Multidimensional Extension of Buffon's Needle Problem

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Abstract

Consider a line segment randomly placed on a two-dimensional plane ruled with a set of regularly spaced parallel lines. The classical Buffon's needle problem asks what the probability is that the line segment intersects at least 1 of these lines. This paper extends this problem by considering a line segment randomly placed in $\mathbb{R}^D$ and its probability of intersection with a set of regularly spaced parallel hyperplanes.

Keywords Buffon's needle problem · Geometric Probability

1 Introduction

Buffon's needle problem was originally posed in the 18th century with the following premise. Given a line segment, or "needle", of length r randomly dropped on a two-dimensional plane ruled with a set of parallel lines regularly spaced s units apart, what is the probability that the needle crosses at least 1 of the lines? The solution, it turns out, is $\frac{2r}{s\pi}$ when $r < s$. Variations and extensions of this problem have been investigated as well, including

- Laplace's Extension - Investigating when the plane is gridded with 2 orthogonal sets of parallel lines with spacings s_1 and s_2.
- Buffon's Noodle - Instead of being rigidly straight, the needle is permitted to bend (a "noodle").
- Pivot Needle - The needle is constructed of two line segments that hinge together. Each crossing is considered.

In this paper, we investigate a particular extension that allows the needle to be dropped into a space with dimension greater than 2. In these higher dimensions, we will rule the space with parallel hyperplanes rather than lines. Additionally, we will look at gridding the space with orthogonal sets of hyperplanes, thereby extending Laplace's extension into higher dimensions.

Given $D \in \mathbb{N}_{>0}$ and $N \in [1, 2, \ldots, D]$, consider a grid on $\mathbb{R}^D$ formed by N orthogonal sets of regularly spaced hyperplanes where each set of hyperplanes has a potentially unique spacing of S_i. For example, if $D = 2, N = 1, S_1 = 2$, the grid would match the original Buffon Needle problem and would have only a single set of parallel lines 2 units apart as seen in 1a. If $D = 2, N = 2, S = [1, 2]$, the grid would have 2 sets of parallel lines that are orthogonal to each other, matching the problem in Laplace's extension as seen in 1b. One set of lines would have a spacing of 1 unit and the other would have a spacing of 2 units.

A line segment of length $r \in \mathbb{R}^+$ is randomly located in the space such that one of its end points, P_0, is uniformly distributed across the entire domain. The line segment's orientation is independently distributed such that when considering P_0 as the center of a $(D-1)$-sphere of radius r, the other point, P_1, is uniformly distributed on the surface of that hypersphere. This line segment may intersect with $C \in \mathbb{N}$ unique hyperplanes. This paper studies the probability that the line segment intersects with at least c hyperplanes, $P(C \geq c | r, D, N, S)$. From there, solutions for crossing less than c hyperplanes and exactly c hyperplanes can be derived.

(a) $D = 2, N = 1, S_1 = 2$, the parameter set matching Buffon's Needle Problem. In this case the number of crossings, C, is 1.

(b) $D = 2, N = 2, S = [2, 1]$, the parameter set matching Laplace's Extension. In this case the number of crossings, C, is 2.

Figure 1: Examples of different parameter sets

We will define the coordinates of line segment using $\vec{x} \in \mathbb{R}^D$ for the location of P_0 and spherical coordinates for the location of P_1 with respect to P_0.

$$
\begin{aligned}
y_1 &= r\cos\phi_1 \\
y_2 &= r\sin\phi_1\cos\phi_2 \\
&\vdots \\
y_{D-1} &= r\sin\phi_1\ldots\sin\phi_{D-2}\cos\phi_{D-1} \\
y_D &= r\sin\phi_1\ldots\sin\phi_{D-2}\sin\phi_{D-1} \\
P_1 &= \vec{x} + \vec{y} \\
\phi_j &\in \begin{cases} [0, \pi] & j < D-1 \\ [0, 2\pi] & j = D-1 \end{cases}
\end{aligned}
$$

Translational symmetry of the grid of hyperplanes allows us to consider the domain of P_0 to be $x_i \in [0, S_i]$ as the origin can be moved to any point on the grid. Reflectional symmetry of the grid also allows us to consider the domain of $\vec{y}$ to be a single orthant of the hypersphere. For convenience, we will pick the orthant where $\phi_i \in [0, \pi/2]$.

The rest of the paper is organized as follows. A derivation of the joint probability density function for P_0 and P_1 will be provided in §2. The derivation and validation of the crossing probabilities for $N = 1$ will be given in §3. The derivation and validation of the crossing probabilities for any N and $r < \min(S)$ will be given in §4. Numeric simulation of the crossing probabilities will be compared to modeled probabilities in §5, along with analysis of the limits and extrema of the probabilities.

2 Joint Probability Density of the Line Segment

Each coordinate for P_0 can be defined as a uniformly distributed random variable $X_i \sim \text{Uniform}(0, S_i)$. Due to independence, the joint PDF for P_0 is the product $\prod_{i=1}^{D} \frac{1}{S_i}$. By the definition of the problem, the coordinates $\vec{x}$ do not influence the orientation of the line segment defined by $\vec{\phi}$. The probability density function for the uniform distribution of points on an orthant of the hypersphere can be determined by calculating the area element in terms of spherical coordinates.

Proposition 1. *In spherical coordinates, the probability density function for a uniform distribution on an orthant of a hypersphere is* $\frac{2^D}{A_{D-1}} \prod_{j=1}^{D-1} r\sin^{D-1-j}\phi_j$ *where* A_{D-1} *is the surface area of a* $(D-1)$*-sphere.*

This doesn't seem like it needs to be a prop as it's only really used once. Maybe the prop should be the PDF of the whole segment

Proof. The area element of an $(D-1)$-sphere of radius r can be expressed as

$$d\Omega = \left(\prod_{j=1}^{D-1} r\sin^{D-1-j}\phi_j\right) d\phi_1 \ldots d\phi_{D-1} \tag{1}$$

The probability that a point lies in this differential element can be expressed as follows.

$$f_\Omega(\Omega)d\Omega = f_{\vec{\phi}}(\phi_1, \ldots, \phi_{D-1})d\phi_1 \ldots d\phi_{D-1} \tag{2}$$

