A SEARCH FOR THE STANDARD MODEL HIGGS DECAYING TO TWO MUONS AT THE CMS EXPERIMENT

By

ANDREW CARNES

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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By

Andrew Carnes

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In 2012 two collaborations at the Large Hadron Collider announced the discovery of a new boson at 125 GeV with properties similar to the Standard Model Higgs. In order to determine whether the boson is actually the Standard Model Higgs, all of the different decay modes need to be investigated. Insofar, any deviations from the Standard Model in one of these channels would imply new physics.

This dissertation presents the search for the Standard Model Higgs Boson decaying to $\mu^+\mu^-$. The search uses the 35.9 ± 0.9 fb^{-1} of $\sqrt{\mathrm{s}} = 13$ TeV proton-proton collision data recorded by the CMS detector in 2016. The observed and expected upper limits on the rate at a 95 % confidence level are presented for Higgs masses in the range 120 to 130 GeV. The expected and observed upper limits at a mass of 125 GeV are x.xx and $1.98^{+0.81}_{-0.57} \times$ SM respectively. These results provide the best results to date on the Higgs coupling to second generation fermions. No deviations from the Standard Model are observed.

CHAPTER 1
INTRODUCTION

The Standard Model of particle physics is an extremely successful theory shown to correctly predict the behavior of the particles and forces which make up the most basic constituents of the universe. In fact, it correctly describes all of the forces known except for gravity. All of the elementary particles detailed in the Standard Model have been discovered except for one, the Higgs boson. This particle is responsible for breaking the electroweak gauge symmetry, consequently bestowing mass onto the massive elementary particles in the Standard Model. On July 4, 2012 two collaborations at the Large Hadron Collider (LHC), the A Toroidal LHC Apparatus (ATLAS) and Compact Muon Solenoid (CMS), announced the discovery of a new boson at 125 GeV with properties similar to the Standard Model Higgs **???**. This discovery was fueled by the investigation into the Higgs decays to the vector bosons ZZ and $\gamma\gamma$. Soon after, evidence for the Higgs coupling to matter was found through the $\tau^+\tau-$ and $\mathrm{b\bar{b}}$ decays **????**. Whether the newly discovered boson is indeed the expected Standard Model Higgs remains to be determined. Insofar, all of the different decay modes will be investigated to search for deviations from the Standard Model predictions.

This leads to the study of the Higgs decay to $\mu^+\mu^-$. Although this decay is the smallest branching fraction expected to be detected **??**, the dimuon branch offers high efficiency and excellent resolution, which should lead to a narrow peak over the falling background, mostly Drell Yan events. The tiny branching fraction enables greater sensitivity to small deviations from the predicted decay rate and in this respect offers an advantage over other channels where a miniscule deviation could be drowned out. Furthermore, the Higgs coupling to second generation fermions remains to be determined.

This dissertation presents the search for the Standard Model Higgs Boson decaying to $\mu^+\mu^-$ using the proton-proton collision data recorded by the CMS experiment in 2016. In order to maximize the data available for the search, the first machine learning in the L1 Trigger system at the LHC was developed and deployed for 2016 data collection. To further maximize the sensitivity of the search, an additional machine learning technique is invented to categorize events based upon the detector resolution as well as the jet and muon kinematics. The search looks for a Higgs boson with a mass between 120 and 130 GeV and presents the expected and observed upper limits in this range.

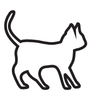
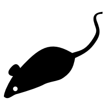
The dissertation first presents an overview of the Standard Model and the resulting symmetry breaking mechanism responsible for the Higgs. After covering the theoretical basis for the Standard Model Higgs, the accelerating apparatus responsible for accelerating and colliding the protons, the LHC, is covered. Then the CMS detector responsible for measuring the paths, momentum, and energy of the particles emerging from every collision is reviewed. Next the machine learning implementation in the L1 trigger that enables the detector to save more of the relevant collisions data is detailed. After, the search for H to $\mu^+\mu^-$ is presented. Finally, the conclusions are presented.

CHAPTER 2
LITERATURE REVIEW

2.1 Dolor Sit Amet

Many of the problems in theses and dissertations involve tables. The UF Graduate Counsel is very specific in the Table Requirements.There should be no vertical lines in tables and only three horizontal lines. No bold text, etc., see the web site for the complete list of requirements. One simple improvement can be incorporated by using tabularx instead of the tabular environment. This allows a table to be stretched the full text width easily, which avoids the centered or left aligned issue. Garfinkle et al. (1991) Table 2-2 is an examble of the tabularx code. Consectetur adipiscing elit. Fusce eget tempus lectus, non porttitor tellus. Aliquam molestie sed urna quis convallis. Aenean nibh eros, aliquam non eros in, tempus lacinia justo. In magna sapien, blandit a faucibus ac, scelerisque nec purus.

Table 2-1. A sample Table using tabularx

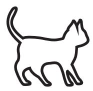
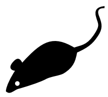
First	Second	Third
12	45	26
17	32	93
text	51	can be there too.
	28	Figures too - a cat.
	000	and a mouse!

Praesent fermentum felis nec massa interdum, vel dapibus mi luctus. Cras id fringilla mauris. Ut molestie eros mi, ut hendrerit nulla tempor et. Pellentesque tortor quam, mattis a scelerisque nec, euismod et odio. Mauris rhoncus metus sit amet risus mattis, eu mattis sem interdum.

2.1.1 Platea Dictumst

Donec convallis scelerisque ante, in sollicitudin orci laoreet eu. Nam arcu magna, semper vel lorem eu, venenatis ultrices est. Nam aliquet ut erat ac scelerisque. Maecenas ut molestie

Table 2-2. A sample Table using standard tablular

First	Second	Third
12	45	26
17	32	93
text	51	can be there too.
	28	Figures too - a cat.
	000	and a mouse!

mi. Phasellus ipsum magna, sollicitudin eu ipsum quis, imperdiet cursus turpis. Etiam pretium enim a fermentum accumsan. Morbi vel vehicula enim.

2.1.2 Long (and/or Wide) Tables

Another problem in LaTeX is the inability to handle long tables. While there are some packages that address this problem none of them quite fit the Editorial Office guidelines. The caption is not repeated but we do need "Table x-y. Continued" on each subsequent page and a repeat of the column headings on each page as well. The following table is the best example of the correct format I can produce. The disadvantage of this method is that much of it is manually set up and changes in the text will cause changes in the table. (**?**) For best results avoid the use of footnotemark and footnotetext commands inside of tables and try to keep your footnotes outside of floats whenever possible.

2.2 Ex id ullamcorper commodo

Augue sapien mattis leo, nec accumsan turpis quam at neque. Ut pellentesque velit sed placerat cursus. Integer congue urna non massa dictum, a pellentesque arcu accumsan. Nulla posuere, elit accumsan eleifend elementum, ipsum massa tristique metus, in ornare neque nisl sed odio. Nullam eget elementum nisi. Duis a consectetur erat, sit amet malesuada sapien. Aliquam nec sapien et leo sagittis porttitor at ut lacus. Vivamus vulputate elit vitae libero condimentum dictum. Nulla facilisi. Quisque non nibh et massa ullamcorper iaculis.

