Multidimensional Extension of Buffon's Needle Problem

A Preprint

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Abstract

Consider a line segment randomly placed on a two-dimensional plane ruled with a set of regularly spaced parallel lines. The classical Buffon's needle problem asks what the probability is that the line segment intersects at least 1 of these lines. This paper extends this problem by considering a line segment randomly placed in $\mathbb{R}^D$ and its probability of intersection with a set of regularly spaced parallel hyperplanes.

Keywords Buffon's needle problem · Geometric Probability

1 Introduction

Buffon's needle problem was originally posed in the 18th century with the following premise. Given a line segment, or "needle", of length r randomly dropped on a two-dimensional plane ruled with a set of parallel lines regularly spaced s units apart, what is the probability that the needle crosses at least 1 of the lines? The solution, it turns out, is $\frac{2r}{s\pi}$ when $r < s$. Variations and extensions of this problem have been investigated as well, including

- Laplace's Extension - Investigating when the plane is gridded with 2 orthogonal sets of parallel lines with spacings s_1 and s_2.
- Buffon's Noodle - Instead of being rigidly straight, the needle is permitted to bend (a "noodle").
- Pivot Needle - The needle is constructed of two line segments that hinge together. Each crossing is considered.

In this paper, we investigate a particular extension that allows the needle to be dropped into a space with dimension greater than 2. In these higher dimensions, we will rule the space with parallel hyperplanes rather than lines. Additionally, we will look at gridding the space with orthogonal sets of hyperplanes, thereby extending Laplace's extension into higher dimensions.

Given $D \in \mathbb{N}_{>0}$ and $N \in [1, 2, \ldots, D]$, consider a grid on $\mathbb{R}^D$ formed by N orthogonal sets of regularly spaced hyperplanes where each set of hyperplanes has a potentially unique spacing of S_i. For example, if $D = 2, N = 1, S_1 = 2$, the grid would match the original Buffon Needle problem and would have only a single set of parallel lines 2 units apart as seen in 1a. If $D = 2, N = 2, S = [1, 2]$, the grid would have 2 sets of parallel lines that are orthogonal to each other, matching the problem in Laplace's extension as seen in 1b. One set of lines would have a spacing of 1 unit and the other would have a spacing of 2 units.

A line segment of length $r \in \mathbb{R}^+$ is randomly located in the space such that one of its end points, P_0, is uniformly distributed across the entire domain. The line segment's orientation is independently distributed such that when considering P_0 as the center of a $(D-1)$-sphere of radius r, the other point, P_1, is uniformly distributed on the surface of that hypersphere. This line segment may intersect with $C \in \mathbb{N}$ unique hyperplanes. This paper studies the probability that the line segment intersects with at least c hyperplanes, $P(C \geq c | r, D, N, S)$. From there, solutions for crossing less than c hyperplanes and exactly c hyperplanes can be derived.

(a) $D = 2, N = 1, S_1 = 2$, the parameter set matching Buffon's Needle Problem. In this case the number of crossings, C, is 1.

(b) $D = 2, N = 2, S = [2, 1]$, the parameter set matching Laplace's Extension. In this case the number of crossings, C, is 2.

Figure 1: Examples of different parameter sets

We will define the coordinates of line segment using $\vec{x} \in \mathbb{R}^D$ for the location of P_0 and spherical coordinates for the location of P_1 with respect to P_0.

$$\begin{aligned}
y_1 &= r\cos\phi_1 \\
y_2 &= r\sin\phi_1\cos\phi_2 \\
&\vdots \\
y_{D-1} &= r\sin\phi_1\ldots\sin\phi_{D-2}\cos\phi_{D-1} \\
y_D &= r\sin\phi_1\ldots\sin\phi_{D-2}\sin\phi_{D-1} \\
P_1 &= \vec{x} + \vec{y} \\
\phi_j &\in \begin{cases} [0,\pi] & j < D-1 \\ [0,2\pi] & j = D-1 \end{cases}
\end{aligned}$$

Translational symmetry of the grid of hyperplanes allows us to consider the domain of P_0 to be $x_i \in [0, S_i]$ as the origin can be moved to any point on the grid. Reflectional symmetry of the grid also allows us to consider the domain of $\vec{y}$ to be a single orthant of the hypersphere. For convenience, we will pick the orthant where $\phi_i \in [0, \pi/2]$.

The rest of the paper is organized as follows. A derivation of the joint probability density function for P_0 and P_1 will be provided in §2. The derivation and validation of the crossing probabilities for $N = 1$ will be given in §3. The derivation and validation of the crossing probabiities for any N and $r < \min(S)$ will be given in §4. Analysis of the limits and extrema of the probabilities is explored in §4.

2 Joint Probability Density of the Line Segment

Each coordinate for P_0 can be defined as a uniformly distributed random variable $X_i \sim \mathrm{Uniform}(0, S_i)$. Due to independence, the joint PDF for P_0 is the product $\prod_{i=1}^{D} \frac{1}{S_i}$. By the definition of the problem, the coordinates $\vec{x}$ do not influence the orientation of the line segment defined by $\vec{\phi}$. The probability density function for the uniform distribution of points on an orthant of the hypersphere can be determined by calculating the area element in terms of spherical coordinates.

Proposition 1. *In spherical coordinates, the probability density function for a uniform distribution on an orthant of a hypersphere is* $\frac{2^D}{A_{D-1}} \prod_{j=1}^{D-1} r\sin^{D-1-j}\phi_j$ *where* A_{D-1} *is the surface area of a* $(D-1)$*-sphere.*

Proof. The area element of an $(D-1)$-sphere of radius r can be expressed as

$$d\Omega = \left(\prod_{j=1}^{D-1} r\sin^{D-1-j}\phi_j\right) d\phi_1\ldots d\phi_{D-1} \tag{1}$$

The probability that a point lies in this differential element can be expressed as follows.

$$f_\Omega(\Omega)d\Omega = f_{\vec{\phi}}(\phi_1,\ldots,\phi_{D-1})d\phi_1\ldots d\phi_{D-1} \tag{2}$$