The points are uniformly distributed over the surface of an orthant of the hypersphere implying that $f_\Omega(\Omega) = \frac{2^D}{A_{D-1}}$. Substituting this and 1 into 2 yields

$$\frac{2^D}{A_{D-1}} \prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j = f_{\vec{\phi}}(\phi_1, \ldots, \phi_{D-1}) \tag{3}$$

□

Then by independence, the joint probability density function for the entire line segment can be expressed as

$$f_{\vec{X},\vec{\phi}}(x_1, \ldots, x_D, \phi_1, \ldots, \phi_{D-1}) = \frac{2^D}{A_{D-1}} \left(\prod_{i=1}^{D} \frac{1}{S_i} \right) \left(\prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j \right) \tag{4}$$

The expression for the surface area of a $D-1$ dimensional hypersphere, $A_{D-1} = \frac{2\pi^{D/2} r^{D-1}}{\Gamma(D/2)}$, can be substituted in and simplified.

$$f_{\vec{X},\vec{\phi}}(\vec{x}, \vec{\phi}) = \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2} \prod_{i=1}^{D} S_i} \left(\prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j \right) \tag{5}$$

3 Probability of Crossing with a Single set of Hyperplanes ($N = 1$)

In general, the probability of meeting some number of crossings given any set of parameters can be described as follows.

$$P(C \geq c | r, D, N, S) = \int \cdots \int_V f_{\vec{X}\vec{\phi}}(\phi_1, \ldots, \phi_{D-1}) dx_1 \ldots dx_D d\phi_1 \ldots d\phi_{D-1} \tag{6}$$

$$= \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2} \prod_{i=1}^{D} S_i} \int \cdots \int_V \prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j dx_1 \ldots dx_D d\phi_1 \ldots d\phi_{D-1} \tag{7}$$

Where V is the hypervolume in which the condition $C \geq c$ is true. The necessary conditions for achieving some number of intersections will be called *crossing conditions*. The definition of these crossing conditions and the solution to the above equation will be explored for a variety of parameters. We start with a simplified set of parameters where there is only a single set of parallel hyperplanes and the needle intersects a hyperplane at least c times. That is, the probability $P(C \geq c | r, D, N = 1, S) \forall c, r, D, S$. For brevity, we will refer to this as $P_{N=1}(c)$.

Due to rotational symmetry of the line segment, it does not matter in which direction the hyperplanes extend. Without loss of generality we assume the planes are in the direction of x_1.

Because P_0 is constrained to be within the gridcell at the origin and because the orientation of the needle is constrained to a single orthant which points in the positive direction of x_1, we know that a crossing occurs whenever the following condition is met.

$$x_1 + r \cos \phi_1 > S_1 c \tag{8}$$

From this crossing condition, we define the bounds of the relevant hypervolume. Note that the constraints above only apply to x_1 and ϕ_1. As such, the hypervolume spans the entire domain of every other variable. Importantly, because the integrand of 6 describes a PDF, the conditions for Fubini's theorem hold. Therefore the order of integration can be freely switched so long as any variable limits of integration are accounted for. All integrals with respect to the translational dimensions $x_2, \ldots, x_D$ can be simplified for all $i > 2$. Taking $f(\vec{\phi})$ as the integrand

$$\int_0^{S_i} f(\vec{\phi}) dx_i = S_i f(\vec{\phi}) \tag{9}$$

Similarly, all of the integrals with respect to $\phi_2, \ldots, \phi_{D-1}$ can be simplified as well using the following identity.

$$\int_0^{\pi/2} \sin^{D-1-j} \phi_j d\phi_j = \frac{B(\frac{D-j}{2}, \frac{1}{2})}{2} \tag{10}$$

Where B is the beta function. Substituting 9 and 10 into 7 results in

$$P_{N=1}(c) = \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2}S_1} \prod_{k=2}^{D-1} \frac{B(\frac{D-k}{2}, \frac{1}{2})}{2} \iint_{\phi_1,x_1} \sin^{D-2}\phi_1 dx_1 d\phi_1 \tag{11}$$

The product of beta functions can be simplified by expanding into gamma functions as follows.

$$\prod_{k=2}^{D-1} \frac{B(\frac{D-k}{2}, \frac{1}{2})}{2} = \frac{1}{2^{D-2}} \frac{\Gamma(\frac{D-2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{D-1}{2})} \frac{\Gamma(\frac{D-3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{D-2}{2})} \cdots \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{2})} \tag{12}$$

$$= \frac{1}{2^{D-2}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})^{D-2}}{\Gamma(\frac{D-1}{2})} = \frac{\pi^{(D-1)/2}}{2^{D-2}\Gamma(\frac{D-1}{2})} \tag{13}$$

Substituting 13 into 11 yields

$$P_{N=1}(c) = \frac{2}{S_1 B(\frac{D-1}{2}, \frac{1}{2})} \iint_{\phi_1,x_1} \sin^{D-2}\phi_1 dx_1 d\phi_1 \tag{14}$$

For the remaining integrals, the limits of integration are defined by the domain in which the crossing conditions are satisfied. This can be derived by combining the domains for the variables and the domain for the crossing condition. Recall the following

$$0 < x_1 < S_1$$
$$0 < \phi_1 < \frac{\pi}{2}$$
$$x_1 + r\cos\phi_1 > S_1 c$$

Combining these inequalities results in the following domain where the crossing conditions are satisfied.

$$\max(0, S_1 c - r\cos\phi_1) < x_1 < S_1 \tag{15}$$

$$0 < \phi_1 < \min(\frac{\pi}{2}, \arccos\frac{S_1 c - x_1}{r}) \tag{16}$$

$$r > \frac{S_1 c - x_1}{\cos\phi_1} \tag{17}$$

The min function in 16 can be simplified to $\arccos\frac{S_1 c - x_1}{r}$ as the conditions for the alternative are only possible in the trivial case where $c = 0$ as shown below.

$$m_{\phi_1}(x_1) = \min(\frac{\pi}{2}, \arccos\frac{S_1 c - x_1}{r}) = \begin{cases} \frac{\pi}{2} & x_1 > S_1 c \\ \arccos\frac{S_1 c - x_1}{r} & \text{otherwise} \end{cases} \tag{18}$$

$$= \arccos\frac{S_1 c - x_1}{r} \tag{19}$$

The final inequality, 17, provides a lower bound for the parameter r. The minimum of $\frac{S_1 c - x_1}{\cos\phi_1}$ occurs at $x_1 = S_1, \phi_1 = 0$ with a value of $S_1(c-1)$. Therefore if $r \leq S_1(c-1)$ we can guarantee that the crossing condition cannot be satisfied. This is equivalent to the scenario where the needle is too short to cross the necessary number of hyperplanes, even when it is oriented orthogonally to the hyperplanes.

