Complexity in Finite Games with Imperfect Recall

Adrian Marple and Yoav Shoham

Stanford University, Computer Science Department,
353 Serra Street Stanford, CA 94305
{amarple,shoham}@stanford.edu
http://http://ai.stanford.edu/~shoham/

Abstract. We investigate the computational complexity of three decision problems for extensive form games with imperfect recall: whether a Nash equilibrium (NE) exists, whether a given strategy profile is a NE, and whether a NE exists in which a given player gets a payoff of at least λ. We consider both single-player and multi-player games. Among other results, we show that for single-player games it is NP-complete to decide if a NE exists that achieves a payoff of λ, and for multi-player games it is Σ_2^{P}-complete to decide whether any NE exists.

1 Introduction

We consider the complexity of following three decision problems for finite games:

1. Does a given game have a Nash equilibrium (NE)?
2. Is a given strategy profile a NE of a given game?
3. Does a given game have a NE in which a given player gets a payoff of at least λ?

For normal-form games the complexity of these problems is well understood: The complexity of (1) is obviously O(1), since every game has a NE [14]; (2) is trivially O(n); and (3) is NP-complete [8, 4]. Our goal in this paper is to investigate these problems in the context of extensive-form games with behavioral strategies. It is not hard to extend the results for the normal form to extensive-form games with perfect recall, that is, games in which players remember the entire history of the game (the information sets visited and the actions taken). Specifically, since with perfect recall behavioral strategies are strategically equivalent to mixed-strategies in games [12], in this class every game has a NE, and the trivial O(1) complexity of (1) remains. Similarly, since polynomial time algorithms exist for finding best-response strategies in games with perfect recall [11], the complexity of (2) is in P. Finally, it is not hard to see that (3) is NP-complete; hardness follows from the normal-form case, and inclusion in NP follows from the fact that (2) is in P.

When it comes to games of *imperfect* recall with *behavioral strategies*, however, all bets are off. In a companion paper [13], we discuss in detail solution concepts with imperfect recall. In that paper we also discuss the importance of this class of games. Briefly, humans and computer agents alike do not have unlimited memory, and requiring players to remember the entire history of play is often unrealistic and can even be counterproductive (as seen, for example, in the computer poker literature [10]). Here, we will discuss solution concepts only to the extent needed to present the complexity questions. The first fact to point out is that games with imperfect recall may not have a NE in behavioral strategies, as seen for example in Figure 1 (this was informally shown by [2] and later formally shown by [17]). We can recover the guaranteed existence of NE if we restrict ourselves to single-player games (also called extensive decision problems), the special case that has attracted much of the literature on solution concepts with imperfect recall [16, 3, 1, 7, 9], in which case the O(1) complexity of (1) remains, but the complexity of (2) and (3) remain open.

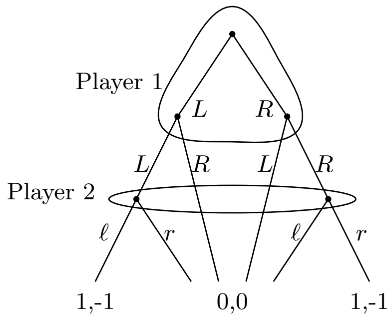

Fig. 1. An extensive form game with no Nash equilibrium in behavioral strategies.

In this paper, we answer all of these open problems; the resulting complexities are summarized in Table 1.

Decision Problem	Existence	Verification	At Least λ
Perfect Recall NE	O(1)	P	NP-complete
1-player NE	O(1)	**co-NP-complete**	**NP-complete**
n-player NE	$\boldsymbol{\Sigma_2^{\mathrm{P}}}$**-complete**	**co-NP-complete**	$\boldsymbol{\Sigma_2^{\mathrm{P}}}$**-complete**

Table 1. Table of complexity of decision problems for extensive form games. **Bold** entries indicate novel contribution of this paper. Recall that $\Sigma_2^{\mathrm{P}} = \mathrm{NP}^{\mathrm{NP}}$ (that is, polynomial time with a non-deterministic Turing machine with access to a SAT oracle).

We find that verifying a NE is co-NP-complete for both single- and multi-player games of imperfect recall. Furthermore, in single-player games it is NP-complete to find a NE that achieves a payoff of at least λ, while it is Σ_2^{P} to find a NE of any type in the multi-player setting. One further point of note, we show this using only games without Nature and with polynomially many pure strategies. That these hardness results utilize such a limited class of games is especially surprising since it is easy to create extensive games even with perfect recall that have exponentially many pure strategies.[1]

We can additionally extend Table 1 to include the complexity of decisions problems associated with equilibrium notions from our companion paper [13] (the most fundamental being the Distributed Agent Equilibrium (DAE)). However, our focus is on Nash equilibrium, and we leave discussion of this addition for Section 5.

