

Similarity preserving compressions of high dimensional sparse data

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ABSTRACT

The rise of internet has resulted in an explosion of data consisting of millions of articles, images, songs, and videos. Most of this data is high dimensional and sparse. The need to perform an efficient search for similar objects in such high dimensional big datasets is becoming increasingly common. Even with the rapid growth in computing power, the brute-force search for such a task is impractical and at times impossible. Therefore algorithmic solutions such as Locality Sensitive Hashing (LSH) are required to achieve the desired efficiency in search.

Any similarity search method that achieves the efficiency uses one (or both) of the following methods: 1. Compress the data by reducing its dimension while preserving the similarities between any pair of data-objects 2. Limit the search space by grouping the data-objects based on their similarities. Typically 2 is obtained as a consequence of 1.

Our focus is on high dimensional sparse data, where the standard compression schemes, such as LSH for Hamming distance (Gionis, Indyk and Motwani [7]), become inefficient in both 1 and 2 due to at least one of the following reasons: 1. No efficient compression schemes mapping binary vectors to binary vectors 2. Compression length is nearly linear in the dimension and grows inversely with the sparsity 3. Randomness used grows linearly with the product of dimension and compression length.

We propose an efficient compression scheme mapping binary vectors into binary vectors and simultaneously preserving Hamming distance and Inner Product. Our schemes avoid all the above mentioned drawbacks for high dimensional sparse data. The length of our compression depends only on the sparsity and is independent of the dimension of the data. Moreover our schemes provide one-shot solution for Hamming distance and Inner Product, and work in the streaming setting as well. In contrast with the “local projection” strategies used by most of the previous schemes, our scheme combines (using sparsity) the following two strategies: 1. Partitioning the dimensions into several buckets, 2. Then obtaining “global linear summaries” in each of these buckets. We generalize our scheme for real-valued data and obtain compressions for Euclidean distance, Inner Product, and k -way Inner Product.

1. INTRODUCTION

The technological advancements have led to the generation of huge amount of data over the web such as texts, images, audios, and videos. Needless to say that

most of these datasets are high dimensional. Searching for similar data-objects in such massive and high dimensional datasets is becoming a fundamental subroutine in many scenarios like clustering, classification, nearest neighbors, ranking etc. However, due to the “*curse of dimensionality*” a brute-force way to compute the similarity scores on such data sets is very expensive and at times infeasible. Therefore it is quite natural to investigate the techniques that compress the dimension of dataset while preserving the similarity between data objects. There are various compressing schemes that have been already studied for different similarity measures. We would like to emphasize that any such compressing scheme is useful only when it satisfies the following guarantee, *i.e.* when data objects are “nearby” (under the desired similarity measure), then they should remain near-by in the compressed version, and when they are “far”, they should remain far in the compressed version. In the case of probabilistic compression schemes the above should happen with high probability. Below we discuss a few such notable schemes. In this work we consider binary and real-valued datasets. For binary data we focus on Hamming distance and Inner product, while for real-valued data we focus on Euclidean distance and Inner product.

1.1 Examples of similarity preserving compressions

Data objects in a datasets can be considered as points (vectors) in high dimensional space. Let we have n vectors (binary or real-valued) in d -dimensional space.

- Gionis, Indyk, Motwani [7] proposed a data structure to solve approximate nearest neighbor (c -NN) problem in binary data for **Hamming distance**. Their scheme popularly known as **Locality Sensitive Hashing** (LSH). Intuitively, their data structure can be viewed as a compression of a binary vector, which is obtained by projecting it on a randomly chosen bit positions.
- **JL transform** [10] suggests a compressing scheme for real-valued data. For any $\epsilon > 0$, it compresses the dimension of the points from d to $O\left(\frac{1}{\epsilon^2} \log n\right)$ while preserving the **Euclidean distance** between any pair of points within factor of $(1 \pm \epsilon)$.
- Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, the **inner product**

similarity between them is defined as $\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{i=1}^d \mathbf{u}[i]\mathbf{v}[i]$. Ata Kabán [11] suggested a compression schemes for real data which preserves inner product *via random projection*. On the contrary, if the input data is binary, and it is desirable to get the compression only in binary data, then to the best of our knowledge no such compression scheme is available which achieves a non-trivial compression. However, with some sparsity assumption (bound on the number of 1's), there are some schemes available which *via asymmetric padding* (adding a few extra bits in the vector) reduce the inner product similarity (of the original data) to the Hamming [3], and Jaccard similarity (see Preliminaries for a definition) [14]. Then the compression scheme for Hamming or Jaccard can be applied on the padded version of the data.

- Binary data can also be viewed as a collection of sets, then the underlying similarity measure of interest can be the **Jaccard similarity**. Broder *et. al.* [5, 6, 4] suggested a compression scheme for preserving Jaccard similarity between sets which is popularly known as **Minwise permutations**.

1.2 Our focus: High dimensional (sparse) data

In this work, we focus on High Dimensional Sparse Data. In many real-life scenarios, data object is represented as very high-dimensional but sparse vectors, *i.e.* number of all possible attributes (features) is huge, however, each data object has only a very small subset of attributes. For example, in bag-of-word representation of text data, the number of dimensions equals to the size of vocabulary, which is large. However for each data point, say a document, contains only a small number of words in the vocabulary, leading to a sparse vector representation. The bag-of-words representation is also commonly used for image data. Data-sparsity is commonly prevalent in audio and video-data as well.

1.3 Shortcomings of earlier schemes for high dimensional (sparse) data

The quality of any compression scheme can be evaluated based on the following two parameters - 1) the *compression-length*, and 2) the amount of *randomness* required for the compression. The compression-length is defined as the dimension of the data after compression. Ideally, it is desirable to have both of these to be small while preserving a desired accuracy in the compression. Below we will notice that most of the above mentioned compression schemes become in-feasible in the case of high dimensional sparse datasets as 1) their compression-length is very high, and 2) the amount of randomness required for the compression is quite huge.

- **Hamming distance:** Consider the problem of finding c -NN (see Definition 10) for Hamming distance in binary data. In the LHS scheme, the size of hashtable determines the compression-length.

The size of hashtable $K = O\left(\log_{\frac{1}{p_2}} n\right)$ (see Definition 11). If $r = O(1)$, then the size of hashtable

$K = O\left(\log_{\frac{1}{p_2}} n\right) = O\left(\frac{d}{cr} \log n\right) = O(d \log n)$, which is linear in the dimension. Further, in order to randomly choose a bit position (between 1 to d), it is require to generate $O(\log d)$ many random bits. Moreover, as the size of hash table is K , and the number of hash tables is L , it is required to generate $O(KL \log d)$ many random bits to create the hashtable, which become quite large specially when K is linear in d .

- **Euclidean distance:** In order to achieve compression that preserve the distance between any pair of points, due to JL transform [10, 1], it is required to project the input matrix on a random matrix of dimensions $d \times k$, where $k = O\left(\frac{1}{\epsilon^2} \log n\right)$. Each entry of the random matrix is chosen from $\{\pm 1\}$ with probability $\frac{1}{2}$ (see [1]), or from a normal distribution (see [10]). The compression-length in this scheme is $O\left(\frac{1}{\epsilon^2} \log n\right)$, and it requires $O\left(\frac{1}{\epsilon^2} d \log n\right)$ randomness.

- **Inner product:** Compression schemes which compress binary data into binary data while preserving Inner product is not known. However using “*asymmetric padding scheme*” of [3, 14] it is possible to get a compression via Hamming or Jaccard Similariy measure, then shortcomings of Jaccard and Hamming will get carry forward in such scheme. Further, in case of real valued data the compression scheme of Ata Kabán [11] has compression-length = $O\left(\frac{1}{\epsilon^2} \log n\right)$, and requires $O\left(\frac{1}{\epsilon^2} d \log n\right)$ randomness.

- **Jaccard Similarity:** Minhash permutations [5, 6, 4] suggest a compression scheme for preserving Jaccard similarity for a collection of sets. A major disadvantage of this scheme is that for high dimensional data computing permutations are very expensive, and further in order to achieve a reasonable accuracy in the compression a larger number of repetition might be required. A major disadvantage of this scheme is that it requires substantially large amount of randomness that grows polynomially in the dimension.

Lack of good binary to binary compression schemes.