$$P_{N=1}(C \geq c | r < S_1(c-1)) = 0 \tag{20}$$

The max function found in 15 also depends on the length of the needle, r.

$$m_{x_1}(\phi_1) = \max(0, S_1 c - r\cos\phi_1) = \begin{cases} 0 & r > \frac{S_1 c}{\cos\phi_1} \\ S_1 c - r\cos\phi_1 & \text{otherwise} \end{cases} \tag{21}$$

This partitions the problem into three regions depending on the value of r.

$$0 < r \leq S_1(c-1) \implies P_{N=1}(c) = 0 \tag{22}$$

$$S_1(c-1) < r \leq S_1 c \implies m_{x_1}(\phi_1) = S_1 c - r\cos\phi_1 \forall\phi_1 \tag{23}$$

$$S_1 c < r \implies m_{x_1}(\phi_1) = \max(0, S_1 c - r\cos\phi_1) \tag{24}$$

The probability $P_{N=1}(c)$ will be derived for the two cases given in 23 and 24 in §3.1 and §3.2 respectively. Numeric simulation and comparison will be explored in §5

3.1 $S_1(c-1) < r < S_1 c$

Using the conditions 15 and 16, the limits of integration can be defined for the expression found in 14.

$$P_{N=1}(c) = \frac{2}{S_1 B(\frac{D-1}{2}, \frac{1}{2})} \int_0^{m_{\phi_1}(x_1)} \int_{m_{x_1}(\phi_1)}^{S_1} \sin^{D-2}\phi_1 dx_1 d\phi_1 \tag{25}$$

Simplifying with 23 and the upper bound for x_1 yields the following.

$$P_{N=1}(c) = \frac{2}{S_1 B(\frac{D-1}{2}, \frac{1}{2})} \int_0^{\arccos\frac{S_1(c-1)}{r}} \int_{S_1 c - r\cos\phi_1}^{S_1} \sin^{D-2}\phi_1 dx_1 d\phi_1 \tag{26}$$

$$= \frac{2}{S_1 B(\frac{D-1}{2}, \frac{1}{2})} \int_0^{\arccos\frac{S_1(c-1)}{r}} (S_1(1-c) + r\cos\phi_1)\sin^{D-2}\phi_1 d\phi_1 \tag{27}$$

Substituting in $\gamma = \frac{S_1(c-1)}{r}$ yields the following

$$P_{N=1}(c) = \frac{2r}{S_1 B(\frac{D-1}{2}, \frac{1}{2})} \int_0^{\arccos\gamma} (-\gamma + \cos\phi_1)\sin^{D-2}\phi_1 d\phi_1 \tag{28}$$

$$= \frac{2r}{S_1 B(\frac{D-1}{2}, \frac{1}{2})} \left[-\gamma \int_0^{\arccos\gamma} \sin^{D-2}\phi_1 d\phi_1 + \frac{1}{D-1}(1-\gamma^2)^{\frac{D-1}{2}} \right] \tag{29}$$

To solve, we derive an expression for the remaining integral. Before doing so, it is convenient to derive an expression for the following ratio of double factorials.

Proposition 2. *When given the ratio $(k-1)!!/k!!$ where the double exclam represents the double factorial function, it is equivalent the following.*

$$= \begin{cases} \frac{1}{\pi} B(\frac{k+1}{2}, \frac{1}{2}) & k \bmod 2 = 0 \\ \frac{1}{2} B(\frac{k+1}{2}, \frac{1}{2}) & k \bmod 2 = 1 \end{cases} \tag{30}$$

Proof. Every double factorial can be written in terms of single factorials depending on the parity of the natural number in question.

$$n!! = \begin{cases} 2^{n/2}\left(\frac{n}{2}\right)! & n \bmod 2 = 0 \\ \frac{n!}{2^{(n-1)/2}(\frac{n-1}{2})!} & n \bmod 2 = 1 \end{cases} \tag{31}$$

This single factorial representation can be used to simplify $(k-1)!!/k!!$. First, assuming that k is even

$$\frac{(k-1)!!}{k!!} = \frac{(k-1)!}{2^{(k-2)/2}(\frac{k-2}{2})!} \frac{1}{2^{k/2}(\frac{k}{2})!} \tag{32}$$

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{\frac{1}{2}\frac{2}{2}\cdots\frac{k-2}{2}\frac{k-1}{2}}{\frac{2}{2}\frac{4}{2}\cdots\frac{k-4}{2}\frac{k-2}{2}(\frac{k}{2}!)} \tag{33}$$

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{\frac{1}{2}\frac{3}{2}\cdots\frac{k-3}{2}\frac{k-1}{2}}{\frac{k}{2}!} \tag{34}$$

Using the property $n\Gamma(n) = \Gamma(n+1)$ and $n! = \Gamma(n+1)$, the sequence of fractions can be reduced iteratively.

$$\frac{(k-1)!!}{k!!} = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{k+2}{2})} \tag{35}$$

Finally, using $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $\Gamma(1/2) = \sqrt{\pi}$ results in the following.

$$\frac{(k-1)!!}{k!!} = \frac{1}{\pi} B\left(\frac{k+1}{2}, \frac{1}{2}\right) \tag{36}$$

We now repeat the process for the case where k is odd.

$$
\begin{aligned}
\frac{(k-1)!!}{k!!} &= \left(\frac{k-1}{2}\right)!2^{(k-1)/2}\frac{(\frac{k-1}{2})!2^{(k-1)/2}}{k!} && (37)\\
&= \frac{2\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2})}\frac{2^{k-1}(\frac{k-1}{2})!^2}{k!} = \frac{\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2})}\frac{\frac{2}{2}\frac{4}{2}\dots\frac{k-3}{2}\frac{k-1}{2}(\frac{k-1}{2}!)}{\frac{1}{2}\frac{2}{2}\dots\frac{k-1}{2}\frac{k}{2}} && (38)\\
&= \frac{\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2})}\frac{\frac{k-1}{2}!}{\frac{1}{2}\frac{3}{2}\dots\frac{k-2}{2}\frac{k}{2}} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{k+1}{2})}{2\Gamma(\frac{k+2}{2})} && (39)\\
&= \frac{1}{2}B\left(\frac{k+1}{2},\frac{1}{2}\right) && (40)
\end{aligned}
$$

□

The solution for the remaining integral in 29 can now be solved in general for any whole number exponential of the sin function.

