MULTIDIMENSIONAL EXTENSION OF BUFFON'S NEEDLE PROBLEM

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ABSTRACT

Consider a line segment randomly placed on a two-dimensional plane ruled with a set of regularly spaced parallel lines. The classical Buffon's needle problem asks what the probability is that the line segment intersects at least 1 of these lines. This paper extends this problem by considering a line segment randomly placed in $\mathbb{R}^D$ and its probability of intersection with a set of regularly spaced parallel hyperplanes.

Keywords Buffon's needle problem · Geometric Probability

1 Introduction

Given $D \in \mathbb{N}_{>0}$ and $N \in [1, 2, \ldots, D]$, consider a grid on $\mathbb{R}^D$ formed by N orthogonal sets of regularly spaced hyperplanes. Each set of hyperplanes has a potentially unique spacing of S_i. A line segment of length $r \in \mathbb{R}^+$ is randomly located in the space such that one of its end points, P_0, is uniformly distributed across the entire domain. The line segment's orientation is independently distributed such that when considering P_0 as the center of a $(D-1)$-sphere of radius r, the other point, P_1, is uniformly distributed on the surface of that hypersphere. This line segment may intersect with $C \in \mathbb{N}$ unique hyperplanes. This paper studies the probability of the line segment intersecting more than c hyperplanes, $P(C > c|r, D, N, S)$. From there, solutions for crossing less than c hyperplanes and exactly c hyperplanes can be derived. As an example, the classical Buffon's needle problem would be $P(C > 0|r, 2, 1, S)$. Laplace's extension would be represented as $P(C > 0|r, 2, 2, S)$.

We will define the coordinates of line segment using $\vec{x} \in \mathbb{R}^D$ for the location of P_0 and spherical coordinates for the location of P_1 with respect to P_0.

$$\begin{aligned}
y_1 &= r\cos\phi_1 \\
y_2 &= r\sin\phi_1\cos\phi_2 \\
&\vdots \\
y_{D-1} &= r\sin\phi_1\ldots\sin\phi_{D-2}\cos\phi_{D-1} \\
y_D &= r\sin\phi_1\ldots\sin\phi_{D-2}\sin\phi_{D-1} \\
P_1 &= \vec{x} + \vec{y} \\
\phi_j &\in \begin{cases} [0,\pi] & j < D-1 \\ [0,2\pi] & j = D-1 \end{cases}
\end{aligned}$$

Translational symmetry of the grid of hyperplanes allows us to consider the domain of P_0 to be $x_i \in [0, S_i]$ as the origin can be moved to any point on the grid. Reflectional symmetry of the grid also allows us to consider the domain of $\vec{y}$ to be a single orthant of the hypersphere. For convenience, we will pick the orthant where $\phi_i \in [0, \pi/2]$.

The rest of the paper is organized as follows. A derivation of the joint probability density function for P_0 and P_1 will be provided in §2. The derivation and validation of the crossing probabilities will be given in §3. Analysis of the limits and extrema of the probabilities is explored in §4.

2 Joint Probability Density of the Line Segment

Each coordinate for P_0 can be defined as a uniformly distributed random variable $X_i \sim \mathrm{Uniform}(0, S_i)$. Due to independence, the joint PDF for P_0 is the product $\prod_{i=1}^{D} \frac{1}{S_i}$. By the definition of the problem, the coordinates $\vec{x}$ do not influence the orientation of the line segment defined by $\vec{\phi}$. The probability density function for the uniform distribution of points on an orthant of the hypersphere can be determined by calculating the area element in terms of spherical coordinates.

Proposition 1. *In spherical coordinates, the probability density function for a uniform distribution on an orthant of a hypersphere is* $\frac{2^D}{A_{D-1}} \prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j$ *where* A_{D-1} *is the surface area of a* $(D-1)$*-sphere.*

Proof. The area element of an $(D-1)$-sphere of radius r can be expressed as

$$d\Omega = \left(\prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j \right) d\phi_1 \dots d\phi_{D-1} \tag{1}$$

The probability that a point lies in this differential element can be expressed as follows.

$$f_\Omega(\Omega) d\Omega = f_{\vec{\phi}}(\phi_1, \dots, \phi_{D-1}) d\phi_1 \dots d\phi_{D-1} \tag{2}$$

