2 = 4/15, \quad \alpha_3 = -6/5, \; \alpha_4 = -1/30, \quad \alpha_5 = 1/10, \\ \beta_1 = 26/35, \; \beta_2 = 2/105, \; \beta_3 = 9/70, \;\; \beta_4 = -1/140, \; \beta_5 = -13/420, \end{cases} \tag{2.4}$$

we have the following formulas (cf., Section 2.3 in [9]):

$$\begin{cases} (v_i', v_i') = h^{-1}\alpha_1, & (s_i', s_i') = h^{-1}\alpha_2, \\ (v_i', s_i') = 0, & (v_i', v_{i+1}') = h^{-1}\alpha_3, \\ (s_i', s_{i+1}') = h^{-1}\alpha_4, & (v_i', s_{i+1}') = -(s_i', v_{i+1}') = h^{-1}\alpha_5, \\ (v_0', v_0') = (v_N', v_N') = h^{-1}\alpha_1/2, & (s_0', s_0') = (s_N', s_N') = h^{-1}\alpha_2/2, \\ (v_0', s_0') = -(v_N', s_N') = h^{-1}\alpha_5, & \end{cases} \tag{2.5}$$

$$\begin{cases} (v_i, v_i) = h\beta_1, & (s_i, s_i) = h\beta_2, \\ (v_i, s_i) = 0, & (v_i, v_{i+1}) = h\beta_3, \\ (s_i, s_{i+1}) = h\beta_4, & (v_i, s_{i+1}) = -(s_i, v_{i+1}) = h\beta_5, \\ (v_0, v_0) = (v_N, v_N) = h\beta_1/2, & (s_0, s_0) = (s_N, s_N) = h\beta_2/2. \end{cases} \tag{2.6}$$

3 Dirichlet boundary conditions

The basis $\{\phi_i\}_{i=1}^{2N}$ for $S_h^{\mathcal{D}}$ is given by

$$\{\phi_1, \ldots, \phi_{2N}\} = \{v_1, \ldots, v_{N-1}, s_0, \ldots, s_N\}. \tag{3.1}$$

For any τ_i, $i = 1, \ldots, 5$, we introduce the tridiagonal matrices

$$T_\tau(i, j) = \begin{bmatrix} \tau_i & \tau_j & & & \\ \tau_j & \tau_i & \tau_j & & \\ & \ddots & \ddots & \ddots & \\ & & \tau_j & \tau_i & \tau_j \\ & & & \tau_j & \tau_i \end{bmatrix}, \quad R_\tau(i, j) = \begin{bmatrix} \tau_i/2 & \tau_j & & & \\ \tau_j & \tau_i & \tau_j & & \\ & \ddots & \ddots & \ddots & \\ & & \tau_j & \tau_i & \tau_j \\ & & & \tau_j & \tau_i/2 \end{bmatrix}, \tag{3.2}$$

which are of orders $N-1$ and $N+1$, respectively, and the $(N-1)\times(N+1)$ rectangular matrix $S_\tau(5)$ given by

$$S_\tau(5) = \begin{bmatrix} -\tau_5 & 0 & \tau_5 & & & & \\ & -\tau_5 & 0 & \tau_5 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & -\tau_5 & 0 & \tau_5 & \\ & & & & -\tau_5 & 0 & \tau_5 \end{bmatrix}. \tag{3.3}$$

Using (2.2), (3.1), (2.5)–(2.6), and (3.2)–(3.3), it is easy to verify that

$$A_1 = h^{-1} \begin{bmatrix} T_\alpha(1,3) & S_\alpha(5) \\ S_\alpha^T(5) & R_\alpha(2,4) \end{bmatrix}, \quad B_1 = h \begin{bmatrix} T_\beta(1,3) & S_\beta(5) \\ S_\beta^T(5) & R_\beta(2,4) \end{bmatrix}, \tag{3.4}$$

where the α_i and β_i are given in (2.4). Thus (1.13) becomes

$$\begin{bmatrix} T_\gamma(1,3) & S_\gamma(5) \\ S_\gamma^T(5) & R_\gamma(2,4) \end{bmatrix} \mathbf{z} = \mathbf{0}, \tag{3.5}$$

where

$$\gamma_i = \alpha_i - \lambda h^2 \beta_i, \qquad i = 1, \ldots, 5. \tag{3.6}$$

In [11], explicit formulas are given for the eigenvalues only. Explicit formulas for the corresponding normalized eigenvectors $\{\mathbf{z}_k^\pm\}_{k=1}^N$ can be derived in the following manner [10]. We introduce the vectors

$$\mathbf{x}^{(k)} = \left[x_1^{(k)}, \ldots, x_{N-1}^{(k)}\right]^T, \qquad k = 1, \ldots, N-1, \tag{3.7}$$

$$\mathbf{y}^{(k)} = \left[y_0^{(k)}, \ldots, y_N^{(k)}\right]^T, \qquad k = 0, \ldots, N, \tag{3.8}$$

where

$$x_i^{(k)} = \sin \frac{ik\pi}{N}, \qquad y_i^{(k)} = \cos \frac{ik\pi}{N}. \tag{3.9}$$

Let

$$\mathbf{z}_k^\pm = \begin{bmatrix} \mathbf{x}^{(k)} \\ c_k^\pm \mathbf{y}^{(k)} \end{bmatrix}, \qquad k = 1, \ldots, N-1, \tag{3.10}$$

where the constants $c_k^\pm$, $k = 1, \ldots, N-1$, are to be specified, and let

$$\mathbf{z}_N^- = \begin{bmatrix} \mathbf{0} \\ \mathbf{y}^{(0)} \end{bmatrix}, \qquad \mathbf{z}_N^+ = \begin{bmatrix} \mathbf{0} \\ \mathbf{y}^{(N)} \end{bmatrix}, \tag{3.11}$$

where $\mathbf{0}$ is the zero vector with $N-1$ components. The vectors $\mathbf{z}_k^\pm$ of (3.10) satisfy (3.5) provided

$$\begin{cases} T_\gamma(1,3)\mathbf{x}^{(k)} + c_k^\pm S_\gamma(5)\mathbf{y}^{(k)} = \mathbf{0}, \\ S_\gamma^T(5)\mathbf{x}^{(k)} + c_k^\pm R_\gamma(2,4)\mathbf{y}^{(k)} = \mathbf{0}, \end{cases} \qquad k = 1, \ldots, N-1. \tag{3.12}$$

It is easy to show that

$$\begin{cases} T_\gamma(1,3)\mathbf{x}^{(k)} = (2\gamma_3\mu_k + \gamma_1)\mathbf{x}^{(k)}, & k = 1, \ldots, N-1, \\ S_\gamma(5)\mathbf{y}^{(k)} = -2\gamma_5\nu_k\mathbf{x}^{(k)}, & k = 0, \ldots, N, \\ S_\gamma^T(5)\mathbf{x}^{(k)} = -2\gamma_5\nu_k D\mathbf{y}^{(k)}, & k = 1, \ldots, N-1, \\ R_\gamma(2,4)\mathbf{y}^{(k)} = (2\gamma_4\mu_k + \gamma_2) D\mathbf{y}^{(k)}, & k = 0, \ldots, N, \end{cases} \tag{3.13}$$

where the γ_i are defined in (3.6), the diagonal matrix D of order $N+1$ is given by (1.14), and

$$\nu_k = \sin\frac{k\pi}{N}, \quad \mu_k = \cos\frac{k\pi}{N}. \tag{3.14}$$

Then, using (3.12)–(3.13), we obtain

$$\begin{cases} (2\gamma_3\mu_k + \gamma_1)\mathbf{x}^{(k)} - c_k^{\pm} 2\gamma_5\nu_k\mathbf{x}^{(k)} = \mathbf{0}, \\ -2\gamma_5\nu_k D\mathbf{y}^{(k)} + c_k^{\pm}(2\gamma_4\mu_k + \gamma_2)D\mathbf{y}^{(k)} = \mathbf{0}, \end{cases} \quad k = 1, \ldots, N-1. \tag{3.15}$$

