Multi-wavefront sensor single-conjugate transfer function derivations

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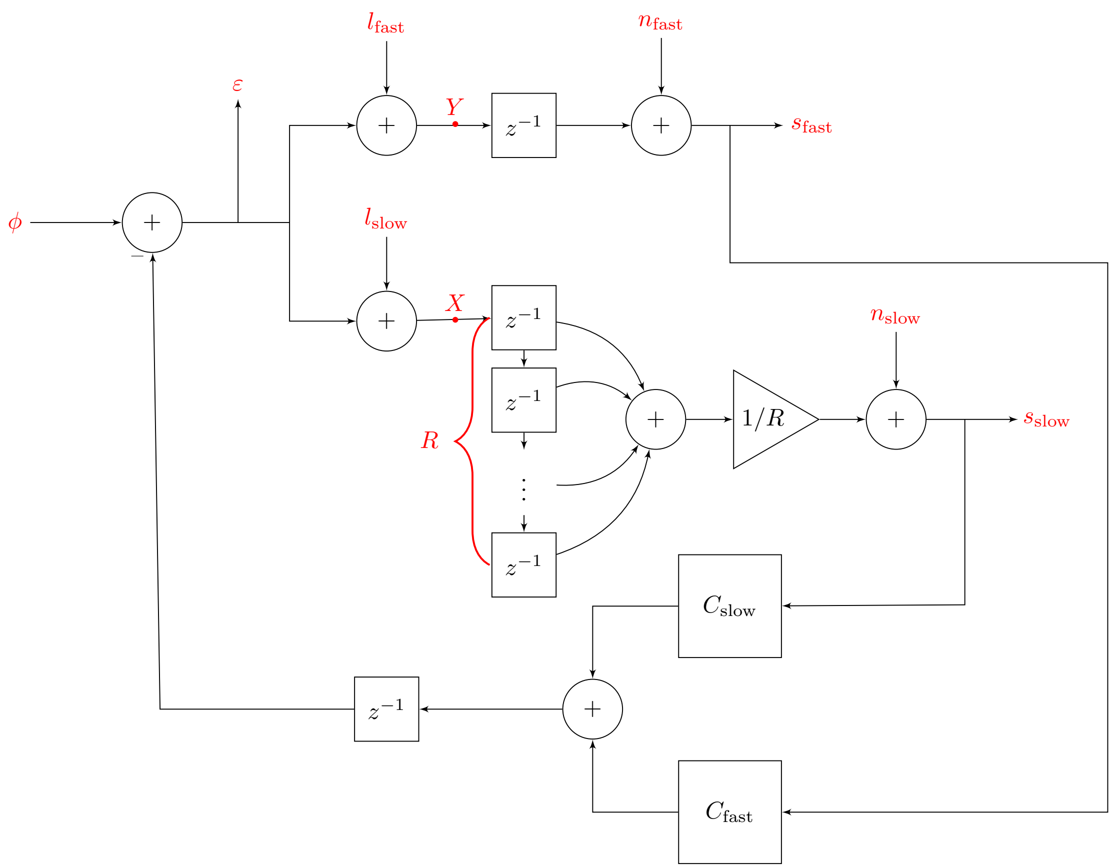

We'd like to derive transfer functions between each input (ϕ, L_{fast}, L_{slow}, n_{fast}, n_{slow}) and output (X, the signal seen by the slow WFS; and Y, the signal seen by the fast WFS). We do this by writing down relationships between the intermediate named signals, and eliminating any signals other than the input and output.

Going "backwards", we have

$$\begin{aligned}
\epsilon &= \phi - D \\
D &= z^{-1}\left(C_{\text{fast}} s_{\text{fast}} + C_{\text{slow}} s_{\text{slow}}\right) \\
s_{\text{fast}} &= n_{\text{fast}} + z^{-1} Y \\
Y &= L_{\text{fast}} + \epsilon \\
s_{\text{slow}} &= n_{\text{slow}} + \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right) X \\
X &= L_{\text{slow}} + \epsilon
\end{aligned}$$

To calculate X/ϕ and equivalently Y/ϕ, we set the L and n terms to zero. We're left with $X = Y = \epsilon$.