The rest of the paper is organized as follows. Section 2 covers background information needed for the rest of the paper. Section 3 covers the complexity of NE in extensive decision problems and results that follow immediately from this. Section 4 covers the complexity of NE in general extensive form games. Finally, Section 5 discusses the relation to this work to other relevant prior work and proffers open complexity questions that are not addressed in this paper.

2 Background

This section defines the established background information needed for this paper and gives the decision problem abbreviations that will be used throughout the paper.

Definition 1 (Extensive Form). *An* extensive form game *is a six-tuple $\Gamma = \langle N, H, P, \rho, u, I \rangle$, where*

- *N is a finite set of players.*
- *H is a finite set of sequences that represent the possible* histories *of actions. If $(a_1, \ldots, a_K) \in H$ and $K \neq 0$ then $(a_1, \ldots, a_{K-1}) \in H$. A history $(a_1, \ldots, a_K) = z \in Z$ is terminal if there is no a such that $(z, a) \in H$. The set of actions available at history h is defined as $A(h) = \{a : (h, a) \in H\}$.*
- *$P : H/Z \to N \cup \{c\}$ (where H/Z is the set of all non-terminal histories) is the player function that assigns a player or Nature to each non-terminal history. We will denote C to be the set of non-terminal histories assigned to Nature and D to be the set of non-terminal histories assigned to players.*[2]
- *ρ is a prior on Nature's actions (i.e. for all $h \in C$, $\rho(h) \in \Delta(A(h))$).*
- *$u_i : Z \to \mathbb{R}$ is the utility function for player i.*
- *I is the set of information sets. I is a partition of D such that if $X \in I$ then for all $h, h' \in X$, $A(h) = A(h')$ and $P(h) = P(h')$. Because of this requirement we will overload the action and player functions; for an information set X we define $A(X) = A(h)$, $P(X) = P(h)$ where $h \in X$. For a player i, $I_i = \{X : P(X) = i\}$ denotes the set of information sets assigned to player i.*

[1] Note that the completeness results still apply to the full class of games in question even though only a limited subset of games is required to show hardness.

[2] Although our games will not require the use of Nature, most standard definitions include Nature, and for this reason we decided not to extract it from our definition.

We additionally call an extensive form game Γ with the property that $|N| = 1$ an *extensive decision problem.*

The *experience* of player i at history h, denoted $\exp_i(h)$, is the sequence of information sets and actions of player i along the history h. An extensive form game has *perfect recall* if for every information set X and for every $h, h' \in X$ it is the case that $\exp_i(h) = \exp_i(h')$, where i is the player assigned to X. A game of *imperfect recall* is one without perfect recall. Given a history $h = (a_1, \ldots, a_K)$ and $L \leq K$ the history $h' = (a_1, \ldots, a_L)$ is said to *precede* h, which is denoted $h' \leq h$. Furthermore, a history h' *strictly precedes* history h, written $h' < h$, if $h' \leq h$ and $h' \neq h$. A game exhibits *absentmindedness* if there is an information set X and $h, h' \in X$ such that $h' < h$; thus, absentmindedness is a special case of imperfect recall.

A *behavioral strategy*[3] for player i, b_i, is a distribution for each information set assigned to i over the actions available at that information set (i.e. $b_i \in \prod_{X \in I_i} \Delta(A(X))$). The set of strategies for player i will be denoted Σ_i. A *strategy profile* $b = \prod_{i \in N} b_i$ is a set of strategies, one for each player. Likewise the set of strategy profiles for a game will be denoted Σ. Given a strategy profile b and an information set X we let $b(X)$ denote the distribution over $A(X)$ defined by b (that is, $b(X) = b_i(X)$ where $b_i \in B$ and i is the player assigned to X). Thus given a particular action $a \in A(X)$, $b(X)(a)$ gives the probability of playing a upon reaching information set X. We can also break up a strategy profile into two parts; $b = (b_i, b_{-i})$ where b_i is strategy for player i and b_{-i} is the set of strategies for all other players.

A strategy profile b induces a probability distribution over terminal histories, $p_b \in \Delta(Z)$. We remind the reader that for behavioral strategies this distribution is determined by sampling from distribution $b(X)$ independently every time a history from information set X is reached (as opposed to randomizing each information set once and for all before play). p_b can be extended to all (not necessarily terminal) histories $h \in H$ by

$$p_b(h) = \sum_{z \in Z : h \leq z} p_b(z)$$

We use this distribution to extend the utility function for player i to a strategy profile b linearly as follows

$$u_i(b) = \sum_{z \in Z} p_b(z) u_i(z)$$

Using this, we can give a simple definition of NE.