Proposition 3. *Any integral of the form $\int_0^{\arccos(\gamma)}\sin^m\phi d\phi$ has two possible solutions depending on the parity of m.*

$$
= \frac{B(\frac{m+1}{2},\frac{1}{2})}{2}\left(g(\gamma,m)-\gamma(1-\gamma^2)^{(m+1)/2}\sum_{i=1}^{\lfloor m/2\rfloor}\frac{B(\frac{m+2-2i}{2},\frac{1}{2})}{\pi(1-\gamma^2)^i}\right) \tag{41}
$$

$$
g(\gamma,m) = \begin{cases}\frac{2}{\pi}\arccos\gamma & m \bmod 2 = 0\\ 1-\gamma & m \bmod 2 = 1\end{cases} \tag{42}
$$

Proof. We start with the following integration by reduction identity

$$
\begin{aligned}
\int_0^{\arccos\gamma}\sin^m\phi d\phi &= -\frac{1}{m}\sin^{m-1}\phi\cos\phi\Big|_0^{\arccos\gamma} + \frac{m-1}{m}\int_0^{\arccos\gamma}\sin^{m-2}\phi d\phi && (43)\\
&= -\frac{1}{m}(1-\gamma^2)^{(m-1)/2}\gamma \\
&\quad + \frac{m-1}{m}\left(-\frac{1}{m-2}\sin^{m-3}\phi\cos\phi\Big|_0^{\arccos\gamma} + \frac{m-3}{m-2}\int_0^{\arccos\gamma}\sin^{m-4}\phi d\phi\right) && (44)
\end{aligned}
$$

This pattern continues until the sin in the final integrand is raised to either the first or zeroth power. This depends on whether m is even or odd. If m is even

$$
\begin{aligned}
&= -\frac{1}{m}(1-\gamma^2)^{(m-1)/2}\gamma - \frac{m-1}{m(m-2)}(1-\gamma^2)^{(m-3)/2}\gamma - \dots \\
&\quad - \frac{(m-1)(m-3)\dots(3)}{(m)(m-2)\dots(2)}(1-\gamma^2)^{1/2}\gamma + \frac{(m-1)!!}{m!!}\int_0^{\arccos\gamma}d\phi && (45)\\
&= \frac{(m-1)!!}{m!!}\left(\arccos\gamma - \gamma(1-\gamma^2)^{(m+1)/2}\sum_{i=1}^{m/2}\frac{(m-2i)!!}{(m+1-2i)!!}(1-\gamma^2)^{-i}\right) && (46)
\end{aligned}
$$

Using 2 we can reduce to the following.

$$
\begin{aligned}
&= \frac{B(\frac{m+1}{2},\frac{1}{2})}{\pi}\left(\arccos\gamma - \gamma(1-\gamma^2)^{(m+1)/2}\sum_{i=1}^{m/2}\frac{B(\frac{m+2-2i}{2},\frac{1}{2})}{2}(1-\gamma^2)^{-i}\right) && (47)\\
&= \frac{B(\frac{m+1}{2},\frac{1}{2})}{2}\left(\frac{2}{\pi}\arccos\gamma - \gamma(1-\gamma^2)^{(m+1)/2}\sum_{i=1}^{m/2}\frac{B(\frac{m+2-2i}{2},\frac{1}{2})}{\pi(1-\gamma^2)^i}\right) && (48)
\end{aligned}
$$

Repeating for the case where m is odd

$$= -\frac{1}{m}(1-\gamma^2)^{(m-1)/2}\gamma - \frac{m-1}{m(m-2)}(1-\gamma^2)^{(m-3)/2}\gamma - \dots \\ - \frac{(m-1)(m-3)\dots(3)}{(m)(m-2)\dots(2)}(1-\gamma^2)^{1/2}\gamma + \frac{(m-1)!!}{m!!}\int_0^{\arccos\gamma}\sin\phi d\phi \tag{49}$$

$$= \frac{(m-1)!!}{m!!}\left(1-\gamma-\gamma(1-\gamma^2)^{(m+1)/2}\sum_{i=1}^{(m-1)/2}\frac{(m-2i)!!}{(m+1-2i)!!}(1-\gamma^2)^{-i}\right) \tag{50}$$

$$= \frac{B(\frac{m+1}{2},\frac{1}{2})}{2}\left(1-\gamma-\gamma(1-\gamma^2)^{(m+1)/2}\sum_{i=1}^{\lfloor m/2\rfloor}\frac{B(\frac{m+2-2i}{2},\frac{1}{2})}{\pi(1-\gamma^2)^i}\right) \tag{51}$$

□

Using the result of 3 on 29 yields the following

$$P_{N=1}(c) = \frac{2r}{S_1B(\frac{D-1}{2},\frac{1}{2})}\left[-\gamma\frac{B(\frac{D-1}{2},\frac{1}{2})}{2}\left(g(\gamma,D-2)-\gamma(1-\gamma^2)^{\frac{D-1}{2}}\sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2i}{2},\frac{1}{2})}{\pi(1-\gamma^2)^i}\right)+\frac{1}{D-1}(1-\gamma^2)^{\frac{D-1}{2}}\right] \tag{52}$$

$$= \frac{r}{S_1}\left[-\gamma\left(g(\gamma,D-2)-\gamma(1-\gamma^2)^{\frac{D-1}{2}}\sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2i}{2},\frac{1}{2})}{\pi(1-\gamma^2)^i}\right)+\frac{B(\frac{D}{2},\frac{1}{2})}{\pi}(1-\gamma^2)^{\frac{D-1}{2}}\right] \tag{53}$$

$$= \frac{r}{S_1}\left[\frac{(1-\gamma^2)^{\frac{D-1}{2}}}{\pi}\left(B\left(\frac{D}{2},\frac{1}{2}\right)+\gamma^2\sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2i}{2},\frac{1}{2})}{(1-\gamma^2)^i}\right)-\gamma g(\gamma,D-2)\right] \tag{54}$$

