Multidimensional Extension of Buffon's Needle Problem

A Preprint

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Abstract

Consider a line segment randomly placed on a two-dimensional plane ruled with a set of regularly spaced parallel lines. The classical Buffon's needle problem asks what the probability is that the line segment intersects at least 1 of these lines. This paper extends this problem by considering a line segment randomly placed in $\mathbb{R}^D$ and its probability of intersection with a set of regularly spaced parallel hyperplanes.

Keywords Buffon's needle problem · Geometric Probability

1 Introduction

Buffon's needle problem was originally posed in the 18th century with the following premise. Given a line segment, or "needle", of length r randomly dropped on a two-dimensional plane ruled with a set of parallel lines regularly spaced s units apart, what is the probability that the needle crosses at least 1 of the lines? The solution, it turns out, is $\frac{2r}{s\pi}$ when the needle is short, $r < s$. Variations and extensions of this problem have been investigated as well, including cite all below

- Laplace's Extension - Investigating when the plane is gridded with 2 orthogonal sets of parallel lines with spacings s_1 and s_2.
- Buffon's Noodle - Instead of being rigidly straight, the needle is permitted to bend (a "noodle") [Ramaley, 1969].

In this paper, we investigate a particular extension that allows the needle to be dropped into a space with dimension greater than 2. In these higher dimensions, we will rule the space with parallel hyperplanes rather than lines. Additionally, we will look at gridding the space with orthogonal sets of hyperplanes, thereby extending Laplace's extension into higher dimensions.

Given $D \in \mathbb{N}_{>0}$, consider a grid on $\mathbb{R}^D$ formed by D orthogonal sets of regularly spaced hyperplanes. Each set of hyperplanes has a potentially unique spacing of S_i. Consider the intersections that the needle may make with the first N sets of hyperplanes where $N \in [1, 2, \ldots, D]$. For example, if $D = 2, N = 1, S_1 = 2$, the grid would match the original Buffon Needle problem and would have only a single set of relevant parallel lines 2 units apart as seen in 1a. If $D = 2, N = 2, S = [2, 1]$, needle intersections with either set of parallel lines would be counted, matching the problem in Laplace's extension as seen in 1b. One set of lines would have a spacing of 2 units and the other would have a spacing of 1 unit.

A line segment of length $r \in \mathbb{R}^+$ is randomly located in the space such that one of its end points, P_0, is uniformly distributed across the entire domain. The line segment's orientation is independently distributed such that when considering P_0 as the center of a $(D-1)$-sphere of radius r, the other point, P_1, is uniformly distributed on the surface of that hypersphere. This line segment may intersect with $C \in \mathbb{N}$ unique hyperplanes. This paper studies the probability that the line segment intersects with at least c hyperplanes, $P(C \geq c | r, D, N, S)$. From there, solutions for crossing at most c hyperplanes and exactly c hyperplanes can be derived.

(a) $D = 2, N = 1, S_1 = 2$, the parameter set matching Buffon's Needle Problem. In this case the number of crossings, C, is 1.

(b) $D = 2, N = 2, S = [2, 1]$, the parameter set matching Laplace's Extension. In this case the number of crossings, C, is 2.

Figure 1: Examples of different parameter sets

We will define the coordinates of line segment using $\vec{x} \in \mathbb{R}^D$ for the location of P_0 and spherical coordinates [Blumenson, 1960] for the location of P_1 with respect to P_0.

$$\begin{aligned}
y_1 &= r \cos \phi_1 \\
y_2 &= r \sin \phi_1 \cos \phi_2 \\
&\vdots \\
y_{D-1} &= r \sin \phi_1 \dots \sin \phi_{D-2} \cos \phi_{D-1} \\
y_D &= r \sin \phi_1 \dots \sin \phi_{D-2} \sin \phi_{D-1} \\
P_1 &= \vec{x} + \vec{y} \\
\phi_j &\in \begin{cases} [0, \pi] & j < D - 1 \\ [0, 2\pi] & j = D - 1 \end{cases}
\end{aligned}$$

The selection of basis vectors provide simplifications to the problem. Translational symmetry of the selection of P_0 allows the origin to be freely moved without loss of generaliztion. By placing the origin of the coordinate system on a vertex of the grid cell containing P_0 and constraining the basis vectors to be collinear with the edges of that grid cell, the domain of x_i is constrained to $[0, S_i]$. The particular vertex can be chosen to ensure that the needle is oriented in the direction of a particular orthant. For convenience, we will choose the orthant such that $\phi_i \in [0, \pi/2]$.

The rest of the paper is organized as follows. A derivation of the joint probability density function for P_0 and P_1 will be provided in §2. The derivation and validation of the crossing probabilities for $N = 1$ will be given in §3. The derivation and validation of the crossing probabilities for all N and $r < \min(S)$ will be given in §4. Numeric simulation of the crossing probabilities will be compared to modeled probabilities in §5, along with analysis of the limits and extrema of the probabilities.

2 Joint Probability Density of the Line Segment

In general, the probability of the needle crossing some number of hyperplanes given a set of parameters can be described as follows.

$$P(C \geq c | r, D, N, S) = \int \cdots \int_V f_{\vec{X}\vec{\Phi}}(\vec{x}, \vec{\phi}) dx_1 \dots dx_D d\phi_1 \dots d\phi_{D-1} \tag{1}$$

Where V is the hypervolume in which the condition $C \geq c$ is true and f is the probability density function of the needle. The necessary conditions for achieving some number of intersections will be called *crossing conditions*. The definition of these crossing conditions and the solution to the above equation will be explored for a variety of parameters.

Proposition 1. *The probability density function of the needle's location and orientation $(\vec{x}, \vec{\phi})$ can be expressed as follows.*

$$f_{\vec{X},\vec{\Phi}}(\vec{x}, \vec{\phi}) = \frac{2^{D-1} \Gamma(\frac{D}{2})}{\pi^{D/2} \prod_{i=1}^{D} S_i} \left(\prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j \right) \tag{2}$$

Proof. Each coordinate for P_0 can be defined as a uniformly distributed random variable $X_i \sim \text{Uniform}(0, S_i)$. Due to independence, the joint PDF for P_0 is the product $\prod_{i=1}^{D} \frac{1}{S_i}$.

By the definition of the problem, the coordinates $\vec{x}$ do not influence the orientation of the line segment defined by $\vec{\phi}$. The probability density function for the uniform distribution of points on an orthant of the hypersphere can be determined by calculating the area element in terms of spherical coordinates.

The area element of an $(D-1)$-sphere of radius r is cite

$$d\Omega = \left(\prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j \right) d\phi_1 \dots d\phi_{D-1} \tag{3}$$

The probability that a point lies in this differential element can be expressed as follows.

$$f_\Omega(\Omega) d\Omega = f_{\vec{\phi}}(\phi_1, \dots, \phi_{D-1}) d\phi_1 \dots d\phi_{D-1} \tag{4}$$

The points are uniformly distributed over the surface of an orthant of the hypersphere implying that $f_\Omega(\Omega) = \frac{2^D}{A_{D-1}}$. Substituting this and 3 into 4 yields

$$\frac{2^D}{A_{D-1}} \prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j = f_{\vec{\phi}}(\phi_1, \dots, \phi_{D-1}) \tag{5}$$

By independence, the joint probability density function for the entire needle can be expressed as the product of the two PDFs.

$$f_{\vec{X},\vec{\phi}}(x_1, \dots, x_D, \phi_1, \dots, \phi_{D-1}) = \frac{2^D}{A_{D-1}} \left(\prod_{i=1}^{D} \frac{1}{S_i} \right) \left(\prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j \right) \tag{6}$$

The expression for the surface area of a $D-1$ dimensional hypersphere, $A_{D-1} = \frac{2\pi^{D/2} r^{D-1}}{\Gamma(D/2)}$, can be substituted in and the expression can be simplified.

$$f_{\vec{X},\vec{\Phi}}(\vec{x}, \vec{\phi}) = \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2} \prod_{i=1}^{D} S_i} \left(\prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j \right) \tag{7}$$

□

The PDF from Proposition 1 can be substituted into the expression from 1 to get the following.

$$P(C \geq c | r, D, N, S) = \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2} \prod_{i=1}^{D} S_i} \int \cdots \int_V \prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j dx_1 \dots dx_D d\phi_1 \dots d\phi_{D-1} \tag{8}$$

3 Probability of Crossing with a Single set of Hyperplanes ($N = 1$)

We start with a simplified set of parameters where there is only a single set of parallel hyperplanes and the needle intersects a hyperplane at least c times. That is, the probability $P(C \geq c | r, D, N = 1, S) \forall c, r, D, S$. For brevity, we will refer to this as $P_{N=1}(c)$.

