Multi-wavefront sensor single-conjugate transfer function derivations

Aditya R. Sengupta

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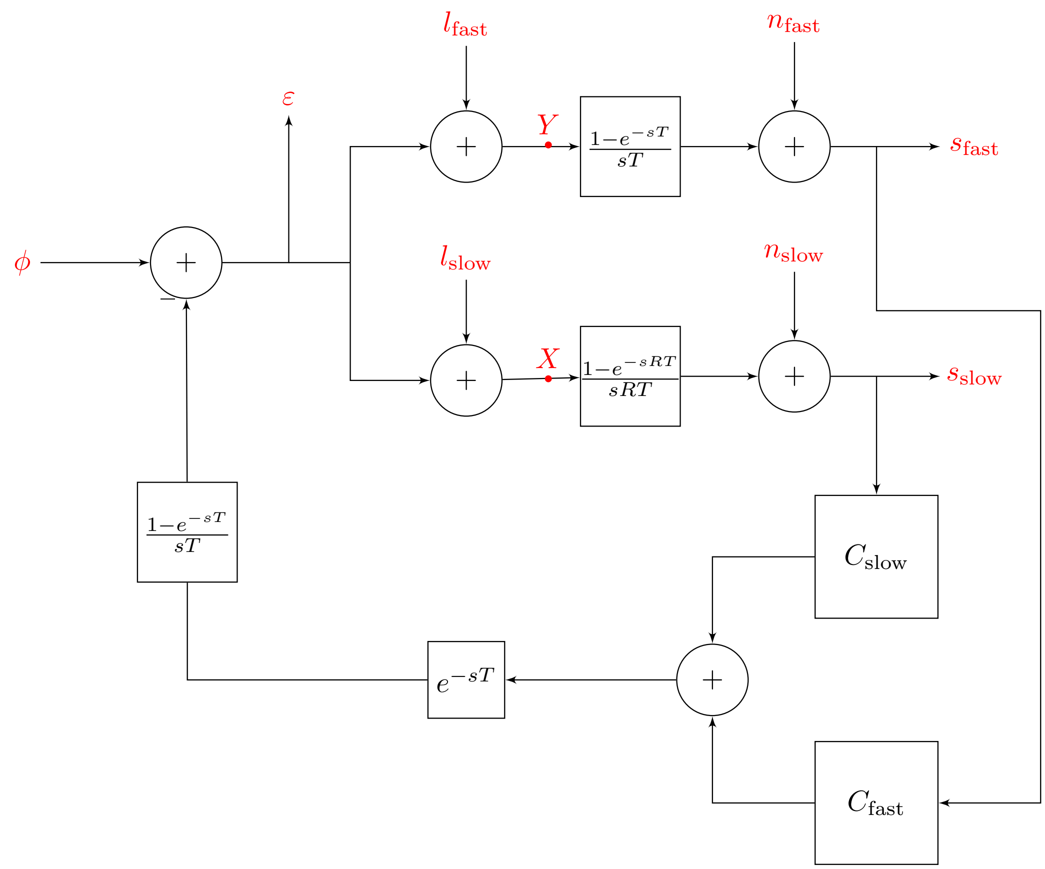

We'd like to derive transfer functions between each input (ϕ, L_{fast}, L_{slow}, n_{fast}, n_{slow}) and output (X, the signal seen by the slow WFS; and Y, the signal seen by the fast WFS). We do this by writing down relationships between the intermediate named signals, and eliminating any signals other than the input and output.

Going "backwards", we have

$$\begin{aligned}
\epsilon &= \phi - D \\
D &= \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} s_{\text{fast}} + C_{\text{slow}} s_{\text{slow}}\right) \\
s_{\text{fast}} &= n_{\text{fast}} + \frac{1-e^{-sT}}{sT} Y \\
Y &= L_{\text{fast}} + \epsilon \\
s_{\text{slow}} &= n_{\text{slow}} + \frac{1-e^{-sRT}}{sRT} X \\
X &= L_{\text{slow}} + \epsilon
\end{aligned}$$

To calculate X/ϕ and equivalently Y/ϕ, we set the L and n terms to zero. We're left with $X = Y = \epsilon$.

$$\begin{aligned}
X &= \phi - D \\
D &= \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} s_{\text{fast}} + C_{\text{slow}} s_{\text{slow}}\right) \\
s_{\text{fast}} &= \frac{1-e^{-sT}}{sT} X \\
s_{\text{slow}} &= \frac{1-e^{-sRT}}{sRT} X
\end{aligned}$$

This simplifies to

$$\begin{aligned}
X &= \phi - \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}\right) X \\
X &\left(1 + \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}\right)\right) = \phi \\
\frac{X}{\phi} &= \frac{1}{1 + \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}\right)}
\end{aligned}$$

For convenience we'll use the name plant $= \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}\right)$. This means $X/\phi = Y/\phi = 1/(1 + \text{plant})$.

