# Multidimensional Extension of Buffon's Needle Problem

A Preprint

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**Abstract**

Consider a line segment randomly placed on a two-dimensional plane ruled with a set of regularly spaced parallel lines. The classical Buffon's needle problem asks what the probability is that the line segment intersects at least 1 of these lines. This paper extends this problem by considering a line segment randomly placed in $\mathbb{R}^D$ and its probability of intersection with a set of regularly spaced parallel hyperplanes.

***Keywords*** Buffon's needle problem · Geometric Probability

## 1 Introduction

Given $D \in \mathbb{N}_{>0}$ and $N \in [1, 2, \ldots, D]$, consider a grid on $\mathbb{R}^D$ formed by $N$ orthogonal sets of regularly spaced hyperplanes. Each set of hyperplanes has a potentially unique spacing of $S_i$. A line segment of length $r \in \mathbb{R}^+$ is randomly located in the space such that one of its end points, $P_0$, is uniformly distributed across the entire domain. The line segment's orientation is independently distributed such that when considering $P_0$ as the center of a $(D-1)$-sphere of radius $r$, the other point, $P_1$, is uniformly distributed on the surface of that hypersphere. This line segment may intersect with $C \in \mathbb{N}$ unique hyperplanes. This paper studies the probability of the line segment intersecting more than $c$ hyperplanes, $P(C > c|r, D, N, S)$. From there, solutions for crossing less than $c$ hyperplanes and exactly $c$ hyperplanes can be derived. As an example, the classical Buffon's needle problem would be $P(C > 0|r, 2, 1, S)$. Laplace's extension would be represented as $P(C > 0|r, 2, 2, S)$.

We will define the coordinates of line segment using $\vec{x} \in \mathbb{R}^D$ for the location of $P_0$ and spherical coordinates for the location of $P_1$ with respect to $P_0$.

$$
\begin{aligned}
y_1 &= r \cos \phi_1 \\
y_2 &= r \sin \phi_1 \cos \phi_2 \\
&\vdots \\
y_{D-1} &= r \sin \phi_1 \ldots \sin \phi_{D-2} \cos \phi_{D-1} \\
y_D &= r \sin \phi_1 \ldots \sin \phi_{D-2} \sin \phi_{D-1} \\
P_1 &= \vec{x} + \vec{y} \\
\phi_j &\in \begin{cases} [0, \pi] & j < D-1 \\ [0, 2\pi] & j = D-1 \end{cases}
\end{aligned}
$$

Translational symmetry of the grid of hyperplanes allows us to consider the domain of $P_0$ to be $x_i \in [0, S_i]$ as the origin can be moved to any point on the grid. Reflectional symmetry of the grid also allows us to consider the domain of $\vec{y}$ to be a single orthant of the hypersphere. For convenience, we will pick the orthant where $\phi_i \in [0, \pi/2]$.

The rest of the paper is organized as follows. A derivation of the joint probability density function for $P_0$ and $P_1$ will be provided in §2. The derivation and validation of the crossing probabilities will be given in §3. Analysis of the limits and extrema of the probabilities is explored in §4.

## 2 Joint Probability Density of the Line Segment

Each coordinate for $P_0$ can be defined as a uniformly distributed random variable $X_i \sim \mathrm{Uniform}(0, S_i)$. Due to independence, the joint PDF for $P_0$ is the product $\prod_{i=1}^{D} \frac{1}{S_i}$. By the definition of the problem, the coordinates $\vec{x}$ do not influence the orientation of the line segment defined by $\vec{\phi}$. The probability density function for the uniform distribution of points on an orthant of the hypersphere can be determined by calculating the area element in terms of spherical coordinates.

**Proposition 1.** *In spherical coordinates, the probability density function for a uniform distribution on an orthant of a hypersphere is $\frac{2^D}{A_{D-1}} \prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j$ where $A_{D-1}$ is the surface area of a $(D-1)$-sphere.*

*Proof.* The area element of an $(D-1)$-sphere of radius $r$ can be expressed as

$$d\Omega = \left( \prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j \right) d\phi_1 \dots d\phi_{D-1} \tag{1}$$

The probability that a point lies in this differential element can be expressed as follows.

$$f_\Omega(\Omega) d\Omega = f_{\vec{\phi}}(\phi_1, \dots, \phi_{D-1}) d\phi_1 \dots d\phi_{D-1} \tag{2}$$

The points are uniformly distributed over the surface of an orthant of the hypersphere implying that $f_\Omega(\Omega) = \frac{2^D}{A_{D-1}}$. Substituting this and 1 into 2 yields

$$\frac{2^D}{A_{D-1}} \prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j = f_{\vec{\phi}}(\phi_1, \dots, \phi_{D-1}) \tag{3}$$