The points are uniformly distributed over the surface of an orthant of the hypersphere implying that $f_\Omega(\Omega) = \frac{2^D}{A_{D-1}}$. Substituting this and 1 into 2 yields

$$\frac{2^D}{A_{D-1}} \prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j = f_{\vec{\phi}}(\phi_1, \ldots, \phi_{D-1}) \tag{3}$$

□

Then by independence, the joint probability density function for the entire line segment can be expressed as

$$f_{\vec{X},\vec{\phi}}(x_1, \ldots, x_D, \phi_1, \ldots, \phi_{D-1}) = \frac{2^D}{A_{D-1}} \left(\prod_{i=1}^{D} \frac{1}{S_i} \right) \left(\prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j \right) \tag{4}$$

The expression for the surface area of a $D-1$ dimensional hypersphere, $A_{D-1} = \frac{2\pi^{D/2} r^{D-1}}{\Gamma(D/2)}$, can be substituted in and simplified.

$$f_{\vec{X},\vec{\phi}}(\vec{x}, \vec{\phi}) = \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2} \prod_{i=1}^{D} S_i} \left(\prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j \right) \tag{5}$$

3 Probability of crossing $N = 1$

In general, the probability of meeting some number of crossings given any set of parameters can be described as follows.

$$P(C \geq c | r, D, N, S) = \int \cdots \int_V f_{\vec{X}\vec{\phi}}(\phi_1, \ldots, \phi_{D-1}) dx_1 \ldots dx_D d\phi_1 \ldots d\phi_{D-1} \tag{6}$$

$$= \frac{2^{D-1}\Gamma(\frac{D}{2})}{\pi^{D/2} \prod_{i=1}^{D} S_i} \int \cdots \int_V \prod_{j=1}^{D-1} \sin^{D-1-j} \phi_j dx_1 \ldots dx_D d\phi_1 \ldots d\phi_{D-1} \tag{7}$$

Where V is the hypervolume in which the condition $C \geq c$ is true. The necessary conditions for achieving some number of intersections will be called *crossing conditions*. The definition of these crossing conditions and the solution to the above equation will be explored for a variety of parameters.

We start with a simplified set of parameters where there is only a single set of parallel hyperplanes and the needle intersects a hyperplane at least c times. That is, the probability $P(C \geq c | r, D, N = 1, S) \forall c, r, D, S$. For brevity, we will refer to this as $P_{N=1}(c)$.

Due to rotational symmetry of the line segment, it does not matter in which direction the hyperplanes extend. Without loss of generality we assume the planes are in the direction of x_1.

Because P_0 is constrained to be within the gridcell at the origin and because the orthant we are investigating is in the direction of x_1, we know that a crossing occurs whenever the following equivalent conditions are met.

$$x_1 + r \cos \phi_1 > S_1 c \tag{8}$$

$$r > \frac{S_1 c - x_1}{\cos \phi_1} \tag{9}$$

$$x_1 > S_1 c - r \cos \phi_1 \tag{10}$$

$$\phi_1 < \arccos \frac{S_1 c - x_1}{r} \tag{11}$$

From these crossing conditions, we define the bounds of the relevant hypervolume. Note that the constraints above only apply to x_1 and ϕ_1. As such, the hypervolume spans the entire domain of every other variable.

The minimum of $\frac{S_1 c - x_1}{\cos \phi_1}$ occurs at $x_1 = S_1, \phi_1 = 0$ with a value of $S_1(c-1)$. Therefore if $r < S_1(c-1)$ we can guarantee that the crossing condition cannot be satisfied.

$$P_{N=1}(C \geq c | r < S_1(c-1)) = 0 \tag{12}$$

This is equivalent to the scenario where the needle is too short to cross the necessary number of hyperplanes, even when it is oriented orthogonally to the hyperplanes.

The domains of x_1 can then be used to define the space in which a valid crossing has occured

$$m(\phi_1) < x_1 < S_1 \tag{13}$$
$$m(\phi_1) = \max(0, S_1 c - r\cos\phi_1) \tag{14}$$

The domain of x_1 also provides a maximum bound for the maximum acceptable value of ϕ_1 when $x_1 = S_1$.

$$0 < \phi_1 < \min\left(\frac{\pi}{2}, \arccos\frac{S_1(c-1)}{r}\right) \tag{15}$$
$$0 < \phi_1 < \arccos\frac{S_1(c-1)}{r} \tag{16}$$

Maybe call out Fubini directly here?

We can now express our volume integral in terms of these conditions and solve for the location dimensions. Because our probability density function is finite across the entire domain, we may arbitrarily choose the order of integration except for x_1 and ϕ_1 whose bounds are dependent and will require the bounds of integration to change if their order is swapped.

Use fubini to simplify the unimportant integrals first before tackling the crossing conditions

$$P_1(c) = \int_0^{\pi/2}\dots\int_0^{\pi/2}\int_0^{\arccos\frac{S_1(c-1)}{r}}\int_0^{S_D}\dots\int_0^{S_2}\int_{m(\phi_1)}^{S_1} f_{\vec{\phi}}(\phi_1,\dots,\phi_{D-1})dx_1dx_2\dots dx_Dd\phi_1d\phi_2\dots d\phi_{D-1} \tag{17}$$

$$= \int_0^{\pi/2}\dots\int_0^{\pi/2}\int_0^{\arccos\frac{S_1(c-1)}{r}}\left(\prod_{i=2}^{D}S_i\right)(S_1c - m(\phi_1))\, f_{\vec{\phi}}(\phi_1,\dots,\phi_{D-1})d\phi_1d\phi_2\dots d\phi_{D-1} \tag{18}$$

$$= \frac{2^D r^{D-1}\prod_{i=2}^{D}S_i}{A_{D-1}\prod_{i=1}^{D}S_i}\int_0^{\pi/2}\dots\int_0^{\pi/2}\int_0^{\arccos\frac{S_1(c-1)}{r}}(S_1c - m(\phi_1))\prod_{j=1}^{D-1}\sin^{D-1-j}\phi_j d\phi_1d\phi_2\dots d\phi_{D-1} \tag{19}$$

$$= \frac{2^D r^{D-1}}{A_{D-1}S_1}\int_0^{\pi/2}\dots\int_0^{\pi/2}\int_0^{\arccos\frac{S_1(c-1)}{r}}(S_1c - m(\phi_1))\prod_{j=1}^{D-1}\sin^{D-1-j}\phi_j d\phi_1d\phi_2\dots d\phi_{D-1} \tag{20}$$