Integer laoreet bibendum arcu non pulvinar. Curabitur ac magna nibh. Phasellus sed nisi semper, molestie neque at, tempus lacus. Aenean vitae lacinia est. Phasellus aliquam lacus sit

Table 2-3. Feasible triples for highly variable Grid, MLMMH.

Time (s)	Triple chosen	Other feasible triples
0	(1, 11, 13725)	(1, 12, 10980), (1, 13, 8235), (2, 2, 0), (3, 1, 0)
2745	(1, 12, 10980)	(1, 13, 8235), (2, 2, 0), (2, 3, 0), (3, 1, 0)
5490	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
8235	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
10980	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
13725	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
16470	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
19215	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
21960	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
24705	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
27450	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
30195	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
32940	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
35685	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
38430	(1, 13, 10980)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
41175	(1, 12, 13725)	(1, 13, 10980), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
43920	(1, 13, 10980)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
46665	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
49410	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
52155	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
54900	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
57645	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
60390	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
63135	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
65880	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
68625	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
71370	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
74115	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
76860	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
79605	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
82350	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
85095	(1, 12, 13725)	(1, 13, 10980), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
87840	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
90585	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
93330	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
96075	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
98820	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
101565	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
104310	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
107055	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
109800	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
112545	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)

Table 2-3. Continued

Time (s)	Triple chosen	Other feasible triples
115290	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
118035	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
120780	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
123525	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
126270	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
129015	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
131760	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
134505	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
137250	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
139995	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
142740	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
145485	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
148230	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
150975	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
153720	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
156465	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
159210	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
161955	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
164700	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)

amet placerat molestie. Sed sit amet bibendum lectus, ac ornare ligula. Curabitur porttitor interdum tortor a dignissim. Quisque a placerat nibh. Phasellus lobortis imperdiet augue, non congue est bibendum eu. Vivamus tincidunt quam eu fringilla laoreet.

Maecenas efficitur dolor et ipsum convallis, ut fringilla neque luctus. Donec ac nisl quis leo gravida accumsan sit amet sed tellus. Quisque placerat hendrerit augue sit amet aliquet. Vestibulum laoreet consequat nunc, et egestas nisl auctor et. Duis scelerisque vulputate placerat. Proin tempus ligula ac tempor eleifend. Nullam est odio, commodo quis nisl eu, feugiat efficitur purus.

Duis egestas in mauris vel efficitur. Sed a faucibus sem, non euismod enim. Maecenas nec nulla justo. Suspendisse ut orci ac mi aliquet tincidunt ac eget quam. Quisque ac mi sagittis, dapibus dui a, facilisis neque. Aenean euismod orci sem, non imperdiet ipsum pulvinar ac. Proin eu vestibulum magna, eu ullamcorper nulla. Etiam enim felis, dignissim eget commodo ac, faucibus nec justo. Nulla condimentum velit imperdiet ligula aliquam semper. Nulla facilisi. Ut in lobortis metus, at dictum ipsum. Suspendisse facilisis nec eros eget mollis. Vestibulum eget dolor ac mauris lobortis gravida. Suspendisse consectetur orci in risus pharetra, sed eleifend nisl lacinia. Mauris augue nibh, commodo sed sem at, congue molestie massa. Suspendisse sodales aliquet tellus, a tristique nunc aliquam id.

CHAPTER 3
MATERIALS ANS METHODS

3.1 Consectetur Adipiscing Elit

Fusce eget tempus lectus, non porttitor tellus. Aliquam molestie sed urna quis convallis. Aenean nibh eros, aliquam non eros in, tempus lacinia justo. In magna sapien, blandit a faucibus ac, scelerisque nec purus. Praesent fermentum felis nec massa interdum, vel dapibus mi luctus. Cras id fringilla mauris. Ut molestie eros mi, ut hendrerit nulla tempor et. Pellentesque tortor quam, mattis a scelerisque nec, euismod et odio. Mauris rhoncus metus sit amet risus mattis, eu mattis sem interdum.

3.1.1 This Is an Isolated Heading

Either promote this to a section heading, add another subsection heading, or delete this heading.

3.2 Augue sapien mattis leo

Nec accumsan turpis quam at neque. Ut pellentesque velit sed placerat cursus. Integer congue urna non massa dictum, a pellentesque arcu accumsan. Nulla posuere, elit accumsan eleifend elementum, ipsum massa tristique metus, in ornare neque nisl sed odio. Nullam eget elementum nisi. Duis a consectetur erat, sit amet malesuada sapien. Aliquam nec sapien et leo sagittis porttitor at ut lacus. Vivamus vulputate elit vitae libero condimentum dictum. Nulla facilisi. Quisque non nibh et massa ullamcorper iaculis.

CHAPTER 4
RESULTS

4.1 Fusce Eget Tempus Lectus,

Algorithm 4.1. *Euclids algorithm*

```
procedure Euclid(a, b)                          ▷ The g.c.d. of a and b
    r ← a mod b
    while r ≠ 0 do                              ▷ We have the answer if r is 0
        a ← b
        b ← r
        r ← a mod b
    end while
    return b                                    ▷ The gcd is b
end procedure
```

Proposition 4.1. *The Upsilon Function*

(1) If $\beta > 0$ and $\alpha \neq 0$, then for all $n \geq -1$,

$$I_n(c;\alpha;\beta;\delta) = -\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}(\frac{\beta}{\alpha})^{n-i}Hh_i(\beta c-\delta)$$

$$+(\frac{\beta}{\alpha})^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\phi(-\beta c+\delta+\frac{\alpha}{\beta})$$

(2) If $\beta < 0$ and $\alpha < 0$, then for all $x \geq -1$

$$I_n(c;\alpha;\beta;\delta) = -\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}(\frac{\beta}{\alpha})^{n-i}Hh_i(\beta c-\delta)$$

$$-(\frac{\beta}{\alpha})^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\phi(\beta c-\delta-\frac{\alpha}{\beta})$$

Proof. Case 1.