□

Then by independence, the joint probability density function for the entire line segment can be expressed as

$$f_{\vec{X},\vec{\phi}}(x_1, \dots, x_D, \phi_1, \dots, \phi_{D-1}) = \frac{2^D}{A_{D-1}} \left( \prod_{i=1}^{D} \frac{1}{S_i} \right) \left( \prod_{j=1}^{D-1} r \sin^{D-1-j} \phi_j \right) \tag{4}$$

## 3 Probability of crossing

In general, the probability of any number of crossings given any set of parameters can be described as follows.

$$P(C > c | r, D, N, S) = \int \cdots \int_V f_{\vec{\phi}}(\phi_1, \dots, \phi_{D-1}) dx_1 \dots dx_D d\phi_1 \dots d\phi_{D-1} \tag{5}$$

Where $V$ is the hypervolume in which some sort of crossing condition is met. The definition of these crossing conditions and the solution to the above equation will be explored in the following subsections for a variety of parameters.

### 3.1 C>0, N=1

We start with a simplified set of parameters where there is only a single set of parallel hyperplanes and we are interested in the condition where at least 1 crossing happens. Due to rotational symmetry of the line segment, it should not matter in which direction the hyperplanes extend. Without loss of generality we assume the planes are in the direction of $x_1$.

Because $P_0$ is constrained to be within the gridcell at the origin and because the orthant we are investigating is in the direction of $x_1$, we know that a crossing occurs whenever the following condition is met

$$x_1 + r \cos \phi_1 > S_1 \tag{6}$$

$$x_1 > S_1 - r \cos \phi_1 \tag{7}$$

The domain of $x_1$ can then be used to define the space in which a valid crossing has occured

$$\max(0, S_1 - r\cos\phi_1) < x_1 < S_1 \tag{8}$$

We can now express our volume integral in terms of these conditions.

$$P(C > 0|r, D, N = 1, S) = \int_0^{\pi/2} \cdots \int_0^{\pi/2} \int_0^{S_D} \cdots \int_0^{S_2} \int_{m(\phi_1)}^{S_1} f_{\vec{\phi}}(\phi_1, \ldots, \phi_{D-1}) dx_1 dx_2 \ldots dx_D d\phi_1 \ldots d\phi_{D-1} \tag{9}$$

Where $m(\phi_1)$ is the maximum function from 8. There are now two options for how to represent the bounds of integration for $x_1$. If $r < S_1$, then $m(\phi_1) = 0 \forall \phi_1$. Otherwise we will need to partition the interval of integration into two regions, one where $S_1 - r\cos\phi_1$ is greater than 0 and one where it is less than zero. This occurs at the value $\phi_1 = \arccos\frac{S_1}{r}$.

# 4 Headings: first level

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## 4.1 Headings: second level

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$$\xi_{ij}(t) = P(x_t = i, x_{t+1} = j|y, v, w; \theta) = \frac{\alpha_i(t) a_{ij}^{w_t} \beta_j(t+1) b_j^{v_{t+1}}(y_{t+1})}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i(t) a_{ij}^{w_t} \beta_j(t+1) b_j^{v_{t+1}}(y_{t+1})} \tag{10}$$

### 4.1.1 Headings: third level

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# 5 Examples of citations, figures, tables, references

## 5.1 Citations

Citations use `natbib`. The documentation may be found at

`http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf`

Here is an example usage of the two main commands (`citet` and `citep`): Some people thought a thing [Kour and Saabne, 2014a, Hadash et al., 2018] but other people thought something else [Kour and Saabne, 2014b]. Many people have speculated that if we knew exactly why Kour and Saabne [2014b] thought this...

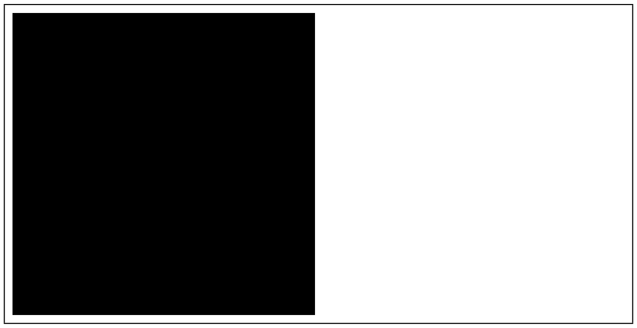

Figure 1: Sample figure caption.

Table 1: Sample table title

| Part | | |
|---|---|---|
| Name | Description | Size ($\mu$m) |
| Dendrite | Input terminal | $\sim$100 |
| Axon | Output terminal | $\sim$10 |
| Soma | Cell body | up to $10^6$ |

### 5.2 Figures

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### 5.3 Tables

See awesome Table 1.

The documentation for `booktabs` ('Publication quality tables in LaTeX') is available from:

`https://www.ctan.org/pkg/booktabs`

### 5.4 Lists

- Lorem ipsum dolor sit amet
- consectetur adipiscing elit.
- Aliquam dignissim blandit est, in dictum tortor gravida eget. In ac rutrum magna.

## References

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[1] Sample of the first footnote.

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