The points are uniformly distributed over the surface of an orthant of the hypersphere implying that $f_\Omega(\Omega) = \frac{2^D}{A_{D-1}}$. Substituting this and 1 into 2 yields

$$\frac{2^D}{A_{D-1}} \prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j = f_{\vec{\phi}}(\phi_1, \dots, \phi_{D-1}) \tag{3}$$

□

Then by independence, the joint probability density function for the entire line segment can be expressed as

$$f_{\vec{X}, \vec{\phi}}(x_1, \dots, x_D, \phi_1, \dots, \phi_{D-1}) = \frac{2^D}{A_{D-1}} \left(\prod_{i=1}^{D} \frac{1}{S_i} \right) \left(\prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j \right) \tag{4}$$

3 Probability of crossing

In general, the probability of any number of crossings given any set of parameters can be described as follows.

$$P(C > c | r, D, N, S) = \int \cdots \int_V f_{\vec{\phi}}(\phi_1, \dots, \phi_{D-1}) dx_1 \dots dx_D d\phi_1 \dots d\phi_{D-1} \tag{5}$$

$$= \frac{2^D r^{D-1}}{A_{D-1} \prod_{i=1}^{D} S_i} \int \cdots \int_V \prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j dx_1 \dots dx_D d\phi_1 \dots d\phi_{D-1} \tag{6}$$

Where V is the hypervolume in which some sort of crossing condition is met. The definition of these crossing conditions and the solution to the above equation will be explored in the following subsections for a variety of parameters.

3.1 C>0, N=1

We start with a simplified set of parameters where there is only a single set of parallel hyperplanes and we are interested in the condition where at least 1 crossing happens. Due to rotational symmetry of the line segment, it should not matter in which direction the hyperplanes extend. Without loss of generality we assume the planes are in the direction of x_1.

Because P_0 is constrained to be within the gridcell at the origin and because the orthant we are investigating is in the direction of x_1, we know that a crossing occurs whenever the following condition is met

$$x_1 + r\cos\phi_1 > S_1 \tag{7}$$
$$x_1 > S_1 - r\cos\phi_1 \tag{8}$$

The domain of x_1 can then be used to define the space in which a valid crossing has occured

$$m(\phi_1) = \max(0, S_1 - r\cos\phi_1) < x_1 < S_1 \tag{9}$$

We can now express our volume integral in terms of these conditions and solve for the location dimensions.

$$P(C > 0|r, D, N = 1, S) = \int_0^{\pi/2} \dots \int_0^{\pi/2} \int_0^{S_D} \dots \int_0^{S_2} \int_{m(\phi_1)}^{S_1} f_{\vec{\phi}}(\phi_1, \dots, \phi_{D-1}) dx_1 dx_2 \dots dx_D d\phi_1 \dots d\phi_{D-1} \tag{10}$$

$$= \int_0^{\pi/2} \dots \int_0^{\pi/2} \left(\prod_{i=2}^{D} S_i\right) (S_1 - m(\phi_1))\, f_{\vec{\phi}}(\phi_1, \dots, \phi_{D-1}) d\phi_1 \dots d\phi_{D-1} \tag{11}$$

$$= \frac{2^D r^{D-1} \prod_{i=2}^D S_i}{A_{D-1} \prod_{i=1}^D S_i} \int_0^{\pi/2} \dots \int_0^{\pi/2} (S_1 - m(\phi_1)) \prod_{j=1}^{D-1} \sin^{D-1-j}\phi_j d\phi_1 \dots d\phi_{D-1} \tag{12}$$

$$= \frac{2^D r^{D-1}}{A_{D-1} S_1} \int_0^{\pi/2} \dots \int_0^{\pi/2} (S_1 - m(\phi_1)) \prod_{j=1}^{D-1} \sin^{D-1-j}\phi_j d\phi_1 \dots d\phi_{D-1} \tag{13}$$

The value of $m(\phi_1)$ depends on the value of r. If $r < S_1$, then $m(\phi_1) = S_1 - r\cos\phi_1 \forall \phi_1$. If $r > S_1$ we will need to partition the interval of integration into two regions, one where $S_1 - r\cos\phi_1$ is greater than 0 and one where it is less than zero. The transition occurs at the value $\phi_1 = \arccos\frac{S_1}{r}$.