Since, by (3.7)–(3.9) and (1.14),

$$D\mathbf{y}^{(k)} \neq \mathbf{0}, \quad \mathbf{x}^{(k)} \neq \mathbf{0}, \quad k = 1, \ldots, N-1, \tag{3.16}$$

equations (3.15) are equivalent to

$$(2\gamma_3\mu_k + \gamma_1) - c_k^{\pm} 2\gamma_5\nu_k = 0, \tag{3.17}$$

$$-2\gamma_5\nu_k + c_k^{\pm}(2\gamma_4\mu_k + \gamma_2) = 0, \tag{3.18}$$

for $k = 1, \ldots, N-1$. On solving (3.17) for $c_k^{\pm}$, we obtain

$$c_k^{\pm} = \frac{2\gamma_3\mu_k + \gamma_1}{2\gamma_5\nu_k}, \quad k = 1, \ldots, N-1, \tag{3.19}$$

where we assume the denominator of (3.19) to be non-zero. We substitute (3.19) into (3.18) to obtain

$$(2\gamma_3\mu_k + \gamma_1)(2\gamma_4\mu_k + \gamma_2) - 4\gamma_5^2\nu_k^2 = 0, \quad k = 1, \ldots, N-1. \tag{3.20}$$

Substituting (3.6) into (3.20) yields the quadratic equation

$$a(\lambda h^2)^2 + b\lambda h^2 + c = 0, \quad k = 1, \ldots, N-1,$$

the solutions of which are given by

$$\lambda_k^{\pm} = h^{-2}\Phi^{\pm}(\mu_k, \nu_k), \quad k = 1, \ldots, N-1, \tag{3.21}$$

where

$$\Phi^{\pm}(\mu, \nu) = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a, \tag{3.22}$$

and

$$\begin{cases} a = b_1 b_2 - b_3^2, & b = 2a_3 b_3 - a_1 b_2 - b_1 a_2, & c = a_1 a_2 - a_3^2, \\ a_1 = 2\alpha_3\mu + \alpha_1, & a_2 = 2\alpha_4\mu + \alpha_2, & a_3 = 2\alpha_5\nu, \\ b_1 = 2\beta_3\mu + \beta_1, & b_2 = 2\beta_4\mu + \beta_2, & b_3 = 2\beta_5\nu. \end{cases} \tag{3.23}$$

Thus $\{\lambda_k^{\pm}\}_{k=1}^{N-1}$ of (3.21) are eigenvalues of (1.13) and $\{z_k^{\pm}\}_{k=1}^{N-1}$ of (3.10) with $c_k^{\pm}$ of (3.19) and $\gamma_1, \gamma_3, \gamma_5$ of (3.6) with $\lambda = \lambda_k^{\pm}$ are the corresponding eigenvectors.

For $\mathbf{z}_N^-$ and $\mathbf{z}_N^+$ of (3.11), (3.5) is equivalent to

$$R_\gamma(2,4)\mathbf{y}^{(0)} = \mathbf{0}, \quad R_\gamma(2,4)\mathbf{y}^{(N)} = \mathbf{0}, \tag{3.24}$$

which becomes

$$(2\gamma_4 + \gamma_2) D\mathbf{y}^{(0)} = \mathbf{0}, \quad (\gamma_2 - 2\gamma_4) D\mathbf{y}^{(N)} = \mathbf{0}.$$

Since $D\mathbf{y}^{(0)}$ and $D\mathbf{y}^{(N)}$ are nonzero vectors, it follows from (3.6) that

$$\lambda_N^- = h^{-2}\frac{\alpha_2 + 2\alpha_4}{\beta_2 + 2\beta_4}, \quad \lambda_N^+ = h^{-2}\frac{\alpha_2 - 2\alpha_4}{\beta_2 - 2\beta_4}, \tag{3.25}$$

are eigenvalues of (1.13) and $\mathbf{z}_N^-$ and $\mathbf{z}_N^+$ of (3.11) are the corresponding eigenvectors. It has been shown in [10] that the eigenvalues $\{\lambda_k^\pm\}_{k=1}^N$ are positive and distinct. Note that the denominator of (3.19) is non-zero since by (3.6) with $\lambda = \lambda_k^\pm$, (2.4) and (3.14), we have

$$\gamma_5 \nu_k = \left(1/10 + \lambda_k^\pm h^2 13/420\right) \nu_k > 0, \qquad k = 1, \ldots, N-1.$$

We now normalize the eigenvectors so that (1.11) is satisfied. Since B_1 is symmetric and positive definite, there exists symmetric and positive definite $B_1^{1/2}$ such that $B_1 = B_1^{1/2} B_1^{1/2}$. Therefore, the equation

$$A_1 \mathbf{z}_k^\pm = \lambda_k^\pm B_1 \mathbf{z}_k^\pm, \qquad k = 1, \ldots, N,$$

can be written as

$$B_1^{-1/2} A_1 B_1^{-1/2} \mathbf{w}_k^\pm = \lambda_k^\pm \mathbf{w}_k^\pm, \qquad k = 1, \ldots, N,$$

where $\mathbf{w}_k^\pm = B_1^{1/2} \mathbf{z}_k^\pm$. Since $B_1^{-1/2} A_1 B_1^{-1/2}$ is symmetric and $\{\lambda_k^\pm\}_{k=1}^N$ are distinct, we have

$$\left(B_1 \mathbf{z}_k^\pm, \mathbf{z}_l^\pm\right)_{\mathbb{R}^{2N}} = \left(B_1^{1/2} \mathbf{z}_k^\pm, B_1^{1/2} \mathbf{z}_l^\pm\right)_{\mathbb{R}^{2N}} = \left(\mathbf{w}_k^\pm, \mathbf{w}_l^\pm\right)_{\mathbb{R}^{2N}} = 0, \qquad k \neq l, \tag{3.26}$$

where here and throughout $(\cdot,\cdot)_{\mathbb{R}^{2N}}$ denotes the standard inner product in $\mathbb{R}^{2N}$.