Proposition 2. *For the case of $N = 1$, the volume integral representing the probability of a needle intersecting at least c hyperplanes can be represented as follows.*

$$P_{N=1}(c) = \frac{2}{S_1 B(\frac{D-1}{2}, \frac{1}{2})} \iint_{\phi_1, x_1} \sin^{D-2} \phi_1 dx_1 d\phi_1$$

Proof. Due to rotational symmetry of the line segment, it does not matter in which direction the hyperplanes extend. Without loss of generality we assume the planes are in the direction of x_1.

Because P_0 is constrained to be within the gridcell at the origin and because the orientation of the needle is constrained to a single orthant which points in the positive direction of x_1, a crossing occurs whenever the following condition is met.

$$x_1 + r\cos\phi_1 > S_1 c \tag{9}$$

From this crossing condition, the bounds of the relevant hypervolume can be defined. Note that the constraints above only apply to x_1 and ϕ_1. As such, the hypervolume spans the entire domain of every other variable. Importantly, because the integrand of 1 describes a PDF, the conditions for Fubini's theorem hold. Therefore the order of integration can be freely switched so long as any variable limits of integration are accounted for. All integrals with respect to the translational dimensions $x_2, \ldots, x_D$ can be simplified for all $i > 2$. Taking $f(\vec{\phi})$ as the integrand

$$\int_0^{S_i} f(\vec{\phi})dx_i = S_i f(\vec{\phi}) \tag{10}$$

Similarly, all of the integrals with respect to $\phi_2, \ldots, \phi_{D-1}$ can be simplified as well using the following identity.

$$\int_0^{\pi/2} \sin^{D-1-j}\phi_j d\phi_j = \frac{B(\frac{D-j}{2}, \frac{1}{2})}{2} \tag{11}$$

Where B is the beta function. Substituting 10 and 11 into 8 results in

$$P_{N=1}(c) = \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2}S_1} \prod_{k=2}^{D-1} \frac{B(\frac{D-k}{2}, \frac{1}{2})}{2} \iint_{\phi_1, x_1} \sin^{D-2}\phi_1 dx_1 d\phi_1 \tag{12}$$

The product of beta functions can be simplified by expanding into gamma functions as follows.

$$\prod_{k=2}^{D-1} \frac{B(\frac{D-k}{2}, \frac{1}{2})}{2} = \frac{1}{2^{D-2}} \frac{\Gamma(\frac{D-2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{D-1}{2})} \frac{\Gamma(\frac{D-3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{D-2}{2})} \cdots \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{2})} \tag{13}$$

$$= \frac{1}{2^{D-2}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})^{D-2}}{\Gamma(\frac{D-1}{2})} = \frac{\pi^{(D-1)/2}}{2^{D-2}\Gamma(\frac{D-1}{2})} \tag{14}$$

Substituting 14 into 12 yields

$$P_{N=1}(c) = \frac{2}{S_1 B(\frac{D-1}{2}, \frac{1}{2})} \iint_{\phi_1, x_1} \sin^{D-2}\phi_1 dx_1 d\phi_1 \tag{15}$$

□

For the remaining integrals, the limits of integration are defined by the domain in which the crossing conditions are satisfied. This can be derived by combining the domains for the variables and the domain for the crossing condition. Recall the following

$$0 < x_1 < S_1$$
$$0 < \phi_1 < \frac{\pi}{2}$$
$$x_1 + r\cos\phi_1 > S_1 c$$

Combining these inequalities results in the following domain where the crossing conditions are satisfied.

$$\max(0, S_1 c - r\cos\phi_1) < x_1 < S_1 \tag{16}$$

$$0 < \phi_1 < \min(\frac{\pi}{2}, \arccos\frac{S_1 c - x_1}{r}) \tag{17}$$

$$r > \frac{S_1 c - x_1}{\cos\phi_1} \tag{18}$$

The min function in 17 can be simplified to $\arccos \frac{S_1 c - x_1}{r}$ as the conditions for the alternative are only possible in the trivial case where $c = 0$ as shown below.

$$m_{\phi_1}(x_1) = \min(\frac{\pi}{2}, \arccos \frac{S_1 c - x_1}{r}) = \begin{cases} \frac{\pi}{2} & x_1 > S_1 c \\ \arccos \frac{S_1 c - x_1}{r} & \text{otherwise} \end{cases} \tag{19}$$

$$= \arccos \frac{S_1 c - x_1}{r} \tag{20}$$

The final inequality, 18, provides a lower bound for the parameter r. The minimum of $\frac{S_1 c - x_1}{\cos \phi_1}$ occurs at $x_1 = S_1, \phi_1 = 0$ with a value of $S_1(c-1)$. Therefore if $r \leq S_1(c-1)$ we can guarantee that the crossing condition cannot be satisfied. This is equivalent to the scenario where the needle is too short to cross the necessary number of hyperplanes, even when it is oriented orthogonally to the hyperplanes.

$$P_{N=1}(C \geq c | r < S_1(c-1)) = 0 \tag{21}$$

The max function found in 16 also depends on the length of the needle, r.

$$m_{x_1}(\phi_1) = \max(0, S_1 c - r\cos\phi_1) = \begin{cases} 0 & r > \frac{S_1 c}{\cos \phi_1} \\ S_1 c - r\cos\phi_1 & \text{otherwise} \end{cases} \tag{22}$$

This partitions the problem into three regions depending on the value of r.

$$0 < r \leq S_1(c-1) \implies P_{N=1}(c) = 0 \tag{23}$$

$$S_1(c-1) < r \leq S_1 c \implies m_{x_1}(\phi_1) = S_1 c - r\cos\phi_1 \forall \phi_1 \tag{24}$$

$$S_1 c < r \implies m_{x_1}(\phi_1) = \max(0, S_1 c - r\cos\phi_1) \tag{25}$$

The probability $P_{N=1}(c)$ will be derived for the two cases given in 24 and 25 in §3.1 and §3.2 respectively. Numeric simulation and comparison will be explored in §5

3.1 $S_1(c-1) < r < S_1 c$

Lemmas 1 and 2 will be used in the proof of Theorem 1 which gives a derivation of the probability $P_{N=1}(c)$ for a short needle, $S_1(c-1) r < S_1 c$.

Lemma 1. *When given the ratio $(k-1)!!/k!!$ where the double exclam represents the double factorial function, it is equivalent the following.*

$$\frac{(k-1)!!}{k!!} = \begin{cases} \frac{1}{\pi} B(\frac{k+1}{2}, \frac{1}{2}) & k \bmod 2 = 0 \\ \frac{1}{2} B(\frac{k+1}{2}, \frac{1}{2}) & k \bmod 2 = 1 \end{cases} \tag{26}$$

Proof. Every double factorial can be written in terms of single factorials depending on the parity of the natural number in question.

$$n!! = \begin{cases} 2^{n/2} \left(\frac{n}{2}\right)! & n \bmod 2 = 0 \\ \dfrac{n!}{2^{(n-1)/2} (\frac{n-1}{2})!} & n \bmod 2 = 1 \end{cases} \tag{27}$$

This single factorial representation can be used to simplify $(k-1)!!/k!!$. First, assuming that k is even

$$\frac{(k-1)!!}{k!!} = \frac{(k-1)!}{2^{(k-2)/2} (\frac{k-2}{2})!} \frac{1}{2^{k/2} (\frac{k}{2})!} \tag{28}$$

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{\frac{1}{2}\frac{2}{2} \dots \frac{k-2}{2}\frac{k-1}{2}}{\frac{2}{2}\frac{4}{2} \dots \frac{k-4}{2}\frac{k-2}{2}(\frac{k}{2}!)} \tag{29}$$

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{\frac{1}{2}\frac{3}{2} \dots \frac{k-3}{2}\frac{k-1}{2}}{\frac{k}{2}!} \tag{30}$$

Using the property $n\Gamma(n) = \Gamma(n+1)$ and $n! = \Gamma(n+1)$, the sequence of fractions can be reduced iteratively.

$$\frac{(k-1)!!}{k!!} = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{k+2}{2})} \tag{31}$$

Finally, using $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ and $\Gamma(1/2) = \sqrt{\pi}$ results in the following.

$$\frac{(k-1)!!}{k!!} = \frac{1}{\pi} B\left(\frac{k+1}{2}, \frac{1}{2}\right) \tag{32}$$