The value of $m(\phi_1)$ depends on the value of r. If $r < S_1c$, then $m(\phi_1) = S_1c - r\cos\phi_1 \forall\phi_1$. If $r > S_1c$ we will need to partition the interval of integration into two regions, one where $S_1c - r\cos\phi_1$ is greater than 0 and one where it is less than zero. The transition occurs at the value $\phi_1 = \arccos\frac{S_1c}{r}$.

$$m(\phi_1) = \begin{cases} 0 & \frac{S_1c}{\cos\phi_1} > r > S_1c \\ S_1c - r\cos\phi_1 & \text{otherwise} \end{cases} \tag{21}$$

3.1 $S_1(c-1) < r < S_1c$

When $r < S_1c$ we have the following expression by substituting $S_1c - r\cos\phi_1$ for $m(\phi_1)$.

$$P_1(c) = \frac{2^D r^{D-1}}{A_{D-1}S_1}\int_0^{\pi/2}\dots\int_0^{\pi/2}\int_0^{\arccos\frac{S_1(c-1)}{r}}(S_1(1-c) + r\cos\phi_1)\prod_{j=1}^{D-1}\sin^{D-1-j}\phi_j d\phi_1d\phi_2\dots d\phi_{D-1} \tag{22}$$

$$= \frac{2^D r^{D-1}}{A_{D-1}S_1}\int_0^{\pi/2}\dots\int_0^{\pi/2}\int_0^{\arccos\frac{S_1(c-1)}{r}}(S_1(1-c) + r\cos\phi_1)\sin^{D-2}\phi_1\prod_{j=2}^{D-1}\sin^{D-1-j}\phi_j d\phi_1d\phi_2\dots d\phi_{D-1} \tag{23}$$

$$\begin{aligned} = \frac{2^D r^{D}}{A_{D-1}S_1}\int_0^{\pi/2}\dots\int_0^{\pi/2}\prod_{j=2}^{D-1}\sin^{D-1-j}\phi_j\Bigg(\frac{S_1(1-c)}{r}\int_0^{\arccos\frac{S_1(c-1)}{r}}\sin^{D-2}\phi_1 d\phi_1 \\ + \int_0^{\arccos\frac{S_1(c-1)}{r}}\cos\phi_1\sin^{D-2}\phi_1 d\phi_1\Bigg)d\phi_2\dots d\phi_{D-1} \end{aligned} \tag{24}$$

The two interior integrals can be solved via integration by reduction and u-substitution respectively. It is convenient if we first define the following proposition.

This prop can be condensed by maybe just citing the gamma representation of a multi-factorial

Proposition 2. *When given the ratio $(k-1)!!/k!!$ where the double exclam represents the double factorial function, it is equivalent the following.*

$$= \begin{cases} \frac{1}{\pi}B(\frac{k+1}{2},\frac{1}{2}) & k \bmod 2 = 0 \\ \frac{1}{2}B(\frac{k+1}{2},\frac{1}{2}) & k \bmod 2 = 1 \end{cases} \tag{25}$$

Proof. We start by deriving a value for $n!!$ in terms of factorials. If $n \bmod 2 = 0$

$$n!! = n(n-2)\dots(4)(2) \tag{26}$$

$$= 2^{n/2}\frac{n}{2}\frac{n-2}{2}\cdots\frac{4}{2}\frac{2}{2} \tag{27}$$

$$= 2^{n/2}\frac{n}{2}! \tag{28}$$

If $n \bmod 2 = 1$

$$n!! = n(n-2)\dots(3)(1) \tag{29}$$

$$= \frac{n!}{(n-1)!!} \tag{30}$$

$$= \frac{n!}{2^{(n-1)/2}(\frac{n-1}{2})!} \tag{31}$$

Using 28 and 31 we can simplify $(k-1)!!/k!!$. First, assuming that k is even

$$\frac{(k-1)!!}{k!!} = \frac{(k-1)!}{2^{(k-2)/2}(\frac{k-2}{2})!}\frac{1}{2^{k/2}(\frac{k}{2})!} \tag{32}$$

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{\frac{1}{2}\frac{2}{2}\cdots\frac{k-2}{2}\frac{k-1}{2}}{\frac{2}{2}\frac{4}{2}\cdots\frac{k-4}{2}\frac{k-2}{2}(\frac{k}{2}!)} \tag{33}$$

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{\frac{1}{2}\frac{3}{2}\cdots\frac{k-3}{2}\frac{k-1}{2}}{\frac{k}{2}!} \tag{34}$$

Now using the property $n\Gamma(n) = \Gamma(n+1)$ and $n! = \Gamma(n+1)$, we get the following.

$$\frac{(k-1)!!}{k!!} = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{k+2}{2})} \tag{35}$$

Finally, using $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $\Gamma(1/2) = \sqrt{\pi}$ we get

$$\frac{(k-1)!!}{k!!} = \frac{1}{\pi}B\left(\frac{k+1}{2},\frac{1}{2}\right) \tag{36}$$

We now repeat the process for the case where k is odd.

$$\frac{(k-1)!!}{k!!} = \left(\frac{k-1}{2}\right)!2^{(k-1)/2}\frac{(\frac{k-1}{2})!2^{(k-1)/2}}{k!} \tag{37}$$

$$= \frac{2\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2})}\frac{2^{k-1}(\frac{k-1}{2})!^2}{k!} \tag{38}$$

$$= \frac{\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2})}\frac{\frac{2}{2}\frac{4}{2}\cdots\frac{k-3}{2}\frac{k-1}{2}(\frac{k-1}{2}!)}{\frac{1}{2}\frac{2}{2}\cdots\frac{k-1}{2}\frac{k}{2}} \tag{39}$$

$$= \frac{\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2})}\frac{\frac{k-1}{2}!}{\frac{1}{2}\frac{3}{2}\cdots\frac{k-2}{2}\frac{k}{2}} \tag{40}$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{k+1}{2})}{2\Gamma(\frac{k+2}{2})} \tag{41}$$

$$= \frac{1}{2}B\left(\frac{k+1}{2},\frac{1}{2}\right) \tag{42}$$

□

We now define the following proposition for the initial integral in 24.