To satisfy (1.11), we must select $\{d_k^\pm\}_{k=1}^N$ so that

$$\left(B_1 d_k^\pm \mathbf{z}_k^\pm, d_k^\pm \mathbf{z}_k^\pm\right)_{\mathbb{R}^{2N}} = 1, \qquad k = 1, \ldots, N. \tag{3.27}$$

Using (3.10), (3.7)–(3.9), (1.14), and the identity

$$\sum_{i=1}^{N-1} \sin^2\left(\frac{ik\pi}{N}\right) = \frac{N}{2}, \quad k = 1, \ldots, N-1,$$

we have

$$\begin{aligned}
\left(h^{-1} B_1 \mathbf{z}_k^\pm, \mathbf{z}_k^\pm\right)_{\mathbb{R}^{2N}} &= \left(2\beta_3\mu_k + \beta_1 - c_k^\pm 2\beta_5\nu_k\right) \sum_{i=1}^{N-1} \sin^2\left(\frac{ik\pi}{N}\right) \\
&\quad + \left[c_k^\pm\left(2\beta_4\mu_k + \beta_2\right) - 2\beta_5\nu_k\right] c_k^\pm \left[1 + \sum_{i=1}^{N-1} \cos^2\left(\frac{ik\pi}{N}\right)\right] \\
&= \frac{N}{2}\left(\left[\left(2\beta_4\mu_k + \beta_2\right) c_k^\pm - 4\beta_5\nu_k\right] c_k^\pm + 2\beta_3\mu_k + \beta_1\right), \\
&\quad k = 1, \ldots, N-1.
\end{aligned} \tag{3.28}$$

Also, we have

$$\left(h^{-1}B_1\mathbf{z}_N^-, \mathbf{z}_N^-\right)_{I\!R^{2N}} = (\beta_2+2\beta_4)\,N, \quad \left(h^{-1}B_1\mathbf{z}_N^+, \mathbf{z}_N^+\right)_{I\!R^{2N}} = (\beta_2-2\beta_4)N. \tag{3.29}$$

Clearly, it follows from (3.28), (3.29), and $Nh = 1$ that, to satisfy (3.27), we must take

$$d_k^\pm = \left\{([(2\beta_4\mu_k+\beta_2)c_k^\pm - 4\beta_5\nu_k]c_k^\pm + 2\beta_3\mu_k+\beta_1)/2\right\}^{-1/2}, \quad k=1,\ldots,N-1,$$

and $d_N^\pm = (\beta_2 \mp 2\beta_4)^{-1/2}$. Now let $\Lambda = \mathrm{diag}(\lambda_1^-,\ldots,\lambda_N^-,\lambda_1^+,\ldots,\lambda_N^+)$ and

$$Z = \left[d_1^-\mathbf{z}_1^-,\ldots,d_N^-\mathbf{z}_N^-, d_1^+\mathbf{z}_1^+,\ldots,d_N^+\mathbf{z}_N^+\right]^T,$$

where $\{\lambda_k^\pm\}_{k=1}^N$ and $\{\mathbf{z}_k^\pm\}_{k=1}^N$ are given by (3.21)–(3.23), (3.25), (3.10), (3.19), and (3.11). Then it follows from (1.13), (3.26) and (3.27) that (1.10) and (1.11) are satisfied.

With

$$\begin{cases} \Lambda_1^- = \mathrm{diag}\left(d_1^-,\ldots,d_{N-1}^-\right), \quad \Lambda_1^+ = \left[\mathbf{0}, \mathrm{diag}\left(d_1^+,\ldots,d_{N-1}^+\right), \mathbf{0}\right], \\ \Lambda_2^- = \begin{bmatrix} \mathbf{0}^T \\ \mathrm{diag}(c_1^-d_1^-,\ldots,c_{N-1}^-d_{N-1}^-) \\ \mathbf{0}^T \end{bmatrix}, \\ \Lambda_2^+ = \mathrm{diag}(d_N^-, c_1^+d_1^+,\ldots,c_{N-1}^+d_{N-1}^+, d_N^+), \end{cases} \tag{3.30}$$

using (3.7)–(3.11), we can write Z in the form

$$Z = \begin{bmatrix} \mathcal{S}_{N-1} & \\ & \mathcal{C}_{N+1} \end{bmatrix} \begin{bmatrix} \Lambda_1^- & \Lambda_1^+ \\ \Lambda_2^- & \Lambda_2^+ \end{bmatrix}, \tag{3.31}$$

where $\mathcal{S}_{N-1}$ and $\mathcal{C}_{N+1}$ are given in (1.15).

In the MDA, the FFT routines SINT and COST of [14] (initialized by SINTI and COSTI) are used in steps 1 and 3, respectively.

4 Neumann boundary conditions

The basis $\{\phi_i\}_{i=1}^{2N}$ for $S_h^{\mathcal{N}}$ is given by

$$\{\phi_1,\ldots,\phi_{2N}\} = \{v_0,\ldots,v_N, s_1,\ldots,s_{N-1}\}. \tag{4.1}$$

Using (2.2), (4.1), (2.5)–(2.6) and the matrices in (3.2)–(3.3), with the γ_i as in (3.6), (1.13) becomes

$$\begin{bmatrix} R_\gamma(1,3) & -S_\gamma^T(5) \\ -S_\gamma(5) & T_\gamma(2,4) \end{bmatrix} \mathbf{z} = \mathbf{0}. \tag{4.2}$$

If the vectors $\mathbf{x}^{(k)}$ and $\mathbf{y}^{(k)}$ are as in (3.7)–(3.9), then substituting

$$\mathbf{z}_k^\pm = \begin{bmatrix} c_k^\pm \mathbf{y}^{(k)} \\ \mathbf{x}^{(k)} \end{bmatrix}, \quad k = 1,\ldots,N-1, \tag{4.3}$$

into (4.2) leads to

$$c_k^{\pm}(2\gamma_3\mu_k+\gamma_1)+2\gamma_5\nu_k=0, \tag{4.4}$$

$$c_k^{\pm}2\gamma_5\nu_k+(2\gamma_4\mu_k+\gamma_2)=0, \tag{4.5}$$

for $k=1,\ldots,N-1$. Solving (4.5) for $c_k^{\pm}$ gives

$$c_k^{\pm}=-\frac{2\gamma_4\mu_k+\gamma_2}{2\gamma_5\nu_k},\quad k=1,\ldots,N-1. \tag{4.6}$$

On substituting (4.6) into (4.4), we obtain (3.20). Thus the corresponding eigenvalues $\{\lambda_k^{\pm}\}_{k=1}^{N-1}$ are given by (3.21)–(3.23). (Note that these $\{\lambda_k^{\pm}\}_{k=1}^{N-1}$ are the same as those for the Dirichlet boundary conditions.) It can be shown that two additional eigenvectors and eigenvalues are given by

$$\mathbf{z}_N^-=\begin{bmatrix}\mathbf{y}^{(0)}\\ \mathbf{0}\end{bmatrix},\quad \mathbf{z}_N^+=\begin{bmatrix}\mathbf{y}^{(N)}\\ \mathbf{0}\end{bmatrix}, \tag{4.7}$$

and

$$\lambda_N^-=h^{-2}\frac{\alpha_1+2\alpha_3}{\beta_1+2\beta_3}=0,\quad \lambda_N^+=h^{-2}\frac{\alpha_1-2\alpha_3}{\beta_1-2\beta_3}. \tag{4.8}$$

As in the Dirichlet case, to satisfy (1.11), we need to determine $\{d_k^{\pm}\}_{k=1}^{N}$ such that

$$\left(B_1d_k^{\pm}\mathbf{z}_k^{\pm},d_k^{\pm}\mathbf{z}_k^{\pm}\right)_{I\!R^{2N}}=1,\quad k=1,\ldots,N.$$

Then following the same procedure as in the Dirichlet case, we find that

$$d_k^{\pm}=\left\{\left(\left[(2\beta_3\mu_k+\beta_1)\,c_k^{\pm}+4\beta_5\nu_k\right]c_k^{\pm}+2\beta_4\mu_k+\beta_2\right)/2\right\}^{-1/2},\quad k=1,\ldots,N-1,$$

and $d_N^{\pm}=(\beta_1\mp 2\beta_3)^{-1/2}$.