$$
\begin{aligned}
X &= \phi - D \\
D &= z^{-1}\left(C_{\text{fast}} s_{\text{fast}} + C_{\text{slow}} s_{\text{slow}}\right) \\
s_{\text{fast}} &= z^{-1} X \\
s_{\text{slow}} &= \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right) X
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
X &= \phi - z^{-2} C_{\text{fast}} X - z^{-1} C_{\text{slow}} \left(\sum_{k=1}^{R} z^{-k}\right) X \\
X &\left(1 + z^{-2} C_{\text{fast}} + z^{-1} C_{\text{slow}} \left(\sum_{k=1}^{R} z^{-k}\right)\right) = \phi \\
\frac{X}{\phi} &= \frac{1}{1 + z^{-2} C_{\text{fast}} + z^{-1} C_{\text{slow}} \left(\sum_{k=1}^{R} z^{-k}\right)}
\end{aligned}
$$

For convenience we'll use the name $\text{plant} = z^{-2} C_{\text{fast}} + z^{-1} C_{\text{slow}} \left(\sum_{k=1}^{R} z^{-k}\right)$. This means $X/\phi = Y/\phi = 1/(1 + \text{plant})$. To calculate X/L_{fast} and Y/L_{fast}, we set ϕ, L_{slow}, and the n terms to zero.

$$
\begin{aligned}
X &= -z^{-1}\left(C_{\text{fast}} z^{-1} Y + C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right) X\right) \\
Y &= L_{\text{fast}} + X
\end{aligned}
$$

We'll first eliminate Y from this:

$$
\begin{aligned}
X &= -z^{-1}\left(C_{\text{fast}} z^{-1}\left(L_{\text{fast}} + X\right) + C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right) X\right) \\
X &\left(1 + z^{-2} C_{\text{fast}} X + z^{-1} C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)\right) = -z^{-2} C_{\text{fast}} L_{\text{fast}} \\
\frac{X}{L_{\text{fast}}} &= \frac{-z^{-2} C_{\text{fast}}}{1 + z^{-2} C_{\text{fast}} X + z^{-1} C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)} = \frac{-z^{-2} C_{\text{fast}}}{1 + \text{plant}}.
\end{aligned}
$$

Then, we'll eliminate X:

$$
\begin{aligned}
Y - L_{\text{fast}} &= -z^{-1}\left(C_{\text{fast}} z^{-1} Y + C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)\left(Y - L_{\text{fast}}\right)\right) \\
Y &\left(1 + z^{-2} C_{\text{fast}} + z^{-1} C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)\right) = L_{\text{fast}}\left(1 + z^{-1} C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)\right) \\
\frac{Y}{L_{\text{fast}}} &= \frac{1 + z^{-1} C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)}{1 + \text{plant}}.
\end{aligned}
$$

To calculate X/L_{slow} and Y/L_{slow}, we set ϕ, L_{fast}, and the n terms to zero.

$$
\begin{aligned}
Y &= -z^{-1}\left(C_{\text{fast}} z^{-1} Y + C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right) X\right) \\
X &= L_{\text{slow}} + Y
\end{aligned}
$$

We'll first eliminate Y from this:

$$\begin{aligned}
X - L_{\text{slow}} &= -z^{-1}\left(C_{\text{fast}} z^{-1}\left(X - L_{\text{slow}}\right) + C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right) X\right) \\
X\left(1 + z^{-2} C_{\text{fast}} + z^{-1} C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)\right) &= L_{\text{slow}}\left(1 + z^{-2} C_{\text{fast}}\right) \\
\frac{X}{L_{\text{slow}}} &= \frac{1 + z^{-2} C_{\text{fast}}}{1 + \text{plant}}.
\end{aligned}$$

Then, we'll eliminate X:

$$\begin{aligned}
Y &= -z^{-1}\left(C_{\text{fast}} z^{-1} Y + C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)\left(L_{\text{slow}} + Y\right)\right) \\
Y\left(1 + z^{-2} C_{\text{fast}} + z^{-1} C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)\right) &= L_{\text{slow}}\left(-z^{-1} C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)\right) \\
\frac{Y}{L_{\text{slow}}} &= \frac{-z^{-1} C_{\text{slow}} \frac{1}{R}\left(\sum_{k=1}^{R} z^{-k}\right)}{1 + \text{plant}}
\end{aligned}$$

I'll do the n transfer functions later.