As an example, setting the parameters relevant for the classical 2 dimensional Buffon needle problem results in the expected probability.

$$c = 1, D = 2, N = 1, S = s \tag{55}$$

$$\gamma = \frac{s(c-1)}{r} = 0 \tag{56}$$

$$P_{N=1}(c) = \frac{r}{s}\left[\frac{1}{\pi}\left(\frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}+0\right)-0\right] \tag{57}$$

$$= \frac{2r}{\pi s} \tag{58}$$

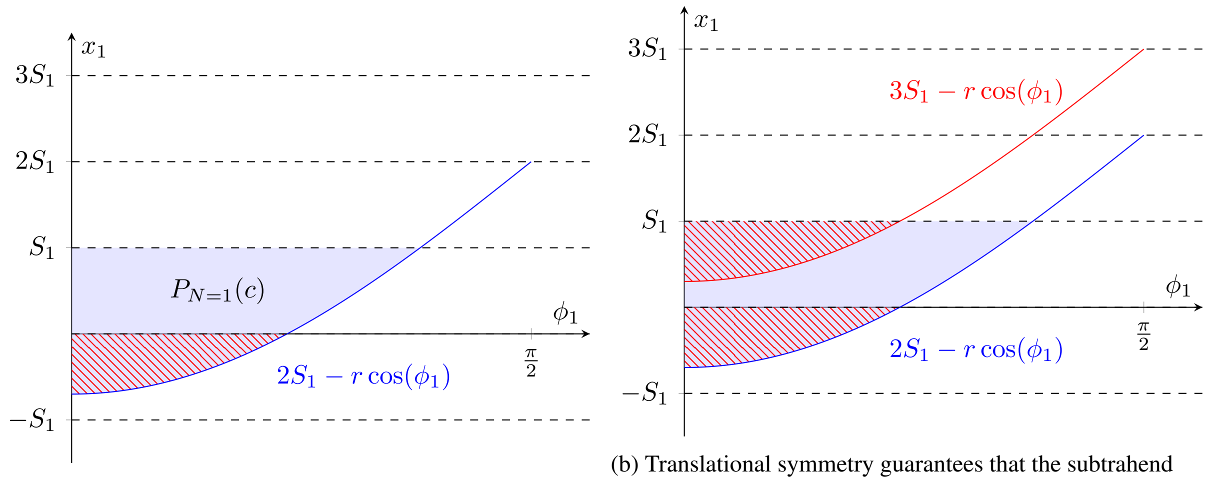
Additional validation and comparison to numeric simulation will be explored in §5

3.2 $r > S_1c$

When $r > S_1c$, the value of $m_{x_1}(\phi_1)$ is no longer constant for all ϕ_1. Below is a restatement of the general equation for $P(C \geq c)$ when $N = 1$, 14, and the domain of the crossing conditions.

$$P_{N=1}(c) = \frac{2}{S_1B(\frac{D-1}{2},\frac{1}{2})}\iint_{\phi_1,x_1}\sin^{D-2}\phi_1 dx_1 d\phi_1$$
$$\max(0, S_1c - r\cos\phi_1) < x_1 < S_1$$
$$0 < \phi_1 < \arccos\frac{S_1c - x_1}{r}$$

There are now two things to note. First, the integrand only varies with ϕ_1, suggesting that the directional derivative of the PDF in the direction of x_1 is zero. Second, when $r > S_1c$, the domain of the crossing condition can be calculated as the integral spanning the domain $S_1c - r\cos\phi_1 < x_1 < 0$ subtracted from the integral of the domain spanning $S_1c - r\cos\phi_1 < x_1 < S_1$ as seen in Figure 2a. This difference is explicitly written as follows.



(a) The integral spanning the domain of the crossing condition can be found as a difference between two regions.

(b) Translational symmetry guarantees that the subtrahend is equivalent to the "short needle" probability of crossing an additional hyperplane.

Figure 2

$$\int_0^{\phi_{1,h}} \int_{m_{x_1}(\phi_1)}^{S_1} \sin^{D-2}\phi_1 dx_1 d\phi_1 = \int_0^{\phi_{1,h}} \int_{S_1 c - r\cos\phi_1}^{S_1} \sin^{D-2}\phi_1 dx_1 d\phi_1 - \int_0^{\phi_{1,l}} \int_{S_1 c - r\cos\phi_1}^{0} \sin^{D-2}\phi_1 dx_1 d\phi_1 \tag{59}$$

$$I_1 = I_2 - I_3 \tag{60}$$

$$\phi_{1,h} = \arccos\frac{S_1(c-1)}{r} \tag{61}$$

$$\phi_{1,l} = \arccos\frac{S_1 c}{r} \tag{62}$$

The integral I_2 is of the same form as the short needle case and can therefore be substituted with the solution 54. Using symmetry of the PDF in the x_1 direction, I_3 can be rewritten with the limits of integration spanning $S_1(c+1) - r\cos\phi_1 < x_1 < S_1$. This is, again, exactly the problem of the short needle case but with $c+1$ instead of c. This is visualized in Figure 2b As such, the result is

$$P_{N=1}(c|r > S_1 c) = A(c) - A(c+1) \tag{63}$$

$$A(k) = \frac{r}{S_1}\left[\frac{(1-\gamma^2)^{\frac{D-1}{2}}}{\pi}\left(B\left(\frac{D}{2},\frac{1}{2}\right) + \gamma^2 \sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor} \frac{B(\frac{D-2i}{2},\frac{1}{2})}{(1-\gamma^2)^i}\right) - \gamma g(\gamma, D-2)\right] \tag{64}$$

$$\gamma = \frac{S_1(k-1)}{r} \tag{65}$$

$$g(n,m) = \begin{cases} \frac{2}{\pi}\arccos(n) & m \bmod 2 = 0 \\ 1-n & m \bmod 2 = 1 \end{cases} \tag{66}$$

4 Probability of crossing $N \geq 1$

When there is only a single set of parallel hyperplanes, there is only one way for a needle to make c intersections. The needle would have to go through c hyperplanes in a single direction. When we increase the number of orthogonal sets of hyperplanes then we must deal with the fact that there are now many ways to cross c hyperplanes due to the many combinations of directions available.