We now repeat the process for the case where k is odd.

$$\begin{aligned}
\frac{(k-1)!!}{k!!} &= \left(\frac{k-1}{2}\right)!2^{(k-1)/2} \frac{(\frac{k-1}{2})!2^{(k-1)/2}}{k!} && (33)\\
&= \frac{2\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2})} \frac{2^{k-1}(\frac{k-1}{2})!^2}{k!} = \frac{\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2})} \frac{\frac{2}{2}\frac{4}{2}\dots\frac{k-3}{2}\frac{k-1}{2}(\frac{k-1}{2}!)}{\frac{1}{2}\frac{2}{2}\dots\frac{k-1}{2}\frac{k}{2}} && (34)\\
&= \frac{\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2})} \frac{\frac{k-1}{2}!}{\frac{1}{2}\frac{3}{2}\dots\frac{k-2}{2}\frac{k}{2}} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{k+1}{2})}{2\Gamma(\frac{k+2}{2})} && (35)\\
&= \frac{1}{2} B\left(\frac{k+1}{2}, \frac{1}{2}\right) && (36)
\end{aligned}$$

□

Lemma 2. *Any integral of the form $\int_0^{\arccos(\gamma)} \sin^m \phi d\phi$ has two possible solutions depending on the parity of m.*

$$\int_0^{\arccos\gamma} \sin^m \phi d\phi = \frac{B(\frac{m+1}{2}, \frac{1}{2})}{2} \left(g(\gamma, m) - \gamma(1-\gamma^2)^{(m+1)/2} \sum_{i=1}^{\lfloor m/2 \rfloor} \frac{B(\frac{m+2-2i}{2}, \frac{1}{2})}{\pi(1-\gamma^2)^i} \right) \tag{37}$$

$$g(\gamma, m) = \begin{cases} \frac{2}{\pi} \arccos\gamma & m \bmod 2 = 0 \\ 1 - \gamma & m \bmod 2 = 1 \end{cases} \tag{38}$$

Proof. We start with the following integration by reduction identity

$$\int_0^{\arccos\gamma} \sin^m \phi d\phi = -\frac{1}{m} \sin^{m-1}\phi \cos\phi \Big|_0^{\arccos\gamma} + \frac{m-1}{m} \int_0^{\arccos\gamma} \sin^{m-2}\phi d\phi \tag{39}$$

$$\begin{aligned}
&= -\frac{1}{m}(1-\gamma^2)^{(m-1)/2}\gamma \\
&\quad + \frac{m-1}{m}\left(-\frac{1}{m-2} \sin^{m-3}\phi\cos\phi \Big|_0^{\arccos\gamma} + \frac{m-3}{m-2} \int_0^{\arccos\gamma} \sin^{m-4}\phi d\phi \right)
\end{aligned} \tag{40}$$

This pattern continues until the $\sin$ in the final integrand is raised to either the first or zeroth power. This depends on whether m is even or odd. If m is even

$$\begin{aligned}
&= -\frac{1}{m}(1-\gamma^2)^{(m-1)/2}\gamma - \frac{m-1}{m(m-2)}(1-\gamma^2)^{(m-3)/2}\gamma - \dots \\
&\qquad - \frac{(m-1)(m-3)\dots(3)}{(m)(m-2)\dots(2)}(1-\gamma^2)^{1/2}\gamma + \frac{(m-1)!!}{m!!} \int_0^{\arccos\gamma} d\phi
\end{aligned} \tag{41}$$

$$= \frac{(m-1)!!}{m!!} \left(\arccos\gamma - \gamma(1-\gamma^2)^{(m+1)/2} \sum_{i=1}^{m/2} \frac{(m-2i)!!}{(m+1-2i)!!}(1-\gamma^2)^{-i} \right) \tag{42}$$

Using Lemma 1 we can reduce to the following.

$$= \frac{B(\frac{m+1}{2}, \frac{1}{2})}{\pi} \left(\arccos\gamma - \gamma(1-\gamma^2)^{(m+1)/2} \sum_{i=1}^{m/2} \frac{B(\frac{m+2-2i}{2}, \frac{1}{2})}{2}(1-\gamma^2)^{-i} \right) \tag{43}$$

$$= \frac{B(\frac{m+1}{2}, \frac{1}{2})}{2} \left(\frac{2}{\pi} \arccos\gamma - \gamma(1-\gamma^2)^{(m+1)/2} \sum_{i=1}^{m/2} \frac{B(\frac{m+2-2i}{2}, \frac{1}{2})}{\pi(1-\gamma^2)^i} \right) \tag{44}$$

Repeating for the case where m is odd

$$\begin{aligned}
&= -\frac{1}{m}(1-\gamma^2)^{(m-1)/2}\gamma - \frac{m-1}{m(m-2)}(1-\gamma^2)^{(m-3)/2}\gamma - \dots \\
&\qquad - \frac{(m-1)(m-3)\dots(3)}{(m)(m-2)\dots(2)}(1-\gamma^2)^{1/2}\gamma + \frac{(m-1)!!}{m!!}\int_0^{\arccos\gamma}\sin\phi d\phi
\end{aligned} \tag{45}$$

$$= \frac{(m-1)!!}{m!!}\left(1-\gamma-\gamma(1-\gamma^2)^{(m+1)/2}\sum_{i=1}^{(m-1)/2}\frac{(m-2i)!!}{(m+1-2i)!!}(1-\gamma^2)^{-i}\right) \tag{46}$$

$$= \frac{B(\frac{m+1}{2},\frac{1}{2})}{2}\left(1-\gamma-\gamma(1-\gamma^2)^{(m+1)/2}\sum_{i=1}^{\lfloor m/2\rfloor}\frac{B(\frac{m+2-2i}{2},\frac{1}{2})}{\pi(1-\gamma^2)^i}\right) \tag{47}$$

□

Theorem 1. *The probability that a short needle, $S_1(c-1) < r < S_1 c$, crosses at least c hyperplanes given that there is only a single set of parallel hyperplanes ruling D dimensional space is as follows.*

$$P_{N=1}(c|S_1(c-1)<r<S_1c) = \frac{r}{S_1}\left[\frac{(1-\gamma^2)^{\frac{D-1}{2}}}{\pi}\left(B\left(\frac{D}{2},\frac{1}{2}\right)+\gamma^2\sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2i}{2},\frac{1}{2})}{(1-\gamma^2)^i}\right)-\gamma g(\gamma,D-2)\right] \tag{48}$$

where

$$\gamma = \frac{S_1(c-1)}{r} \tag{49}$$

$$g(n,m) = \begin{cases}\frac{2}{\pi}\arccos n & m \bmod 2 = 0\\ 1-n & m \bmod 2 = 1\end{cases} \tag{50}$$

Proof. Let r_s be the condition that $S_1(c-1) < r < S_1c$. Using the conditions 16 and 17, the limits of integration can be defined for the expression found in 15.

$$P_{N=1}(c|r_s) = \frac{2}{S_1B(\frac{D-1}{2},\frac{1}{2})}\int_0^{m_{\phi_1}(x_1)}\int_{m_{x_1}(\phi_1)}^{S_1}\sin^{D-2}\phi_1 dx_1 d\phi_1 \tag{51}$$

Simplifying with 24 and the upper bound for x_1 yields the following.

$$P_{N=1}(c|r_s) = \frac{2}{S_1B(\frac{D-1}{2},\frac{1}{2})}\int_0^{\arccos\frac{S_1(c-1)}{r}}\int_{S_1c-r\cos\phi_1}^{S_1}\sin^{D-2}\phi_1 dx_1 d\phi_1 \tag{52}$$

$$= \frac{2}{S_1B(\frac{D-1}{2},\frac{1}{2})}\int_0^{\arccos\frac{S_1(c-1)}{r}}(S_1(1-c)+r\cos\phi_1)\sin^{D-2}\phi_1 d\phi_1 \tag{53}$$

Substituting in $\gamma = \frac{S_1(c-1)}{r}$ yields the following

$$P_{N=1}(c|r_s) = \frac{2r}{S_1B(\frac{D-1}{2},\frac{1}{2})}\int_0^{\arccos\gamma}(-\gamma+\cos\phi_1)\sin^{D-2}\phi_1 d\phi_1 \tag{54}$$

$$= \frac{2r}{S_1B(\frac{D-1}{2},\frac{1}{2})}\left[-\gamma\int_0^{\arccos\gamma}\sin^{D-2}\phi_1 d\phi_1 + \frac{1}{D-1}(1-\gamma^2)^{\frac{D-1}{2}}\right] \tag{55}$$