Definition 2 (Nash Equilibrium). *A strategy profile b is a NE if for every player i and every strategy b'_i.*

$$u_i(b'_i, b_{-i}) \leq u_i(b)$$

[3] We will refer to behavioral strategies simply as strategies for the remainder of the paper.

Finally, there are a few decision problems that we will return to more than once. For this reason, we give abbreviations for the following problems:

- EDP-λ: For a given extensive decision problem, does there exist a (behavioral strategy) NE with value at least λ?
- EDP-λ!: For a given extensive decision problem, does there exist a NE with value *strictly* greater than λ?
- EFG-λ: For a given extensive form game, does there exist a NE such that player 1 (without loss of generality) gets at least λ?
- EFG-NE: For a given extensive form game, does there exist a NE?

3 Extensive Decision Problems

This section contains the proof that EDP-λ and EDP-λ! are NP-complete and results that follow immediately from these two completeness results. Note that much of the material in this section is phrased to be leveraged in the next section.

To do this we will first show that k-SAT can be reduced to both EDP-λ and EDP-λ!. We will first show the form of this reduction without proof and explain the intuition with a 2-SAT example.

Let a k-SAT instance have n variables $x_1, \ldots, x_n$. Using this k-SAT instance we can define an extensive decision problem, Γ, as follows:

> The decision maker absentmindedly (i.e. has only 1 information set) chooses k literals from $\{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$. The utility she gets at a terminal node $z = (a_1, \ldots, a_k)$ is
>
> $$u(z) = \begin{cases} \text{-1} & : \text{if any two literals are negations of each other} \\ c_y < 1 & : \text{if all literals are distinct and these literals violate} \\ & \quad \text{clause } y \text{ [4]} \\ 0 & : \text{if } a_1 = \ldots = a_k \\ 1 & : \text{otherwise} \end{cases}$$

In other words, if the decision maker manages to play inconsistently (play a variable and its negation) she is penalized; if the decision maker plays literals that violate a clause, she in penalized; and if the decision maker plays the same literal k times, she is penalized. The respective goals of these penalties are to incentivize the decision maker to be consistent, avoid violating clauses, and play as many literals as possible. For instance, if the k-SAT instance is statisfiable the decision maker's optimal strategy will be to uniformly play n consistent literals, that don't violated any clauses, i.e. a satisfying assignment.

To make the technical implementation of this definition a bit clearer, we give a 2-SAT example. Consider the following 2-SAT instance over 3 variables:

$$(x_1 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_2 \vee x_3)$$

[4] For the purposes of reducing k-SAT to EDP-$\lambda(')$, we can take c_y in the above definition to be -1. However, it will be important for later sections to allow for different values of $c < 1$ that can depend on the violated clause.

Figure 2 gives a matrix visualization of the extensive decision problem reduction for this 2-SAT instance. Note that only symmetric strategies of this matrix game correspond to strategies of the extensive decision problem. It is not hard to see that the support of this symmetric equilibrium corresponds to a satisfying assignment to the corresponding 2-SAT instance.

	x_1	$\neg x_1$	x_2	$\neg x_2$	x_3	$\neg x_3$
x_1	**0**	-1	-1	**1**	**1**	1
$\neg x_1$	-1	0	1	1	-1	1
x_2	-1	1	0	-1	1	-1
$\neg x_2$	**1**	1	-1	**0**	**1**	-1
x_3	**1**	-1	1	**1**	**0**	-1
$\neg x_3$	1	1	-1	-1	-1	0

Fig. 2. The row move represents the decision maker's first literal choice and the column her second. Notice that the decision maker can only play symmetric strategies. The highlighted cells show the outcomes with support in the (only) optimal strategy.

We will now start developing the tools needed in order to prove the reduction from k-SAT to EDP-λ.

Lemma 1. *Let Γ be the reduction of a k-SAT instance and let x be some variable in this instance. If a strategy profile σ assigns positive probability to both x and $\neg x$, it is not a Nash equilibrium.*

Proof. Assume a strategy profile σ assigns positive probability to both x and $\neg x$ (say p and q respectively). Now consider a derivative game where the player is only allowed to play x, $\neg x$, or a third action, a, that corresponds to playing the actions that are not x or $\neg x$ with the same proportions as in σ_1. It is key to realize that strategy profiles of this derivative game correspond to strategy profiles of Γ and the utility of a strategy profile in this game is the same as the utility in of the corresponding strategy profile in Γ. For example, if all players play the strategy of x, $\neg x$, and a with probabilities p, q, and $r = (1 - p - q)$ respectively, this corresponds to playing σ in Γ. Using the utilities from Γ, we can partition outcomes of this game by the fact that they have the same utility:

1. E_1: the event that only x is played, or only $\neg x$ is played (utility of 0).
2. E_2: the event that x is played between 2 and $k-1$ times and $\neg x$ is not played, or $\neg x$ is played between 2 and $k-1$ times and x is not played (utility of 1).
3. E_3: the event that both x and $\neg x$ are played (utility of -1).
4. E_4: the event that x is played exactly once and $\neg x$ is not played (utility of $\alpha = u(x, a, \ldots, a)$).
5. E_5: the event that $\neg x$ is played exactly once and x is not played (utility of $\beta = u(\neg x, a, \ldots, a)$).
6. E_6: the event that neither x nor $\neg x$ is played (utility of $\gamma = u(a, \ldots, a)$).