$$m(\phi_1) = \begin{cases} 0 & \frac{S_1}{\cos\phi_1} > r > S_1 \\ S_1 - r\cos\phi_1 & \text{otherwise} \end{cases} \tag{14}$$

3.1.1 $r < S_1$

When $r < S_1$ we have the following expression by substituting $S_1 - r\cos\phi_1$ for $m(\phi_1)$.

$$P(C > 0|r, D, N = 1, S) = \frac{2^D r^{D-1}}{A_{D-1} S_1} \int_0^{\pi/2} \dots \int_0^{\pi/2} r\cos\phi_1 \prod_{j=1}^{D-1} \sin^{D-1-j}\phi_i d\phi_1 \dots d\phi_{D-1} \tag{15}$$

$$= \frac{2^D r^D}{A_{D-1} S_1} \int_0^{\pi/2} \dots \int_0^{\pi/2} \cos\phi_1 \sin^{D-2}\phi_1 \prod_{j=2}^{D-1} \sin^{D-1-j}\phi_i d\phi_1 \dots d\phi_{D-1} \tag{16}$$

Applying u-substitution where $u = \sin\phi_1$ we get the following

$$= \frac{2^D r^D}{A_{D-1} S_1} \int_0^{\pi/2} \dots \int_0^{\pi/2} \frac{1}{D-1} \prod_{j=2}^{D-1} \sin^{D-1-j}\phi_i d\phi_2 \dots d\phi_{D-1} \tag{17}$$

To solve the remaining $D - 2$ integrals, we start by noting the method of integration by reduction.

$$I(m) = \int_0^{\pi/2} \sin^m\phi d\phi = -\frac{1}{m}\sin^{m-1}\phi\cos\phi\Big|_0^{\pi/2} + \frac{m-1}{m}\int_0^{\pi/2} \sin^{m-2}\phi d\phi \tag{18}$$

$$= 0 + \frac{m-1}{m}\left(-\frac{1}{m-2}\sin^{m-3}\phi\cos\phi\Big|_0^{\pi/2} + \frac{m-3}{m-2}\int_0^{\pi/2}\sin^{m-4}\phi d\phi\right) \tag{19}$$

Note that this is valid only when $m > 0$. This reduction continues until one of two things happens. If m is even, then the final integrand becomes $sin^0\phi = 1$. In this case we get the following

$$I(m) = \frac{(m-1)!!}{m!!}\int_0^{\pi/2} d\phi = \frac{\pi(m-1)!!}{2m!!} \tag{20}$$

In the case where m is odd, we get the following

$$I(m) = \frac{(m-1)!!}{m!!}\int_0^{\pi/2} \sin\phi d\phi = \frac{(m-1)!!}{m!!} \tag{21}$$

For every integral in 17 we can use either 20 or 21 as a solution where m is equal to the exponent. Each integral contributes to a product of the form

$$\int_0^{\pi/2} \dots \int_0^{\pi/2} \prod_{j=2}^{D-1} \sin^{D-1-j}\phi_i d\phi_2 \dots d\phi_{D-1} = \int_0^{\pi/2} \prod_{m=1}^{D-3} I(m) d\phi_{D-1} \tag{22}$$

Because of the two possible values for $I(m)$, we get the following expression

$$\int_0^{\pi/2} \prod_{m=1}^{D-3} I(m) d\phi_{D-1} = \int_0^{\pi/2} \frac{1}{(D-3)!!}\left(\frac{\pi}{2}\right)^{\frac{D-3-k}{2}} d\phi_{D-1} \tag{23}$$

$$k = 1 - D \bmod 2 \tag{24}$$

The final integration simply adds a factor of $\pi/2$. Substituting 23 into 17 we get

$$P(C>0|r,D,N=1,S) = \frac{2^D r^D}{A_{D-1}S_1(D-1)}\frac{1}{(D-3)!!}\left(\frac{\pi}{2}\right)^{\frac{D-3-k}{2}}\frac{\pi}{2} \tag{25}$$

$$= \frac{2^{\frac{D+3+k}{2}-1} r^D \pi^{\frac{D-3-k}{2}+1}}{A_{D-1}S_1(D-1)!!} \tag{26}$$