To summarize the above, there are two main compression schemes currently available for binary to binary compression. The first one is LSH and the second one is JL-transform. The LSH requires the compression size to be linear in the dimension and the JL-transform can achieve logarithmic compression size but it will compress binary vectors to real vectors. The analogue of JL-transform which compresses binary vectors to binary vectors requires the compression-length to be linear in the number of data points (see Lemma 7.) Since both dimension as well as the number of data points can be large, these schemes are inefficient. In this paper we propose an efficient binary to binary compression scheme for sparse data which works simultaneously for both Hamming distance and Inner Product.

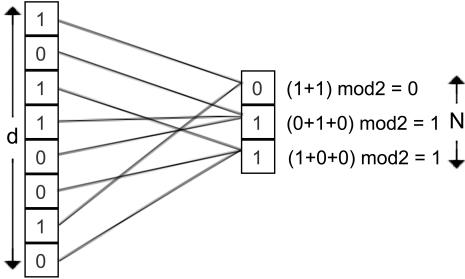
1.4 Our contribution

In this work we present a compressing scheme for high dimensional sparse data. In contrast with the “local projection” strategies used by most of the previous schemes such as LSH [9, 7] and JL [10], our scheme combines (using sparsity) the following two step approach 1. Partitioning the dimensions into several buckets, 2. Then obtaining “global linear summaries” of each of these buckets. We present our result below:

1.4.1 For binary data

For binary data, our compression scheme provides one-shot solution for both Hamming and Inner product – compressed data preserves both Hamming distance and Inner product. Moreover, the compression-length depends only on the sparsity of data and is independent of the dimension of data. We first informally state our compression scheme for binary data, see Definition 12 for a formal definition.

Given a binary vector $\mathbf{u} \in \{0, 1\}^d$, our scheme compress it into a N -dimensional binary vector (say) $\mathbf{u}' \in \{0, 1\}^N$ as follows, where N to be specified later. We randomly map each bit position (say) $\{i\}_{i=1}^d$ of the original data to an integer $\{j\}_{j=1}^N$. To compute the j -th bit of the compressed vector \mathbf{u}' we check which bits positions have been mapped to j , we compute the parity of bits located at those positions, and assign it to $\mathbf{u}'[j]$. The following figure illustrate an example of the compression. In the following theorems let ψ denote the maximum



number of 1 in any vector. We state our result for binary data as follows:

THEOREM 1. Consider a set U of binary vectors $\{\mathbf{u}_i\}_{i=1}^n \subseteq \{0, 1\}^d$, a positive integer r , and $\epsilon > 0$. If $\epsilon r > 3 \log n$, we set $N = O(\psi^2)$; if $\epsilon r < 3 \log n$, we set $N = O(\psi^2 \log^2 n)$, and compress them into a set U' of binary vectors $\{\mathbf{u}'_i\}_{i=1}^n \subseteq \{0, 1\}^N$ using our Binary Compression Scheme. Then for all $\mathbf{u}_i, \mathbf{u}_j \in U$,

- if $d_H(\mathbf{u}_i, \mathbf{u}_j) < r$, then $\Pr[d_H(\mathbf{u}'_i, \mathbf{u}'_j) < r] = 1$,
- if $d_H(\mathbf{u}_i, \mathbf{u}_j) \geq (1+\epsilon)r$, then $\Pr[d_H(\mathbf{u}'_i, \mathbf{u}'_j) < r] < \frac{1}{n}$.

THEOREM 2. Consider a set U of binary vectors $\{\mathbf{u}_i\}_{i=1}^n \subseteq \{0, 1\}^d$, a positive integer r , and $\epsilon > 0$. If $\epsilon r > 3 \log n$, we set $N = O(\psi^2)$; if $\epsilon r < 3 \log n$, we set $N = O(\psi^2 \log^2 n)$, and compress them into a set U' of binary vectors $\{\mathbf{u}'_i\}_{i=1}^n \subseteq \{0, 1\}^N$ using our Binary

Compression Scheme. Then for all $\mathbf{u}_i, \mathbf{u}_j \in U$ the following is true with probability at least $1 - \frac{1}{n}$,

$$(1 - \epsilon)IP(\mathbf{u}_i, \mathbf{u}_j) \leq IP(\mathbf{u}'_i, \mathbf{u}'_j) \leq (1 + \epsilon)IP(\mathbf{u}_i, \mathbf{u}_j).$$

In the following theorem, we strengthen our result of Theorem 1, and shows a compression bound which is independent of the dimension and the sparsity, but depends only on the Hamming distance between the vectors. However, we could show our result in the Expectation, and only for a pair of vectors.

THEOREM 3. Consider two binary vectors $\mathbf{u}, \mathbf{v} \in \{0, 1\}^d$, which get compressed into vectors $\mathbf{u}', \mathbf{v}' \in \{0, 1\}^N$ using our Binary Compression Scheme. If we set $N = O(r^2)$, then

- if $d_H(\mathbf{u}, \mathbf{v}) < r$, then $\Pr[d_H(\mathbf{u}', \mathbf{v}') < r] = 1$, and
- if $d_H(\mathbf{u}, \mathbf{v}) \geq 4r$, then $\mathbb{E}[d_H(\mathbf{u}', \mathbf{v}')] > 2r$.

REMARK 1. To the best of our knowledge, ours is the first efficient binary to binary compression scheme for preserving Hamming distance and Inner product. For Hamming distance in fact our scheme obtains the “no-false-negative” guarantee analogous to the one obtained in recent paper by Pagh [12].

REMARK 2. When r is constant, as mentioned above, LSH [7] requires compression length linear in the dimension. However, due to Theorem 3, our compression length is only constant.

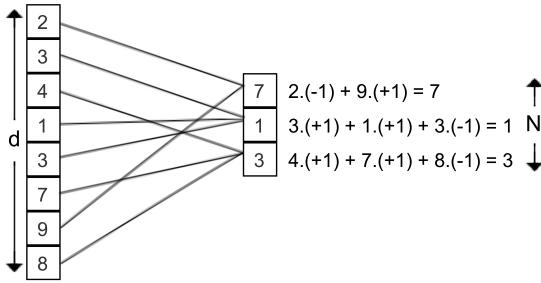
REMARK 3. Our compression length is $O(\psi \log^2 n)$, which is independent of the dimension d ; whereas other schemes such as LSH may require the compression length growing linearly in d and the analogue of JL-transform for binary to binary compression requires compression length growing linearly in n (see Lemma 7).

REMARK 4. The randomness used by our compression scheme is $O(d \log N)$ which grows logarithmically in the compression length N whereas the JL-transform uses randomness growing linearly in the compression length. For all-pair compression for n data points we use $O(d(\log \psi + \log \log n))$ randomness, which grows logarithmically in the sparsity and sub-logarithmically in terms of number of data points.

1.4.2 For real-valued data

We generalize our scheme for real-valued data also and obtain compressions for Euclidean distance, Inner product, and k -way Inner product. We first state our compression scheme as follows:

Given a vector $\mathbf{a} \in \mathbb{R}^d$, our scheme compress it into a N -dimensional vector (say) α^N as follows. We randomly map each coordinate position (say) $\{i\}_{i=1}^d$ of the original data to an integer $\{j\}_{j=1}^N$. To compute the j -th coordinate of the compressed vector α we check which coordinates of the original data have been mapped to j , we multiply the numbers located at those positions with a random variable x_i , compute their summation,



and assign it to $\alpha[j]$, where x_i takes a value between $\{-1, +1\}$ with probability $1/2$. The following figure illustrate an example of the compression. In the following we present our main result for real valued data which is compression bound for preserving k -way inner product. For a set of k vectors $\{\alpha_i\}_{i=1}^k \in \mathbb{R}^d$, their k -way inner product is defined as

$$\langle \alpha_1 \alpha_2 \dots \alpha_k \rangle = \sum_{j=1}^d \alpha_1[j] \alpha_2[j] \dots \alpha_k[j],$$

where $\alpha_i[j]$ denote the j -th coordinate of the vector α_i .