Without loss of generality let $\alpha \geq \beta$. Now let θ be the strategy profile where the decision maker plays x, $\neg x$, and a with probabilities p, q, and r respectively. As already mentioned this corresponds to σ in Γ. Furthermore, let θ' be the strategy profile where the decision maker plays x and a with probabilities $(p+q)$ and r respectively (i.e. giving all the weight from $\neg x$ to x). Note that $p_\theta(E_2) \leq p_{\theta'}(E_2)$, $p_\theta(E_6) = p_{\theta'}(E_6)$, and $p_{\theta'}(E_3) = 0$. Using the events outlined above we can describe the utility θ by

$$\begin{aligned}
u(\theta) &= p_\theta(E_2) - p_\theta(E_3) + p_\theta(E_4)\alpha + p_\theta(E_5)\beta + p_\theta(E_6)\gamma \\
&= p_\theta(E_2) - p_\theta(E_3) + kpr^{k-1}\alpha + kqr^{k-1}\beta + p_\theta(E_6)\gamma \\
&\leq p_\theta(E_2) - p_\theta(E_3) + k(p+q)r^{k-1}\alpha + p_\theta(E_6)\gamma \\
&< p_{\theta'}(E_2) + p_{\theta'}(E_4)\alpha + p_{\theta'}(E_6)\gamma \\
&= u(\theta')
\end{aligned}$$

We therefore know that θ' has strictly higher utility than θ and because θ' corresponds to a symmetric strategy profile of Γ with the same utility, σ cannot maximize utility and hence cannot be a NE.

Lemma 2. *Let Γ be the reduction for a k-SAT instance with n variables and b be a NE (optima) of Γ. Then for every action a in the support of b there does not exist an action a' such that:*

$$\sum_{h=(a_1,\ldots,a_{k-1})} p(h|b)u(h,a') - p(h|b)u(h,a) > 0$$

Proof. It is not difficult to see that the above property is equivalent to Piccione and Rubinstein's Modified Multiselves Consistency for the game Γ [16]. Since they show that every NE is Modified Multiselves Consistent, it follows that every NE has the above property.

The following two values will be used for the remainder of the paper:

$$v(n,k) = \frac{n^k - n}{n^k}$$

$$v(n,k,c) = 1 - \left(n - k + k(2-c)^{\frac{1}{1-k}}\right)^{1-k}$$

Lemma 3. *Let Γ be the reduction for a k-SAT instance with n variables. Consider a strategy σ that has a clause contradiction with payoff $c < 1$ but no literal contradiction (i.e. playing both x and $\neg x$ with positive probability). This strategy can have a value at most* $\max(v(n-1,k), v(n,k,c))$.

Proof. Consider the following game Γ'. Remove the effect of all clauses other than the one that is violated by b with payoff of c (potentially increasing the value of some outcomes). Then remove the actions that are not played in b. Clearly a NE of Γ' has value at least as high as b has in Γ.

Assume a NE of Γ contains the clause violation. Let p be the probability of playing any action without a contradiction (they must have the same probability by symmetry) and let q be the probability of playing one of the k actions that are mutually contradictory (again all have the same probability by symmetry). Lemma 2 implies that in a Nash equilibrium there must be a form of indifference which is captured by the following equation:

$$\begin{aligned} & 1 \cdot (1 - p^{k-1}) + 0 \cdot p^{k-1} = 1 \cdot (1 - 2q^{k-1}) + c \cdot q^{k-1} + 0 \cdot q^{k-1} \\ \implies \quad & 1 - p^{k-1} = 1 - 2q^{k-1} \end{aligned}$$

Furthermore, if the clause is violated all k actions required for the violation must be played, and all non-contradictory actions must be played otherwise lemma 2 is violated (i.e. the expected value of playing such an action in the last round is 1, while it must be less that one for any action with support). In order for p and q define a valid probability distribution over all actions, the following equation must hold:

$$(n-k)p + kq = 1$$

Solving for p yields

$$p = \left(n + k(2-c)^{\frac{1}{1-k}}\right)^{-1}$$

Which means the value of this strategy is

$$1 - \left(n - k + k(2-c)^{\frac{1}{1-k}}\right)^{1-k} = v(n,k,c)$$

Now assume a NE of Γ' does not contain the clause violation. Then it clearly contains all but one action for the same reason a NE with a contradiction contains all actions. The symmetry of the situation means that such a strategy has value

$$1 - \left(\frac{1}{n-1}\right)^{k-1} = v(n-1,k)$$

Therefore the value of b is bounded above by $\max(v(n-1,k), v(n,k,c))$.