Proposition 3. *Any integral of the form $\int_0^{\arccos(\gamma)} \sin^m \phi d\phi$ has two possible solutions depending on the parity of m.*

$$= \frac{B(\frac{m+1}{2}, \frac{1}{2})}{2} \left(g(\gamma, m) - \gamma(1-\gamma^2)^{(m+1)/2} \sum_{i=1}^{\lfloor m/2 \rfloor} \frac{B(\frac{m+2-2i}{2}, \frac{1}{2})}{\pi(1-\gamma^2)^i} \right) \tag{43}$$

$$g(\gamma, m) = \begin{cases} \frac{2}{\pi} \arccos \gamma & m \bmod 2 = 0 \\ 1 - \gamma & m \bmod 2 = 1 \end{cases} \tag{44}$$

Proof. We start with the following integration by reduction identity

$$\int_0^{\arccos \gamma} \sin^m \phi d\phi = -\frac{1}{m} \sin^{m-1} \phi \cos \phi \Big|_0^{\arccos \gamma} + \frac{m-1}{m} \int_0^{\arccos \gamma} \sin^{m-2} \phi d\phi \tag{45}$$

$$\begin{aligned} &= -\frac{1}{m}(1-\gamma^2)^{(m-1)/2}\gamma \\ &\quad + \frac{m-1}{m}\left(-\frac{1}{m-2} \sin^{m-3} \phi \cos \phi \Big|_0^{\arccos \gamma} + \frac{m-3}{m-2} \int_0^{\arccos \gamma} \sin^{m-4} \phi d\phi \right) \end{aligned} \tag{46}$$

This pattern continues until the $\sin$ in the final integrand is raised to either the first or zeroth power. This depends on whether m is even or odd. If m is even

$$\begin{aligned} &= -\frac{1}{m}(1-\gamma^2)^{(m-1)/2}\gamma - \frac{m-1}{m(m-2)}(1-\gamma^2)^{(m-3)/2}\gamma - \dots \\ &\quad - \frac{(m-1)(m-3)\dots(3)}{(m)(m-2)\dots(2)}(1-\gamma^2)^{1/2}\gamma + \frac{(m-1)!!}{m!!} \int_0^{\arccos \gamma} d\phi \end{aligned} \tag{47}$$

$$= \frac{(m-1)!!}{m!!} \left(-\frac{(m-2)!!}{(m-1)!!}(1-\gamma^2)^{(m-1)/2}\gamma - \frac{(m-4)!!}{(m-3)!!}(1-\gamma^2)^{(m-3)/2}\gamma - \dots - \frac{0!!}{1!!}(1-\gamma^2)^{1/2} + \arccos \gamma \right) \tag{48}$$

$$= \frac{(m-1)!!}{m!!} \left(\arccos \gamma - \gamma \sum_{i=1}^{m/2} \frac{(m-2i)!!}{(m+1-2i)!!} (1-\gamma^2)^{(m+1-2i)/2} \right) \tag{49}$$

$$= \frac{(m-1)!!}{m!!} \left(\arccos \gamma - \gamma(1-\gamma^2)^{(m+1)/2} \sum_{i=1}^{m/2} \frac{(m-2i)!!}{(m+1-2i)!!} (1-\gamma^2)^{-i} \right) \tag{50}$$

Using 2 we can reduce to the following.

$$= \frac{B(\frac{m+1}{2}, \frac{1}{2})}{\pi} \left(\arccos \gamma - \gamma(1-\gamma^2)^{(m+1)/2} \sum_{i=1}^{m/2} \frac{B(\frac{m+2-2i}{2}, \frac{1}{2})}{2} (1-\gamma^2)^{-i} \right) \tag{51}$$

$$= \frac{B(\frac{m+1}{2}, \frac{1}{2})}{2} \left(\frac{2}{\pi} \arccos \gamma - \gamma(1-\gamma^2)^{(m+1)/2} \sum_{i=1}^{m/2} \frac{B(\frac{m+2-2i}{2}, \frac{1}{2})}{\pi(1-\gamma^2)^i} \right) \tag{52}$$

Repeating for the case where m is odd

$$\begin{aligned} &= -\frac{1}{m}(1-\gamma^2)^{(m-1)/2}\gamma - \frac{m-1}{m(m-2)}(1-\gamma^2)^{(m-3)/2}\gamma - \dots \\ &\quad - \frac{(m-1)(m-3)\dots(3)}{(m)(m-2)\dots(2)}(1-\gamma^2)^{1/2}\gamma + \frac{(m-1)!!}{m!!} \int_0^{\arccos \gamma} \sin \phi d\phi \end{aligned} \tag{53}$$

$$= \frac{(m-1)!!}{m!!} \left(1 - \gamma - \gamma(1-\gamma^2)^{(m+1)/2} \sum_{i=1}^{(m-1)/2} \frac{(m-2i)!!}{(m+1-2i)!!} (1-\gamma^2)^{-i} \right) \tag{54}$$

$$= \frac{B(\frac{m+1}{2}, \frac{1}{2})}{2} \left(1 - \gamma - \gamma(1-\gamma^2)^{(m+1)/2} \sum_{i=1}^{\lfloor m/2 \rfloor} \frac{B(\frac{m+2-2i}{2}, \frac{1}{2})}{\pi(1-\gamma^2)^i} \right) \tag{55}$$

□

We can substitute the solution from 3 int 24 to get the following.

$$P_1(c) = \frac{2^D r^D}{A_{D-1} S_1} \int_0^{\pi/2} \dots \int_0^{\pi/2} \prod_{j=2}^{D-1} \sin^{D-1-j} \phi_j \left(\frac{-\gamma}{2} B\left(\frac{D-1}{2}, \frac{1}{2}\right) \left(g(\gamma, D-2) - \gamma (1-\gamma^2)^{(D-1)/2} \sum_{k=1}^{\lfloor \frac{D-2}{2} \rfloor} \frac{B(\frac{D-2k}{2}, \frac{1}{2})}{\pi (1-\gamma^2)^k} \right) + \int_0^{\arccos \gamma} \cos\phi_1 \sin^{D-2} \phi_1 d\phi_1 \right) d\phi_2 \dots d\phi_{D-1} \tag{56}$$

Where $\gamma = S_1(c-1)/r$.