We can now substitute in a double factorial representation of A_{D-1} as follows

$$A_{D-1} = \left(\frac{\sqrt{2}}{\sqrt{\pi}}\right)^{D \bmod 2} \frac{2^{\frac{D}{2}}\pi^{\frac{D}{2}} r^{D-1}}{(D-2)!!} \tag{27}$$

$$P = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)^{D \bmod 2} \frac{2^{\frac{1+k}{2}}(D-2)!!}{\pi^{\frac{1+k}{2}}(D-1)!!}\frac{r}{S_1} \tag{28}$$

$$= \left(\frac{\pi}{2}\right)^{D \bmod 2} \frac{2(D-2)!!}{\pi(D-1)!!}\frac{r}{S_1} \tag{29}$$

$$= \begin{cases} \frac{2(D-2)!!}{\pi(D-1)!!}\frac{r}{S_1} & D \bmod 2 = 0 \\ \frac{(D-2)!!}{(D-1)!!}\frac{r}{S_1} & D \bmod 2 = 1 \end{cases} \tag{30}$$

Instead of double factorial operations, we can represent this probability in terms of the gamma function.

$$P(C>0|r,D,N=1,S) = \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{D+1}{2})}\frac{r}{S_1} \tag{31}$$

3.1.2 $r > S_1$

When $r > S_1$ we must split the bounds of integration for the conditions where $\phi_1 < \arccos\frac{S_1}{r}$ and $\phi_1 > \arccos\frac{S_1}{r}$.

$$P(C>0|r,D,N=1,S) = \frac{r^D 2^D}{S_1 A_{D-1}}\int_0^{\pi/2} \dots \int_0^{\arccos S_1 r} \cos\phi_1 \prod_{j=1}^{D-1}\sin^{D-1-j}\phi_i d\phi_1 \dots d\phi_{D-1} \tag{32}$$

$$= \frac{r^D 2^D}{S_1 A_{D-1}}\int_0^{\pi/2} \dots \int_0^{\pi/2} \cos\phi_1 \sin^{D-2}\phi_1 \prod_{j=2}^{D-1}\sin^{D-1-j}\phi_i d\phi_1 \dots d\phi_{D-1} \tag{33}$$

4 Headings: first level

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4.1 Headings: second level

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$$\xi_{ij}(t) = P(x_t = i, x_{t+1} = j | y, v, w; \theta) = \frac{\alpha_i(t) a^{w_t}_{ij} \beta_j(t+1) b^{v_{t+1}}_j(y_{t+1})}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i(t) a^{w_t}_{ij} \beta_j(t+1) b^{v_{t+1}}_j(y_{t+1})} \tag{34}$$

4.1.1 Headings: third level

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5 Examples of citations, figures, tables, references

5.1 Citations

Citations use `natbib`. The documentation may be found at

`http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf`

Here is an example usage of the two main commands (`citet` and `citep`): Some people thought a thing [Kour and Saabne, 2014a, Hadash et al., 2018] but other people thought something else [Kour and Saabne, 2014b]. Many people have speculated that if we knew exactly why Kour and Saabne [2014b] thought this...

5.2 Figures

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5.3 Tables

See awesome Table 1.

The documentation for `booktabs` ('Publication quality tables in LaTeX') is available from:

`https://www.ctan.org/pkg/booktabs`

[1] Sample of the first footnote.

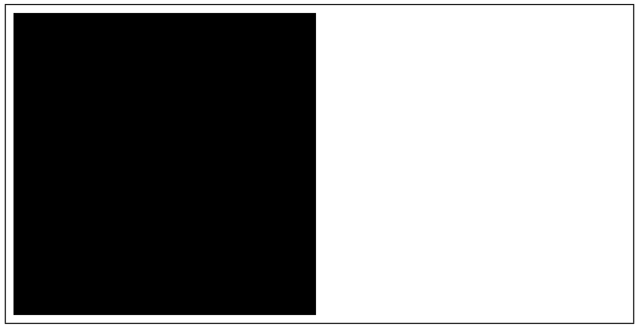

Figure 1: Sample figure caption.

Table 1: Sample table title

Part		
Name	Description	Size (μm)
Dendrite	Input terminal	$\sim$100
Axon	Output terminal	$\sim$10
Soma	Cell body	up to 10^6

5.4 Lists

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- consectetur adipiscing elit.
- Aliquam dignissim blandit est, in dictum tortor gravida eget. In ac rutrum magna.

References

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