$\beta > 0$ and $\alpha \neq 0$. Since, for any constant α and $n \geq 0$, $e^{\alpha x}Hh_n(\beta x-\delta) \to 0$ as $x \to \infty$ thanks to (B4), integration by parts leads to

$$I_n = -\frac{1}{\alpha}Hh(\beta c-\delta)e^{\alpha c} + \frac{\beta}{\alpha}\int_c^{\infty} e^{\alpha x}Hh_{n-1}(\beta c-\delta)dx$$

In other words, we have a recursion, for $\mathrm{n} \geq 0$, $\mathrm{I_n} = -(\mathrm{e}^{\alpha \mathrm{c}}\alpha)\mathrm{Hh_n}(\beta \mathrm{c} - \delta) + (\frac{\beta}{\alpha})\mathrm{I_{n-1}}$ with

$$\mathrm{I_{-1}} = \sqrt{2\pi}\int_{\mathrm{c}} \infty \mathrm{e}^{\alpha \mathrm{x}}\varphi(-\beta \mathrm{x} + \delta)\mathrm{dx}$$

$$= \frac{\sqrt{2\pi}}{\beta}\mathrm{e}^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\phi(-\beta \mathrm{c} + \delta + \frac{\alpha}{\beta})$$

Solving it yields, for $\mathrm{n} \geq -1$,

$$\mathrm{I_n} = -\frac{\mathrm{e}^{\alpha \mathrm{c}}}{\alpha}\sum_{\mathrm{i}=0}^{\mathrm{n}}(\frac{\beta}{\alpha})^{\mathrm{i}}\mathrm{Hh_{n-i}}(\beta \mathrm{c} + \delta) + (\frac{\beta}{\alpha})^{\mathrm{n}+1}\mathrm{I_{-1}}$$

$$= -\frac{\mathrm{e}^{\alpha \mathrm{c}}}{\alpha}\sum_{\mathrm{i}=0}^{\mathrm{n}}(\frac{\beta}{\alpha})^{\mathrm{n-i}}\mathrm{Hh_{i}}(\beta \mathrm{c} + \delta)$$

$$+(\frac{\beta}{\alpha})^{\mathrm{n}+1}\frac{\sqrt{2\pi}}{\beta}\mathrm{e}^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\phi(-\beta \mathrm{c} + \delta + \frac{\alpha}{\beta})$$

where the sum over an empty set is defined to be zero. □

Proof. Case2. $\beta < 0$ and $\alpha < 0$. In this case, we must also have, for $\mathrm{n} \geq 0$ and any constant $\alpha < 0, \mathrm{e}^{\alpha \mathrm{x}}\mathrm{Hh_n}(\beta \mathrm{x} - \delta) \to 0$ as

$\mathrm{x} \to \infty$, thanks to (B5). Using integration by parts, we again have the same recursion, for $\mathrm{n} \geq 0, \mathrm{I_n} = -(\mathrm{e}^{\alpha \mathrm{c}}/\alpha)\mathrm{Hh_n}(\beta \mathrm{c} - \delta) + (\beta/\alpha)\mathrm{I_{n-1}}$, but with a different initial condition

$$\mathrm{I_{-1}} = \sqrt{2\pi}\int_{\mathrm{c}}^{\infty} \mathrm{e}^{\alpha \mathrm{x}}\varphi(-\beta \mathrm{x} + \delta)\mathrm{dx}$$

$$= -\frac{\sqrt{2\pi}}{\beta}\exp\{\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}\}\phi(\beta \mathrm{c} - \delta - \frac{\alpha}{\beta})$$

Solving it yields (B8), for $\mathrm{n} \geq -1$.

□

Finally, we sum the double exponential and the normal random variables

Proposition B.3.

Suppose $\{\xi_1, \xi_2, ...\}$ *is a sequence of i.i.d. exponential random variables with rate* $\eta > 0$*, and Z is a normal variable with distribution* $\mathrm{N}(0, \sigma^2)$*. Then for every* $\mathrm{n} \geq 1$*, we have: (1) The density functions are given by:*

$$\mathrm{f}_{\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}}}(\mathrm{t}) = (\sigma\eta)^{\mathrm{n}} \frac{\mathrm{e}^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\mathrm{t}\eta} \mathrm{Hh}_{\mathrm{n}-1}(-\frac{\mathrm{t}}{\sigma} + \sigma\eta)$$

$$\mathrm{f}_{\mathrm{Z}-\sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}}}(\mathrm{t}) = (\sigma\eta)^{\mathrm{n}} \frac{\mathrm{e}^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\mathrm{t}\eta} \mathrm{Hh}_{\mathrm{n}-1}(\frac{\mathrm{t}}{\sigma} + \sigma\eta)$$

(2) The tail probabilities are given by

$$\mathrm{P}(\mathrm{Z} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}} \geq \mathrm{x}) = (\sigma\eta)^{\mathrm{n}} \frac{\mathrm{e}^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\mathrm{t}\eta} \mathrm{I}_{\mathrm{n}-1}(\mathrm{x}; -\eta, -\frac{1}{\sigma}, -\sigma\eta)$$

$$\mathrm{P}(\mathrm{Z} - \sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}} \geq \mathrm{x}) = (\sigma\eta)^{\mathrm{n}} \frac{\mathrm{e}^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\mathrm{t}\eta} \mathrm{I}_{\mathrm{n}-1}(\mathrm{x}; \eta, \frac{1}{\sigma}, -\sigma\eta)$$

Proof. Case 1. The densities of $\mathrm{Z} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}}$*, and* $\mathrm{Z} - \sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}}$*. We have*

$$\mathrm{f}_{\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}}}(\mathrm{t}) = \int_{-\infty}^{\infty} \mathrm{f}_{\sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}}}(\mathrm{t} - \mathrm{x}) \mathrm{f}_{\mathrm{Z}}(\mathrm{x}) \mathrm{dx}$$

$$= \mathrm{e}^{-\mathrm{t}\eta}(\eta^{\mathrm{n}}) \int_{-\infty} \mathrm{t} \frac{\mathrm{e}^{\mathrm{x}\eta}(\mathrm{t}-\mathrm{x})^{\mathrm{n}-1}}{(\mathrm{n}-1)!} \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\mathrm{x}^2/(2\sigma^2)} \mathrm{dx}$$

$$= \mathrm{e}^{-\mathrm{t}\eta}(\eta^{\mathrm{n}}) \mathrm{e}^{(\sigma\eta)^2/(2)} \int_{-\infty} \mathrm{t} \frac{(\mathrm{t}-\mathrm{x})^{\mathrm{n}-1}}{(\mathrm{n}-1)!} \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(\mathrm{x}-\sigma^2\eta)^2/(2\sigma^2)} \mathrm{dx}$$

Letting $\mathrm{y} = (\mathrm{x} - \sigma^2\eta)/\sigma$ *yields*

$$\mathrm{f}_{\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}}}(\mathrm{t}) = \mathrm{e}^{-\mathrm{t}\eta}(\eta^{\mathrm{n}}) \mathrm{e}^{(\sigma\eta)^2/(2)} \sigma^{\mathrm{n}-1}$$

$$\times \int_{-\infty}^{\mathrm{t}/\sigma - \sigma\eta} \frac{(\mathrm{t}/\sigma - \mathrm{y} - \sigma\eta)^{\mathrm{n}-1}}{(\mathrm{n}-1)!} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{y}^2/2} \mathrm{dy}$$

$$= \frac{e^{(\sigma\eta)^2/2}}{\sqrt{2\pi}}(\sigma^{n-1}\eta^n)e^{-t\eta}Hh_{n-1}(-t/\sigma+\sigma\eta)$$

because $(1/(n-1)!)\int_{-\infty} a(a-y)^{n-1}e^{-y^2/2}dy = Hh_{n-1}(a)$. *The derivation of* $f_{Z+\sum_{i=1}^n \xi_i}(t)$ *is similar.*