Applying u-substitution where $u = \sin\phi_1$ we get the following

$$P_1(c) = \frac{2^D r^D \xi}{A_{D-1} S_1} \int_0^{\pi/2} \dots \int_0^{\pi/2} \prod_{j=2}^{D-1} \sin^{D-1-j} \phi_i d\phi_2 \dots d\phi_{D-1} \tag{57}$$

$$\xi = \frac{-\gamma}{2} B\left(\frac{D-1}{2}, \frac{1}{2}\right) \left(g(\gamma, D-2) - \gamma (1-\gamma^2)^{(D-1)/2} \sum_{k=1}^{\lfloor \frac{D-2}{2} \rfloor} \frac{B(\frac{D-2k}{2}, \frac{1}{2})}{\pi (1-\gamma^2)^k} \right) + \frac{1}{D-1} (1-\gamma^2)^{(D-1)/2} \tag{58}$$

To solve the remaining $D-2$ integrals, we start by noting that we can simplify the result from 3 by noting that the remaining upper bounds of integration are all $\pi/2$.

Restating the result, we have the following

$$\int_0^{\arccos\gamma} sin^m \phi d\phi = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{1}{2}\right) \left(g(\gamma, m) - \gamma \sum_{i=1}^{\lfloor m/2 \rfloor} \frac{B(\frac{m+2-2i}{2}, \frac{1}{2})}{\pi} (1-\gamma^2)^{(m+1)/2-i} \right) \tag{59}$$

$$\int_0^{\arccos 0} sin^m \phi d\phi = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{1}{2}\right) (1-0) \tag{60}$$

$$= \frac{1}{2} B\left(\frac{m+1}{2}, \frac{1}{2}\right) \tag{61}$$

For every integral in 57 we get the following product.

$$P_1(c) = \frac{2^D r^D \xi}{A_{D-1} S_1} \prod_{j=2}^{D-1} \frac{1}{2} B\left(\frac{D-j}{2}, \frac{1}{2}\right) \tag{62}$$

$$= \frac{2^D r^D \xi}{A_{D-1} S_1} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{D-1}{2})} \left(\frac{\sqrt{\pi}}{2}\right)^{D-2} \tag{63}$$

We can now substitute in an expression of A_{D-1} as follows

$$A_{D-1} = \frac{2\pi^{D/2} r^{n-1}}{\Gamma(\frac{D}{2})} \tag{64}$$

$$P_1(c) = \frac{2^D r^D \xi}{2\pi^{D/2} r^{D-1} S_1} \frac{\Gamma(\frac{D}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{D+1}{2})} \left(\frac{\sqrt{\pi}}{2}\right)^{D-2} \tag{65}$$

$$= \frac{2r}{\pi S_1} \frac{\xi \pi}{B(\frac{D-1}{2}, \frac{1}{2})} \tag{66}$$

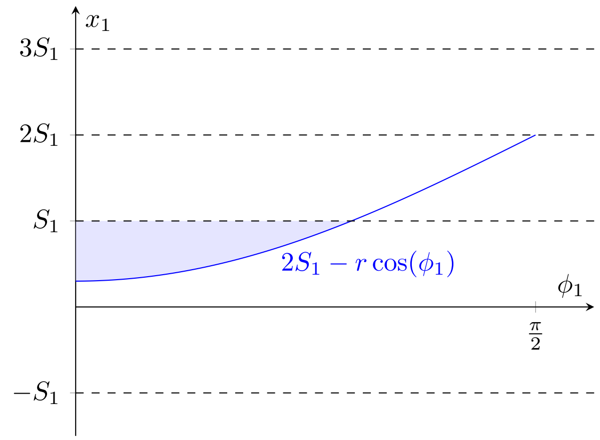

Figure 2: Domain of integration when $r < S_1 c$

We get a solution reminiscent of the original Buffon needle problem $(2r/\pi S)$ with an extra factor that is dependent on the dimension of the space. Substituting in our function for ξ in 58 and simplifying, we get

$$\begin{aligned} P_1(c) &= \frac{2r}{\pi S_1}\Bigg(-\frac{\gamma\pi B(\frac{D-1}{2},\frac{1}{2})}{2B(\frac{D-1}{2},\frac{1}{2})}\left(g(\gamma, D-2) - \gamma(1-\gamma^2)^{(D-1)/2}\sum_{k=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2k}{2},\frac{1}{2})}{\pi(1-\gamma^2)^k}\right) \\ &\quad + \frac{\pi}{B(\frac{D-1}{2},\frac{1}{2})(D-1)}(1-\gamma^2)^{(D-1)/2}\Bigg) \end{aligned} \tag{67}$$

$$= \frac{2r}{\pi S_1}\left(-\frac{\gamma\pi}{2}\left(g(\gamma, D-2) - \gamma(1-\gamma^2)^{(D-1)/2}\sum_{k=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2k}{2},\frac{1}{2})}{\pi(1-\gamma^2)^k}\right) + \frac{B(\frac{D}{2},\frac{1}{2})}{2}(1-\gamma^2)^{(D-1)/2}\right) \tag{68}$$

$$= \frac{r}{S_1}\left(\frac{(1-\gamma^2)^{(D-1)/2}}{\pi}\left(B(\frac{D}{2},\frac{1}{2}) + \gamma^2\sum_{k=1}^{\lfloor\frac{D-2}{2}\rfloor}\frac{B(\frac{D-2k}{2},\frac{1}{2})}{(1-\gamma^2)^k}\right) - \gamma g(\gamma, D-2)\right) \tag{69}$$

3.2 $r > S_1 c$

When $r > S_1 c$, the value of $m(\phi_1)$ is no longer constant for all ϕ_1. Normally this would require the splitting of the bounds of integration for the conditions where $\phi_1 < \arccos\frac{S_1}{r}$ and $\phi_1 > \arccos\frac{S_1}{r}$. However, there is an alternative method which can avoid additional integration.