Now let

$$\Lambda=\operatorname{diag}\left(\lambda_1^-,\ldots,\lambda_N^-,\lambda_1^+,\ldots,\lambda_N^+\right)$$

and

$$Z=\left[d_1^-\mathbf{z}_1^-,\ldots,d_N^-\mathbf{z}_N^-,d_1^+\mathbf{z}_1^+,\ldots,d_N^+\mathbf{z}_N^+\right],$$

where $\{\lambda_k^{\pm}\}_{k=1}^{N}$ and $\{\mathbf{z}_k^{\pm}\}_{k=1}^{N}$ are given in (3.21)–(3.23), (4.8), (4.3), (4.6), and (4.7). Using (4.3) and (4.7), we can write Z as

$$Z=\begin{bmatrix}\mathcal{C}_{N+1} & \\ & \mathcal{S}_{N-1}\end{bmatrix}\begin{bmatrix}\Lambda_2^- & \Lambda_2^+\\ \Lambda_1^- & \Lambda_1^+\end{bmatrix},$$

where $\mathcal{S}_{N-1}$ and $\mathcal{C}_{N+1}$ are given in (1.15) and $\Lambda_1^{\pm}$ and $\Lambda_2^{\pm}$ are as in (3.30).

In the corresponding MDA, the FFT routines SINT and COST of [14] (initialized by SINTI and COSTI) are used in steps 1 and 3, respectively.

5 Dirichlet-Neumann boundary conditions

The basis $\{\phi_i\}_{i=1}^{2N}$ for $S_h^{\mathcal{D},\mathcal{N}}$ is given by

$$\{\phi_1, \ldots, \phi_{2N}\} = \{v_1, \ldots, v_N, s_0, \ldots, s_{N-1}\}. \tag{5.1}$$

For any τ_i, $i = 1, \ldots, 5$, we introduce the $N \times N$ matrices

$$T_\tau(i, j) = \begin{bmatrix} \tau_i & \tau_j & & & \\ \tau_j & \tau_i & \tau_j & & \\ & \ddots & \ddots & \ddots & \\ & & \tau_j & \tau_i & \tau_j \\ & & & \tau_j & \tau_i/2 \end{bmatrix}, \quad R_\tau(i, j) = \begin{bmatrix} \tau_i/2 & \tau_j & & & \\ \tau_j & \tau_i & \tau_j & & \\ & \ddots & \ddots & \ddots & \\ & & \tau_j & \tau_i & \tau_j \\ & & & \tau_j & \tau_i \end{bmatrix}, \tag{5.2}$$

and

$$S_\tau(5) = \begin{bmatrix} -\tau_5 & 0 & \tau_5 & & \\ & \ddots & \ddots & \ddots & \\ & & -\tau_5 & 0 & \tau_5 \\ & & & -\tau_5 & 0 \\ & & & & -\tau_5 \end{bmatrix}. \tag{5.3}$$

Using (2.2), (5.1), (2.5)–(2.6), and (5.2)–(5.3), (1.13) becomes

$$\begin{bmatrix} T_\gamma(1, 3) & S_\gamma(5) \\ S_\gamma^T(5) & R_\gamma(2, 4) \end{bmatrix} \mathbf{z} = \mathbf{0}, \tag{5.4}$$

where the γ_i are given in (3.6). We set

$$\mathbf{x}^{(k)} = \left[x_1^{(k)}, \ldots, x_N^{(k)}\right]^T, \quad \mathbf{y}^{(k)} = \left[y_1^{(k)}, \ldots, y_N^{(k)}\right]^T, \quad k = 1, \ldots, N, \tag{5.5}$$

where

$$x_i^{(k)} = \cos\frac{(2k-1)(i-1)\pi}{2N}, \quad y_i^{(k)} = \sin\frac{(2k-1)i\pi}{2N}, \tag{5.6}$$

and define $\mathbf{z}_k^\pm$ as

$$\mathbf{z}_k^\pm = \begin{bmatrix} \mathbf{y}^{(k)} \\ c_k^\pm \mathbf{x}^{(k)} \end{bmatrix}, \quad k = 1, \ldots, N. \tag{5.7}$$

Substituting (5.7) into (5.4) leads to

$$(2\gamma_3\rho_k + \gamma_1) - c_k^\pm 2\gamma_5\sigma_k = 0, \tag{5.8}$$

$$-2\gamma_5\sigma_k + c_k^\pm(2\gamma_4\rho_k + \gamma_2) = 0, \tag{5.9}$$

for $k = 1, \ldots, N$, where

$$\rho_k = \cos\frac{(2k-1)\pi}{2N}, \quad \sigma_k = \sin\frac{(2k-1)\pi}{2N}. \tag{5.10}$$

Solving (5.8) for $c_k^{\pm}$, we obtain

$$c_k^{\pm} = \frac{2\gamma_3\rho_k + \gamma_1}{2\gamma_5\sigma_k}, \quad k = 1, \dots, N. \tag{5.11}$$

Substituting (5.11) into (5.9) gives

$$(2\gamma_3\rho_k + \gamma_1)(2\gamma_4\rho_k + \gamma_2) - 4\gamma_5^2\sigma_k^2 = 0, \quad k = 1, \dots, N. \tag{5.12}$$

Thus the corresponding eigenvalues $\{\lambda_k^{\pm}\}_{k=1}^N$ are given by

$$\lambda_k^{\pm} = h^{-2}\Phi^{\pm}(\rho_k, \sigma_k), \quad k = 1, \dots, N, \tag{5.13}$$

where $\Phi^{\pm}$ is defined in (3.22)–(3.23).

Now we require $\{d_k^{\pm}\}_{k=1}^N$ such that

$$\left(B_1 d_k^{\pm}\mathbf{z}_k^{\pm}, d_k^{\pm}\mathbf{z}_k^{\pm}\right)_{I\!R^{2N}} = 1, \quad k = 1, \dots, N.$$

Using the identity

$$\sum_{i=1}^{N-1} \cos^2\left(\frac{i(2k-1)\pi}{2N}\right) = \sum_{i=1}^{N-1} \sin^2\left(\frac{i(2k-1)\pi}{2N}\right) = \frac{N-1}{2}, \quad k = 1, \dots, N,$$

we obtain

$$d_k^{\pm} = \left\{\left(\left[(2\beta_4\rho_k + \beta_2)\, c_k^{\pm} - 4\beta_5\sigma_k\right] c_k^{\pm} + 2\beta_3\rho_k + \beta_1\right)/2\right\}^{-1/2}, \quad k = 1, \dots, N.$$