Given these two lemmas we can now show the main result of this section.

Theorem 1. *EDP-λ and EDP-λ! are NP-complete.*

Proof. Clearly EDP-λ and EDP-λ! are in NP since it takes polynomial time to evaluate the payoff of a particular strategy and the existence of a strategy with payoff at least λ implies the existence of a NE with payoff at least λ.

Let Γ be the reduction for a k-SAT instance with n variables. To show that EDP-λ and EDP-λ! are NP-hard, we must simply show that there is a strategy with payoff strictly greater than $\lambda' = \max(v(n-1,k), v(n,k,-1)) < \lambda = v(n,k)$ if and only if the k-SAT instance is satisfiable.

$\implies$: Assume the k-SAT instance is satisfiable. Simply consider the strategy of playing the literals of a satisfying assignment uniformly. Clearly this strategy achieves a value of $v(n,k) = \lambda > \lambda'$.

$\Longleftarrow$: Assume the k-SAT instance is not satisfiable. Let b be a NE. By Lemma 1, it follows that b cannot contain a literal contradiction (i.e. play both x and $\neg x$ with positive probability). Therefore b must fall into one of the following two cases:

- *Case 1: b contains a clause contradiction.*
  By Proposition 3, the best that can be done in this case is $\max(v(n,k), v(n,k,-1))$ which is not strictly greater than λ' and is clearly strictly less than λ.
- *Case 2: b contains no contradictions.*
  In this case, the support of a NE must contain at most $n-1$ literals (if there were n mutually non-contradictory literals that would constitute a satisfying variable assignment of a non-satisfiable k-SAT instance). By symmetry all of these must be played uniformly for a value of $v(n-1,k) \leq \lambda' < \lambda$.

Several corollaries immediately follow from Theorem 1, many of which directly answer complexity questions raised in Section 1.

Corollary 1. *The problem of deciding whether a strategy in an extensive decision problem is a Nash equilibrium is co-NP-complete.*

Proof. This can be shown by reducing to the complement of EDP-λ!. The reduction itself is obvious since a strategy is a NE (with payoff of λ) if and only if there does not exist a NE that does strictly better than λ.

Corollary 2. *The problem of deciding whether a strategy in an extensive form game is a Nash equilibrium is co-NP-complete.*

Proof. Clearly this problem is co-NP-hard by Corollary 1. It is also clear that this problem is in co-NP since a best response that does strictly better than a player's current payoff proves the strategy in question is not a NE (and is therefore a polynomial time verifiable counter-example).

Corollary 3. *EFG-NE and EFG-λ are in Σ_2^{P}.*

Proof. By Corollary 2 a Nash equilibrium of an extensive form game can be verified in polynomial time using an NP oracle. Therefore both EFG-NE and EFG-λ are in Σ_2^{P}.

4 Extensive Form Games

In this section we will prove that EFG-λ and EFG-NE are both Σ_2^{P}-complete. We will first show that EFG-λ and EFG-NE are equivalent and will finish by reducing a canonical Σ_2^{P}-complete problem to EFG-λ.

4.1 Equivalence of EFG-λ and EFG-NE

Proposition 1. *EFG-λ and EFG-NE are polynomial time reducible to each other.*

Proof. Clearly EFG-λ can solve EFG-NE simply by setting λ below the minimum player 1 payoff for any outcome.

Let Γ be an extensive form game. Now consider the derivative game, Γ', in Figure 3. We will show that there is a NE in Γ' if and only if there exists a NE such that player 1 receives a payoff of λ in Γ.

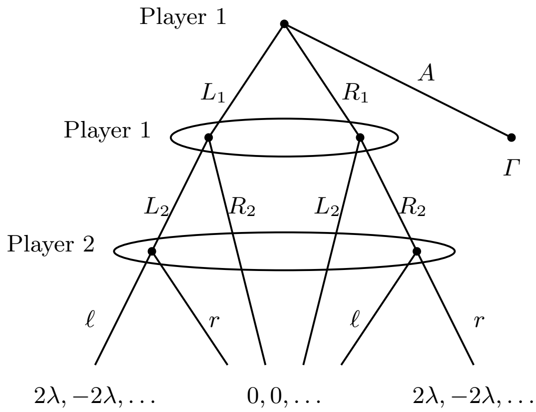

Fig. 3. The game Γ' for the proof of Proposition 1. Note that Γ represents the entire game tree of the game Γ. For outcomes outside of Γ and for $i > 2$ player i gets a payoff of 0.