THEOREM 4. Consider a set of k vectors $\{\mathbf{a}_i\}_{i=1}^k \in \mathbb{R}^d$, which get compressed into vectors $\{\alpha_i\}_{i=1}^k \in \mathbb{R}^N$ using our Real Compression Scheme. If we set $N = \frac{10\Psi^k}{\epsilon^2}$, where $\Psi = \max\{\|\mathbf{a}_i\|^2\}_{i=1}^k$ and $\epsilon > 0$, then the following holds

$$\Pr[|\langle \alpha_1 \alpha_2 \dots \alpha_k \rangle - \langle \mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_k \rangle| > \epsilon] < 1/10.$$

REMARK 5. An advantage of our compression scheme is that it can be constructed in the streaming model quite efficiently. The only requirement is that in the case of binary data the maximum number of 1's the vectors in the stream should be bounded, and in the case of real valued data norm of the vectors should be bounded.

1.5 Comparison with previous work

A major advantage of our compression scheme is that it provides a one-shot solution for different similarity measures – Binary compression scheme preserves both Hamming distance and Inner product, and Real valued data compression scheme preserves both Euclidean distance, Inner product, and k -way Inner product. The second main advantage of our compression scheme for binary data it gives a binary to binary compression as opposed to the binary to real compression by JL-transform. Third main advantage is that our compression scheme is that its compression size is independent of the dimensions and depends only on the sparsity as opposed to Gionis, Indyk, Motwani [7] scheme which requires linear size compression. For real-valued data our results are weaker compared to previous known works but they generalize to k -way inner product, which none of the previous work does. Another advantage of our real valued compression scheme is that when the number of points are small (constant), then for preserving a

pairwise Inner Product or Euclidean distance, we have a clear advantage on the amount of randomness required for the compression, the randomness required by our scheme grows logarithmically in the compression length, whereas the other schemes require randomness which grows linearly in the compression length.

Potential applications

A potential use of our result is to improve approximate nearest neighbor search via composing with LSH. Due to the “curse of dimensionality” many search algorithms scale poorly in high dimensional data. So, if it is possible to get a succinct compression of data while preserving the similarity score between pair of data points, then such compression naturally helps for efficient search. One can first compress the input such that it preserve the desired similarity measure, and then can apply a collision based hashing algorithm such as LSH [7, 9] for efficient approximate nearest neighbor (c -NN) on the compressed data. As our compression scheme provides a similar guarantee as of Definition 11, then one can construct data structure for LSH for approximate nearest neighbor problem. Thus, our similarity preserving compression scheme leads to an efficient approximate nearest neighbor search.

There are many similarity based algorithmic methods used in large scale learning and information retrieval, e.g., Frequent itemset mining [2], ROCK clustering [8]. One could potentially obtain algorithmic speed up in these methods via our compression schemes. Recently compression based on LSH for inner-product is used to speed up the forward and back-propagation in neural networks [15]. One could potentially use our scheme to take advantage of sparsity and obtain further speed up.

Organization of the paper

In Section 2, we present the necessary background which helps to understand the paper. In Section 3, we present our compression scheme for high dimensional sparse binary data. In Section 4, we present our compression scheme for high dimensional sparse real data. Finally in Section 5, we conclude our discussion, and state some possible extensions of the work.

2. BACKGROUND

Notations	
N	number of coordinates/bit positions in the compressed data
ψ	upper bound on the number of 1's in any binary vector.
Ψ	upper bound on the norm of any real-valued vector.
$\ \mathbf{a}\ $	l_2 norm of the vector \mathbf{a}
$\mathbf{a}[i]$	i -th bit position (coordinate) of binary (real-valued) vector \mathbf{a} .
$d_H(\mathbf{u}, \mathbf{v})$	Hamming distance between binary vectors \mathbf{u} and \mathbf{v} .
$IP(\mathbf{a}, \mathbf{b})$	Inner product between binary/real-valued vectors \mathbf{a} and \mathbf{b} .

2.1 Probability background

DEFINITION 1. *The Variance of a random variable X , denoted $\text{Var}(X)$, is defined as the expected value of the squared deviation of X from its mean.*

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}(X^2) - \mathbb{E}(X)^2.$$

DEFINITION 2. *Let X and Y be jointly distributed random variables. The Covariance of X and Y , denoted $\text{Cov}(X, Y)$, is defined as*

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))].$$

FACT 3. *Let X be a random variable and λ be a constant. Then, $\text{Var}(\lambda + X) = \text{Var}(X)$ and $\text{Var}(\lambda X) = \lambda^2 \text{Var}(X)$.*

FACT 4. *Let X_1, X_2, \dots, X_n be a set of n random variables. Then,*

$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j).$$

FACT 5. *Let X and Y be a pair of random variables and λ be a constant. Then, $\text{Cov}(\lambda X, \lambda Y) = \lambda^2 \text{Cov}(X, Y)$.*

FACT 6 (CHEBYSHEV'S INEQUALITY). *Let X be a random variable having finite mean and finite non-zero variance σ^2 . Then for any real number $\lambda > 0$,*

$$\Pr[|X - \mathbb{E}(X)| \geq \lambda\sigma] \leq \frac{1}{\lambda^2}.$$

2.2 Similarity measures and their respective compression schemes

Hamming distance.

Let $\mathbf{u}, \mathbf{v} \in \{0, 1\}^d$ be two binary vectors, then the Hamming distance between these two vectors is the number of bit positions where they differ. To the best of our knowledge, there does not exist any non-trivial compression scheme which provide similar compression guarantees such as JL-lemma provides for Euclidean distance. In the following lemma, we show that for a set of n -binary vectors an analogous JL-type binary to binary compression (if it exist) may require compression length linear in n . Further collision¹ based hashing scheme such as LSH (due to Gionis *et al.* [7], see Subsection 2.3) can be considered as a binary to binary compression scheme, where the size of hashtable determines the compression-length. Their techniques includes randomly choosing bit positions and checking if the query and input vectors are matching exactly at those bit positions.

LEMMA 7. *Consider a set of n -binary vectors, then an analogous JL-type binary to binary compression (if it exist) may require compression length linear in n .*

¹A collision occurs when two object hash to the same hash value.

PROOF. Consider a set of n binary vectors $\{e_i\}_{i=1}^n$ – standard unit vectors, and the zero vector e_0 . The Hamming distance between e_0 and any e_i is 1, and the Hamming distance between any pair of vectors e_i and e_j for $i \neq j$ is 2. Let f be a map which map these points into binary vectors of dimension k by preserving the distance between any pair of vectors within a factor of $1 \pm \varepsilon$, for a parameter $\varepsilon > 0$. Thus, these n points $\{f(e_i)\}_{i=1}^n$ are within a distance at most $(1 + \varepsilon)$ from $f(e_0)$, and any two points $f(e_i)$ and $f(e_j)$ for $i \neq j$ are at distance at least $2(1 - \varepsilon)$. However, the total number of points at distance at most $(1 + \varepsilon)$ from $f(e_0)$ is $O(k^{1+\varepsilon})$, and distance between any two points $f(e_i)$ and $f(e_j)$ for $i \neq j$ is non-zero so each point $\{e_i\}_{i=1}^n$ has its distinct image. Thus $O(k^{1+\varepsilon})$ should be equal to n , which gives $k = \Omega(n^{\frac{1}{1+\varepsilon}})$. Thus the compression length can be linear in n . \square

Euclidean distance.

Given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, the Euclidean distance between them is denoted as $\|\mathbf{a}, \mathbf{b}\|$ and defined as $\sqrt{\sum_{i=1}^d (\mathbf{a}[i] - \mathbf{b}[i])^2}$. A classical result by Johnson and Lindenstrauss [10] suggest a compressing scheme which for any set \mathcal{D} of n vectors in \mathbb{R}^d preserve pairwise Euclidean distance between any pair of vectors in \mathcal{D} .

LEMMA 8 (JL TRANSFORM [10]). *For any $\epsilon \in (0, 1)$, and any integer n , let k be a positive integer such that $k = O(\frac{1}{\epsilon^2} \log n)$. Then for any set \mathcal{D} of n vectors in \mathbb{R}^d , there is a map $f : \mathbb{R}^d \rightarrow \mathbb{R}^k$ such that for any pair of vectors \mathbf{a}, \mathbf{b} in \mathcal{D} :*

$$(1 - \epsilon)\|\mathbf{a}, \mathbf{b}\|^2 \leq \|f(\mathbf{a}), f(\mathbf{b})\|^2 \leq (1 + \epsilon)\|\mathbf{a}, \mathbf{b}\|^2$$

Furthermore, the mapping f can be found in randomized polynomial time.