For instance, if $N = 2$ and we want to know when $C = 2$, then a valid number of crossings occurs if the needle crosses 2 hyperplanes in x_1 and 0 in x_2, or 1 hyperplane in each direction, or 0 hyperplanes in x_1 and 2 in x_2.

For simplicity, we begin with the assumption that $r < \min(S)$ to ensure that the needle can never cross more than 1 hyperplane in any given direction.

4.1 $N \geq 1, r < \min(S)$

Let $P_{1\cap 2\cap \ldots \cap h}(C = h|r, D, N, S) = P_{1\cap \ldots \cap h}$ be the probability that the needle crosses exactly 1 hyperplane in every direction $x_1, x_2, \ldots, x_h$. Similarly, let $P_{1\cup 2\cup \ldots \cup h}(C \geq 1|r, D, N, S) = P_{1\cup \ldots \cup h}$ be the probability that the needle crosses at least 1 hyperplane total in any direction $x_1, x_2, \ldots, x_h$. This probability is equivalent to the probability that $P(C \geq 1|r, D, N, S)$ as crossing a hyperplane in any direction is sufficient to meet the condition $C \geq 1$. Using the inclusion-exclusion principle, this probability can be written as the following sum.

$$P(C \geq 1|r, D, N, S) = P_{1\cup 2\cup \ldots \cup N} = \sum_{k=1}^{N}(-1)^{k+1}\left(\sum_{1\leq i_1<\ldots<i_k\leq N} P_{i_1\cap \ldots \cap i_k}\right) \tag{67}$$

Similarly, for $C \geq c$, we can define the set of events E_c^N which consists of each of the $\binom{N}{c}$ hyperplane crossing combinations. For example, $E_2^3 = \{(1, 2), (1, 3), (2, 3)\}$. If the needle crosses hyperplanes in all of the directions listed in any element of E_c^N, then the crossing condition for the criteria $C \geq c$ has been met.

$$P(C \geq c|r, D, N, S) = P_{(\cap E_c^N[1])\cup(\cap E_c^N[2])\cup \ldots \cup(\cap E_c^N[\binom{N}{c}])} = \sum_{k=1}^{N}(-1)^{k+1}\left(\sum_{1\leq i_1<\ldots<i_k\leq N} P_{E_c^N[i_1]\cap \ldots \cap E_c^N[i_k]}\right) \tag{68}$$

This expression requires an equation for the probability of having at least 1 crossing in each direction listed.

Proposition 4. *For any given set of hyperplane directions,* H*, the probability that a needle would cross at least 1 hyperplane in each of the specified directions can be represented as follows.*

$$P_{H_1\cap \ldots \cap H_h} = \frac{r^h}{\pi^{h/2}(\prod_{i=1}^{h} S_{H,i})}\frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D+h}{2})} \tag{69}$$

Proof. The set of hyperplane directions, H, with spacings S_H is a subset of all the hyperplanes that grid the space. Without loss of generalization, the axes can be relabeled to align H_1 with x_1, H_2 with x_2 and so on. All other hyperplanes that are not included in the set H can be ignored as any intersections with them are irrelevant.

The necessary conditions for crossings to occur in each direction specified in H is as follows

$$S_1 \leq x_1 + r\cos\phi_1 \tag{70}$$
$$S_2 \leq x_2 + r\sin\phi_1\cos\phi_2 \tag{71}$$
$$\vdots \tag{72}$$
$$S_{h-1} \leq x_{h-1} + r\sin\phi_1\sin\phi_2\ldots\sin\phi_{h-2}\cos\phi_{h-1} \tag{73}$$
$$S_h \leq x_h + \begin{cases} r\sin\phi_1\ldots\sin\phi_{h-1}\cos\phi_h & h < D \\ r\sin\phi_1\ldots\sin\phi_{h-2}\sin\phi_{h-1} & h = D \end{cases} \tag{74}$$

These conditions, along with the domain of $x_i \forall i \in 1, \ldots, h$, define the bounds of the volume where the needle crosses a hyperplane in each direction H.

$$S_1 \geq x_1 \geq m_1(\phi_1) = \max\{0, S_1 - r\cos\phi_1\} \tag{75}$$
$$S_2 \geq x_2 \geq m_2(\phi_2) = \max\{0, S_2 - r\sin\phi_1\cos\phi_2\} \tag{76}$$
$$\vdots \tag{77}$$
$$S_{h-1} \geq x_{h-1} \geq m_{h-1}(\phi_{h-1}) = \max\{0, S_{h-1} - r\sin\phi_1\ldots\sin\phi_{h-2}\cos\phi_{h-1}\} \tag{78}$$
$$S_h \geq x_h \geq m_h(\phi_h) = \max\left\{0, S_h - \begin{cases} r\sin\phi_1\ldots\sin\phi_{h-1}\cos\phi_h & h < D \\ r\sin\phi_1\ldots\sin\phi_{h-2}\sin\phi_{h-1} & h = D \end{cases}\right\} \tag{79}$$

Starting with 7, the crossing conditions above are encoded into the bounds of integration. As in the $N = 1$ case, the order of integration is arbitrary so long as the limits of integration are not variable. Therefore the integrals with respect to the spatial dimensions greater than h are reduced to a coefficient.

$$P_{H_1\cap\ldots\cap H_h} = \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{D} S_i}\int\cdots\int_V \prod_{j=1}^{D-1}\sin^{D-1-j}\phi_j dV \tag{80}$$

$$= \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{h} S_i}\int\cdots\int_\phi \int_{m_h(\phi_h)}^{S_h}\cdots\int_{m_1(\phi_1)}^{S_1}\prod_{j=1}^{D-1}\sin^{D-1-j}\phi_j dx_1\ldots dx_h d\phi_1\ldots d\phi_{D-1} \tag{81}$$

Given that r is less than every spacing S_i, every function $m_i(\phi_i)$ is guaranteed to be greater than zero. Every spatial integral will reduce to the polar representation of the corresponding x_i. This simplifies to the following.