$\Longrightarrow$: Let b' be a NE of Γ'. Assume there is no NE such that player 1 receives a payoff of λ in Γ. Now assume that b' gives positive probability to action A in player 1's initial information set. Because the least player 1 can get by playing one of $(L_1, L_2, *)$ or $(R_1, R_2, *)$ is λ, the NE induced in Γ must receive a payoff of at least λ, which contradicts our earlier assumption. Therefore player 1 must play A in her initial information set with probability 0. Given this, playing $(L_1, L_2, *)$ and $(R_1, R_2, *)$ are player 1's only possible best responses to any player 2 strategy. However, player 2's best response to $(L_1, L_2, *)$ is $(r, *)$ and player 1's best response to this is $(R_1, R_2, *)$, so this cannot be a NE. Likewise, player 2's best response to $(R_1, R_2, *)$ is $(\ell, *)$ and player 1's best response to this is $(L_1, L_2, *)$, so this also cannot be a NE. Therefore, assumption that there exist no NE such that player 1 receives a payoff of λ is Γ is false.

$\Longleftarrow$: Let b be a NE of Γ such that player 1 receives a payoff of λ. Now choose the Γ' strategy profile b' to be $b'_1 = (A, L_2, b_1)$, $b'_2 = ((\frac{1}{2}\ell, \frac{1}{2}r), b_2)$, and for all $i > 2$, $b'_i = b_i$. Clearly for $i > 2$ player i is playing a best response. Similarly player 2 is playing a best response because the probability of encountering a player 2 information set outside of Γ is 0. Finally, player 1 is also playing a best response because the best payoff she can achieve by not playing A in her initial information set is λ, while the payoff for playing A is at least λ.

4.2 Reduction from QSAT_2 to EFG-NE

As we did in Section 3 we will first give the form of the reduction without proof. But first we must specify the Σ_2^{P}-complete problem we are reducing from. An instance of the problem, abbreviated QSAT_2, has the following form [15]:

> Let X and Y be sets of boolean variables and f be a boolean formula with variables from X and Y. Is the following statement true?
>
> $$\exists X.\forall Y.\ f$$

To start we will recast this as a question about a boolean game.[5] This game will have two players. Player 1 will control X (containing n_1 variables) and player 2 will control Y (containing n_2 variables). Furthermore, player 1's target formula is f and player 2's target formula is $\neg f$. The question is then: is there a NE such that player 1 receives a payoff of 1? Without much effort it is easy to verify that this question is in fact equivalent to the QSAT_2 instance.

We can now tweak the boolean game just defined to produce another boolean game that is in a sense equivalent. First consider the canonical reduction from SAT to 3-SAT. In this reduction the original set of variables, say X, is supplemented with additional variables, X', and the original boolean formula, f, is replaced by a formula containing only clauses composed of the disjunction of 3 variables from $X \cup X'$. Furthermore f and f' are equi-satisfiable and a satisfying assignment to f' corresponds directly to a satisfying assignment to f (by simply ignoring the X' component). The tweak to our existing boolean game is that we will apply this reduction to both players' variables and formulas independently. So now player 1 will have variables $X \cup X'$ (with a total of n_1' variables) and will try to satisfy f' and player 2 will have variables $Y \cup Y'$ (with a total of n_2' variables) and will try to satisfy $(\neg f)'$. Again we will ask "Is there a NE such that player 1 receives a payoff of 1?" and again it is not difficult to see that this question is equivalent to the QSAT_2 instance.

The final step of the reduction is the actual extensive form game. Using this second version of the boolean game we can define the extensive form game, Γ as follows:

> First, player 1 absentmindedly chooses 3 literals from $X \cup X'$. Then player 2, without knowing anything about player 1's selections absentmindedly chooses 3 literals from $Y \cup Y'$ (note that both players have a single

[5] In a boolean game, players are given a set of boolean variables and a boolean formula which may depend on any players' variables. Players then play a (pure) assignment of their variables. Each player receives a payoff of 1 if their formula is true and 0 if it is false.

information set). The utilities for players 1 and 2 respectively for terminal node $z = (a_1, a_2, a_3, b_1, b_2, b_3)$ are:

$$u_1(z) = \begin{cases} \text{-1} : \text{if any player 1 literals are negations of each other} \\ \text{-1} : \text{if player 1's literals are distinct and both players'} \\ \quad\ \ \text{literals violate a clause in } f' \\ 0\ \ : \text{if } a_1 = a_2 = a_3 \\ 1\ \ : \text{otherwise} \end{cases}$$

$$u_2(z) = \begin{cases} \text{-1} : \text{if any player 2 literals are negations of each other} \\ \text{-1} : \text{if player 2's literals are distinct and both players'} \\ \quad\ \ \text{literals violate a clause in } (\neg f)' \\ 0\ \ : \text{if } b_1 = b_2 = b_3 \\ 1\ \ : \text{otherwise} \end{cases}$$

The most important thing to notice about this game is that if we take the strategy of one of the players as fixed the remaining extensive decision problem is of the same form as the reduction from Section 3, and we can therefore apply the proofs from that section to this induced decision problem.