In several followup works on JL lemma, the function f has been regarded as a random projection matrix $R \in \mathbb{R}^{d \times k}$, and can be constructed element-wise using Gaussian due to Indyk and Motwani [9], or uniform $\{+1, -1\}$ due to Achlioptas [1].

Inner product.

Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, the Inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ between them is defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{i=1}^d \mathbf{u}[i]\mathbf{v}[i].$$

Compression schemes which preserves Inner product has been studied quite a lot in the recent time. In the case of binary data, along with some sparsity assumption (bound on the number of 1's), there are some schemes available which by padding (add a few extra bits in the vector) reduce the Inner product (of the original data) to the Hamming [3], and Jaccard similarity [14]. Then the compression scheme for Hamming or Jaccard can be applied on the padded version of the data. Similarly, in the case of real-valued data, a similar padding technique is known that due padding reduces Inner product to Euclidean distance [13]. Recently, an interesting work

by Ata Kabán [11] suggested a compression schemes *via* random projection method. Their scheme approximately preserve Inner Product between any pair of input points and their compression bound matches the bound of JL-transform [10].

Jaccard similarity.

Binary vectors can also be considered as sets over the universe of all possible features, and a set contain only those elements which have non-zero entries in the corresponding binary vector. For example two vectors $\mathbf{u}, \mathbf{v} \in \{0, 1\}^d$ can be viewed as two sets $\mathbf{u}, \mathbf{v} \subseteq \{1, 2, \dots, d\}$. Here, the underlying similarity measure of interest is the Jaccard similarity which is defined as follows $JS(\mathbf{u}, \mathbf{v}) = \frac{|\mathbf{u} \cap \mathbf{v}|}{|\mathbf{u} \cup \mathbf{v}|}$.

A celebrated work by Broder *et al.* [5, 6, 4] suggested a technique to compress a collection of sets while preserving the Jaccard similarity between any pair of sets. Their technique includes taking a random permutation of $\{1, 2, \dots, d\}$ and assigning a value to each set which maps to minimum under that permutation. This compression scheme is popularly known as Minwise hashing.

DEFINITION 9 (MINWISE HASH FUNCTION). Let π be a permutations over $\{1, \dots, d\}$, then for a set $\mathbf{u} \subseteq \{1, \dots, d\}$ $h_\pi(\mathbf{u}) = \arg \min_i \pi(i)$ for $i \in \mathbf{u}$. Then due to [5, 6, 4],

$$\Pr[h_\pi(\mathbf{u}) = h_\pi(\mathbf{v})] = \frac{|\mathbf{u} \cap \mathbf{v}|}{|\mathbf{u} \cup \mathbf{v}|}.$$

2.3 Locality Sensitive Hashing

LSH suggest an algorithm or alternatively a data structure for efficient approximate nearest neighbor (c -NN) search in high dimensional space. We formally state it as follows:

DEFINITION 10. (c -Approximate Nearest Neighbor (c -NN)). Let \mathcal{D} be set of points in \mathbb{R}^d , and $\text{Sim}(\cdot, \cdot)$ be a desired similarity measure. Then for parameters $S, c > 0$, the c -NN problem is to construct a data structure that given any query point $\mathbf{q} \in \mathcal{D}$ reports a cS -near neighbor of \mathbf{q} in \mathcal{D} if there is an S -near neighbor of \mathbf{q} in \mathcal{D} . Here, we say a point $\mathbf{x} \in \mathcal{D}$ is S -near neighbor of \mathbf{q} if $\text{Sim}(\mathbf{q}, \mathbf{x}) > S$.

In the following we define the concept of locality sensitive hashing (LSH) which suggest a data structure to solve c -NN problem.

DEFINITION 11 (LOCALITY SENSITIVE HASHING [9]). Let \mathcal{D} be a set of n vectors in \mathbb{R}^d , and U be the hashing universe. Then, a family \mathcal{H} of functions from \mathcal{D} to U is called as (S, cS, p_1, p_2) -sensitive for a similarity measure $\text{Sim}(\cdot, \cdot)$ if for any $\mathbf{x}, \mathbf{y} \in \mathcal{D}$,

- if $\text{Sim}(\mathbf{x}, \mathbf{y}) \geq S$, then $\Pr_{h \in \mathcal{H}}[h(\mathbf{x}) = h(\mathbf{y})] \geq p_1$,
- if $\text{Sim}(\mathbf{x}, \mathbf{y}) \leq cS$, then $\Pr_{h \in \mathcal{H}}[h(\mathbf{x}) = h(\mathbf{y})] \leq p_2$.

Clearly, any such scheme is interesting only when $p_1 > p_2$, and $c < 1$. Let K, L be the parameters of the data structure for LSH, where K is the number of hashes in each hash table, and L is the number of hash tables, then due to [9, 7], we have $K = O\left(\log \frac{1}{p_2}\right)$ and $L = O(n^\rho \log n)$, where $\rho = \frac{\log p_1}{\log p_2}$. Thus, given a family of (S, cS, p_1, p_2) -sensitive hash functions, and using result of [9, 7], one can construct a data structure for c -NN with $O(n^\rho \log n)$ query time and space $O(n^{1+\rho})$.

2.3.1 How to convert similarity preserving compression schemes to LSH ?

LSH schemes for various similarity measures can be viewed as first compressing the input such that it preserve the desired similarity measure, and then applying collision based hashing on top of it. If any similarity preserving compression scheme provides a similar guarantee as of Definition 11, then for parameters – similarity threshold S , and c , one can construct data structure for LSH (hash-tables with parameters K and L) for the c -NN problem via [9, 7].

3. A COMPRESSION SCHEME FOR HIGH DIMENSIONAL SPARSE BINARY DATA

We first formally define our Compression Scheme as follows:

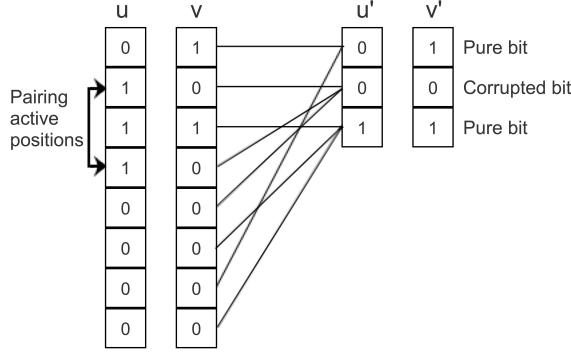
DEFINITION 12 (BINARY COMPRESSION SCHEME). Let N be the number of buckets, for $i = 1$ to d , we randomly assign the i -th position to a bucket number $b(i) \in \{1, \dots, N\}$. Then a vector $\mathbf{u} \in \{0, 1\}^d$, compressed into a vector $\mathbf{u}' \in \{0, 1\}^N$ as follows:

$$\mathbf{u}'[j] = \sum_{i:b(i)=j} \mathbf{u}[i] \pmod{2}.$$

NOTE 13. For brevity we denote the Binary Compression Scheme as BCS.

Some intuition.

Consider two binary vectors $\mathbf{u}, \mathbf{v} \in \{0, 1\}^d$, we call a bit position “active” if at least one of the vector between \mathbf{u} and \mathbf{v} has value 1 in that position. Let ψ be the maximum number of 1 in any vector, then there could be at most 2ψ active positions shared between vectors \mathbf{u} and \mathbf{v} . Further, using the BCS, let \mathbf{u} and \mathbf{v} get compressed into binary vectors $\mathbf{u}', \mathbf{v}' \in \{0, 1\}^N$. In the compressed vectors, we call a particular bit position “pure” if the number of active positions mapped to that position is at most one, otherwise we call it “corrupted”. It is easy to see that the contribution of pure bit positions in \mathbf{u}', \mathbf{v}' towards Hamming distance (or Inner product similarity), is exactly equal to the contribution of the bit positions in \mathbf{u}, \mathbf{v} which get mapped to the pure bit positions. The number of maximum possible corrupted bits in the compressed data is ψ because in the worst case it is possible that all the 2ψ active bit position got paired up while compression. The deviation of Hamming distance (or Inner product similarity) between \mathbf{u}'



and \mathbf{v}' from that of \mathbf{u} and \mathbf{v} , corresponds to the number of corrupted bit positions shared between \mathbf{u}' and \mathbf{v}' . The above figure illustrate this with an example, and the lemma below analyse it.