Case 2. $P(Z+\sum_{i=1}^n \xi_i \geq x)$ *and* $P(Z-\sum_{i=1}^n \xi_i \geq x)$. *From (B9), it is clear that*

$$P(Z+\sum_{i=1}^n \xi_i \geq x) = \frac{(\sigma\eta)^n e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}}\int_x^\infty e^{(-i\eta)}Hh_{n-1}(-\frac{t}{\sigma}+\sigma\eta)dt$$

$$= \frac{(\sigma\eta)^n e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} I_{n-1}(x;-\eta,-\frac{1}{\sigma},-\sigma\eta)dt$$

by (B6). We can compute $P(Z-\sum_{i=1}^n \xi_i \geq x)$ *similarly.*

Theorem 4.1. *Theorem With* $\pi_n := P(N(t)=n) = e^{-\lambda T}(\lambda T)^n/n!$ *and* I_n *in Proposition 2-2.*

, we have

$$P(Z(T)\geq a) = \frac{e^{(\sigma\eta_1)^2 T/2}}{\sigma\sqrt{2\pi T}}\sum_{n=1}^\infty \pi_n \sum_{k=1}^n P_{n,k}(\sigma\sqrt{T}\eta_1)^k \times I_{k-1}(a-\mu T;-\eta_1,-\frac{1}{\sigma\sqrt{T}},-\sigma\eta_1\sqrt{T})$$

$$+\frac{e^{(\sigma\eta_2)^2 T/2}}{\sigma\sqrt{2\pi T}}\sum_{n=1}^\infty \pi_n \sum_{k=1}^n Q_{n,k}(\sigma\sqrt{T}\eta_2)^k$$

$$\times I_{k-1}(a-\mu T;\eta_2,\frac{1}{\sigma\sqrt{T}},-\sigma\eta_2\sqrt{T})$$

$$+\pi_0\phi(-\frac{a-\mu T}{\sigma\sqrt{T}})$$

Proof by the decomposition (B2)

$$P(Z(T)\geq a) = \sum_{n=0}^\infty \pi_n P(\mu T+\sigma\sqrt{T}Z+\sum_{j=1}^n Y_j \geq a)$$

$$= \pi_0 \mathrm{P}(\mu \mathrm{T} + \sigma\sqrt{\mathrm{T}}\mathrm{Z} \geq \mathrm{a})$$

$$+ \sum_{\mathrm{n}=1}^{\infty} \pi_{\mathrm{n}} \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{P}_{\mathrm{n,k}} \mathrm{P}(\mu \mathrm{T} + \sigma\sqrt{\mathrm{T}}\mathrm{Z} + \sum_{\mathrm{j}=1}^{\mathrm{n}} \xi_{\mathrm{j}}^{+} \geq \mathrm{a})$$

$$+ \sum_{\mathrm{n}=1}^{\infty} \pi_{\mathrm{n}} \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{Q}_{\mathrm{n,k}} \mathrm{P}(\mu \mathrm{T} + \sigma\sqrt{\mathrm{T}}\mathrm{Z} - \sum_{\mathrm{j}=1}^{\mathrm{n}} \xi_{\mathrm{j}}^{-} \geq \mathrm{a})$$

The result now follows via (B11) and (B12) for $\eta_1 > 1$ *and* $\eta_2 > 0$*.*

CHAPTER 5
SUMMARY AND CONCLUSIONS

5.1 Non Porttitor Tellus

Aliquam molestie sed urna quis convallis. Aenean nibh eros, aliquam non eros in, tempus lacinia justo. In magna sapien, blandit a faucibus ac, scelerisque nec purus. Praesent fermentum felis nec massa interdum, vel dapibus mi luctus. Cras id fringilla mauris. Ut molestie eros mi, ut hendrerit nulla tempor et. Pellentesque tortor quam, mattis a scelerisque nec, euismod et odio. Mauris rhoncus metus sit amet risus mattis, eu mattis sem interdum.

5.1.1 Nam Arcu Magna

Semper vel lorem eu, venenatis ultrices est. Nam aliquet ut erat ac scelerisque. Maecenas ut molestie mi. Phasellus ipsum magna, sollicitudin eu ipsum quis, imperdiet cursus turpis. Etiam pretium enim a fermentum accumsan. Morbi vel vehicula enim.

5.1.1.1 Ut pellentesque velit sede

Placerat cursus. Integer congue urna non massa dictum, a pellentesque arcu accumsan. Nulla posuere, elit accumsan eleifend elementum, ipsum massa tristique metus, in ornare neque nisl sed odio. Nullam eget elementum nisi. Duis a consectetur erat, sit amet malesuada sapien. Aliquam nec sapien et leo sagittis porttitor at ut lacus. Vivamus vulputate elit vitae libero condimentum dictum. Nulla facilisi. Quisque non nibh et massa ullamcorper iaculis.

APPENDIX A
THIS IS THE FIRST APPENDIX

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Maecenas eget magna. Aenean et lorem. Ut dignissim neque at nisi. In hac habitasse platea dictumst. In porta ornare eros. Nunc eu ante. In non est vehicula tellus cursus suscipit. Proin sed libero. Sed risus enim, eleifend in, pellentesque ac, nonummy quis, nulla. Phasellus imperdiet libero nec massa. Ut sapien libero, adipiscing eu, volutpat porttitor, ultricies eget, nisi. Sed odio. Suspendisse potenti. Duis dolor augue, viverra id, porta in, dignissim id, nisl. Vivamus blandit cursus eros. Maecenas sit amet urna sit amet orci nonummy pharetra.

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Nullam mi. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Duis vitae metus in massa hendrerit rhoncus. Fusce tortor justo, laoreet eu, facilisis at, gravida et, felis. Donec imperdiet mollis erat. Integer tempus nulla ac lorem. Fusce porttitor. Aenean quis arcu. Morbi consectetuer, leo eu mollis elementum, urna massa malesuada risus, euismod tempor lorem elit ut mauris. Cras elit orci, facilisis ac, mattis iaculis, cursus ac, augue. Donec eget nisl. Pellentesque fermentum sodales nibh. Vivamus non risus. Donec est libero, tincidunt sit amet, pretium vitae, blandit sed, tellus. Nunc diam risus, interdum sed, laoreet quis, varius ac, turpis. In et purus eget nibh vehicula rhoncus. Aenean et neque. Praesent nisl nisi, tempus quis, nonummy ac, auctor a, neque. Suspendisse et metus. Suspendisse non metus eu mauris auctor sagittis.