$$P_{H_1\cap\ldots\cap H_h} = \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{h} S_i}\int\cdots\int_\phi r^h\left(\prod_{k=1}^{\min(D-1,h)}\cos\phi_k\sin^{h-k}\phi_k\right)\prod_{j=1}^{D-1}\sin^{D-1-j}\phi_j d\phi_1\ldots d\phi_{D-1} \tag{82}$$

$$= \frac{2^{D-1}r^h\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{h} S_i}\int\cdots\int_\phi\left(\prod_{k=1}^{\min(D-1,h)}\cos\phi_k\sin^{D+h-2k-1}\phi_k\right)\prod_{j=h+1}^{D-1}\sin^{D-1-j}\phi_j d\phi_1\ldots d\phi_{D-1} \tag{83}$$

The product from $k = 1$ to $\min(D-1, h)$ can be reduced by using u-substitution where $u = \sin\phi_k$. Assuming that $h \le D-1$, this results as follows

$$P_{H_1\cap\ldots\cap H_h} = \frac{2^{D-1}r^h\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{h} S_i}\int\cdots\int_0^{\pi/2}\frac{1}{(D+h-2)(D+h-4)\ldots(D-h)}\prod_{j=h+1}^{D-1}\sin^{D-1-j}\phi_j d\phi_{h+1}\ldots d\phi_{D-1} \tag{84}$$

$$= \frac{2^{D-1}r^h\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{h} S_i}\frac{(D-h-2)!!}{(D+h-2)!!}\int\cdots\int_0^{\pi/2}\prod_{j=h+1}^{D-1}\sin^{D-1-j}\phi_j d\phi_{h+1}\ldots d\phi_{D-1} \tag{85}$$

$$= \frac{2^{D-1}r^h\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{h} S_i}\frac{\Gamma(\frac{D-h}{2})}{2^h\Gamma(\frac{D+h}{2})}\prod_{j=h+1}^{D-1}\frac{B(\frac{D-j}{2},\frac{1}{2})}{2} \tag{86}$$

$$= \frac{r^h}{\pi^{h/2}\prod_{i=1}^{h} S_i}\frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D+h}{2})} \tag{87}$$

$$\tag{88}$$

If h is greater than $D-1$ (ie. $h = D$), the result remains the same.

$$P_{H_1\cap\ldots\cap H_h} = \frac{2^D r^{D+h-1}}{A_{D-1}\prod_{i=1}^{h} S_i}\frac{1}{(D+h-2)(D+h-4)\ldots(4)(2)} \tag{89}$$

$$= \frac{2^D r^h\Gamma(\frac{D}{2})}{2\pi^{D/2}\prod_{i=1}^{h} S_i}\frac{1}{(D+h-2)!!} \tag{90}$$

$$= \frac{2^D r^h\Gamma(\frac{D}{2})}{2\pi^{D/2}\prod_{i=1}^{h} S_i}\frac{1}{2^{(D+h-2)/2}\Gamma(\frac{D+h}{2})} \tag{91}$$

$$= \frac{r^h}{\pi^{h/2}\prod_{i=1}^{h} S_i}\frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D+h}{2})} \tag{92}$$

$$\tag{93}$$

□

5 Numeric Simulation of Crossing and Other Conditions when $N = 1$

To summarize, the probability that a randomly placed line segment will cross at least c hyperplanes given that there is 1 set of parallel hyperplanes with spacing S_1 is as follows

$$P(C \geq c|r, D, N = 1, S) = \begin{cases} 0 & r < S(c-1) \\ A(c) & S_1(c-1) < r < S_1 c \\ A(c) - A(c+1) & r > S_1 c \end{cases}$$

$$A(k) = \frac{r}{S_1}\left[\frac{(1-\gamma^2)^{\frac{D-1}{2}}}{\pi}\left(B\left(\frac{D}{2}, \frac{1}{2}\right) + \gamma^2 \sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor} \frac{B(\frac{D-2i}{2}, \frac{1}{2})}{(1-\gamma^2)^i}\right) - \gamma g(\gamma, D-2)\right]$$

$$\gamma = \frac{S_1(k-1)}{r}, \; g(n, m) = \begin{cases} \frac{2}{\pi}\arccos(n) & m \bmod 2 = 0 \\ 1 - n & m \bmod 2 = 1 \end{cases}$$

To compare this against numeric simulation, many samples of a randomly placed needle must be generated. The initial point of the needle, P_0, is simulated using a uniformly distributed random variable in the domain $(\vec{0}, S)$ and has coordinates $\vec{x} \in \mathbb{R}^D$. For the other point of the needle, P_1, samples with uniform spherical distribution must be generated for higher dimensional space. We use the method proposed by Marsaglia of normalizing a rotationally symmetric distribution (such as a D-dimensional gaussian variable) and label these coordinates as $\vec{y} \in \mathbb{R}^D$. For a needle of length r, the point P_1 is then at the point $\vec{x} + r\vec{y}$.

ref

The number of hyperplanes crossed by the needle in each coordinate is then of the following form.

$$c = \sum_{n=1}^{N} \left| \left\lfloor \frac{x_n + r y_n}{S_n} \right\rfloor \right| \tag{94}$$

The expected value of the probability can then be approximated by simulating many needles, checking how many have satisfied the number of crossings, and dividing by the total number of simulations. The above procedure was simulated for 100,000 randomly generated needles for each permutation of parameters listed below[1] and results shown in Figure 3. The probabilities were calculated for $c \in \{1, 2, 3, 4\}$, needle length $r \in (0, 10]$, dimensions $D \in [2, 10]$, a single hyperplane with spacing $S_1 = 1$.

Similarly, the count of intersections can be compared to an exact number of crossings as well. The modeled probability can be calculated as follows.

$$P(C = c|r, D, N = 1, S) = P(C \geq c|r, D, N = 1, S) - P(C \geq c + 1|r, D, N = 1, S) \tag{95}$$

The simulated and modeled probabilities can be found in Figure 4. In the exact crossing condition, there is a needle length that maximizes the probability that a certain number of crossings occur. This can be calculated by taking the derivative of the expression for $P(C = c)$ with respect to r and setting it equal to zero.

$$\frac{d}{dr}\gamma(k) = -\frac{S_1(k-1)}{r^2} \tag{96}$$

$$\frac{d}{dr}A(k) = \frac{A(k)}{r} + \frac{r}{S_1}\left(\right) \tag{97}$$

$$\frac{d}{dr}P(C = c|r, D, N = 1, S) = 0 = \begin{cases} 0 & r < S_1(c-1) \\ \frac{d}{dr}A(c) - 0 & S_1(c-1) < r < S_1 \\ \frac{d}{dr}A(c) - \frac{d}{dr}(2A(c+1)) & S_1 c < r < S_1(c+1) \\ \frac{d}{dr}A(c) - \frac{d}{dr}(2A(c+1)) + \frac{d}{dr}A(c+2) & r > S_1(c+1) \end{cases} \tag{98}$$