LEMMA 14. Consider two binary vectors $\mathbf{u}, \mathbf{v} \in \{0, 1\}^d$, which get compressed into vectors $\mathbf{u}', \mathbf{v}' \in \{0, 1\}^N$ using the BCS, and suppose ψ is the maximum number of 1 in any vector. Then for an integer $r \geq 1$, and $\epsilon > 0$, probability that \mathbf{u}' and \mathbf{v}' share more than ϵr corrupted positions is at most $\left(\frac{2\psi}{\sqrt{N}}\right)^{\epsilon r}$.

PROOF. We first calculate the probability that a particular bit position gets corrupted between \mathbf{u}' and \mathbf{v}' . As there are at most 2ψ active positions shared between vectors \mathbf{u} and \mathbf{v} , the number of ways of pairing two active positions from 2ψ active positions is at most $\binom{2\psi}{2}$, and this pairing will result a corrupted bit position in \mathbf{u}' or \mathbf{v}' . Then, the probability that a particular bit position in \mathbf{u}' or \mathbf{v}' gets corrupted is at most $\frac{\binom{2\psi}{2}}{N} \leq \left(\frac{4\psi^2}{N}\right)$. Further, if the deviation of Hamming distance (or Inner product similarity) between \mathbf{u}' and \mathbf{v}' from that of \mathbf{u} and \mathbf{v} is more than ϵr , then at least ϵr corrupted positions are shared between \mathbf{u}' and \mathbf{v}' , which implies that at least $\frac{\epsilon r}{2}$ pair of active positions in \mathbf{u} and \mathbf{v} got paired up while compression. The number of possible ways of pairing $\frac{\epsilon r}{2}$ active positions from 2ψ active positions is at most $\binom{2\psi}{\frac{\epsilon r}{2}} \binom{2\psi - \frac{\epsilon r}{2}}{\frac{\epsilon r}{2}} \frac{\epsilon r}{2}! \leq (2\psi)^{\epsilon r}$. Since the probability that a pair of active positions got mapped in the same bit position in the compressed data is $\frac{1}{N}$, the probability that $\frac{\epsilon r}{2}$ pair of active positions got mapped in $\frac{\epsilon r}{2}$ distinct bit positions in the compressed data is at most $(\frac{1}{N})^{\frac{\epsilon r}{2}}$. Thus, by union bound, the probability that at least ϵr corrupted bit position shared between \mathbf{u}' and \mathbf{v}' is at most $\frac{(2\psi)^{\epsilon r}}{N^{\frac{\epsilon r}{2}}} = \left(\frac{2\psi}{\sqrt{N}}\right)^{\epsilon r}$. \square

In the following lemma we generalize the above result on a set of n binary vectors. We suggest a compression bound such that any pair of compressed vectors share only a very small number of corrupted bits, with high probability.

LEMMA 15. Consider a set \mathcal{U} of n binary vectors $\{\mathbf{u}_i\}_{i=1}^n \subseteq \{0, 1\}^d$, which get compressed into a set \mathcal{U}' of binary vectors $\{\mathbf{u}'_i\}_{i=1}^n \subseteq \{0, 1\}^N$ using the BCS. Then for any positive integer r , and $\epsilon > 0$,

- if $\epsilon r > 3 \log n$, and we set $N = 16\psi^2$, then probability that for all $\mathbf{u}'_i, \mathbf{u}'_j \in \mathcal{U}'$ share more than ϵr corrupted positions is at most $\frac{1}{n}$.
- If $\epsilon r < 3 \log n$, and we set $N = 144\psi^2 \log^2 n$, then probability that for all $\mathbf{u}'_i, \mathbf{u}'_j \in \mathcal{U}'$ share more than ϵr corrupted positions is at most $\frac{1}{n}$.

PROOF. In the first case, for a fixed pair of compressed vectors \mathbf{u}'_i and \mathbf{u}'_j , due to lemma 14, probability that they share more than ϵr corrupted positions is at most $\left(\frac{2\psi}{\sqrt{N}}\right)^{\epsilon r}$. If $\epsilon r > 3 \log n$, and $N = 16\psi^2$, then the above probability is at most $\left(\frac{2\psi}{\sqrt{N}}\right)^{\epsilon r} < \left(\frac{2\psi}{4t}\right)^{3 \log n} = \left(\frac{1}{2}\right)^{3 \log n} < \frac{1}{n^3}$. As there are at most $\binom{n}{2}$ pairs of vectors, then the probability of every pair of compressed vectors share more than ϵr corrupted positions is at most $\frac{\binom{n}{2}}{n^3} < \frac{1}{n}$.

In the second case, as $\epsilon r < 3 \log n$, we cannot upper bound the desired probability similar to the first case. Here we use a trick, in the input data we replicate each bit position $3 \log n$ times, which makes a d dimensional vector to a $3d \log n$ dimensional, and as a consequence the Hamming distance (or Inner product similarity) is also scaled up by a multiplicative factor of $3 \log n$. We now apply the compression scheme on these scaled vectors, then for a fixed pair of compressed vectors \mathbf{u}'_i and \mathbf{u}'_j , probability that they have more than $3\epsilon r \log n$ corrupted positions is at most $\left(\frac{6\psi \log n}{\sqrt{N}}\right)^{3\epsilon r \log n}$. As we set $N = 144\psi^2 \log^2 n$, the above probability is at most $\left(\frac{6\psi \log n}{\sqrt{144\psi^2 \log^2 n}}\right)^{3\epsilon r \log n} < \left(\frac{1}{2}\right)^{3 \log n} < \frac{1}{n^3}$. The final probability follows by applying union bound over all $\binom{n}{2}$ pairs. \square

REMARK 6. We would like to emphasize that using the BCS, for any pair of vectors, the Hamming distance between them in the compressed version is always less than or equal to their original Hamming distance. Thus, this compression scheme has only one-sided-error for the Hamming case. However, in the case of inner product similarity this compression scheme can possibly have two-sided-error – as the inner product in the compressed version can be smaller or higher than the inner product of original input. We illustrate this by the following example, where the compression scheme assigns both bit positions of the input to one bit of the compressed data.

- If $\mathbf{u} = [1, 0]$ and $\mathbf{v} = [0, 1]$, then $\text{IP}(\mathbf{u}, \mathbf{v}) = 0$; and after compression $\mathbf{u}' = [1]$ and $\mathbf{v}' = [1]$ which gives $\text{IP}(\mathbf{u}', \mathbf{v}') = 1$.
- If $\mathbf{u} = [1, 1]$ and $\mathbf{v} = [1, 1]$, then $\text{IP}(\mathbf{u}, \mathbf{v}) = 2$, and after compression $\mathbf{u}' = [0]$ and $\mathbf{v}' = [0]$ which gives $\text{IP}(\mathbf{u}', \mathbf{v}') = 0$.

As a consequence of Lemma 15 and the above remark, we present our compression guarantee for the Hamming distance and Inner product similarity.

THEOREM 1. Consider a set U of binary vectors $\{\mathbf{u}_i\}_{i=1}^n \subseteq \{0, 1\}^d$, a positive integer r , and $\epsilon > 0$. If $\epsilon r > 3 \log n$, we set $N = O(\psi^2)$; if $\epsilon r < 3 \log n$, we set $N = O(\psi^2 \log^2 n)$, and compress them into a set U' of binary vectors $\{\mathbf{u}'_i\}_{i=1}^n \subseteq \{0, 1\}^N$ using BCS. Then for all $\mathbf{u}_i, \mathbf{u}_j \in U$,

- if $d_H(\mathbf{u}_i, \mathbf{u}_j) < r$, then $\Pr[d_H(\mathbf{u}'_i, \mathbf{u}'_j) < r] = 1$,
- if $d_H(\mathbf{u}_i, \mathbf{u}_j) \geq (1+\epsilon)r$, then $\Pr[d_H(\mathbf{u}'_i, \mathbf{u}'_j) < r] < \frac{1}{n}$.