APPENDIX B
AN EXAMPLE OF A HALF TITLE PAGE

LaTeX 2_ε

Figure B-1. LaTeX2ϵ. logo

This is how a section should look if the first page is a landscape page. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut sit amet nulla. Integer mauris turpis, dapibus ac, auctor non, vehicula sit amet, magna. Suspendisse eu tellus. Etiam porta. Donec magna. Donec ut dui. In hac habitasse platea dictumst. Nullam suscipit, mi at adipiscing commodo, lorem erat scelerisque erat, non pulvinar leo mi eu metus. Phasellus id felis. Sed quam purus, molestie quis, ultrices nec, dictum at, magna. Proin viverra viverra ante.

Maecenas sagittis magna quis ligula. Duis vestibulum mi a felis. Aenean accumsan mattis massa. Nullam lacus sem, consectetuer non, condimentum sit amet, pharetra ac, odio. Morbi nisi magna, tincidunt sed, placerat nec, tincidunt id, lectus. Donec ac dui non mauris vulputate aliquam. Nullam scelerisque congue pede. Integer ipsum. Vestibulum auctor. Suspendisse eget leo id libero cursus dictum. Sed malesuada. Aliquam imperdiet. Donec dui metus, porta eu, aliquet vel, vulputate vitae, lacus.

Nulla quis purus id turpis luctus feugiat. Fusce feugiat. Proin felis. Morbi elit est, fermentum in, tincidunt vitae, convallis vel, orci. Vestibulum justo. Suspendisse non nisl. Pellentesque pretium adipiscing elit. Phasellus fermentum consequat augue. Sed pede nisl, fermentum vel, vulputate id, sollicitudin sed, ligula. Cras suscipit, quam et euismod sagittis, nisl felis gravida felis, quis pulvinar purus est vel pede. Suspendisse mattis est ac nunc. Curabitur rutrum, turpis sit amet commodo tempus, metus lorem commodo lectus, eget fringilla justo nisi et purus. Ut quam sapien, vehicula quis, rhoncus non, sagittis nec, risus.

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APPENDIX C
DERIVATION OF THE Υ FUNCTION

Proposition C.1. *The Upsilon Function*

(1) If $\beta > 0$ and $\alpha \neq 0$, then for all $\mathrm{n} \geq -1$,

$$\mathrm{I_n}(\mathrm{c};\alpha;\beta;\delta) = -\frac{\mathrm{e}^{\alpha \mathrm{c}}}{\alpha}\sum_{\mathrm{i}=0}^{\mathrm{n}}(\frac{\beta}{\alpha})^{\mathrm{n-i}}\mathrm{Hh_i}(\beta \mathrm{c} - \delta)$$

$$+(\frac{\beta}{\alpha})^{\mathrm{n}+1}\frac{\sqrt{2\pi}}{\beta}\mathrm{e}^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\phi(-\beta \mathrm{c} + \delta + \frac{\alpha}{\beta})$$

(2) If $\beta < 0$ and $\alpha < 0$, then for all $\mathrm{x} \geq -1$

$$\mathrm{I_n}(\mathrm{c};\alpha;\beta;\delta) = -\frac{\mathrm{e}^{\alpha \mathrm{c}}}{\alpha}\sum_{\mathrm{i}=0}^{\mathrm{n}}(\frac{\beta}{\alpha})^{\mathrm{n-i}}\mathrm{Hh_i}(\beta \mathrm{c} - \delta)$$

$$-(\frac{\beta}{\alpha})^{\mathrm{n}+1}\frac{\sqrt{2\pi}}{\beta}\mathrm{e}^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\phi(\beta \mathrm{c} - \delta - \frac{\alpha}{\beta})$$

Proof. Case 1.

$\beta > 0$ and $\alpha \neq 0$. Since, for any constant α and $\mathrm{n} \geq 0$, $\mathrm{e}^{\alpha \mathrm{x}}\mathrm{Hh_n}(\beta \mathrm{x} - \delta) \to 0$ as $\mathrm{x} \to \infty$ thanks to (B4), integration by parts leads to

$$\mathrm{I_n} = -\frac{1}{\alpha}\mathrm{Hh}(\beta \mathrm{c} - \delta)\mathrm{e}^{\alpha \mathrm{c}} + \frac{\beta}{\alpha}\int_{\mathrm{c}}^{\infty}\mathrm{e}^{\alpha \mathrm{x}}\mathrm{Hh_{n-1}}(\beta \mathrm{c} - \delta)\mathrm{dx}$$

In other words, we have a recursion, for $\mathrm{n} \geq 0$, $\mathrm{I_n} = -(\mathrm{e}^{\alpha \mathrm{c}}\alpha)\mathrm{Hh_n}(\beta \mathrm{c} - \delta) + (\frac{\beta}{\alpha})\mathrm{I_{n-1}}$ with

$$\mathrm{I_{-1}} = \sqrt{2\pi}\int_{\mathrm{c}}\infty \mathrm{e}^{\alpha \mathrm{x}}\varphi(-\beta \mathrm{x} + \delta)\mathrm{dx}$$

$$= \frac{\sqrt{2\pi}}{\beta}\mathrm{e}^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\phi(-\beta \mathrm{c} + \delta + \frac{\alpha}{\beta})$$

Solving it yields, for $\mathrm{n} \geq -1$,

$$\mathrm{I_n} = -\frac{\mathrm{e}^{\alpha\mathrm{c}}}{\alpha}\sum_{\mathrm{i}=0}^{\mathrm{n}}(\frac{\beta}{\alpha})^{\mathrm{i}}\mathrm{Hh_{n-i}}(\beta\mathrm{c}+\delta) + (\frac{\beta}{\alpha})^{\mathrm{n}+1}\mathrm{I_{-1}}$$

$$= -\frac{\mathrm{e}^{\alpha\mathrm{c}}}{\alpha}\sum_{\mathrm{i}=0}^{\mathrm{n}}(\frac{\beta}{\alpha})^{\mathrm{n-i}}\mathrm{Hh_i}(\beta\mathrm{c}+\delta)$$

$$+(\frac{\beta}{\alpha})^{\mathrm{n}+1}\frac{\sqrt{2\pi}}{\beta}\mathrm{e}^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\phi(-\beta\mathrm{c}+\delta+\frac{\alpha}{\beta})$$

where the sum over an empty set is defined to be zero. □

Case2. $\beta < 0$ *and* $\alpha < 0$*. In this case, we must also have, for* $\mathrm{n} \geq 0$ *and any constant* $\alpha < 0, \mathrm{e}^{\alpha\mathrm{x}}\mathrm{Hh_n}(\beta\mathrm{x}-\delta) \to 0$ *as*

$\mathrm{x} \to \infty$*, thanks to (B5). Using integration by parts, we again have the same recursion, for* $\mathrm{n} \geq 0, \mathrm{I_n} = -(\mathrm{e}^{\alpha\mathrm{c}}/\alpha)\mathrm{Hh_n}(\beta\mathrm{c}-\delta) + (\beta/\alpha)\mathrm{I_{n-1}}$*, but with a different initial condition*

$$\mathrm{I_{-1}} = \sqrt{2\pi}\int_{\mathrm{c}}^{\infty}\mathrm{e}^{\alpha\mathrm{x}}\varphi(-\beta\mathrm{x}+\delta)\mathrm{dx}$$

$$= -\frac{\sqrt{2\pi}}{\beta}\exp\{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}\}\phi(\beta\mathrm{c}-\delta-\frac{\alpha}{\beta})$$

Solving it yields (B8), for $\mathrm{n} \geq -1$*.*

Finally, we sum the double exponential and the normal random variables

Proposition B.3.