Using the result of Lemma 2 on 55 yields the following

$$
\begin{aligned}
P_{N=1}(c|r_s) &= \frac{2r}{S_1 B(\frac{D-1}{2},\frac{1}{2})}\left[-\gamma\frac{B(\frac{D-1}{2},\frac{1}{2})}{2}\left(g(\gamma,D-2)-\gamma(1-\gamma^2)^{\frac{D-1}{2}}\sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2i}{2},\frac{1}{2})}{\pi(1-\gamma^2)^i}\right)\right. \\
&\qquad\left.+\frac{1}{D-1}(1-\gamma^2)^{\frac{D-1}{2}}\right] && (56) \\
&= \frac{r}{S_1}\left[-\gamma\left(g(\gamma,D-2)-\gamma(1-\gamma^2)^{\frac{D-1}{2}}\sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2i}{2},\frac{1}{2})}{\pi(1-\gamma^2)^i}\right)+\frac{B(\frac{D}{2},\frac{1}{2})}{\pi}(1-\gamma^2)^{\frac{D-1}{2}}\right] && (57) \\
&= \frac{r}{S_1}\left[\frac{(1-\gamma^2)^{\frac{D-1}{2}}}{\pi}\left(B\left(\frac{D}{2},\frac{1}{2}\right)+\gamma^2\sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2i}{2},\frac{1}{2})}{(1-\gamma^2)^i}\right)-\gamma g(\gamma,D-2)\right] && (58)
\end{aligned}
$$

□

As an example, setting the parameters relevant for the classical 2 dimensional Buffon needle problem results in the expected probability.

$$
\begin{aligned}
c = 1, D = 2, N = 1, S = s && (59) \\
\gamma = \frac{s(c-1)}{r} = 0 && (60) \\
P_{N=1}(c) = \frac{r}{s}\left[\frac{1}{\pi}\left(\frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}+0\right)-0\right] && (61) \\
= \frac{2r}{\pi s} && (62)
\end{aligned}
$$

Additional validation and comparison to numeric simulation will be explored in §5

3.2 $r > S_1 c$

When the needle is long, $r > S_1 c$, the value of $m_{x_1}(\phi_1)$ is no longer constant for all ϕ_1. Below is a restatement of the general equation for $P(C \geq c)$ when $N = 1$, 15, and the domain of the crossing conditions.

$$
\begin{gathered}
P_{N=1}(c) = \frac{2}{S_1 B(\frac{D-1}{2},\frac{1}{2})}\iint_{\phi_1,x_1}\sin^{D-2}\phi_1 dx_1 d\phi_1 \\
\max(0, S_1 c - r\cos\phi_1) < x_1 < S_1 \\
0 < \phi_1 < \arccos\frac{S_1 c - x_1}{r}
\end{gathered}
$$

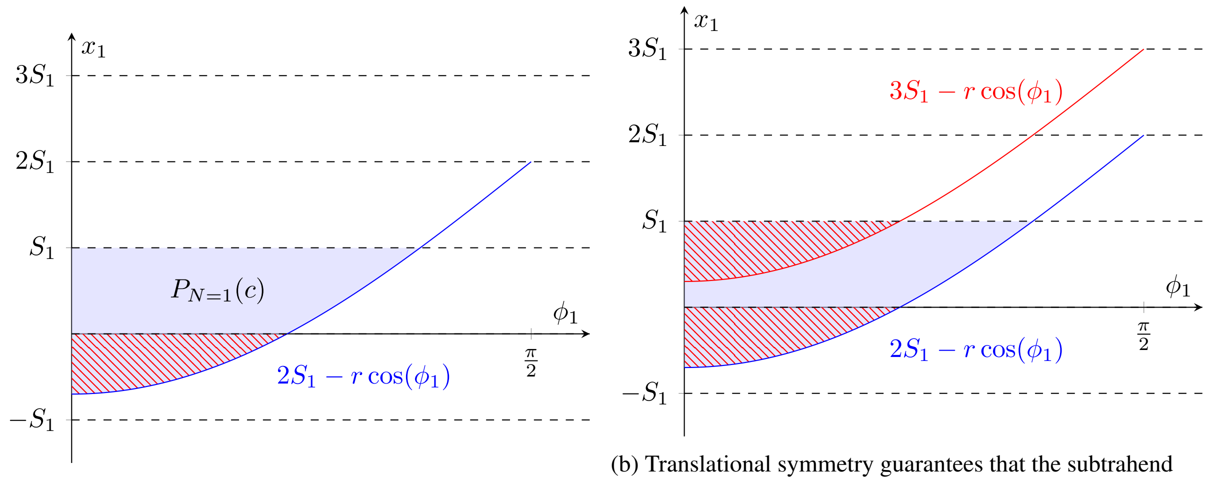
Theorem 2. *The probability that a long needle, $S_1 c < r$, crosses at least c hyperplanes given that there is only a single set of parallel hyperplanes ruling D dimensional space is as follows.*

$$P_{N=1}(c|r > S_1 c) = A(c) - A(c+1) \tag{63}$$

where A is the function used to define $P_{N=1}(c|S_1(c-1) < r < S_1 c)$. That is

$$
\begin{aligned}
A(k) = \frac{r}{S_1}\left[\frac{(1-\gamma^2)^{\frac{D-1}{2}}}{\pi}\left(B\left(\frac{D}{2},\frac{1}{2}\right)+\gamma^2\sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2i}{2},\frac{1}{2})}{(1-\gamma^2)^i}\right)-\gamma g(\gamma,D-2)\right] && (64) \\
\gamma = \frac{S_1(k-1)}{r} && (65) \\
g(n,m) = \begin{cases}\frac{2}{\pi}\arccos(n) & m \bmod 2 = 0 \\ 1-n & m \bmod 2 = 1\end{cases} && (66)
\end{aligned}
$$

Proof. There are now two things to note. First, the PDF only varies with ϕ_1, indicating that the directional derivative of the PDF in the direction of x_1 is zero. Second, when $r > S_1 c$, the domain of the crossing condition can be calculated



(a) The integral spanning the domain of the crossing condition can be found as a difference between two regions.

(b) Translational symmetry guarantees that the subtrahend is equivalent to the "short needle" probability of crossing an additional hyperplane.

Figure 2

as the integral spanning the domain $S_1c - r\cos\phi_1 < x_1 < 0$ subtracted from the integral of the domain spanning $S_1c - r\cos\phi_1 < x_1 < S_1$ as seen in Figure 2a. This difference is explicitly written as follows.

$$\int_0^{\phi_{1,h}} \int_{m_{x_1}(\phi_1)}^{S_1} \sin^{D-2}\phi_1 dx_1 d\phi_1 = \int_0^{\phi_{1,h}} \int_{S_1c - r\cos\phi_1}^{S_1} \sin^{D-2}\phi_1 dx_1 d\phi_1 - \int_0^{\phi_{1,l}} \int_{S_1c - r\cos\phi_1}^{0} \sin^{D-2}\phi_1 dx_1 d\phi_1 \tag{67}$$

$$I_1 = I_2 - I_3 \tag{68}$$

$$\phi_{1,h} = \arccos\frac{S_1(c-1)}{r} \tag{69}$$

$$\phi_{1,l} = \arccos\frac{S_1c}{r} \tag{70}$$

The integral I_2 is of the same form as the short needle case and can therefore be substituted with the solution 58. Using symmetry of the PDF in the x_1 direction, I_3 can be rewritten with the limits of integration spanning $S_1(c+1) - r\cos\phi_1 < x_1 < S_1$. This is, again, exactly the problem of the short needle case but with $c+1$ instead of c. This is visualized in Figure 2b As such, the result is

$$P_{N=1}(c|r > S_1c) = A(c) - A(c+1) \tag{71}$$

$$A(k) = \frac{r}{S_1}\left[\frac{(1-\gamma^2)^{\frac{D-1}{2}}}{\pi}\left(B\left(\frac{D}{2},\frac{1}{2}\right) + \gamma^2 \sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor} \frac{B(\frac{D-2i}{2},\frac{1}{2})}{(1-\gamma^2)^i}\right) - \gamma g(\gamma, D-2)\right] \tag{72}$$

$$\gamma = \frac{S_1(k-1)}{r} \tag{73}$$

$$g(n,m) = \begin{cases} \frac{2}{\pi}\arccos(n) & m \bmod 2 = 0 \\ 1-n & m \bmod 2 = 1 \end{cases} \tag{74}$$

□

4 Probability of crossing $N \geq 1$

When there is only a single set of parallel hyperplanes, there is only one way for a needle to make c intersections. The needle would have to go through c hyperplanes in a single direction. When we increase the number of orthogonal sets of hyperplanes then we must deal with the fact that there are now many ways to cross c hyperplanes due to the many combinations of directions available.