THEOREM 2. Consider a set U of binary vectors $\{\mathbf{u}_i\}_{i=1}^n \subseteq \{0, 1\}^d$, a positive integer r , and $\epsilon > 0$. If $\epsilon r > 3 \log n$, we set $N = O(\psi^2)$; if $\epsilon r < 3 \log n$, we set $N = O(\psi^2 \log^2 n)$, and compress them into a set U' of binary vectors $\{\mathbf{u}'_i\}_{i=1}^n \subseteq \{0, 1\}^N$ using BCS. Then for all $\mathbf{u}_i, \mathbf{u}_j \in U$ the following is true with probability at least $1 - \frac{1}{n}$,

$$(1 - \epsilon)IP(\mathbf{u}_i, \mathbf{u}_j) \leq IP(\mathbf{u}'_i, \mathbf{u}'_j) \leq (1 + \epsilon)IP(\mathbf{u}_i, \mathbf{u}_j).$$

3.1 A tighter analysis for Hamming distance

In this subsection, we strengthen our analysis for the Hamming case, and shows a compression bound which is independent of the dimension and the sparsity, and depends only on the Hamming distance between the vectors. However, we could show our result in expectation, and only for a pair of vectors.

For a pair of vectors $\mathbf{u}, \mathbf{v} \in \{0, 1\}^d$, we say that a bit position is “unmatched” if exactly one of the vector has value 1 in that position and the other one has value 0. We say that a bit position in the compressed data is “odd-bit” if odd number of unmatched positions get mapped to that bit. Let \mathbf{u} and \mathbf{v} get compressed into vectors \mathbf{u}' and \mathbf{v}' using the BCS. Our observation is that each odd bit position in the compressed data contributes to Hamming distance in 1 in the compressed data. We illustrate this with an example: let $\mathbf{u}[i, j, k] = [1, 0, 1]$, $\mathbf{v}[i, j, k] = [0, 1, 0]$ and let i, j, k get mapped to bit position i' (say) in the compressed data, then $\mathbf{u}[i'] = 0, \mathbf{v}[i'] = 1$, then clearly $d_H(\mathbf{u}[i'], \mathbf{v}[i']) = 1$.

THEOREM 3. Consider two binary vectors $\mathbf{u}, \mathbf{v} \in \{0, 1\}^d$, which get compressed into vectors $\mathbf{u}', \mathbf{v}' \in \{0, 1\}^N$ using BCS. If we set $N = O(r^2)$, then

- if $d_H(\mathbf{u}, \mathbf{v}) < r$, then $\Pr[d_H(\mathbf{u}', \mathbf{v}') < r] = 1$, and
- if $d_H(\mathbf{u}, \mathbf{v}) \geq 4r$, then $\mathbb{E}[d_H(\mathbf{u}', \mathbf{v}')] > 2r$.

PROOF. Let ψ_u denote the number of unmatched bit positions between \mathbf{u} and \mathbf{v} . As mentioned earlier, if odd number of unmatched bit positions gets mapped to a particular bit in the compressed data, then that bit position corresponds to the Hamming distance 1. Let we call that bit position as “odd-bit” position. In order to give a bound on the Hamming distance in the compressed data we need to give a bound on number of such odd-bit positions. We first calculate the probability that a particular bit position say k -th position in the

compressed data is odd. Let we denote this by $\Pr_{odd}^{(k)}$. We do it using the following binomial distribution:

$$\Pr_{odd}^{(k)} = \sum_{i \bmod 2=1}^{\psi_u} \frac{1}{N^i} \binom{\psi_u}{i} \left(1 - \frac{1}{N}\right)^{\psi_u-1}.$$

Similarly, we compute the probability that the k -th bit is even:

$$\Pr_{even}^{(k)} = \sum_{i \bmod 2=0}^{\psi_u} \frac{1}{N^i} \binom{\psi_u}{i} \left(1 - \frac{1}{N}\right)^{\psi_u-1}.$$

We have,

$$\Pr_{even}^{(k)} + \Pr_{odd}^{(k)} = 1. \quad (1)$$

Further,

$$\begin{aligned} & \Pr_{even}^{(k)} - \Pr_{odd}^{(k)} \\ &= \sum_{i \bmod 2=0}^{\psi_u} \frac{1}{N^i} \binom{\psi_u}{i} \left(1 - \frac{1}{N}\right)^{\psi_u-1} \\ &\quad - \sum_{i \bmod 2=1}^{\psi_u} \frac{1}{N^i} \binom{\psi_u}{i} \left(1 - \frac{1}{N}\right)^{\psi_u-1} \\ &= \left(1 - \frac{1}{N} - \frac{1}{N}\right)^{\psi_u} \\ &= \left(1 - \frac{2}{N}\right)^{\psi_u}. \end{aligned} \quad (2)$$

Thus, we have the following from Equation 1 and Equation 2

$$\begin{aligned} \Pr_{odd}^{(k)} &= \frac{1}{2} \left(1 - \left(1 - \frac{2}{N}\right)^{\psi_u}\right) \\ &\geq \frac{1}{2} \left(1 - \exp\left(-\frac{2\psi_u}{N}\right)\right). \end{aligned} \quad (3)$$

The last inequality follows as $(1 - x) \leq e^{-x}$ for $x < 1$. Thus expected number of odd-bits is at least

$$\frac{N}{2} \left(1 - \exp\left(-\frac{2\psi_u}{N}\right)\right).$$

We now split here in two cases: 1) $\psi_u < 20r$, and 2) $\psi_u \geq 20r$. We address them one-by-one.

Case 1: $\psi_u < 20r$. We complete this case using Lemma 14. It is easy to verify that in the case of Hamming distance the analysis of Lemma 14 also holds if we consider “unmatched” bits instead of “active” bits in the analysis. Thus, the probability that at least r corrupted bit position shared between \mathbf{u}' and \mathbf{v}' is at most $\left(\frac{2\psi_u}{\sqrt{N}}\right)^r$. We wish to set the value of N such that with probability at most $1/3$ that \mathbf{u}' and \mathbf{v}' share more than r corrupted positions. If we set the value of $N = 4\psi_u^2 3^{\frac{2}{r}}$, then the above probability is at most $\left(\frac{2\psi_u}{\sqrt{4\psi_u^2 3^{\frac{2}{r}}}}\right)^r = \frac{1}{3}$.

Thus, when $N = 4\psi_u^2 3^{\frac{2}{r}} = O(\psi_u^2) = O(r^2)$ as $\psi_u < 20r$ and $r \geq 2$, with probability at most $1/3$, at most

r corrupted bits are shared between \mathbf{u}' and \mathbf{v}' . As a consequence to this, we have $\mathbb{E}[d_H(\mathbf{u}', \mathbf{v}')] > \frac{2}{3} \cdot 3r = 2r$.

Case 2: $\psi_u \geq 20r$. We continue here from Equation 3

Expected number of odd buckets

$$\begin{aligned} &\geq \frac{N}{2} \left(1 - \exp \left(-\frac{2\psi_u}{N} \right) \right) \\ &\geq \frac{N}{2} \left(1 - \exp \left(-\frac{40r}{N} \right) \right) \\ &= 4r^2 \left(1 - \exp \left(-\frac{5}{r} \right) \right) \end{aligned} \quad (4)$$

$$\begin{aligned} &> 4r^2 \left(\frac{1}{2r} \right) \\ &= 2r. \end{aligned} \quad (5)$$

Equality 4 follows by setting $N = 8r^2$ and Inequality 5 holds as $1 - \exp(-\frac{5}{r}) > \frac{1}{2r}$ for $r \geq 2$.

Finally, Case 1 and Case 2 complete a proof of the theorem. \square

4. A COMPRESSION SCHEME FOR HIGH DIMENSIONAL SPARSE REAL DATA

We first define our compression scheme for the real valued data.

DEFINITION 16. (*Real-valued Compression Scheme*)
Let N be the number of buckets, for $i = 1$ to d , we randomly assign the i -th position to the bucket number $b(i) \in \{1, \dots, N\}$. Then, for $j = 1$ to N , the j -th coordinate of the compressed vector $\boldsymbol{\alpha}$ is computed as follows:

$$\boldsymbol{\alpha}[j] = \sum_{i:b(i)=j} \mathbf{a}[i]x_i,$$

where each x_i is a random variable that takes a value between $\{-1, +1\}$ with probability $1/2$.