This section feels pretty loosy goosy. Seems like need to be more rigorous

Other than the double integral involving x_1 and ϕ_1, all other terms stay the same. Because we are able to change the order of integration, we can claim the following.

$$P_1(c) = \frac{2r}{S_1 B(\frac{D-1}{2},\frac{1}{2})}\iint_A \sin^{D-2}\phi_1 dA \tag{70}$$

There are now two things to note. First, the integrand only varies with ϕ_1. Second, the formula for $P_1(c)$ calculated for when $r < S_1 c$ took the integral from the curve $S_1 c - r\cos(\phi_1)$ to the line $S_1 c$ and resulted in ξ. This is shown in figures 2 and 3 as the blue shaded region.

fix figure spacing, maybe put the figs side by side instead

When r exceeds the value of $S_1 c$, the region enclosed by the curve exceeds the domain of interest. Specifically, the region where $x_1 < 0$. One way to correct for this is to realize that the area between the curve and the axis is identical to the area between $x_1 = S_1$ and the same curve translated up by S_1. This is convenient as we have an expression for the integrals in the region between curves of the form $S_1 c - r cos\phi_1$ and S_1. Because the integrand is invariant to changes in x_1, we can guarantee that the integrals evaluate to the same value.

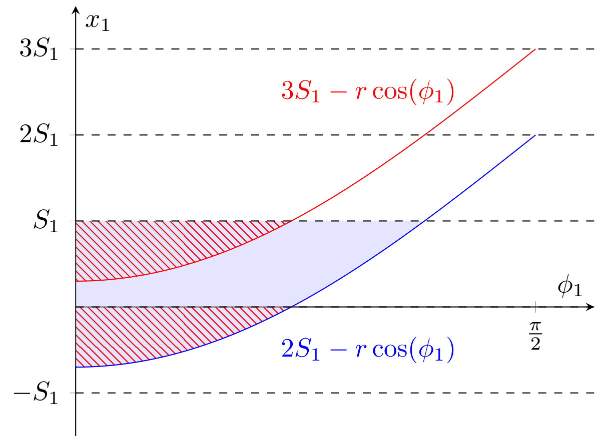

Figure 3

As such, the result is simply

$$P_1(c|r > S_1c) = \frac{2r}{S_1 B(\frac{D-1}{2}, \frac{1}{2})}(\xi(c) - \xi(c+1)) \tag{71}$$
$$= P_1(c|r < S_1c) - P_1(c+1|r < S_1c) \tag{72}$$

3.3 Numeric Validation of Crossing $N = 1$

To summarize, the probability that a randomly placed line segment will cross at least c hyperplanes given that there is 1 set of parallel hyperplanes with spacing S_1 is as follows

$$P(C \geq c|r, D, N = 1, S) = \begin{cases} 0 & r < S(c-1) \\ \frac{2r}{S_1 B(\frac{D-1}{2}, \frac{1}{2})}\xi(c) & S_1(c-1) < r < S_1c \\ \frac{2r}{S_1 B(\frac{D-1}{2}, \frac{1}{2})}(\xi(c) - \xi(c+1)) & r > S_1c \end{cases} \tag{73}$$

$$\xi(c) = \frac{-\gamma}{2} B\left(\frac{D-1}{2}, \frac{1}{2}\right)\left(g(\gamma, D-2) - \gamma(1-\gamma^2)^{(D-1)/2} \sum_{k=1}^{\lfloor\frac{D-2}{2}\rfloor} \frac{B(\frac{D-2k}{2}, \frac{1}{2})}{\pi(1-\gamma^2)^k}\right) + \frac{1}{D-1}(1-\gamma^2)^{(D-1)/2} \tag{74}$$

$$\gamma = \frac{S_1(c-1)}{r} \tag{75}$$

To compare this against numeric simulation, we must generate many samples with uniform spherical distribution. We use the method proposed by Marsaglia of normalizing rotationally symmetric distribution (such as a D-dimensional gaussian variable).

4 Probability of crossing $N \geq 1$

When there is only a single set of parallel hyperplanes, there is only one way for a needle to make c intersections. The needle would have to go through c hyperplanes in a single direction. When we increase the number of orthogonal sets of hyperplanes then we must deal with the fact that there are now many ways to cross c hyperplanes due to the many combinations of directions available.

For instance, if $N = 2$ and we want to know when $C = 2$, then a valid number of crossings occurs if the needle crosses 2 hyperplanes in x_1 and 0 in x_2, or 1 hyperplane in each direction, or 0 hyperplanes in x_1 and 2 in x_2.

For simplicity, we begin with the assumption that $r < \min(S)$ to ensure that the needle can never cross more than 1 hyperplane in any given direction. We will then investigate what happens as r grows in size.