For instance, if $N = 2$ and we want to know when $C = 2$, then a valid number of crossings occurs if the needle crosses 2 hyperplanes in x_1 and 0 in x_2, or 1 hyperplane in each direction, or 0 hyperplanes in x_1 and 2 in x_2.

Let $P(C_i \geq c_i|r, D, N, S) = P(C_i \geq c_i)$ be the probability that a needle crosses at least c_i hyperplanes in the direction of x_i. This is representable with a hypervolume where the relevant crossing condition is satisfied. Let $P(C_1 \geq c_1, \ldots, C_N \geq c_N|r, D, N, S) = P(\vec{C} \geq \vec{c})$ be the probability that the needle crosses at least c_i hyperplanes in the direction x_i for all positive integer values of i up to N. This is equivalent to the intersection of each individual hypervolume associated with each $P(C_i \geq c_i)$. Additionally, let $P(C_1 \geq c_1 \cup \ldots \cup C_N \geq c_N|r, D, N, S) = P(\cup\vec{c})$ be the probability that any of the inequalities are satisfied. This is equivalent to the union of each hypervolume. Finally, let $\hat{e}_i$ be a vector where the ith entry is 1 and all other entries are zero.

Using this notation, the probability $P(C \geq 1|r, D, N, S)$ is equivalent to the probability $P(\cup\vec{c})$ when $\vec{c}$ is an N length list of ones, $\vec{1}_N$. This can be written in terms of intersections using the inclusion-exclusion principle.

$$P(C \geq 1|r, D, N, S) = P(\cup\vec{1}_N) = \sum_{k=1}^{N}(-1)^{k+1}\left(\sum_{1\leq i_1<\ldots<i_k\leq N} P(\hat{e}_{i_1}, \ldots, \hat{e}_{i_k})\right) \tag{75}$$

Similarly, for $C \geq c$, we can define the set of events E_c^N which consists of each of the hyperplane crossings combinations that a needle can make to satisfy the inequality. The size of this set is equivalent to the number of combinations with replacement that can be made of hyperplane directions, $\binom{c+N-1}{c}$. For example, $E_2^3 = \{[2,0,0], [0,2,0], [0,0,2], [1,1,0], [1,0,1], [0,1,1]\}$. When the elements of E_c^N are arbitrarily ordered into a list, we notate the ith element as $E_c^N[i]$. If the needle crosses hyperplanes listed in any element of E_c^N, then the crossing conditions for the criteria $C \geq c$ have been met.

Proposition 3. *The probability that a needle of any length r crosses c hyperplanes is as follows.*

$$P(C \geq c|r, D, N, S) = \sum_{k=1}^{\binom{c+N-1}{c}}(-1)^{k+1}\left(\sum_{1\leq i_1<\ldots<i_k\leq\binom{c+N-1}{c}} P(\vec{C} \geq \bar{c})\right) \tag{76}$$

with

$$\bar{c} = [\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_N] \tag{77}$$

$$\bar{c}_j = \max(E_c^N[i_1][j], E_c^N[i_2][j], \ldots, E_c^N[i_k][j]) \tag{78}$$

Proof. Let $P(\cup E_c^N)$ be the probability that any element of E_c^N has satisfied its crossing conditions. Explicitly, this can be written as follows.

$$P(C \geq c|r, D, N, S) = P(\cup E_c^N) = P\left(\vec{C} \geq E_c^N[1] \cup \ldots \cup \vec{C} \geq E_c^N\left[\binom{c+N-1}{c}\right]\right) \tag{79}$$

Again, using the inclusion-exclusion principle, this can be expressed strictly with intersections as follows.

$$P(\cup E_c^N) = \sum_{k=1}^{\binom{c+N-1}{c}}(-1)^{k+1}\left(\sum_{1\leq i_1<\ldots<i_k\leq\binom{c+N-1}{c}} P(\vec{C} \geq E_c^N[i_1], \ldots, \vec{C} \geq E_c^N[i_k])\right) \tag{80}$$

The condition $\vec{C} \geq E_c^N[i_1], \ldots, \vec{C} \geq E_c^N[i_k]$ can be rewritten to have a single condition for each direction by taking the maximum number of required crossings across all elements of E_c^N for each direction.

$$\begin{aligned} P(\vec{C} \geq E_c^N[i_1], \ldots, \vec{C} \geq E_c^N[i_k]) = P\Big(& C_1 \geq \max(E_c^N[i_1][1], \ldots, E_c^N[i_k][1]), \\ & \vdots \\ & C_N \geq \max(E_c^N[i_1][N], \ldots, E_c^N[i_k][N])\Big) \end{aligned} \tag{81}$$

These maximum required crossings will be notated as $\bar{c} = [\bar{c}_1, \ldots, \bar{c}_N]$, resulting in the following expression.

$$P(C \geq c | r, D, N, S) = \sum_{k=1}^{\binom{c+N-1}{c}} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq \binom{c+N-1}{c}} P(\vec{C} \geq \bar{c}) \right) \tag{82}$$

□

An expression is therefore needed to determine the probability $P(\vec{C} \geq \vec{c})$ for any selection of $\vec{c}$. In §4.1 this probability will be derived for short needles where $r < \min(S)$.

4.1 Short Needle Crossing Probability $N \geq 1, r < \min(S)$

Short needles, where $r < \min(S)$, cannot cross more than one hyperplane in any direction as even if they were orthogonal to the hyperplanes and x_i was maximized, $x_i + r < 2S_i$. As a result, any probability where $\bar{c}$ has an entry greater than one will be zero. If $\bar{c}$ has all entries as zero, then the probability of satisfying the crossing conditions is trivially one. In other words

$$P(\vec{C} \geq \bar{c} | r < \min(S), D, N, S) = \begin{cases} 1 & ||\bar{c}||_\infty = 0 \\ P(\vec{C} \geq \bar{c}) & ||\bar{c}||_\infty = 1 \\ 0 & ||\bar{c}||_\infty \geq 2 \end{cases} \tag{83}$$

Proposition 4. *For any $\vec{c}$ where $||\vec{c}||_\infty = 1$, the probability that a short needle would satisfy the crossing conditions can be represented as follows.*

$$P(\vec{C} \geq \vec{c}) = \frac{r^h}{\pi^{h/2} (\prod_{i=1}^N S_i^{\vec{c}_i})} \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D+h}{2})} \tag{84}$$

with

$$h = ||\vec{c}||_0 \tag{85}$$

Proof. Let H be the list of indices of non-zero elements of $\vec{c}$. Without loss of generalization, the coordinate system can be realigned such that H_1 aligns with x_1, H_2 aligns with x_2 and so on. All other hyperplanes that are not included in H can be arbitrarily asigned directions as any intersections with them are irrelevant.

The necessary conditions for crossings to occur in each direction specified in H is as follows

$$S_1 \leq x_1 + r \cos \phi_1 \tag{86}$$
$$S_2 \leq x_2 + r \sin \phi_1 \cos \phi_2 \tag{87}$$
$$\vdots \tag{88}$$
$$S_{h-1} \leq x_{h-1} + r \sin \phi_1 \sin \phi_2 \ldots \sin \phi_{h-2} \cos \phi_{h-1} \tag{89}$$
$$S_h \leq x_h + \begin{cases} r \sin \phi_1 \ldots \sin \phi_{h-1} \cos \phi_h & h < D \\ r \sin \phi_1 \ldots \sin \phi_{h-1} \sin \phi_h & h = D \end{cases} \tag{90}$$

These conditions, along with the domain of $x_i \forall i \in 1, \ldots, h$, define the bounds of the volume where the needle crosses a hyperplane in each direction H.

$$S_1 \geq x_1 \geq m_1(\phi_1) = \max\{0, S_1 - r \cos \phi_1\} \tag{91}$$
$$S_2 \geq x_2 \geq m_2(\phi_2) = \max\{0, S_2 - r \sin \phi_1 \cos \phi_2\} \tag{92}$$
$$\vdots \tag{93}$$
$$S_{h-1} \geq x_{h-1} \geq m_{h-1}(\phi_{h-1}) = \max\{0, S_{h-1} - r \sin \phi_1 \ldots \sin \phi_{h-2} \cos \phi_{h-1}\} \tag{94}$$
$$S_h \geq x_h \geq m_h(\phi_h) = \max \left\{ 0, S_h - \begin{cases} r \sin \phi_1 \ldots \sin \phi_{h-1} \cos \phi_h & h < D \\ r \sin \phi_1 \ldots \sin \phi_{h-2} \sin \phi_{h-1} & h = D \end{cases} \right\} \tag{95}$$