Suppose $\{\xi_1, \xi_2, ...\}$ *is a sequence of i.i.d. exponential random variables with rate* $\eta > 0$*, and Z is a normal variable with distribution* $\mathrm{N}(0, \sigma^2)$*. Then for every* $\mathrm{n} \geq 1$*, we have: (1) The density functions are given by:*

$$\mathrm{f}_{\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}}(\mathrm{t}) = (\sigma\eta)^{\mathrm{n}}\frac{\mathrm{e}^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\mathrm{t}\eta}\mathrm{Hh_{n-1}}(-\frac{\mathrm{t}}{\sigma}+\sigma\eta)$$

$$\mathrm{f}_{\mathrm{Z}-\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}}(\mathrm{t}) = (\sigma\eta)^{\mathrm{n}}\frac{\mathrm{e}^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\mathrm{t}\eta}\mathrm{Hh_{n-1}}(\frac{\mathrm{t}}{\sigma}+\sigma\eta)$$

(2) The tail probabilities are given by

$$\mathrm{P}(\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}\geq \mathrm{x})=(\sigma\eta)^{\mathrm{n}}\frac{\mathrm{e}^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\mathrm{t}\eta}\mathrm{I}_{\mathrm{n}-1}(\mathrm{x};-\eta,-\frac{1}{\sigma},-\sigma\eta)$$

$$\mathrm{P}(\mathrm{Z}-\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}\geq \mathrm{x})=(\sigma\eta)^{\mathrm{n}}\frac{\mathrm{e}^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\mathrm{t}\eta}\mathrm{I}_{\mathrm{n}-1}(\mathrm{x};\eta,\frac{1}{\sigma},-\sigma\eta)$$

Proof. Case 1. The densities of $\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}$, *and* $\mathrm{Z}-\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}$. *We have*

$$\mathrm{f}_{\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}}(\mathrm{t})=\int_{-\infty}^{\infty}\mathrm{f}_{\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}}(\mathrm{t}-\mathrm{x})\mathrm{f}_{\mathrm{Z}}(\mathrm{x})\mathrm{dx}$$

$$=\mathrm{e}^{-\mathrm{t}\eta}(\eta^{\mathrm{n}})\int_{-\infty}^{\mathrm{t}}\frac{\mathrm{e}^{\mathrm{x}\eta}(\mathrm{t}-\mathrm{x})^{\mathrm{n}-1}}{(\mathrm{n}-1)!}\frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\mathrm{x}^2/(2\sigma^2)}\mathrm{dx}$$

$$=\mathrm{e}^{-\mathrm{t}\eta}(\eta^{\mathrm{n}})\mathrm{e}^{(\sigma\eta)^2/(2)}\int_{-\infty}^{\mathrm{t}}\frac{(\mathrm{t}-\mathrm{x})^{\mathrm{n}-1}}{(\mathrm{n}-1)!}\frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-(\mathrm{x}-\sigma^2\eta)^2/(2\sigma^2)}\mathrm{dx}$$

Letting $\mathrm{y}=(\mathrm{x}-\sigma^2\eta)/\sigma$ *yields*

$$\mathrm{f}_{\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}}(\mathrm{t})=\mathrm{e}^{-\mathrm{t}\eta}(\eta^{\mathrm{n}})\mathrm{e}^{(\sigma\eta)^2/(2)}\sigma^{\mathrm{n}-1}$$

$$\times\int_{-\infty}^{\mathrm{t}/\sigma-\sigma\eta}\frac{(\mathrm{t}/\sigma-\mathrm{y}-\sigma\eta)^{\mathrm{n}-1}}{(\mathrm{n}-1)!}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{y}^2/2}\mathrm{dy}$$

$$=\frac{\mathrm{e}^{(\sigma\eta)^2/2}}{\sqrt{2\pi}}(\sigma^{\mathrm{n}-1}\eta^{\mathrm{n}})\mathrm{e}^{-\mathrm{t}\eta}\mathrm{Hh}_{\mathrm{n}-1}(-\mathrm{t}/\sigma+\sigma\eta)$$

because $(1/(\mathrm{n}-1)!)\int_{-\infty}^{\mathrm{a}}(\mathrm{a}-\mathrm{y})^{\mathrm{n}-1}\mathrm{e}^{-\mathrm{y}^2/2}\mathrm{dy}=\mathrm{Hh}_{\mathrm{n}-1}(\mathrm{a})$. *The derivation of* $\mathrm{f}_{\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}}(\mathrm{t})$ *is similar.*

Case 2. $\mathrm{P}(\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}\geq\mathrm{x})$ *and* $\mathrm{P}(\mathrm{Z}-\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}\geq\mathrm{x})$. *From (B9), it is clear that*

$$\mathrm{P}(\mathrm{Z}+\sum_{\mathrm{i}=1}^{\mathrm{n}}\xi_{\mathrm{i}}\geq \mathrm{x})=\frac{(\sigma\eta)^{\mathrm{n}}\mathrm{e}^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}}\int_{\mathrm{x}}^{\infty}\mathrm{e}^{(-\mathrm{i}\eta)}\mathrm{Hh}_{\mathrm{n}-1}(-\frac{\mathrm{t}}{\sigma}+\sigma\eta)\mathrm{dt}$$

$$= \frac{(\sigma\eta)^{\mathrm{n}} e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} \mathrm{I}_{\mathrm{n}-1}(\mathrm{x}; -\eta, -\frac{1}{\sigma}, -\sigma\eta)\mathrm{dt}$$

by (B6). We can compute $\mathrm{P}(\mathrm{Z} - \sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}} \geq \mathrm{x})$ *similarly.*

Theorem C.1. *Theorem With* $\pi_{\mathrm{n}} := \mathrm{P}(\mathrm{N}(\mathrm{t}) = \mathrm{n}) = \mathrm{e}^{-\lambda \mathrm{T}}(\lambda \mathrm{T})^{\mathrm{n}}/\mathrm{n}!$ *and* I_{n} *in Proposition 2-2.*