Now let

$$\Lambda = \operatorname{diag}\left(\lambda_1^-, \dots, \lambda_N^-, \lambda_1^+, \dots, \lambda_N^+\right) \tag{5.14}$$

and

$$Z = \left[d_1^-\mathbf{z}_1^-, \dots, d_N^-\mathbf{z}_N^-, d_1^+\mathbf{z}_1^+, \dots, d_N^+\mathbf{z}_N^+\right],$$

where $\{\lambda_k^{\pm}\}_{k=1}^N$ and $\{\mathbf{z}_k^{\pm}\}_{k=1}^N$ are given in (5.13), and (5.7), (5.11). With

$$\mathcal{S}_N = \left(\sin\frac{i(2k-1)\pi}{2N}\right)_{i,k=1}^N, \quad \mathcal{C}_N = \left(\cos\frac{(i-1)(2k-1)\pi}{2N}\right)_{i,k=1}^N, \tag{5.15}$$

and

$$\Lambda_1^{\pm} = \operatorname{diag}\left(d_1^{\pm}, \dots, d_N^{\pm}\right), \quad \Lambda_2^{\pm} = \operatorname{diag}\left(c_1^{\pm}d_1^{\pm}, \dots, c_N^{\pm}d_N^{\pm}\right), \tag{5.16}$$

using (5.5)–(5.7), we can write Z as

$$Z = \begin{bmatrix} \mathcal{S}_N & \\ & \mathcal{C}_N \end{bmatrix} \begin{bmatrix} \Lambda_1^- & \Lambda_1^+ \\ \Lambda_2^- & \Lambda_2^+ \end{bmatrix}.$$

In the corresponding MDA, the FFT routines SINQF, COSQF and SINQB, COSQB of [14] (initialized by SINQI and COSQI) are used in steps 1 and 3, respectively.

6 Neumann-Dirichlet boundary conditions

For (1.6), the basis $\{\phi_i\}_{i=1}^{2N}$ for $S_h^{\mathcal{N},\mathcal{D}}$ is given by $\{\phi_1, \ldots, \phi_{2N}\} = \{v_0, \ldots, v_{N-1}, s_1, \ldots, s_N\}$, for which (1.13) becomes

$$\begin{bmatrix} R_\gamma(1,3) & -S_\gamma^T(5) \\ -S_\gamma(5) & T_\gamma(2,4) \end{bmatrix} \mathbf{z} = \mathbf{0},$$

where $T_\tau(i, j)$, $R_\tau(i, j)$ and $S_\tau(5)$ are defined in (5.2) and (5.3). Then, (1.10) and (1.11) are satisfied with the same Λ as in (5.14) and

$$Z = \begin{bmatrix} \mathcal{C}_N & \\ & \mathcal{S}_N \end{bmatrix} \begin{bmatrix} \Lambda_2^- & \Lambda_2^+ \\ \Lambda_1^- & \Lambda_1^+ \end{bmatrix},$$

where $\mathcal{S}_N, \mathcal{C}_N$ are defined in (5.15) and $\Lambda_1^\pm$, $\Lambda_2^\pm$ are as in (5.16) but with

$$d_k^\pm = \left\{\left(\left[(2\beta_3\rho_k + \beta_1)\, c_k^\pm + 4\beta_5\sigma_k\right] c_k^\pm + 2\beta_4\rho_k + \beta_2\right)/2\right\}^{-1/2}, \quad c_k^\pm = -\frac{2\gamma_4\rho_k + \gamma_2}{2\gamma_5\sigma_k},$$

and ρ_k and σ_k as in (5.10).

7 Periodic boundary conditions

The basis $\{\phi_i\}_{i=1}^{2N}$ for $S_h^{\mathcal{P}}$ is given by

$$\{\phi_1, \ldots, \phi_{2N}\} = \{v_0 + v_N, v_1, \ldots, v_{N-1}, s_0 + s_N, s_1, \ldots, s_{N-1}\}. \tag{7.1}$$

For any τ_i, $i = 1, \ldots, 5$, we introduce the $N \times N$ matrices

$$R_\tau(i, j) = \begin{bmatrix} \tau_i & \tau_j & & & \tau_j \\ \tau_j & \tau_i & \tau_j & & \\ & \ddots & \ddots & \ddots & \\ & & \tau_j & \tau_i & \tau_j \\ \tau_j & & & \tau_j & \tau_i \end{bmatrix}, \quad S_\tau(5) = \begin{bmatrix} 0 & \tau_5 & & & -\tau_5 \\ -\tau_5 & 0 & \tau_5 & & \\ & \ddots & \ddots & \ddots & \\ & & -\tau_5 & 0 & \tau_5 \\ \tau_5 & & & -\tau_5 & 0 \end{bmatrix}. \tag{7.2}$$

Using (2.2), (7.1), (2.5)–(2.6), and (7.2), (1.13) becomes

$$\begin{bmatrix} R_\gamma(1,3) & S_\gamma(5) \\ -S_\gamma(5) & R_\gamma(2,4) \end{bmatrix} \mathbf{z} = \mathbf{0}, \tag{7.3}$$

where the γ_i are defined by (3.6). For even N, we let

$$\mathbf{x}^{(k)} = \left[x_1^{(k)}, \ldots, x_N^{(k)}\right]^T, \quad \mathbf{y}^{(k)} = \left[y_1^{(k)}, \ldots, y_N^{(k)}\right]^T, \quad k = 0, \ldots, N-1, \tag{7.4}$$

where, for $k = 0, \ldots, N/2$,

$$x_i^{(k)} = \cos\frac{(2i-1)k\pi}{N}, \quad y_i^{(k)} = \sin\frac{(2i-1)k\pi}{N}, \tag{7.5}$$

and, for $k = N/2+1, \ldots, N-1$,

$$x_i^{(k)} = \cos\frac{(2i-1)(k-N/2)\pi}{N}, \quad y_i^{(k)} = \sin\frac{(2i-1)(k-N/2)\pi}{N}. \tag{7.6}$$

We define $\mathbf{z}_k^{\pm}$ as

$$\mathbf{z}_k^{\pm} = \begin{bmatrix} c_k^{\pm}\mathbf{x}^{(k)} \\ \mathbf{y}^{(k)} \end{bmatrix}, \quad k = 1, \ldots, N/2-1, \tag{7.7}$$

and

$$\mathbf{z}_k^{\pm} = \begin{bmatrix} \mathbf{y}^{(k)} \\ c_k^{\pm}\mathbf{x}^{(k)} \end{bmatrix}, \quad k = N/2+1, \ldots, N-1. \tag{7.8}$$

Then, substituting (7.7) into (7.3), we obtain, for $k = 1, \ldots, N/2-1$,

$$c_k^{\pm}(2\gamma_3\mu_{2k} + \gamma_1) + 2\gamma_5\nu_{2k} = 0, \tag{7.9}$$

$$2c_k^{\pm}\gamma_5\nu_{2k} + 2\gamma_4\mu_{2k} + \gamma_2 = 0, \tag{7.10}$$

where μ_k and ν_k are defined in (3.14). Then, substituting (7.8) into (7.3), we obtain, for $k = N/2+1, \ldots, N-1$,

$$(2\gamma_3\mu_{2k-N} + \gamma_1) - c_k^{\pm}(2\gamma_5\nu_{2k-N}) = 0, \tag{7.11}$$

$$-2\gamma_5\nu_{2k-N} + c_k^{\pm}(2\gamma_4\mu_{2k-N} + \gamma_2) = 0. \tag{7.12}$$

Solving (7.10) and (7.11) for $c_k^{\pm}$, we obtain

$$c_k^{\pm} = -\frac{2\gamma_4\mu_{2k} + \gamma_2}{2\gamma_5\nu_{2k}}, \quad k = 1, \ldots, N/2-1, \tag{7.13}$$

and

$$c_k^{\pm} = \frac{2\gamma_3\mu_{2k-N} + \gamma_1}{2\gamma_5\nu_{2k-N}}, \quad k = N/2+1, \ldots, N-1. \tag{7.14}$$