Starting with 8, the crossing conditions above are encoded into the bounds of integration. As in the $N = 1$ case, the order of integration is arbitrary so long as the limits of integration are not variable. Therefore the integrals with respect to the spatial dimensions greater than h are reduced to a coefficient.

$$P(\vec{C} \geq \vec{c}) = \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{D} S_i} \int \cdots \int_V \prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j dV \tag{96}$$

$$= \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{N} S_i^{\vec{c}_i}} \int \cdots \int_\phi \int_{m_h(\phi_h)}^{S_h} \cdots \int_{m_1(\phi_1)}^{S_1} \prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j dx_1 \dots dx_h d\phi_1 \dots d\phi_{D-1} \tag{97}$$

Given that r is less than every spacing S_i, every function $m_i(\phi_i)$ is guaranteed to be greater than zero. Every spatial integral will reduce to the polar representation of the corresponding x_i. This simplifies to the following.

$$P(\vec{C} \geq \vec{c}) = \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{N} S_i^{\vec{c}_i}} \int \cdots \int_\phi r^h \left(\prod_{k=1}^{\min(D-1,h)} \cos\phi_k \sin^{h-k} \phi_k \right) \prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j d\phi_1 \dots d\phi_{D-1} \tag{98}$$

$$= \frac{2^{D-1}r^h\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{N} S_i^{\vec{c}_i}} \int \cdots \int_\phi \left(\prod_{k=1}^{\min(D-1,h)} \cos\phi_k \sin^{D+h-2k-1} \phi_k \right) \prod_{j=h+1}^{D-1} \sin^{D-1-j} \phi_j d\phi_1 \dots d\phi_{D-1} \tag{99}$$

The product from $k = 1$ to $\min(D-1, h)$ can be reduced by using u-substitution where $u = \sin\phi_k$. Assuming that $h \leq D-1$, this results as follows

$$P(\vec{C} \geq \vec{c}) = \frac{2^{D-1}r^h\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1} NS_i^{\vec{c}_i}} \int \cdots \int_0^{\pi/2} \frac{1}{(D+h-2)(D+h-4)\dots(D-h)} \prod_{j=h+1}^{D-1} \sin^{D-1-j} \phi_j d\phi_{h+1} \dots d\phi_{D-1} \tag{100}$$

$$= \frac{2^{D-1}r^h\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{N} S_i^{\vec{c}_i}} \frac{(D-h-2)!!}{(D+h-2)!!} \int \cdots \int_0^{\pi/2} \prod_{j=h+1}^{D-1} \sin^{D-1-j} \phi_j d\phi_{h+1} \dots d\phi_{D-1} \tag{101}$$

$$= \frac{2^{D-1}r^h\Gamma(\frac{D}{2})}{\pi^{D/2}\prod_{i=1}^{N} S_i^{\vec{c}_i}} \frac{\Gamma(\frac{D-h}{2})}{2^h\Gamma(\frac{D+h}{2})} \prod_{j=h+1}^{D-1} \frac{B(\frac{D-j}{2}, \frac{1}{2})}{2} \tag{102}$$

$$= \frac{r^h}{\pi^{h/2}\prod_{i=1}^{N} S_i^{\vec{c}_i}} \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D+h}{2})} \tag{103}$$

$$\tag{104}$$

If h is greater than $D-1$ (ie. $h = D$), the result remains the same.

$$P(\vec{C} \geq \vec{c}) = \frac{2^D r^{D+h-1}}{A_{D-1}\prod_{i=1}^{N} S_i^{\vec{c}_i}} \frac{1}{(D+h-2)(D+h-4)\dots(4)(2)} \tag{105}$$

$$= \frac{2^D r^h\Gamma(\frac{D}{2})}{2\pi^{D/2}\prod_{i=1}^{N} S_i^{\vec{c}_i}} \frac{1}{(D+h-2)!!} \tag{106}$$

$$= \frac{2^D r^h\Gamma(\frac{D}{2})}{2\pi^{D/2}\prod_{i=1}^{N} S_i^{\vec{c}_i}} \frac{1}{2^{(D+h-2)/2}\Gamma(\frac{D+h}{2})} \tag{107}$$

$$= \frac{r^h}{\pi^{h/2}\prod_{i=1}^{N} S_i^{\vec{c}_i}} \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D+h}{2})} \tag{108}$$

□

Theorem 3. *The probability that a short needle, $r < \min(S)$, crosses at least c hyperplanes given that there are multiple orthogonal sets of parallel hyperplanes ruling D dimensional space is as follows.*

$$P(C \geq c | r < \min(S), D, N, S) = \sum_{k=1}^{\binom{N}{c}} (-1)^{k+1} \frac{r^k \Gamma(\frac{D}{2})}{\pi^{k/2}\Gamma(\frac{D+k}{2})} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq \binom{N}{c}} \frac{1}{\prod_{j=1}^{N} S_j^{\hat{\bar{c}}_j}} \right) \tag{109}$$

Proof. From 83, if any element of $\bar{c}$ is greater than 1, then the associated probability is zero. The result of Proposition 3 can be rewritten with this stipulation as follows.

$$P(C \geq c | r, D, N, S) = \sum_{k=1}^{\binom{c+N-1}{c}} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq \binom{c+N-1}{c}} P(\vec{C} \geq \bar{c})(||\bar{c}||_\infty \bmod 2) \right) \tag{110}$$

Recall that $\bar{c}$ is defined as follows.

$$\bar{c}_j = \max(E_c^N[i_1][j], E_c^N[i_2][j], \ldots, E_c^N[i_k][j]) \tag{111}$$

This means that any element of E_c^N that contains a value greater than 1 will ultimately result in an addend being zero. These elements with values greater than 1 result from E_c^N being generated by listing all combinations of c hyperplanes with replacement. To prevent this, it is sufficient to instead generate a set of lists which consist of the combinations of c hyperplanes without replacement. This guarantees that every hyperplane direction is selected a maximum of once.

Let $\hat{E}_c^N$ be such a list of combinations without replacement. Similar to before, let $\hat{\bar{c}}$ be the list of maximum crossing requirements for each direction such that

$$\hat{\bar{c}}_j = \max(\hat{E}_c^N[i_1][j], \ldots, E_c^N[i_k][j]) \tag{112}$$

The result of Proposition 3 can then be reindexed as follows.

$$P(C \geq c | r, D, N, S) = \sum_{k=1}^{\binom{N}{c}} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq \binom{N}{c}} P(\vec{C} \geq \hat{\bar{c}}) \right) \tag{113}$$

$$\hat{\bar{c}}_j = \max(\hat{E}_c^N[i_1][j], \hat{E}_c^N[i_2][j], \ldots, \hat{E}_c^N[i_k][j]) \tag{114}$$

Substituting 108 into 113 yields the following.

$$P(C \geq c | r < \min(S), D, N, S) = \sum_{k=1}^{\binom{N}{c}} (-1)^{k+1} \frac{r^k \Gamma(\frac{D}{2})}{\pi^{k/2}\Gamma(\frac{D+k}{2})} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq \binom{N}{c}} \frac{1}{\prod_{j=1}^{N} S_j^{\hat{\bar{c}}_j}} \right) \tag{115}$$

□

As an example, Laplace's extension of the short needle problem can be evaluated as follows.

$$P(C \geq 1 | r < \min(S), 2, 2, S) = \frac{r\Gamma(1)}{\sqrt{\pi}\Gamma(\frac{3}{2})} \left(\frac{1}{S_1} + \frac{1}{S_2} \right) - \frac{r^2\Gamma(1)}{\pi\Gamma(2)} \frac{1}{S_1 S_2} \tag{116}$$

$$= \frac{2r(S_1 + S_2) - r^2}{\pi S_1 S_2} \tag{117}$$

Additional numeric validation for a variety of parameters is provided in Section §5.