, we have

$$\mathrm{P}(\mathrm{Z}(\mathrm{T}) \geq \mathrm{a}) = \frac{e^{(\sigma\eta_1)^2 \mathrm{T}/2}}{\sigma\sqrt{2\pi \mathrm{T}}} \sum_{\mathrm{n}=1}^{\infty} \pi_{\mathrm{n}} \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{P}_{\mathrm{n},\mathrm{k}} (\sigma\sqrt{\mathrm{T}}\eta_1)^{\mathrm{k}} \times \mathrm{I}_{\mathrm{k}-1}(\mathrm{a} - \mu \mathrm{T}; -\eta_1, -\frac{1}{\sigma\sqrt{\mathrm{T}}}, -\sigma\eta_1\sqrt{\mathrm{T}})$$

$$+\frac{e^{(\sigma\eta_2)^2 \mathrm{T}/2}}{\sigma\sqrt{2\pi \mathrm{T}}} \sum_{\mathrm{n}=1}^{\infty} \pi_{\mathrm{n}} \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{Q}_{\mathrm{n},\mathrm{k}} (\sigma\sqrt{\mathrm{T}}\eta_2)^{\mathrm{k}}$$

$$\times \mathrm{I}_{\mathrm{k}-1}(\mathrm{a} - \mu \mathrm{T}; \eta_2, \frac{1}{\sigma\sqrt{\mathrm{T}}}, -\sigma\eta_2\sqrt{\mathrm{T}})$$

$$+\pi_0 \phi(-\frac{\mathrm{a} - \mu \mathrm{T}}{\sigma\sqrt{\mathrm{T}}})$$

Proof by the decomposition (B2)

$$\mathrm{P}(\mathrm{Z}(\mathrm{T}) \geq \mathrm{a}) = \sum_{\mathrm{n}=0}^{\infty} \pi_{\mathrm{n}} \mathrm{P}(\mu \mathrm{T} + \sigma\sqrt{\mathrm{T}}\mathrm{Z} + \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{Y}_{\mathrm{j}} \geq \mathrm{a})$$

$$= \pi_0 \mathrm{P}(\mu \mathrm{T} + \sigma\sqrt{\mathrm{T}}\mathrm{Z} \geq \mathrm{a})$$

$$+\sum_{\mathrm{n}=1}^{\infty} \pi_{\mathrm{n}} \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{P}_{\mathrm{n},\mathrm{k}} \mathrm{P}(\mu \mathrm{T} + \sigma\sqrt{\mathrm{T}}\mathrm{Z} + \sum_{\mathrm{j}=1}^{\mathrm{n}} \xi_{\mathrm{j}}^{+} \geq \mathrm{a})$$

$$+\sum_{\mathrm{n}=1}^{\infty} \pi_{\mathrm{n}} \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{Q}_{\mathrm{n},\mathrm{k}} \mathrm{P}(\mu \mathrm{T} + \sigma\sqrt{\mathrm{T}}\mathrm{Z} - \sum_{\mathrm{j}=1}^{\mathrm{n}} \xi_{\mathrm{j}}^{-} \geq \mathrm{a})$$

The result now follows via (B11) and (B12) for $\eta_1 > 1$ *and* $\eta_2 > 0$.

APPENDIX D
DERIVATION OF THE Υ FUNCTION

We first decompose the sum of the double exponential random variables.

The memoryless property of exponential random variables yields $(\xi^+ - \xi^- | \xi^+ > \xi^-) =^{\mathrm{d}} \xi^+$ and $(\xi^+ - \xi^- | \xi^+ < \xi^-) =^{\mathrm{d}} -\xi^-$, thus leading to the conclusion that

$$\xi^+ - \xi^- = \left\{ \begin{array}{cc} \xi^+ & \text{with probability } \eta_2/(\eta_1+\eta_2) \\ -\xi^- & \text{with probability } \eta_1/(\eta_1+\eta_2) \end{array} \right\}.$$

because the probabilities of the events $\xi^+ > \xi^-$ and $\xi^+ < \xi^-$ are $\eta_2/(\eta_1+\eta_2)$ and $\eta_1/(\eta_1+\eta_2)$, respectively. The following proposition extends (B.1.)

Proposition B.1. For every $\mathrm{n} \geq 1$, we have the following decomposition

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{Y_i} =^{\mathrm{d}} \left\{ \begin{array}{cc} \sum_{\mathrm{i}=1}^{\mathrm{k}} \xi_{\mathrm{i}}^+ & \text{with probability } \mathrm{P_{n,k}}, \mathrm{k} = 1, 2, ..., \mathrm{n} \\ -\sum_{\mathrm{i}=1}^{\mathrm{k}} \xi_{\mathrm{i}}^- & \text{with probability } \mathrm{Q_{n,k}}, \mathrm{k} = 1, 2, ..., \mathrm{n} \end{array} \right\}.$$

where $\mathrm{P_{n,k}}$ and $\mathrm{Q_{n,k}}$ are given by

$$\mathrm{P_{n,k}} = \sum_{\mathrm{i}=\mathrm{k}}^{\mathrm{n}-1} \binom{\mathrm{n}-\mathrm{k}-1}{\mathrm{i}-\mathrm{k}} \binom{\mathrm{n}}{\mathrm{i}} (\frac{\eta_1}{\eta_1+\eta_2})^{\mathrm{i}-\mathrm{k}} (\frac{\eta_2}{\eta_1+\eta_2})^{\mathrm{n}-\mathrm{i}} \mathrm{p^i q^{n-i}}$$

$$1 \leq \mathrm{k} \leq \mathrm{n}-1$$

$$\mathrm{Q_{n,k}} = \sum_{\mathrm{i}=\mathrm{k}}^{\mathrm{n}-1} \binom{\mathrm{n}-\mathrm{k}-1}{\mathrm{i}-\mathrm{k}} \binom{\mathrm{n}}{\mathrm{i}} (\frac{\eta_1}{\eta_1+\eta_2})^{\mathrm{n}-\mathrm{i}} (\frac{\eta_2}{\eta_1+\eta_2})^{\mathrm{i}-\mathrm{k}} \mathrm{p^{n-i} q^i}$$

$$1 \leq \mathrm{k} \leq \mathrm{n}-1, \mathrm{P_{n,n}} = \mathrm{p^n}, \mathrm{Q_{n,n}} = \mathrm{q^n}$$

and $\binom{0}{0}$ is defined to be one. Hence ξ_{i}^+ and ξ_{i}^- are i.i.d. exponential random variables with rates η_1 and η_2, respectively.

As a key step in deriving closed-form solutions for call and put options, this proposition indicates that the sum of the i.i.d. double exponential random variable can be written, in distribution, as a randomly mixed gamma random variable. To prove Proposition B.1, the following lemma is needed.