From (7.9) and (7.13), we obtain

$$(2\gamma_3\mu_{2k} + \gamma_1)(2\gamma_4\mu_{2k} + \gamma_2) - 4\gamma_5^2\nu_{2k}^2 = 0, \quad k = 1, \ldots, N/2-1.$$

Also, from (7.12) and (7.14),

$$(2\gamma_3\mu_{2k-N} + \gamma_1)(2\gamma_4\mu_{2k-N} + \gamma_2) - 4\gamma_5^2\nu_{2k-N}^2 = 0, \quad k = N/2+1, \ldots, N-1.$$

Thus,

$$\lambda_k^{\pm} = \lambda_{k+N/2}^{\pm} = h^{-2}\Phi^{\pm}(\mu_{2k}, \nu_{2k}), \quad k = 1, \ldots, N/2-1,$$

where $\Phi^{\pm}$ is given in (3.22)–(3.23).

With

$$\mathbf{z}_0 = \begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{z}_{N/2} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y}^{(N/2)} \end{bmatrix}, \tag{7.15}$$

we obtain

$$\lambda_0 = h^{-2}\frac{\alpha_1 + 2\alpha_3}{\beta_1 + 2\beta_3} = 0, \quad \lambda_{N/2} = h^{-2}\frac{\alpha_2 - 2\alpha_4}{\beta_2 - 2\beta_4}. \tag{7.16}$$

To determine two additional eigenvalues, which we label λ_s and λ_t, we take

$$\mathbf{z}_s = \begin{bmatrix} \mathbf{0} \\ \mathbf{x}^{(0)} \end{bmatrix}, \quad \mathbf{z}_t = \begin{bmatrix} \mathbf{y}^{(N/2)} \\ \mathbf{0} \end{bmatrix}, \tag{7.17}$$

and obtain

$$\lambda_s = h^{-2}\frac{2\alpha_4 + \alpha_2}{2\beta_4 + \beta_2}, \quad \lambda_t = h^{-2}\frac{\alpha_1 - 2\alpha_3}{\beta_1 - 2\beta_3}. \tag{7.18}$$

Now we require $\{d_k^{\pm}\}_{k=0}^{N-1}$ such that

$$\left(B_1 d_k^{\pm}\mathbf{z}_k^{\pm}, d_k^{\pm}\mathbf{z}_k^{\pm}\right)_{I\!R^{2N}} = 1, \quad k = 0, \ldots, N-1.$$

Since

$$\sum_{i=1}^{N}\cos^2\left(\frac{(2i-1)k\pi}{N}\right) = \sum_{i=1}^{N}\sin^2\left(\frac{(2i-1)k\pi}{N}\right) = \frac{N}{2}, \quad k = 1, \ldots, N/2-1,$$

we obtain

$$d_k^{\pm} = \left\{\left(2\beta_4\mu_{2k} + \beta_2 + c_k^{\pm}\left[c_k^{\pm}(2\beta_3\mu_{2k} + \beta_1) + 4\beta_5\nu_{2k}\right]\right)/2\right\}^{-1/2}, \tag{7.19}$$

for $k = 1, \ldots, N/2-1$. In a similar manner, we derive

$$d_k^{\pm} = \left\{\left(2\beta_3\mu_{2k-N} + \beta_1 + c_k^{\pm}\left[c_k^{\pm}\left(2\beta_4\mu_{2k-N} + \beta_2\right) - 4\beta_5\nu_{2k-N}\right]\right)/2\right\}^{-1/2}, \tag{7.20}$$

for $k = N/2+1, \ldots, N-1$. Also, we have

$$\left(h^{-1}B_1\mathbf{z}_0, \mathbf{z}_0\right)_{I\!R^{2N}} = (2\beta_3 + \beta_1)\,N, \quad \left(h^{-1}B_1\mathbf{z}_{N/2}, \mathbf{z}_{N/2}\right)_{I\!R^{2N}} = (\beta_2 - 2\beta_4)\,N.$$

Thus, $d_0 = (2\beta_3 + \beta_1)^{-1/2}$, $d_{N/2} = (\beta_2 - 2\beta_4)^{-1/2}$. Similarly, $(h^{-1}B_1\mathbf{z}_s, \mathbf{z}_s)_{I\!R^{2N}} = (2\beta_4 + \beta_2)N$, and $(h^{-1}B_1\mathbf{z}_t, \mathbf{z}_t)_{I\!R^{2N}} = (\beta_1 - 2\beta_3)N$, so that $d_s = (2\beta_4 + \beta_2)^{-1/2}$, and $d_t = (\beta_1 - 2\beta_3)^{-1/2}$. Now let

$$\Lambda = \mathrm{diag}\big(\lambda_0, \lambda_1^-, \ldots, \lambda_{N/2-1}^-, \lambda_{N/2}, \lambda_s, \lambda_1^-, \ldots, \lambda_{N/2-1}^-, \lambda_t,$$
$$\lambda_1^+, \ldots, \lambda_{N/2-1}^+, \lambda_1^+, \ldots, \lambda_{N/2-1}^+\big)$$

and

$$Z = \big[d_0\mathbf{z}_0, d_1^-\mathbf{z}_1^-, \ldots, d_{N/2-1}^-\mathbf{z}_{N/2-1}^-, d_{N/2}\mathbf{z}_{N/2}, d_s\mathbf{z}_s, d_{N/2+1}^-\mathbf{z}_{N/2+1}^-, \ldots,$$
$$d_{N-1}^-\mathbf{z}_{N-1}^-, d_t\mathbf{z}_t, d_1^+\mathbf{z}_1^+, \ldots, d_{N/2-1}^+\mathbf{z}_{N/2-1}^+, d_{N/2+1}^+\mathbf{z}_{N/2+1}^+, \ldots, d_{N-1}^+\mathbf{z}_{N-1}^+\big].$$

With

$$\widetilde{\mathcal{S}}_N^e = \left(\sin\frac{(2i-1)(k-1)\pi}{N}\right)_{i=1,k=1}^{N,N/2+1}, \quad \widetilde{\mathcal{C}}_N^e = \left(\cos\frac{(2i-1)(k-1)\pi}{N}\right)_{i=1,k=1}^{N,N/2+1},$$

$$\mathcal{S}_N^e = \left(\sin\frac{(2i-1)(k-1)\pi}{N}\right)_{i=1,k=2}^{N,N/2}, \quad \mathcal{C}_N^e = \left(\cos\frac{(2i-1)(k-1)\pi}{N}\right)_{i=1,k=2}^{N,N/2},$$

and

$$\begin{aligned}
\Lambda_1^d &= \mathrm{diag}(d_0, d_1^-, \ldots, d_{N/2-1}^-, d_{N/2}), & \Lambda_1^c &= \mathrm{diag}(1, c_1^-, \ldots, c_{N/2-1}^-, 1),\\
\Lambda_2^d &= \mathrm{diag}(d_s, d_{N/2+1}^-, \ldots, d_{N-1}^-, d_t), & \Lambda_2^c &= \mathrm{diag}(1, c_{N/2+1}^-, \ldots, c_{N-1}^-, 1),\\
\Lambda_3^d &= \mathrm{diag}(d_1^+, \ldots, d_{N/2-1}^+), & \Lambda_3^c &= \mathrm{diag}(c_1^+, \ldots, c_{N/2-1}^+),\\
\Lambda_4^d &= \mathrm{diag}(d_{N/2+1}^+, \ldots, d_{N-1}^+), & \Lambda_4^c &= \mathrm{diag}(c_{N/2+1}^+, \ldots, c_{N-1}^+),
\end{aligned}$$