5 Numeric Simulation of Crossing and Other Conditions

To summarize, the probability that a randomly placed line segment will cross at least c hyperplanes given that there is 1 set of parallel hyperplanes with spacing S_1 is as follows

$$P(C \geq c|r, D, N=1, S) = \begin{cases} 0 & r < S(c-1) \\ A(c) & S_1(c-1) < r < S_1 c \\ A(c) - A(c+1) & r > S_1 c \end{cases} \tag{118}$$

$$A(k) = \frac{r}{S_1}\left[\frac{(1-\gamma^2)^{\frac{D-1}{2}}}{\pi}\left(B\left(\frac{D}{2},\frac{1}{2}\right) + \gamma^2 \sum_{i=1}^{\lfloor\frac{D-2}{2}\rfloor} \frac{B(\frac{D-2i}{2},\frac{1}{2})}{(1-\gamma^2)^i}\right) - \gamma g(\gamma, D-2)\right] \tag{119}$$

$$\gamma = \frac{S_1(k-1)}{r},\ g(n,m) = \begin{cases} \frac{2}{\pi}\arccos(n) & m \bmod 2 = 0 \\ 1-n & m \bmod 2 = 1 \end{cases} \tag{120}$$

To compare this against numeric simulation, many samples of a randomly placed needle can be generated and checked to see if they satisfy the inequality $C \geq c$. The initial point of the needle, P_0, is simulated using a uniformly distributed random variable in the domain $(\vec{0}, S)$ and has coordinates $\vec{x} \in \mathbb{R}^D$. For the other point of the needle, P_1, samples with uniform spherical distribution must be generated for higher dimensional space. We use the method proposed by Marsaglia [1972] of normalizing a rotationally symmetric distribution (such as a D-dimensional gaussian variable) and label these coordinates as $\vec{y} \in \mathbb{R}^D$. For a needle of length r, the point P_1 is then at the point $\vec{x} + r\vec{y}$.

The number of hyperplanes crossed by the needle in each coordinate is then of the following form.

$$c = \sum_{n=1}^{N}\left|\left\lfloor\frac{x_n + ry_n}{S_n}\right\rfloor\right| \tag{121}$$

The expected value of the probability can then be approximated by simulating many needles, checking how many have satisfied the number of crossings, and dividing by the total number of simulations. The above procedure was simulated for 100,000 randomly generated needles for each permutation of parameters listed below[1] and results shown in Figure 3. The probabilities were calculated for $c \in \{1,2,3,4\}$, needle length $r \in (0,10]$, dimensions $D \in [2,10]$, a single hyperplane with spacing $S_1 = 1$.

Similarly, the count of intersections can be compared to an exact number of crossings and an upper bound as well. The modeled probability for both can be calculated as follows. The simulated and modeled probabilities can be found in Figure 4 and Figure 5 respectively.

$$P(C = c) = P(C \geq c) - P(C \geq c+1) \tag{122}$$

$$P(C \leq c) = 1 - P(C \geq c+1) \tag{123}$$

The same simulation and comparison can be made with the case $N \geq 1$. The short needle probabilities for $C \geq c$, $C = c$, and $C \leq c$ can be seen in Figures 6, 7, and 8 respectively.

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[1]For a total of 363.6 million needles

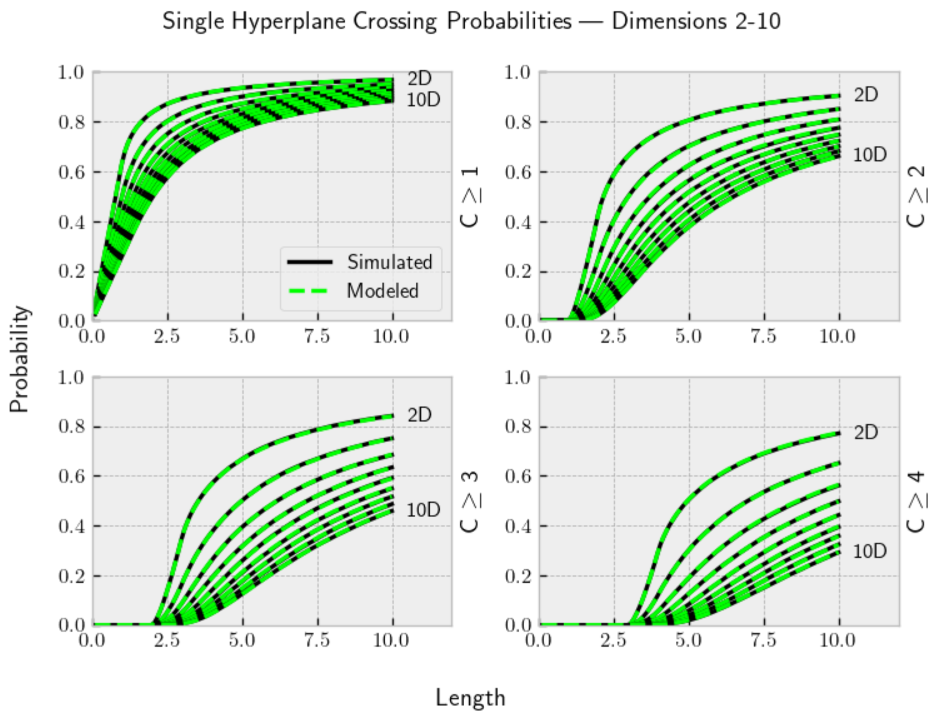

Figure 3: Comparison of numerically simulated and modeled probabilities for a variety of parameters. 100,000 needles were simulated for every parameter permutation.

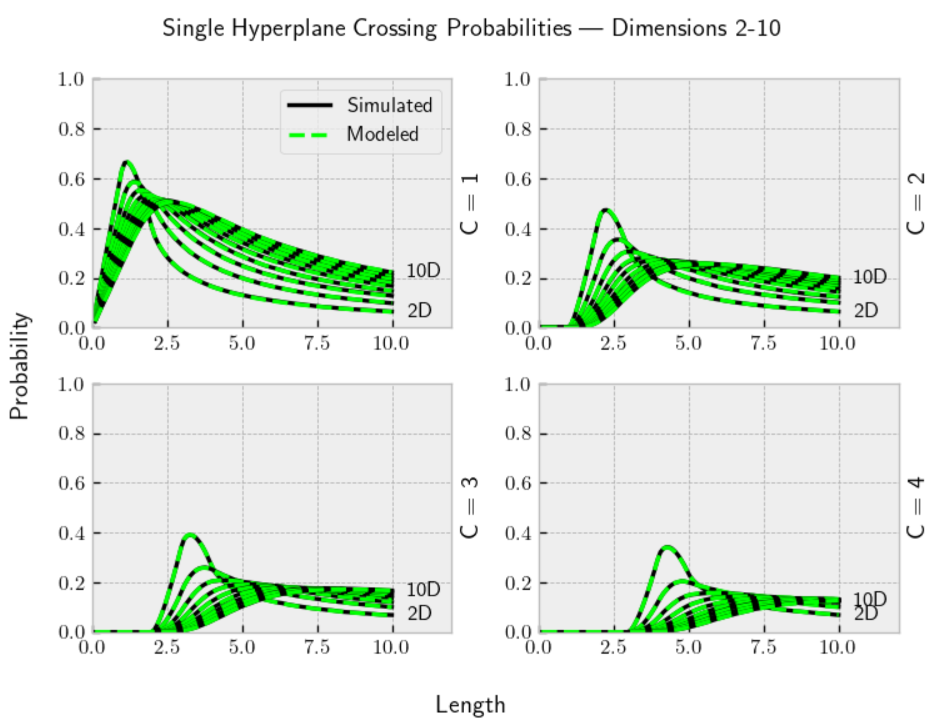

Figure 4: Numeric and modeled probability for exactly c needle crossings.