Lemma B.1.

$$\sum_{i=1}^{n} \xi_i^{+} - \sum_{i=1}^{n} \xi_i^{-}$$

$$=^{d} \left\{ \begin{array}{ll} \sum_{i=1}^{k} \xi_i & \text{with probability } \binom{n-k+m-1}{m-1}(\frac{\eta_1}{\eta_1+\eta_2})^{n-k}(\frac{\eta_2}{\eta_1+\eta_2})^{m}, k = 1, ..., n \\ -\sum_{i=1}^{l} \xi_i & \text{with probability } \binom{n-l+m-1}{n-1}(\frac{\eta_1}{\eta_1+\eta_2})^{n}(\frac{\eta_2}{\eta_1+\eta_2})^{m-l}, l = 1, ..., m \end{array} \right\}.$$

We prove it by introducing the random variables $A(n, m) = \sum_{i=1}^{n} \xi_i - sum_{j=1}^{m} \tilde{\xi}_j$ Then

$$A(n, m) =^{d} \left\{ \begin{array}{ll} A(n-1, m-1) + \xi^{+} & \text{with probability } \eta_2/(\eta_1+\eta_2) \\ A(n-1, m-1) - \xi^{-} & \text{with probability } \eta_1/(\eta_1+\eta_2) \end{array} \right\}.$$

$$=^{d} \left\{ \begin{array}{ll} A(n, m-1) & \text{with probability } \eta_2/(\eta_1+\eta_2) \\ A(n-1, m) & \text{with probability } \eta_1/(\eta_1+\eta_2) \end{array} \right\}.$$

via B.1.. Now suppose horizontal axis that are representing the number of $\{\zeta_i^{+}\}$ and vertical axis representing the number of $\{\zeta_i^{-}\}$. Suppose we have a random walk on the integer lattice points. Starting from any point $(n, m), n, m \geq 1$, the random walk goes either one step to the left with probability $\eta_1/(\eta_1+\eta_2)$ or one step down with probability $\eta_2/(\eta_1+\eta_2)$, and the random walks stops once it reaches the horizontal or vertical axis. For any path from (n,m) to (k,0) , $1 \geq k \geq n$, it must reach (k,1) first before it makes a final move to (k,0). Furthermore, all the paths going from (n,m) to (k,1) must have exactly n-k lefts and m-1 downs, whence the total number of such paths is $\binom{n-k+m-1}{m-1}$. Similarly the total number of paths from (n,m) to (0,l) , $1 \geq l \geq m$, is $\binom{n-l+m-1}{n-1}$. Thus

$$A(n,m) =^d \left\{ \begin{array}{ll} \sum_{i=1}^k \xi_i & \text{with probability } \binom{n-k+m-1}{m-1}(\frac{\eta_1}{\eta_1+\eta_2})^{n-k}(\frac{\eta_2}{\eta_1+\eta_2})^m, k=1,...,n \\ -\sum_{i=1}^l \xi_i & \text{with probability } \binom{n-l+m-1}{n-1}(\frac{\eta_1}{\eta_1+\eta_2})^{n}(\frac{\eta_2}{\eta_1+\eta_2})^{m-l}, l=1,...,m \end{array} \right\}.$$

and the lemma is proven.

Now, let's prove the proposition B.1. By the same analogy used in Lemma B.1 to compute probability $P_{n,m}, 1 \geq k \geq n$, the probability weight assigned to $\sum_{i=1}^k \xi_i^+$ when we decompose $\sum_{i=1}^k Y_i$, it is equivalent to consider the probability of the random walk ever reach (k,0) starting from the point (i,n-i) being $\binom{n}{i} p^i q^{n-i}$. Note that the point (k,0) can only be reached from point (i,n-i) such that $k \geq i \geq n-1$, because the random walk can only go left or down, and stops once it reaches the horizontal axis. Therefore, for $1 \geq k \geq n-1$, (B3) leads to

$$P_{n,k} = \sum_{i=k} n - 1 P(\text{goingfrom}(i, n-i)\text{to}(k,0)).P(\text{startingfrom}(i,n-i))$$

$$= \sum_{i=k}^{n-1} \binom{i+(n-i)-k-1}{(n-i)-1} \binom{n}{i} (\frac{\eta_1}{\eta_1+\eta_2})^{i-k} (\frac{\eta_2}{\eta_1+\eta_2})^{n-i} p^i q^{n-i}$$

$$= \sum_{i=k}^{n-1} \binom{n-k-1}{n-i-1} \binom{n}{i} (\frac{\eta_1}{\eta_1+\eta_2})^{i-k} (\frac{\eta_2}{\eta_1+\eta_2})^{n-i} p^i q^{n-i}$$

$$= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} (\frac{\eta_1}{\eta_1+\eta_2})^{i-k} (\frac{\eta_2}{\eta_1+\eta_2})^{n-i} p^i q^{n-i}$$

Of course $P_{n,n} = p^n$. Similarly, we can compute $Q_{n,k}$:

$$Q_{n,k} = \sum_{i=k} n - 1 P(\text{goingfrom}(n-i, i)\text{to}(0,k)).P(\text{startingfrom}(n-i,i))$$

$$= \sum_{i=k}^{n-1} \binom{i+(n-i)-k-1}{(n-i)-1} \binom{n}{n-i} (\frac{\eta_1}{\eta_1+\eta_2})^{n-i} (\frac{\eta_2}{\eta_1+\eta_2})^{i-k} p^{n-i} q^{i}$$

$$= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} (\frac{\eta_1}{\eta_1+\eta_2})^{n-i} (\frac{\eta_2}{\eta_1+\eta_2})^{i-k} p^{n-i} q^i$$

with $Q_{n,n} = q^n$. Incidentally, we have also got $\sum k = 1n(P_{n,k} + Q_{n,k}) = 1$

B.2. Let's develop now the results on Hh functions. First of all, note that $Hh_n(x) \to 0$, as $x \to \infty$, for $n \geq -1$; and $Hh_n(x) \to \infty$, as $x \to -\infty$, for $n \geq -1$; and $Hh_0(x) = \sqrt{2\pi}\phi(-x) \to \sqrt{2\pi}$, as $x \to -\infty$. Also, for every $n \geq -1$, as $x \to \infty$,

$$\lim Hh_n(x)/\{\frac{1}{x^{n+1}} e^{-\frac{x^2}{2}}\} = 1$$

and as $x \to \infty$

$$Hh_n(x) = O(|x|^n)$$

Here (B4) is clearly true for $n = -1$, while for $n \geq 0$ note that as $x \to_\infty$,

$$Hh_n(x) = \frac{1}{n!} \int_x \infty (t-x)^n e^{-\frac{t^2}{2}} dt$$

$$\leq \frac{2^n}{n!} \int_{-\infty}^{\infty} |t|^n e^{-t^2} 2dt + \frac{2^n}{n!} \int -\infty\infty |x|^n e^{-t^2} 2dt = O(|x|^n)$$

For option pricing it is important to evaluate the integral $I_n(c; \alpha; \beta; \delta)$,

$$I_n(c; \alpha; \beta; \delta) = \int_c \infty e^{\alpha x} Hh_n(\beta x - \delta) dx, n \geq 0$$

for arbitrary constants α, c and β.

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BIOGRAPHICAL SKETCH

This section is where your biographical sketch is typed in the bio.tex file. It should be in third person, past tense. Do not put personal details such as your birthday in the file. Again, to make a full paragraph you must write at least three sentences.