using (7.4)–(7.8), (7.15) and (7.17), we can write

$$Z = \left[\begin{array}{c|c|c|c}
\widetilde{\mathcal{C}}_N^e \Lambda_1^d \Lambda_1^c & \widetilde{\mathcal{S}}_N^e \Lambda_2^d & \mathcal{C}_N^e \Lambda_3^d \Lambda_3^c & \mathcal{S}_N^e \Lambda_4^d \\
\hline
\widetilde{\mathcal{S}}_N^e \Lambda_1^d & \widetilde{\mathcal{C}}_N^e \Lambda_2^d \Lambda_2^c & \mathcal{S}_N^e \Lambda_3^d & \mathcal{C}_N^e \Lambda_4^d \Lambda_4^c
\end{array}\right].$$

In the corresponding MDA, the FFT routines RFFTF and RFFTB of [14] (initialized by RFFTI) are used in steps 1 and 3, respectively.

For odd N, the arguments of this section hold with minor modifications. We obtain

$$\Lambda = \mathrm{diag}\big(\lambda_0, \lambda_1^-, \ldots, \lambda_{(N-1)/2}^-, \lambda_s, \lambda_1^-, \ldots, \lambda_{(N-1)/2}^-,\\ \lambda_1^+, \ldots, \lambda_{(N-1)/2}^+, \lambda_1^+, \ldots, \lambda_{(N-1)/2}^+\big),$$

where λ_0 and λ_s are given in (7.16) and (7.18), respectively, and

$$\lambda_k^\pm = h^{-2}\Phi^\pm(\mu_{2k}, \nu_{2k}), \quad k = 1, \ldots, (N-1)/2,$$

with $\Phi^\pm$ as in (3.22)–(3.23) and the μ_k and ν_k defined in (3.14). Moreover,

$$Z = \left[\begin{array}{c|c|c|c}
\widetilde{\mathcal{C}}_N^o \Lambda_1^d \Lambda_1^c & \widetilde{\mathcal{S}}_N^o \Lambda_2^d & \mathcal{C}_N^o \Lambda_3^d \Lambda_3^c & \mathcal{S}_N^o \Lambda_4^d \\
\hline
\widetilde{\mathcal{S}}_N^o \Lambda_1^d & \widetilde{\mathcal{C}}_N^o \Lambda_2^d \Lambda_2^c & \mathcal{S}_N^o \Lambda_3^d & \mathcal{C}_N^o \Lambda_4^d \Lambda_4^c
\end{array}\right],$$

where

$$\widetilde{\mathcal{S}}_N^o = \left(\sin\frac{(2i-1)(k-1)\pi}{N}\right)_{i=1,k=1}^{N,(N+1)/2}, \quad \widetilde{\mathcal{C}}_N^o = \left(\cos\frac{(2i-1)(k-1)\pi}{N}\right)_{i=1,k=1}^{N,(N+1)/2},$$

$$\mathcal{S}_N^o = \left(\sin\frac{(2i-1)(k-1)\pi}{N}\right)_{i=1,k=2}^{N,(N+1)/2}, \quad \mathcal{C}_N^o = \left(\cos\frac{(2i-1)(k-1)\pi}{N}\right)_{i=1,k=2}^{N,(N+1)/2},$$

$$
\begin{aligned}
\Lambda_1^d &= \mathrm{diag}(d_0, d_1^-, \ldots, d_{(N-1)/2}^-), & \Lambda_1^c &= \mathrm{diag}(1, c_1^-, \ldots, c_{(N-1)/2}^-),\\
\Lambda_2^d &= \mathrm{diag}(d_s, d_{(N+1)/2}^-, \ldots, d_{N-1}^-), & \Lambda_2^c &= \mathrm{diag}(1, c_{(N+1)/2}^-, \ldots, c_{N-1}^-),\\
\Lambda_3^d &= \mathrm{diag}(d_1^+, \ldots, d_{(N-1)/2}^+), & \Lambda_3^c &= \mathrm{diag}(c_1^+, \ldots, c_{(N-1)/2}^+),\\
\Lambda_4^d &= \mathrm{diag}(d_{(N+1)/2}^+, \ldots, d_{N-1}^+), & \Lambda_1^c &= \mathrm{diag}(c_{(N+1)/2}^+, \ldots, c_{N-1}^+),
\end{aligned}
$$

and the $d_k^\pm$ are defined in (7.19) and (7.20) with $N/2-1$ and $N/2+1$ replaced by $(N-1)/2$ and $(N+1)/2$, respectively, and $c_k^\pm$ are defined in (7.13) and (7.14) with $N/2-1$ and $N/2+1$ replaced by $(N-1)/2$ and $(N+1)/2$, respectively.

8 Numerical results

We solved (2.1) with $f(x, y)$ corresponding to the exact solution

$$u(x, y) = v(x)y(1-y)e^y,$$

where

$$v(x) = \begin{cases} x(1-x)e^x \text{ for (1.3)},\\ (x^2-3x+3)e^x \text{ for (1.4)},\\ (2x^2-3x)e^x \text{ for (1.5)},\\ (1-e^2)x^2+e^{2x} \text{ for (1.7)}. \end{cases}$$

Each integral in (2.3) was computed approximately by using, on each cell of the $N \times N$ partition of Ω on which $\phi_i\phi_k$ was not identically equal to 0, the Gauss quadrature rule with 3 nodes per each subinterval in the x and y directions. Then the system (1.8) was solved by the MDA of Section 1. Convergence rates in various norms were determined using the formula

$$\text{rate} = \frac{\log(e_{N/2}/e_N)}{\log 2},$$

where e_N is the error corresponding to the $N \times N$ partition of Ω.

In Tables 1, 2, 3, and 4, we present errors and the corresponding convergence rates using Sobolev norms for the boundary conditions (1.3), (1.4), (1.5), and (1.7), respectively. As expected, the convergence rates for the L^2, H^1, and H^2 norms are 4, 3, and 2, respectively.