4.3 Proof of Reduction

Theorem 2. *EFG-λ is Σ_2^{P}-complete.*

Proof. By Corollary 3, we already know that EFG-λ is in Σ_2^{P}. Given a QSAT$_2$ instance, let Γ be the extensive form game reduction for this instance as described above. To show EFG-λ is Σ_2^{P}-hard it is sufficient to show that Γ has a Nash equilibrium such that player 1 receives a payoff of $v(n'_1, 3)$ if and only if the answer to the QSAT$_2$ instance is "yes."

$\Longleftarrow$: Assume the answer to the QSAT$_2$ instance is "yes." In other words, there exists an assignment to X such that for all assignments to Y, f is true. Now consider a player 1 strategy where she plays the literals for this assignment to X along with $n'_1 - n_1$ consistent literals from X' (which exist because f and f' are equi-satisfiable) with uniform probability. Holding this strategy fixed, we can think of player 2's best response as a NE of the induced extensive decision problem. Therefore by Lemma 1, player 2 cannot give positive probability to both a literal and its negation. This means that player 2's best response can be interpreted as a partial assignment of the variables from Y and (removing unassigned variables from f), the literals from $X \cup Y$ with support in either player 1's or player 2's strategy satisfy f (by assumption). This then implies that the literals from X' can be reassigned in such a way as to satisfy f' and if player 1 modifies her strategy accordingly it will not contradict any clauses from f' and she will receive a payoff of $v(n'_1, 3)$. Finally, because player 2's payoff is unaffected by player 1 shifting probabilities between literals within X', player 2's strategy remains a best response. Since player 1 is already receiving the maximum payoff possible this pair of strategies is a Nash equilibrium.

$\Longrightarrow$: Assume the answer to the QSAT$_2$ instance is "no." In other words, for all assignments to X there exists an assignment to Y such that f is false. For player 1 to possibly achieve a payoff of $v(n'_1, 3)$ she must play the n'_1 literals from some $X \cup X'$ variable assignment uniformly. In particular this implies an assignment of X and therefore there exists an assignment of Y that makes $\neg f$ true and hence an additional assignment of Y' that makes $(\neg f)'$ true. If player 2 plays the literals from these assignments to Y and Y' uniformly she will achieve her maximum payoff of $v(n'_2, 3)$ which makes this type of strategy her only possible best response to player 1's strategy. However, because $\neg f$ is satisfied, f is not and thus neither is f'. In particular, this means that some clause must be violated and player 1 cannot receive her maximal payoff of $v(n'_1, 3)$ by Lemma 3. Therefore, for no NE can player 1 receive a payoff of $v(n'_1, 3)$.

Corollary 4. *EFG-NE is Σ_2^{P}-complete.*

Proof. This follows immediately from Theorem 2 and Proposition 1.

5 Discussion

While we have mentioned a few papers that look at imperfect recall, there is another paper that actually tackles complexity in games with imperfect recall. Koller and Megiddo were primarily concerned with the complexity of two-player zero-sum games with perfect recall (which they showed to be polynomially computable), but they began their paper by talking about imperfect recall [11]. Specifically, they showed that for two-player zero-sum games in which only player 1 has imperfect recall it is NP-complete to decide whether player 1 can ensure a payoff of at least λ for behavioral, mixed, and pure strategies. If we allow ourselves to use Nature, we can take advantage of the methods used by Koller and Megiddo to show that EDP-λ is NP-complete for behavioral, mixed, and pure strategies. The reduction is fairly simple:

> Given a 3-SAT instance, nature chooses a clause uniformly at random. Then the decision maker assigns the variables in that clause without knowing which clause has been chosen or what her prior actions were. The decision maker receives a payoff of 1 if the clause is satisfied and 0 otherwise.

It is straightforward to see that the decision maker can achieve a payoff of 1 if and only if the 3-SAT instance is satisfiable. We believe a similar reduction involving Nature exists for EFG-λ.

In the course of this paper, we have addressed three decision problems for both single and multi-player extensive form games with imperfect recall (summarized in Table 1). However, we can expand along both dimensions by looking at two-player zero-sum games and asking about the complexity of finding a NE. We can immediately show that verifying a NE is a zero-sum games is co-NP-complete, and we can even see that the zero-sum versions of EFG-NE and

EFG-λ are equivalently hard (since Proposition 1 applies equally well to zero-sum games). However, it is unclear how to find this remaining complexity using the methods outlined in this paper.