NOTE 17. For brevity we denote our Real-valued Compression Scheme as RCS.

We first present our compression guarantee for preserving Inner product for a pair of real valued vectors.

LEMMA 18. Consider two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, which get compressed into vectors $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^N$ using the RCS. If we set $N = \frac{10\Psi^2}{\epsilon^2}$, where $\Psi = \max\{\|\mathbf{a}\|^2, \|\mathbf{b}\|^2\}$ and $\epsilon > 0$, then the following holds,

$$\Pr[|\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle - \langle \mathbf{a}, \mathbf{b} \rangle| > \epsilon] < 1/10.$$

PROOF. Let we have two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ such that $\mathbf{a} = [a_1, a_2, \dots, a_d]$ and $\mathbf{b} = [b_1, b_2, \dots, b_d]$. Let $\{x_i\}_{i=1}^d$ be a set of d random variables such that each x_i takes a value between $\{-1, +1\}$ with probability $1/2$, $z_i^{(k)}$ be a random variable that takes the value 1 if i -th dimension of the vector is mapped to the k -th bucket of the compressed vector and 0 otherwise. Using the compression scheme RCS, let vectors \mathbf{a}, \mathbf{b} get compressed into vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, where $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_N]$ such that $\alpha_k = \sum_{i=1}^d a_i x_i z_i^{(k)}$, and $\boldsymbol{\beta} = [\beta_1, \dots, \beta_N]$ such that

$\beta_k = \sum_{i=1}^d b_i x_i z_i^{(k)}$. We now compute the inner product of the compressed vectors $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$.

$$\begin{aligned} \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle &= \sum_{k=1}^N \alpha_k \beta_k = \sum_{k=1}^N \left(\sum_{i=1}^d a_i x_i z_i^{(k)} \right) \left(\sum_{i=1}^d b_i x_i z_i^{(k)} \right) \\ &= \sum_{k=1}^N \left(\sum_{i=1}^d a_i b_i x_i^2 z_i^{(k)2} + \sum_{i \neq j} a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right) \end{aligned} \quad (6)$$

$$= \sum_{k=1}^N \left(\sum_{i=1}^d a_i b_i z_i^{(k)} + \sum_{i \neq j} a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right) \quad (7)$$

$$\begin{aligned} &= \sum_{i=1}^d a_i b_i \sum_{k=1}^N z_i^{(k)} + \sum_{k=1}^N \left(\sum_{i \neq j} a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right) \\ &= \sum_{i=1}^d a_i b_i + \sum_{k=1}^N \left(\sum_{i \neq j} a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right) \\ &= \langle \mathbf{a}, \mathbf{b} \rangle + \sum_{k=1}^N \left(\sum_{i \neq j} a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right). \end{aligned} \quad (8)$$

Equation 7 follows from Equation 6 because $x_i^2 = 1$ as $x_i = \pm 1$, and $z_i^2 = z_i$ as z_i takes value either 1 or 0. We continue from Equation 8 and compute the Expectation and the Variance of the random variable $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$. We first compute the Expectation of the random variable $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$ as follows:

$$\begin{aligned} \mathbb{E}[\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle] &= \mathbb{E} \left[\langle \mathbf{a}, \mathbf{b} \rangle + \sum_{k=1}^N \left(\sum_{i \neq j} a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right) \right] \\ &= \mathbb{E}[\langle \mathbf{a}, \mathbf{b} \rangle] + \mathbb{E} \left[\sum_{k=1}^N \left(\sum_{i \neq j} a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right) \right] \\ &= \langle \mathbf{a}, \mathbf{b} \rangle + \sum_{k=1}^N \left(\sum_{i \neq j} \mathbb{E}[a_i b_j x_i x_j z_i^{(k)} z_j^{(k)}] \right) \end{aligned} \quad (9)$$

$$\begin{aligned} &= \langle \mathbf{a}, \mathbf{b} \rangle + \sum_{k=1}^N \left(\sum_{i \neq j} a_i b_j \mathbb{E}[x_i x_j z_i^{(k)} z_j^{(k)}] \right) \\ &= \langle \mathbf{a}, \mathbf{b} \rangle. \end{aligned} \quad (10)$$

Equation 9 holds due to the linearity of expectation. Equation 11 holds because $\mathbb{E}[x_i x_j z_i^{(k)} z_j^{(k)}] = 0$ as both x_i and x_j take a value between $\{-1, +1\}$ each with probability 0.5 which leads to $\mathbb{E}[x_i x_j] = 0$.

We now compute the Variance of the random variable

$\langle \alpha, \beta \rangle$ as follows:

$$\begin{aligned} \text{Var}[\langle \alpha, \beta \rangle] &= \text{Var} \left[\langle \mathbf{a}, \mathbf{b} \rangle + \sum_{k=1}^N \left(\sum_{i \neq j} a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right) \right] \\ &= \text{Var} \left[\sum_{k=1}^N \left(\sum_{i \neq j} a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right) \right] \quad (11) \end{aligned}$$

$$= \text{Var} \left[\sum_{k=1}^N \left(\sum_{i \neq j} \xi_{ij}^{(k)} \right) \right] \quad (12)$$

$$\begin{aligned} &= \text{Var} \left[\sum_{i \neq j} \sum_{k=1}^N \xi_{ij}^{(k)} \right] \\ &= \sum_{i \neq j} \text{Var} \left[\sum_{k=1}^N \xi_{ij}^{(k)} \right] \dots \\ &\quad + \sum_{i \neq j, i' \neq j', i \neq i', j \neq j'} \text{Cov} \left[\sum_{k=1}^N \xi_{ij}^{(k)}, \sum_{k=1}^N \xi_{i'j'}^{(k)} \right] \quad (13) \end{aligned}$$

Equation 11 holds due to Fact 3; Equation 12 holds as we denote the expression $a_i b_j x_i x_j z_i^{(k)} z_j^{(k)}$ by the variable $\xi_{ij}^{(k)}$; Equation 13 holds due to Fact 4. We now bound the values of the two terms of Equation 13.

$$\begin{aligned} \sum_{i \neq j} \text{Var} \left[\sum_{k=1}^N \xi_{ij}^{(k)} \right] &= \sum_{i \neq j} \sum_{k=1}^N \text{Var} \left[\xi_{ij}^{(k)} \right] \dots \\ &\quad + \sum_{i \neq j} \sum_{k \neq l} \text{Cov} \left[\xi_{ij}^{(k)}, \xi_{ij}^{(l)} \right] \quad (14) \end{aligned}$$

Equation 14 holds due to Fact 4. We bound the values of two terms of Equation 14 one by one as follows.

$$\begin{aligned} \sum_{i \neq j} \sum_{k=1}^N \text{Var} \left[\xi_{ij}^{(k)} \right] &= \sum_{i \neq j} \sum_{k=1}^N \text{Var} \left[a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right] \\ &= \sum_{i \neq j} a_i^2 b_j^2 \sum_{k=1}^N \text{Var} \left[x_i x_j z_i^{(k)} z_j^{(k)} \right] \quad (15) \end{aligned}$$

$$= \sum_{i \neq j} a_i^2 b_j^2 \sum_{k=1}^N \left(\mathbb{E}[x_i^2 x_j^2 z_i^{(k)^2} z_j^{(k)^2}] - \mathbb{E}[x_i x_j z_i^{(k)} z_j^{(k)}]^2 \right) \quad (16)$$

$$= \sum_{i \neq j} a_i^2 b_j^2 \sum_{k=1}^N \mathbb{E} \left[z_i^{(k)} z_j^{(k)} \right] \quad (17)$$

$$= \sum_{i \neq j} a_i^2 b_j^2 / N \leq \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 / N. \quad (18)$$

Equation 15 holds due to Fact 3; Equation 16 holds due to Definition 1; Equation 17 holds as $x_i^2, x_j^2 = 1$, $z_i^{(k)^2} = z_i^{(k)}$, and $\mathbb{E}[x_i x_j] = 0$; finally, Equation 18 holds as $\sum_{i \neq j} a_i^2 b_j^2 \leq \sum_i a_i^2 \sum_i b_i^2 = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2$.