Table 1 Sobolev norm errors and convergence rates for (1.3)

N	$\\|u-U\\|_{L^2(\Omega)}$		$\\|u-U\\|_{H^1(\Omega)}$		$\\|u-U\\|_{H^2(\Omega)}$	
	Error	Rate	Error	Rate	Error	Rate
8	1.99–06		1.06–04		6.48–03	
16	1.34–07	3.890	1.39–05	2.930	1.58–03	2.039
32	8.72–09	3.945	1.79–06	2.963	3.88–04	2.023
64	5.56–10	3.972	2.26–07	2.981	9.61–05	2.013

Table 2 Sobolev norm errors and convergence rates for (1.4)

N	$\|u-U\|_{L^2(\Omega)}$		$\|u-U\|_{H^1(\Omega)}$		$\|u-U\|_{H^2(\Omega)}$	
	Error	Rate	Error	Rate	Error	Rate
8	1.30–05		6.92–04		4.20–02	
16	8.78–07	3.893	9.10–05	2.927	1.03–02	2.032
32	5.70–08	3.945	1.17–05	2.963	2.53–03	2.021
64	3.63–09	3.972	1.48–06	2.981	6.27–04	2.012

Table 3 Sobolev norm errors and convergence rates for (1.5)

N	$\|u-U\|_{L^2(\Omega)}$		$\|u-U\|_{H^1(\Omega)}$		$\|u-U\|_{H^2(\Omega)}$	
	Error	Rate	Error	Rate	Error	Rate
8	8.49–06		4.50–04		2.70–02	
16	5.69–07	3.900	5.89–05	2.933	6.61–03	2.031
32	3.68–08	3.949	7.54–06	2.966	1.63–03	2.020
64	2.34–09	3.974	9.54–07	2.982	4.04–04	2.012

Table 4 Sobolev norm errors and convergence rates for (1.7)

N	$\|u-U\|_{L^2(\Omega)}$		$\|u-U\|_{H^1(\Omega)}$		$\|u-U\|_{H^2(\Omega)}$	
	Error	Rate	Error	Rate	Error	Rate
8	6.18–06		3.25–04		1.91–02	
16	4.09–07	3.918	4.22–05	2.944	4.69–03	2.026
32	2.63–08	3.957	5.38–06	2.970	1.16–03	2.017
64	1.67–09	3.978	6.80–07	2.985	2.87–04	2.010

Table 5 Sobolev norm errors and convergence rates in OSC for (1.3)

N	$\|u-U\|_{L^2(\Omega)}$		$\|u-U\|_{H^1(\Omega)}$		$\|u-U\|_{H^2(\Omega)}$	
	Error	Rate	Error	Rate	Error	Rate
8	4.03–06		1.18–04		6.09–03	
16	2.53–07	3.995	1.47–05	3.001	1.52–03	2.000
32	1.58–08	3.999	1.84–06	3.000	3.81–04	2.000
64	9.87–10	4.000	2.29–07	3.000	9.52–05	2.000

Table 6 Nodal errors and convergence rates in the Galerkin method for (1.3)

N	$\|u-U\|_{C_h}$		$\|(u-U)_x\|_{C_h}$		$\|(u-U)_y\|_{C_h}$		$\|(u-U)_{xy}\|_{C_h}$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
8	7.60–06		4.18–04		4.18–04		7.16–03	
16	4.93–07	3.948	5.34–05	2.968	5.34–05	2.968	9.62–04	2.895
32	3.08–08	4.001	6.76–06	2.982	6.76–06	2.982	1.25–04	2.947
64	1.92–09	4.001	8.51–07	2.991	8.51–07	2.991	1.59–05	2.973

Table 7 Nodal errors and convergence rates in the OSC method for (1.3)

N	$\|u-U\|_{C_h}$		$\|(u-U)_x\|_{C_h}$		$\|(u-U)_y\|_{C_h}$		$\|(u-U)_{xy}\|_{C_h}$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
8	2.88–06		3.12–05		3.12–05		5.65–04	
16	1.85–07	3.961	1.96–06	3.995	1.96–06	3.995	4.33–05	3.706
32	1.15–08	4.000	1.22–07	3.999	1.22–07	3.999	3.19–06	3.764
64	7.22–10	4.000	7.65–09	4.000	7.65–09	4.000	2.29–07	3.801

Table 8 CPU times for the FEG MDA

N	32	64	128	256
CPU time	0.15	0.59	2.40	9.62

In Table 5, we present the Sobolev norm errors and the corresponding convergence rates for the boundary condition (1.3) obtained using the OSC MDA of [5]. The L^2 norm errors in Table 5 are approximately twice as large as those in Table 1, while the H^1 and H^2 norm errors are comparable. In Tables 6 and 7, for the boundary condition (1.3), we give nodal errors and the corresponding convergence rates for the Galerkin and OSC methods, respectively. The nodal error for a function w is defined by

$$\|w\|_{C_h} = \max_{0 \le i, j \le N} |w(x_i, x_j)|.$$

The nodal errors in u for OSC are at least twice as small as those for the Galerkin method. Moreover, in contrast to the Galerkin method, OSC possesses superconvergence properties, namely, the nodal approximations to u_x, u_y, and u_{xy} converge with order 4 rather than 3.

Typically, the solution of Poisson's equation is a component in the solution of a more complex problem. In such a case, the choice of method, OSC or Galerkin, is likely to depend on the discretization used in the remainder of the problem.

Finally, in Table 8, we give the CPU times for our Galerkin MDA recorded on an HP workstation x1100. As N increases by a factor of 2, the CPU time increases approximately by a factor of 4, confirming the theoretical result that the total cost of our algorithm is $O(N^2 \log N)$.

9 Concluding remarks

In this paper, we derive explicit formulas for the generalized eigenvalues and eigenvectors, specifically, the matrices Z and Λ satisfying (1.10), (1.11), associated with the piecewise Hermite cubic FEG discretization of the differential equation

$$-u'' + \lambda u = 0, \quad x \in (0, 1),$$

subject to boundary conditions corresponding to (1.3)–(1.7). Using the matrices Z and Λ, we formulate MDAs employing fast Fourier transforms for the solution of the linear systems arising in the FEG method with piecewise Hermite bicubics for solving (1.1) subject to the various boundary conditions given in Section 1. These methods require $O(N^2 \log N)$ operations on an $N \times N$ uniform partition.

It is clear that the MDA of Section 1 is also applicable to the linear system arising from the FEG discretization with piecewise Hermite cubics on a non-uniform partition in the y-direction. Moreover, in place of (1.1), one could have

$$-u_{xx} - (a(y)u_y)_y + b(y)u_y + c(y)u = f(x, y), \quad (x, y) \in \Omega,$$

and homogeneous Neumann, Dirichlet-Neumann, Neumann-Dirichlet, Robin's or periodic boundary conditions in place of (1.2). Extensions are also possible to non-homogeneous boundary conditions. For any such condition, the approximate solution U is written as $U = U_0 + U_1$, where the unknown piecewise Hermite bicubic U_0 satisfies the corresponding homogeneous boundary conditions and the known piecewise Hermite bicubic U_1 is obtained by interpolating the given functions in the non-homogeneous boundary conditions. For example, in the case of the non-homogeneous Dirichlet boundary condition $u(x, y) = g(x, y)$, $(x, y) \in \partial\Omega$, with $\phi_0 = v_0$ and $\phi_{2N+1} = v_N$, the unknown coefficients $c_{k,l}$ in

$$U_1(x, y) = \sum_{k=0,2N+1} \sum_{l=0}^{2N+1} c_{k,l}\phi_k(x)\phi_l(y) + \sum_{l=0,2N+1} \sum_{k=1}^{2N} c_{k,l}\phi_k(x)\phi_l(y)$$

are determined from the Hermite interpolation conditions

$$\frac{\partial^j (U - g)}{\partial x^j}(x_i, \gamma) = 0, \quad \frac{\partial^j (U - g)}{\partial y^j}(\gamma, x_i) = 0, \quad j = 0, 1, \quad 0 \le i \le N, \quad \gamma = 0, 1.$$

Then the substitution of U into (2.1) leads to a problem for U_0 in which only the right-hand side is different.

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