To calculate X/L_{fast} and Y/L_{fast}, we set ϕ, L_{slow}, and the n terms to zero.

$$\begin{aligned}
X &= -\left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} Y + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT} X\right) \\
Y &= L_{\text{fast}} + X
\end{aligned}$$

We'll first eliminate Y from this:

$$\begin{aligned}
X &= -\left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} \left(L_{\text{fast}} + X\right) + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT} X\right) \\
X &\left(1 + \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}\right)\right) = -L_{\text{fast}} \left(\frac{1-e^{-sT}}{sT}\right)^2 e^{-sT} C_{\text{fast}} \\
\frac{X}{L_{\text{fast}}} &= \frac{-\left(\frac{1-e^{-sT}}{sT}\right)^2 e^{-sT} C_{\text{fast}}}{1 + \text{plant}}.
\end{aligned}$$

Then, we'll eliminate X:

$$\begin{aligned}
Y - L_{\text{fast}} &= -\left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} Y + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT} (Y - L_{\text{fast}})\right) \\
Y &\left(1 + \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}\right)\right) = L_{\text{fast}} \left(1 + \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}\right) \\
\frac{Y}{L_{\text{fast}}} &= \frac{1 + \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}}{1 + \text{plant}}
\end{aligned}$$

To calculate X/L_{slow} and Y/L_{slow}, we set ϕ, L_{fast}, and the n terms to zero.

$$
\begin{aligned}
Y &= -\left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} Y + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT} X\right) \\
X &= L_{\text{slow}} + Y
\end{aligned}
$$

We'll first eliminate Y from this:

$$
\begin{aligned}
X - L_{\text{slow}} &= -\left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} (X - L_{\text{slow}}) + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT} X\right) \\
X \left(1 + \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}\right)\right) &= L_{\text{slow}} \left(1 + \left(\frac{1-e^{-sT}}{sT}\right)^2 e^{-sT} C_{\text{fast}}\right) \\
\frac{X}{L_{\text{slow}}} &= \frac{1 + \left(\frac{1-e^{-sT}}{sT}\right)^2 e^{-sT} C_{\text{fast}}}{1 + \text{plant}}.
\end{aligned}
$$

Then, we'll eliminate X:

$$
\begin{aligned}
Y &= -\left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} Y + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT} (L_{\text{slow}} + Y)\right) \\
Y \left(1 + \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \frac{1-e^{-sT}}{sT} + C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}\right)\right) &= L_{\text{slow}} \left(-\left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}\right) \\
\frac{Y}{L_{\text{slow}}} &= \frac{-\left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} C_{\text{slow}} \frac{1-e^{-sRT}}{sRT}}{1 + \text{plant}}
\end{aligned}
$$

For the n transfer functions, we set $\phi, L_{\text{fast}}, L_{\text{slow}}$ to 0. This gives us $\epsilon = -D = X = Y$, so we can calculate just $X/n_{\text{fast/slow}}$, and the Y ones will be the same.

$$
\begin{aligned}
-X &= \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} s_{\text{fast}} + C_{\text{slow}} s_{\text{slow}}\right) \\
s_{\text{fast}} &= n_{\text{fast}} + \frac{1-e^{-sT}}{sT} X \\
s_{\text{slow}} &= n_{\text{slow}} + \frac{1-e^{-sRT}}{sRT} X
\end{aligned}
$$

This gives us

$$
-X = \left(\frac{1-e^{-sT}}{sT}\right) e^{-sT} \left(C_{\text{fast}} \left(n_{\text{fast}} + \frac{1-e^{-sT}}{sT} X\right) + C_{\text{slow}} \left(n_{\text{slow}} + \frac{1-e^{-sRT}}{sRT} X\right)\right)
$$

which simplifies to

$$
-X \left(1 + C_{\text{fast}} \left(\frac{1-e^{-sT}}{sT}\right)^2 e^{-sT} + C_{\text{slow}} \left(\frac{1-e^{-sT}}{sT}\right) \left(\frac{1-e^{-sRT}}{sRT}\right) e^{-sT}\right) = \frac{1-e^{-sT}}{sT} e^{-sT} \left(C_{\text{fast}} n_{\text{fast}} + C_{\text{slow}} n_{\text{slow}}\right)
$$

This gives us the following solution for all four noise transfer functions, which is the same for fast/slow other than the controller transfer function in the numerator.

$$
\frac{X}{n_{\text{fast/slow}}} = \frac{-\frac{1-e^{-sT}}{sT} e^{-sT} C_{\text{fast/slow}}}{1 + C_{\text{fast}} \left(\frac{1-e^{-sT}}{sT}\right)^2 e^{-sT} + C_{\text{slow}} \left(\frac{1-e^{-sT}}{sT}\right) \left(\frac{1-e^{-sRT}}{sRT}\right) e^{-sT}}
$$