Concerning the complexity of finding a NE, it is not hard to see that finding a NE in a multi-player game is complete for the functional version of Σ_2^{P}, just as the functional version of SAT (FSAT) is complete for FNP. However, the complexity of finding NE in settings other than perfect recall is unknown.

Finally, in our companion paper [13], we present the Distributed Agent Equilibrium (DAE) as an arguably more natural alternative to NE. In fact, it follows from the results in our companion paper that DAE in multi-player games with imperfect recall have similar complexities to NE in games with perfect recall (see the first and third rows of Table 2).

For single-player games, existence still holds and verification still takes polynomial time, but interestingly the problem of deciding whether there exists a DAE that achieves a value of at least λ remains NP-complete. This follows because Piccione and Rubinstein showed that a NE must also be a DAE [16], thus making this problem equivalent to EDP-λ. Because of this, we think it is unlikely that the problem of finding a DAE in a single-player game is in FP; however, it is not obviously PPAD-complete and its complexity remains an open problem.

This gives us a further expanded table as seen in Table 2.

Decision Problem	Existence	Verification	At Least λ	Find One
Perfect Recall NE	O(1)	P	NP-C	PPAD-C [5]
1-player DAE	O(1)	P	**NP-C**	?
n-player DAE	O(1)*	P*	NP-C*	PPAD-C*
1-player NE	O(1)	**co-NP-C**	**NP-C**	?
(2-player) zero-sum NE	?	**co-NP-C**	?	?
n-player NE	$\boldsymbol{\Sigma_2^{\mathrm{P}}}$**-C**	**co-NP-C**	$\boldsymbol{\Sigma_2^{\mathrm{P}}}$**-C**	$\mathbf{F}\boldsymbol{\Sigma_2^{\mathrm{P}}}$**-C**

Table 2. Further extended table of complexity of decision problems for extensive form games. **Bold** entries indicate the contributions of this paper. A "-C" indicates completeness for the preceding complexity class. Entries with a * follow easily from our companion paper and related work [13, 4, 6].

References

1. Robert J. Aumann, Sergiu Hart, and Motty Perry. The absent-minded driver. *Games and Economic Behavior*, 1997.
2. Tamer Başar and Geert Jan Olsder. *Dynamic Noncooperative Game Theory*, pages 52–53. SIAM Series in Classics in Applied Mathematics, third edition, 1999.

3. Pierpaolo Battigalli. Dynamic consistency and imperfect recall. *Games and Economic Behavior*, 20:31–50, 1997.
4. Vincent Conitzer and Tuomas Sandholm. Complexity results about Nash equilibria. In *In Proceedings of the 18th International Joint Conference on Artificial Intelligence (IJCAI-03)*, 2003.
5. Constantinos Daskalakis, Alex Fabrikant, and Cristos H. Papadimitriou. The game world is flat: the complexity of Nash equilibria in succinct games. *Automat, Languages and Programming*, 4051:513–524, 2006.
6. Constantinos Daskalakis, Paul Goldberg, and Christos Papadimitriou. The complexity of computing a nash equilibrium. *SIAM Journal on Computing*, 39(3):489–500, 2009.
7. Itzhak Gilboa. A comment on the absent-minded driver paradox. *Games and Economic Behavior*, 20:25–30, 1997.
8. Itzhak Gilboa and Eitan Zemel. Nash and correlated equilibria: Some complexity considerations. *Games and Economic Behavior*, 1:80–93, 1989.
9. Adam J. Grove and Joseph Y. Halpern. On the expected value of games with absentmindedness. *Games and Economic Behavior*, 1997.
10. Michael Johanson, Kevin Waugh, Michael Bowling, and Martin Zinkevich. Accelerating best response calculation in large extensive games. In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence (IJCAI-11)*, 2011.
11. Daphne Koller and Nimrod Megiddo. The complexity of two-person zero-sum games in extensive form. *Games and Economic Behavior*, 1992.
12. Harold Kuhn. *Entensive Games and the Problem of Information*, volume 2 of *Contributions to the Theory of Games*, pages 193–216. Princeton University Press, 1953.
13. Adrian Marple and Yoav Shoham. Equilibria in finite games with imperfect recall. Available at http://ssrn.com/abstract=2188646, 2013.
14. John Nash. Non-cooperative games. *The Annals of Mathematics*, 54(2):286–295, Sept. 1951.
15. Cristos Papadimitriou. *Computational Complexity.* Addison-Wesley, 1994.
16. Michele Piccione and Ariel Rubinstein. On the interpretation of decision problems with imperfect recall. *Games and Economic Behavior*, 20(1):3–24, July 1997.
17. Philipp C. Wichardt. Existence of Nash equilibria in finite extensive form games with imperfect recall: A counterexample. *Games and Economic Behavior*, 63(1):366–369, May 2008.