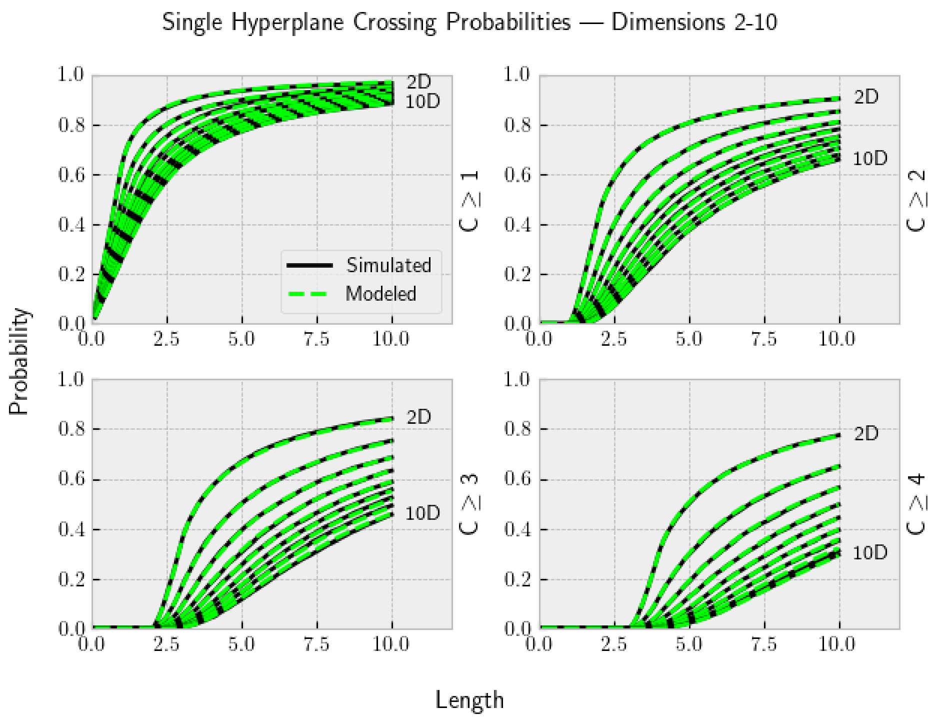

Figure 4: Comparison of numerically simulated crossing probability and modeled probabilities for $c \in [1, 2, 3, 4]$. A hyperplane spacing of 1 is used. Solutions are shown for dimensions 2 through 10. 10,000 samples were used in the numeric simulation.

4.1 $N \geq 1, r < \min(S)$

Let $P_{1\cap 2\cap \ldots \cap h}(C = n|r, D, N, S) = P_{1\cap \ldots \cap h}$ be the probability that the needle crosses at least 1 hyperplane in each of the directions $x_1, x_2, \ldots, x_h$. Similarly, let $P_{1\cup 2\cup \ldots \cup h}(C \geq 1|r, D, N, S) = P_{1\cup \ldots \cup h}$ be the probability that the needle crosses at least 1 hyperplane in any direction $x_1, x_2, \ldots, x_h$. This probability is equivalent to the probability that $P(C \geq 1|r, D, N, S)$ as crossing a hyperplane in any direction is sufficient to meet the condition $C \geq 1$. Using the inclusion-exclusion principle, this probability can be written as the following sum.

$$P(C \geq 1|r, D, N, S) = P_{1\cup 2\cup \ldots \cup N} = \sum_{k=1}^{N} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq N} P_{i_1 \cap \ldots \cap i_k} \right) \tag{76}$$

Similarly, for $C \geq c$, we can define the set of events E_c^N which consists of each of the $\binom{N}{c}$ hyperplane crossing combinations. For example, $E_2^3 = \{(1, 2), (1, 3), (2, 3)\}$. If the needle crosses hyperplanes in all of the directions listed in any element of E_c^N, then the crossing condition for the criteria $C \geq c$ has been met.

$$P(C \geq c|r, D, N, S) = P_{(\cap E_c^N[1]) \cup (\cap E_c^N[2]) \cup \ldots \cup (\cap E_c^N[\binom{N}{c}])} = \sum_{k=1}^{N} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq N} P_{E_c^N[i_1] \cap \ldots \cap E_c^N[i_k]} \right) \tag{77}$$

This expression requires an equation for the probability of having at least 1 crossing in each direction listed.

Proposition 4. *For any given set of hyperplane directions,* H*, the probability that a needle would cross at least 1 hyperplane in each of the specified directions can be represented as follows.*

$$P_{H_1 \cap \ldots \cap H_h} = \frac{r^h}{\pi^{h/2} (\prod_{i=1}^{h} S_{H,i})} \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D+h}{2})} \tag{78}$$

Proof. The set of hyperplane directions, H, with spacings S_H is a subset of all the hyperplanes that grid the space. Without loss of generalization, the axes can be relabeled to align H_1 with x_1, H_2 with x_2 and so on. All other hyperplanes that are not included in the set H can be ignored as any intersections with them are irrelevant.

The necessary conditions for crossings to occur in each direction specified in H is as follows

$$S_1 \le x_1 + r\cos\phi_1 \tag{79}$$
$$S_2 \le x_2 + r\sin\phi_1\cos\phi_2 \tag{80}$$
$$\vdots \tag{81}$$
$$S_{h-1} \le x_{h-1} + r\sin\phi_1\sin\phi_2\ldots\sin\phi_{h-2}\cos\phi_{h-1} \tag{82}$$
$$S_h \le x_h + \begin{cases} r\sin\phi_1\ldots\sin\phi_{h-1}\cos\phi_h & h<D \\ r\sin\phi_1\ldots\sin\phi_{h-2}\sin\phi_{h-1} & h=D \end{cases} \tag{83}$$

These conditions, along with the domain of $x_i \forall i \in 1,\ldots,h$, define the bounds of the volume where the needle crosses a hyperplane in each direction H.

$$S_1 \ge x_1 \ge m_1(\phi_1) = \max\{0, S_1 - r\cos\phi_1\} \tag{84}$$
$$S_2 \ge x_2 \ge m_2(\phi_2) = \max\{0, S_2 - r\sin\phi_1\cos\phi_2\} \tag{85}$$
$$\vdots \tag{86}$$
$$S_{h-1} \ge x_{h-1} \ge m_{h-1}(\phi_{h-1}) = \max\{0, S_{h-1} - r\sin\phi_1\ldots\sin\phi_{h-2}\cos\phi_{h-1}\} \tag{87}$$
$$S_h \ge x_h \ge m_h(\phi_h) = \max\left\{0, S_h - \begin{cases} r\sin\phi_1\ldots\sin\phi_{h-1}\cos\phi_h & h<D \\ r\sin\phi_1\ldots\sin\phi_{h-2}\sin\phi_{h-1} & h=D \end{cases}\right\} \tag{88}$$

Starting with 17, the crossing conditions above are encoded into the bounds of integration. Using [REF], the integrals with respect to the spatial dimensions in dimensions greater than h are reduced to a single coefficient.

turn the "any integration order" thing into a prop

$$P_{H_1\cap\ldots\cap H_h} = \frac{2^D r^{D-1}}{A_{D-1}\prod_{i=1}^h S_i}\int\cdots\int_\phi \int_{m_h(\phi_h)}^{S_h}\cdots\int_{m_1(\phi_1)}^{S_1}\prod_{j=1}^{D-1}\sin^{D-1-j}\phi_j dx_1\ldots dx_h d\phi_1\ldots d\phi_{D-1} \tag{89}$$