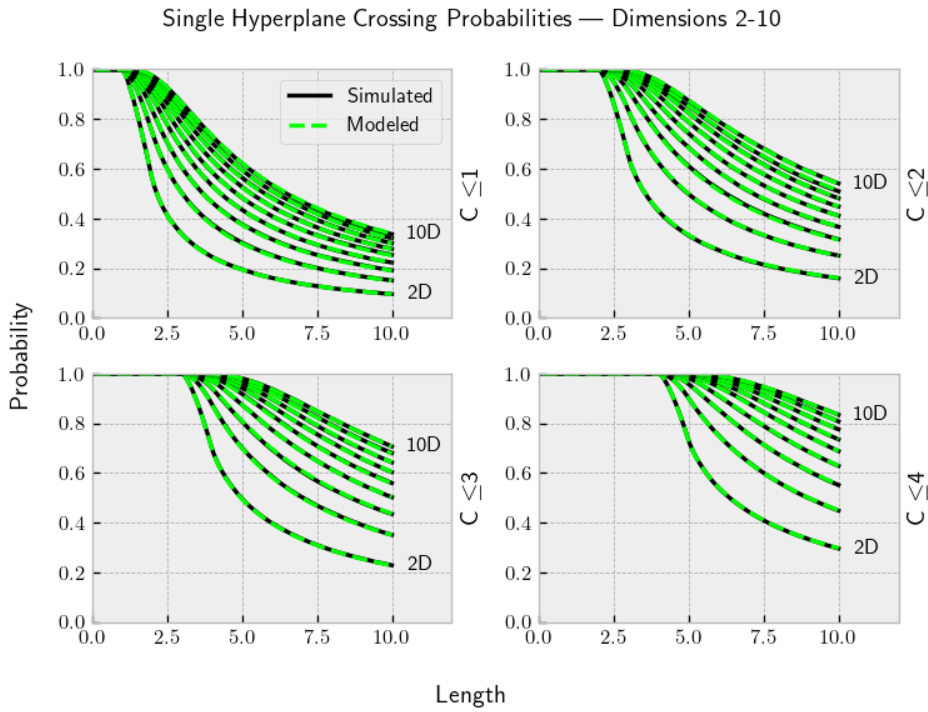

Figure 5: Numeric and modeled probability for at most c needle crossings.

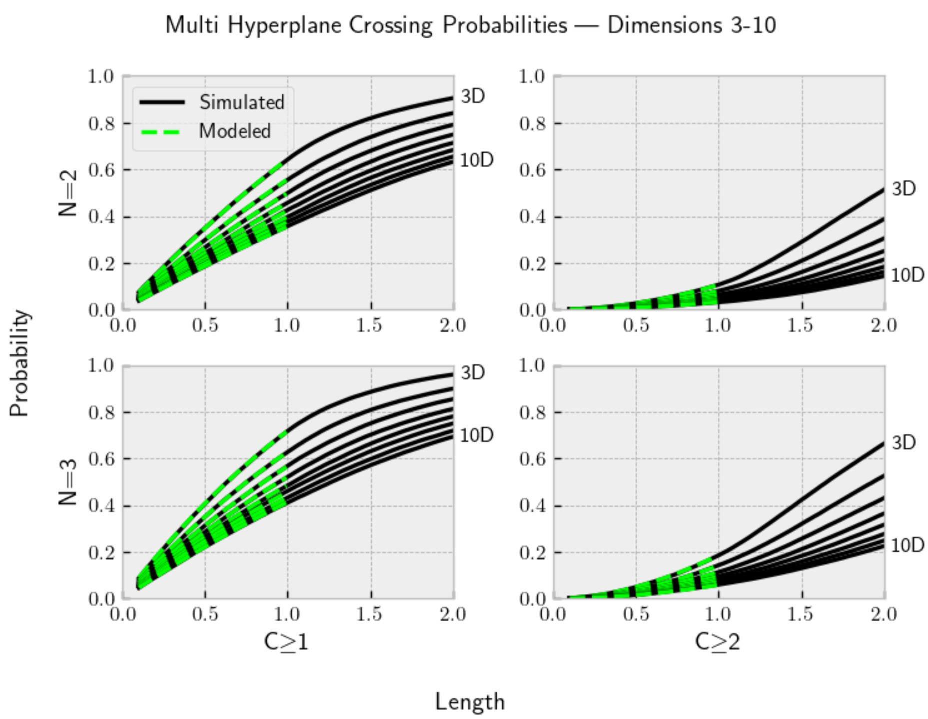

Figure 6: Numeric and modeled probability for at least c needle crossings for various values of N. The left column uses $c = 1$ while the right column shows $c = 2$. The top row uses $N = 2$ while the bottom row uses $N = 3$. The hyperplane spacings, S, were set to have values of $1, 2, \ldots, D$.

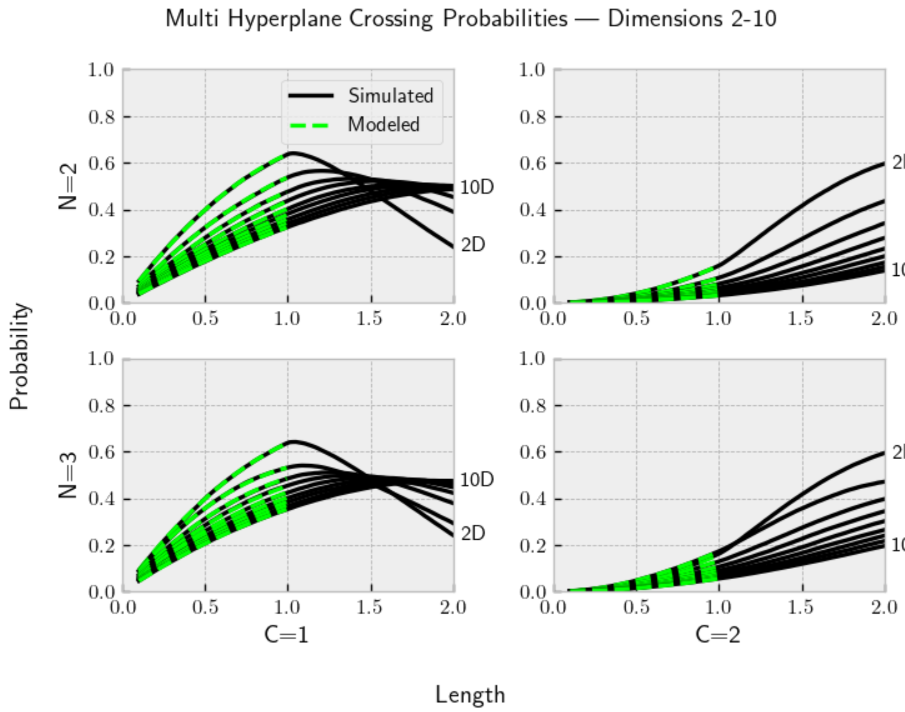

Figure 7: Numeric and modeled probability for exactly c needle crossings for various values of N. The left column uses $c = 1$ while the right column shows $c = 2$. The top row uses $N = 2$ while the bottom row uses $N = 3$. The hyperplane spacings, S, were set to have values of $1, 2, \ldots, D$.

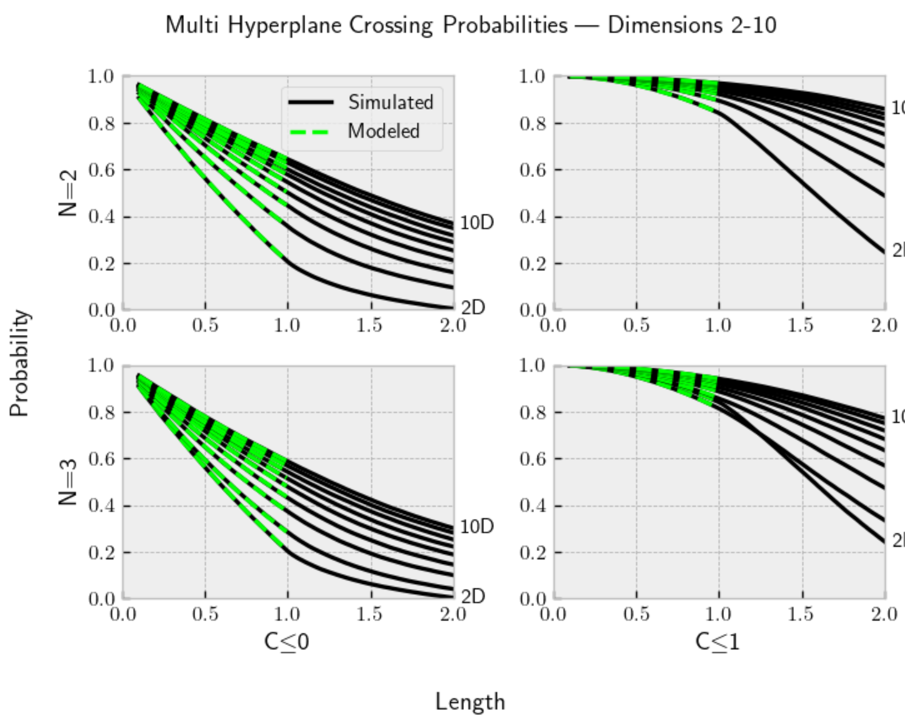

Figure 8: Numeric and modeled probability for at most c needle crossings for various values of N. The left column uses $c = 0$ while the right column shows $c = 1$. The top row uses $N = 2$ while the bottom row uses $N = 3$. The hyperplane spacings, S, were set to have values of $1, 2, \ldots, D$.