We now bound the second term of Equation 14.

$$\begin{aligned} &\text{Cov} \left[\xi_{ij}^{(k)}, \xi_{ij}^{(l)} \right] \\ &= \text{Cov} \left[a_i b_j x_i x_j z_i^{(k)} z_j^{(k)}, a_i b_j x_i x_j z_i^{(l)} z_j^{(l)} \right] \end{aligned} \quad (19)$$

$$\begin{aligned} &= a_i^2 b_j^2 \text{Cov} \left[x_i x_j z_i^{(k)} z_j^{(k)}, x_i x_j z_i^{(l)} z_j^{(l)} \right] \\ &= a_i^2 b_j^2 \mathbb{E}[(x_i x_j z_i^{(k)} z_j^{(k)} - \mathbb{E}(x_i x_j z_i^{(k)} z_j^{(k)}))(x_i x_j z_i^{(l)} z_j^{(l)} - \mathbb{E}(x_i x_j z_i^{(l)} z_j^{(l)}))] \end{aligned} \quad (20)$$

$$= a_i^2 b_j^2 \mathbb{E} \left[x_i^2 x_j^2 z_i^{(k)} z_j^{(k)} z_i^{(l)} z_j^{(l)} \right] \quad (21)$$

$$= a_i^2 b_j^2 \mathbb{E} \left[z_i^{(k)} z_j^{(k)} z_i^{(l)} z_j^{(l)} \right] = 0 \quad (22)$$

Equation 19 holds due to Fact 5; Equation 20 holds due to Definition 2; Equation 21 holds as $\mathbb{E}(x_i x_j) = 0$; finally, Equation 22 holds as in our compression scheme each dimension of the input is mapped to a unique coordinate (bucket) in the compressed vector which implies that at least one of the random variable between $z_i^{(k)}$ and $z_j^{(k)}$ has to be zero.

We now bound the second term of Equation 13.

$$\begin{aligned} &\text{Cov} \left[\sum_{k=1}^N \xi_{ij}^{(k)}, \sum_{k=1}^N \xi_{i'j'}^{(k)} \right] \\ &= \mathbb{E} \left[\left(\sum_{k=1}^N \xi_{ij}^{(k)} - \mathbb{E}(\sum_{k=1}^N \xi_{ij}^{(k)}) \right) \left(\sum_{k=1}^N \xi_{i'j'}^{(k)} - \mathbb{E}(\sum_{k=1}^N \xi_{i'j'}^{(k)}) \right) \right] \\ &= \mathbb{E} \left[(\sum_{k=1}^N \xi_{ij}^{(k)})(\sum_{k=1}^N \xi_{i'j'}^{(k)}) \right] \quad (23) \\ &= \mathbb{E} \left[\left(\sum_{k=1}^N a_i b_j x_i x_j z_i^{(k)} z_j^{(k)} \right) \left(\sum_{k=1}^N a_{i'} b_{j'} x_{i'} x_{j'} z_{i'}^{(k)} z_{j'}^{(k)} \right) \right] \\ &= a_i b_j a_{i'} b_{j'} \mathbb{E} \left[x_i x_j x_{i'} x_{j'} \left(\sum_{k=1}^N z_i^{(k)} z_j^{(k)} \right) \left(\sum_{k=1}^N z_{i'}^{(k)} z_{j'}^{(k)} \right) \right] \\ &= 0 \quad (24) \end{aligned}$$

Equation 23 holds as $\mathbb{E}(\sum_{k=1}^N \xi_{ij}^{(k)})$ and $\mathbb{E}(\sum_{k=1}^N \xi_{i'j'}^{(k)})$ is equal to zero because

$$\mathbb{E}(\sum_{k=1}^N \xi_{ij}^{(k)}) = \sum_{k=1}^N \mathbb{E}(\xi_{ij}^{(k)}) = \sum_{k=1}^N \mathbb{E}(a_i b_j x_i x_j z_i^{(k)} z_j^{(k)}) = 0.$$

A similar argument follows for the other term as well. Equation 24 holds as $\mathbb{E}[x_i x_j x_{i'} x_{j'}]$ is equal to zero because each variable in the expectation term takes a value between $+1$ and -1 with probability 0.5.

Thus, we have

$$\mathbb{E}[\langle \alpha, \beta \rangle] = \langle \mathbf{a}, \mathbf{b} \rangle,$$

and Equation 13 in conjunction with Equations 14, 18, 22, 24 gives

$$\text{Var}[\langle \alpha, \beta \rangle] \leq \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 / N \leq \Psi^2 / N,$$

where $\Psi = \max\{\|\mathbf{a}\|^2, \|\mathbf{b}\|^2\}$.

Thus, by Chebyshev's inequality (see Fact 6), we have

$$\Pr[|\langle \alpha, \beta \rangle - \langle \mathbf{a}, \mathbf{b} \rangle| > \epsilon] < \frac{\Psi^2}{\epsilon^2 N} = 1/10.$$

The last inequality follows as we set $N = \frac{10\Psi^2}{\epsilon^2}$. \square

Using a similar analysis we can generalize our result for k -way inner product. We state our result as follows:

THEOREM 4. Consider a set of k vectors $\{\mathbf{a}_i\}_{i=1}^k \in \mathbb{R}^d$, which get compressed into vectors $\{\alpha_i\}_{i=1}^k \in \mathbb{R}^N$ using the RCS. If we set $N = \frac{10\Psi^k}{\epsilon^2}$, where $\Psi = \max\{\|\mathbf{a}_i\|^2\}_{i=1}^k$ and $\epsilon > 0$, then the following holds

$$\Pr[|\langle \alpha_1 \alpha_2 \dots \alpha_k \rangle - \langle \mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_k \rangle| > \epsilon] < 1/10.$$

We can also generalize the result of Lemma 18 for Euclidean distance as well. Consider a pair of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ which get compressed into vectors $\alpha, \beta \in \mathbb{R}^N$ using the compression scheme RCS. Let $\|\alpha, \beta\|^2$ denote the squared euclidean distance between the vectors α, β . Using a similar analysis of Lemma 18 we can compute Expectation and Variance of the random variable $\|\alpha, \beta\|^2$

$$\mathbb{E}[\|\alpha, \beta\|^2] = \|\mathbf{a}, \mathbf{b}\|^2,$$

and

$$\text{Var}[\|\alpha, \beta\|^2] \leq \frac{(\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2)^2}{N} \leq \frac{\Psi^2}{N},$$

where $\Psi = \max\{\|\mathbf{a}\|^2, \|\mathbf{b}\|^2\}$. Thus, due to Chebyshev's inequality (see Fact 6), we have the following result for Euclidean distance.

THEOREM 5. Consider two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, which get compressed into vectors $\alpha, \beta \in \mathbb{R}^N$ using the RCS. If we set $N = \frac{10\Psi^2}{\epsilon^2}$, where $\Psi = \max\{\|\mathbf{a}\|^2, \|\mathbf{b}\|^2\}$ and $\epsilon > 0$, then the following holds

$$\Pr[|\|\alpha, \beta\|^2 - \|\mathbf{a}, \mathbf{b}\|^2| > \epsilon] < 1/10.$$

REMARK 7. In order to compress a pair of data points our scheme requires $O(d \log N)$ randomness, which grows logarithmically in the compression length, whereas the other schemes require randomness which grows linearly in the compression length. Thus, when the number of points are small (constant), then for preserving a pairwise Inner product or Euclidean distance, we have a clear advantage on the amount of randomness required for the compression. We also believe that using a more sophisticated concentration result (such as Martingale) it is possible to obtain a more tighter concentration guarantee, and as a consequence a smaller compression length.

5. CONCLUSION AND OPEN QUESTIONS

In this work, to the best of our knowledge, we obtain the first efficient binary to binary compression scheme for preserving Hamming distance and Inner Product for high dimensional sparse data. For Hamming distance in fact our scheme obtains the “no-false-negative” guarantee analogous to the one obtained in recent paper by

Pagh [12]. Contrary to the “local” projection approach of previous schemes we first randomly partition the dimension, and then take a “global summary” within a partition. The compression length of our scheme depends only on the sparsity and is independent of the dimension as opposed to previously known schemes. We also obtain a generalization of our result to real-valued setting. Our work leaves the possibility of several open questions – improving the bounds of our compression scheme, and extending it to other similarity measures such as Cosine and Jaccard similarity are major open questions of our work.

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