Given that r is less than every spacing S_i, every function $m_i(\phi_i)$ is guaranteed to be greater than zero. Every spatial integral will reduce to the polar representation of the corresponding x_i. This simplifies to the following.

$$P_{H_1\cap\ldots\cap H_h} = \frac{2^D r^{D-1}}{A_{D-1}\prod_{i=1}^h S_i}\int\cdots\int_\phi r^h\left(\prod_{k=1}^{\min(D-1,h)}\cos\phi_k\sin^{h-k}\phi_k\right)\prod_{j=1}^{D-1}\sin^{D-1-j}\phi_j d\phi_1\ldots d\phi_{D-1} \tag{90}$$
$$= \frac{2^D r^{D+h-1}}{A_{D-1}\prod_{i=1}^h S_i}\int\cdots\int_\phi \left(\prod_{k=1}^{\min(D-1,h)}\cos\phi_k\sin^{D+h-2k-1}\phi_k\right)\prod_{j=h+1}^{D-1}\sin^{D-1-j}\phi_j d\phi_1\ldots d\phi_{D-1} \tag{91}$$

The product from $k=1$ to $\min(D-1,h)$ can be reduced by using u-substitution where $u=\sin\phi_k$. Assuming the minimum function evaluates to h, this results as follows

$$P_{H_1\cap\ldots\cap H_h} = \frac{2^D r^{D+h-1}}{A_{D-1}\prod_{i=1}^h S_i}\int\cdots\int_0^{\pi/2}\frac{1}{(D+h-2)(D+h-4)\ldots(D-h)}\prod_{j=h+1}^{D-1}\sin^{D-1-j}\phi_j d\phi_{h+1}\ldots d\phi_{D-1} \tag{92}$$

$$= \frac{2^D r^{D+h-1}}{A_{D-1}\prod_{i=1}^h S_i}\frac{(D-h-2)!!}{(D+h-2)!!}\int\cdots\int_0^{\pi/2}\prod_{j=h+1}^{D-1}\sin^{D-1-j}\phi_j d\phi_{h+1}\ldots d\phi_{D-1} \tag{93}$$

$$= \frac{2^D r^{D+h-1}}{A_{D-1}\prod_{i=1}^h S_i}\frac{\Gamma(\frac{D-h}{2})}{2^h\Gamma(\frac{D+h}{2})}\prod_{j=h+1}^{D-1}\frac{B(\frac{D-j}{2},\frac{1}{2})}{2} \tag{94}$$

$$= \frac{2^D r^h\Gamma(\frac{D}{2})}{2\pi^{D/2}\prod_{i=1}^h S_i}\frac{\Gamma(\frac{D-h}{2})}{2^h\Gamma(\frac{D+h}{2})}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{D-h}{2})}\frac{\sqrt{\pi}^{D-h-1}}{2^{D-h-1}} \tag{95}$$

$$= \frac{r^h}{\pi^{h/2}\prod_{i=1}^h S_i}\frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D+h}{2})} \tag{96}$$

$$\tag{97}$$

If h is greater than $D-1$ (ie. $h=D$), the result remains the same.

$$P_{H_1\cap\ldots\cap H_h} = \frac{2^D r^{D+h-1}}{A_{D-1}\prod_{i=1}^h S_i}\frac{1}{(D+h-2)(D+h-4)\ldots(4)(2)} \tag{98}$$

$$= \frac{2^D r^h\Gamma(\frac{D}{2})}{2\pi^{D/2}\prod_{i=1}^h S_i}\frac{1}{(D+h-2)!!} \tag{99}$$

$$= \frac{2^D r^h\Gamma(\frac{D}{2})}{2\pi^{D/2}\prod_{i=1}^h S_i}\frac{1}{2^{(D+h-2)/2}\Gamma(\frac{D+h}{2})} \tag{100}$$

$$= \frac{r^h}{\pi^{h/2}\prod_{i=1}^h S_i}\frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D+h}{2})} \tag{101}$$

$$\tag{102}$$

□

5 Headings: first level

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5.1 Headings: second level

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$$\xi_{ij}(t) = P(x_t=i, x_{t+1}=j|y,v,w;\theta) = \frac{\alpha_i(t)a_{ij}^{w_t}\beta_j(t+1)b_j^{v_{t+1}}(y_{t+1})}{\sum_{i=1}^N\sum_{j=1}^N\alpha_i(t)a_{ij}^{w_t}\beta_j(t+1)b_j^{v_{t+1}}(y_{t+1})} \tag{103}$$

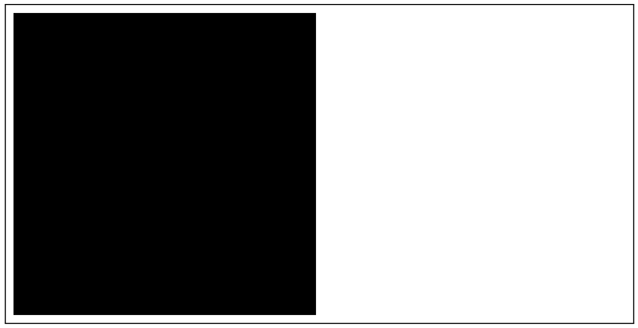

Figure 5: Sample figure caption.

5.1.1 Headings: third level

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6 Examples of citations, figures, tables, references

6.1 Citations

Citations use `natbib`. The documentation may be found at

`http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf`

Here is an example usage of the two main commands (`citet` and `citep`): Some people thought a thing [Kour and Saabne, 2014a, Hadash et al., 2018] but other people thought something else [Kour and Saabne, 2014b]. Many people have speculated that if we knew exactly why Kour and Saabne [2014b] thought this...

6.2 Figures

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6.3 Tables

See awesome Table 1.

The documentation for `booktabs` ('Publication quality tables in LaTeX') is available from:

[1]Sample of the first footnote.

Table 1: Sample table title

Part		
Name	Description	Size (μm)
Dendrite	Input terminal	$\sim$100
Axon	Output terminal	$\sim$10
Soma	Cell body	up to 10^6

`https://www.ctan.org/pkg/booktabs`

6.4 Lists

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- consectetur adipiscing elit.
- Aliquam dignissim blandit est, in dictum tortor gravida eget. In ac rutrum magna.

References

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