[1]For a total of 363.6 million needles

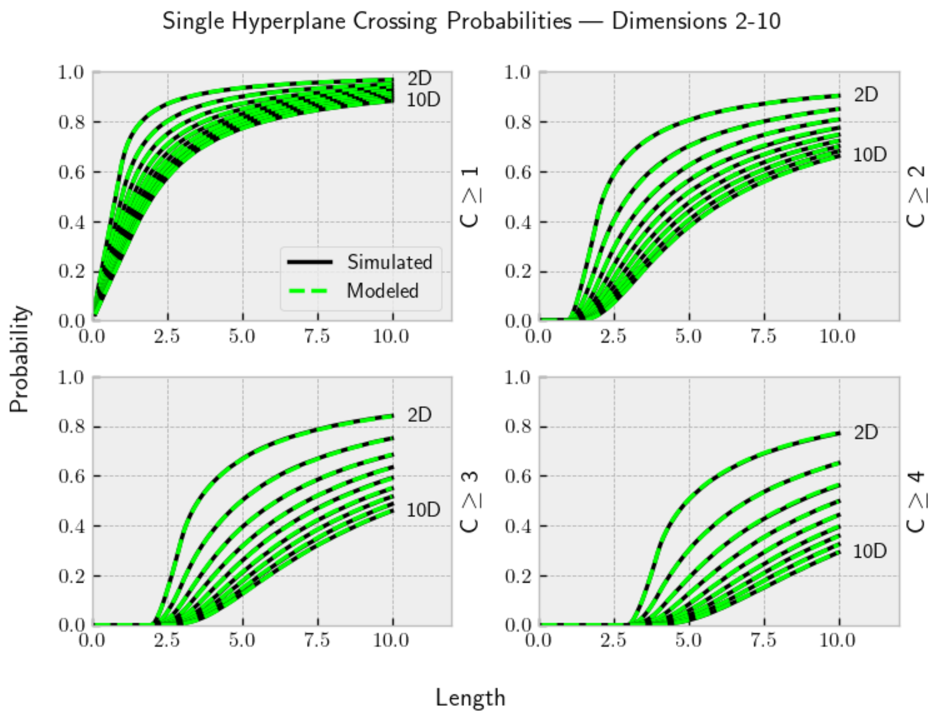


Figure 3: Comparison of numerically simulated and modeled probabilities for a variety of parameters. 100,000 needles were simulated for every parameter permutation.

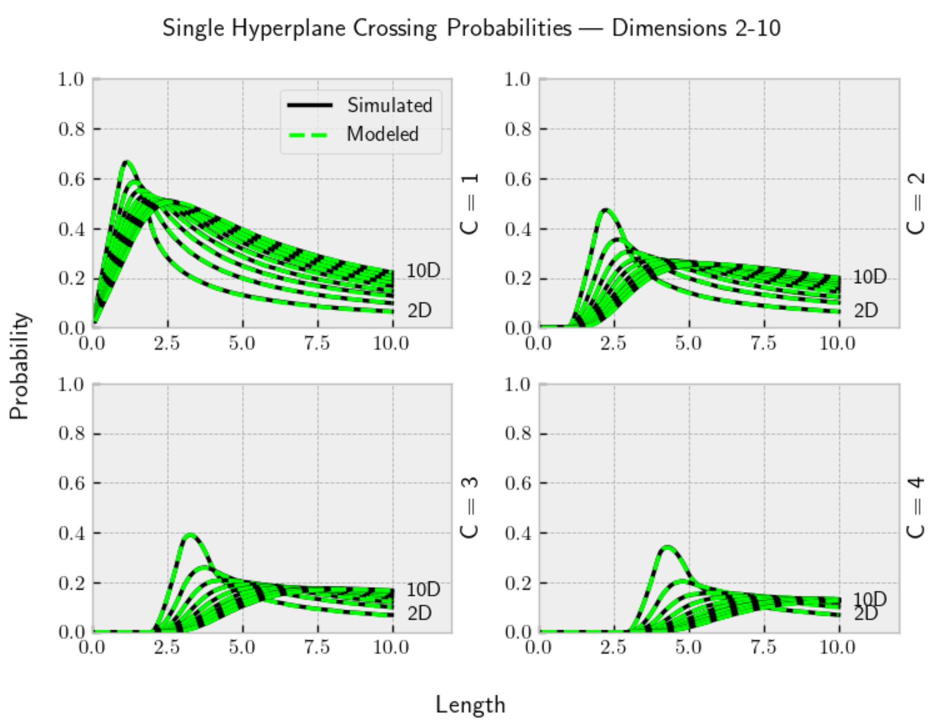


Figure 4: Numeric and modeled probability for exactly c needle crossings.

6 Examples of citations, figures, tables, references

6.1 Citations

Citations use `natbib`. The documentation may be found at

`http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf`

Here is an example usage of the two main commands (`citet` and `citep`): Some people thought a thing [Kour and Saabne, 2014a, Hadash et al., 2018] but other people thought something else [Kour and Saabne, 2014b]. Many people have speculated that if we knew exactly why Kour and Saabne [2014b] thought this…

The documentation for `booktabs` ('Publication quality tables in LaTeX') is available from:

`https://www.ctan.org/pkg/booktabs`

References

George Kour and Raid Saabne. Real-time segmentation of on-line handwritten arabic script. In *Frontiers in Handwriting Recognition (ICFHR), 2014 14th International Conference on*, pages 417–422. IEEE, 2014a.

Guy Hadash, Einat Kermany, Boaz Carmeli, Ofer Lavi, George Kour, and Alon Jacovi. Estimate and replace: A novel approach to integrating deep neural networks with existing applications. *arXiv preprint arXiv:1804.09028*, 2018.

George Kour and Raid Saabne. Fast classification of handwritten on-line arabic characters. In *Soft Computing and Pattern Recognition (SoCPaR), 2014 6th International Conference of*, pages 312–318. IEEE, 2014b. doi:10.1109/SOCPAR.2014.7008025.