MULTIDIMENSIONAL EXTENSION OF BUFFON'S NEEDLE PROBLEM

A PREPRINT

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ABSTRACT

Consider a line segment randomly placed on a two-dimensional plane ruled with a set of regularly spaced parallel lines. The classical Buffon's needle problem asks what the probability is that the line segment intersects at least 1 of these lines. This paper extends this problem by considering a line segment randomly placed in $\mathbb{R}^D$ and its probability of intersection with a set of regularly spaced parallel hyperplanes.

Keywords Buffon's needle problem · Geometric Probability

1 Introduction

Given $D \in \mathbb{N}_{>0}$ and $N \in [1, 2, \dots, D]$, consider a grid on $\mathbb{R}^D$ formed by N orthogonal sets of regularly spaced hyperplanes. Each set of hyperplanes has a potentially unique spacing of S_i. A line segment of length $r \in \mathbb{R}^+$ is randomly located in the space such that one of its end points, P_0, is uniformly distributed across the entire domain. The line segment's orientation is independently distributed such that when considering P_0 as the center of a $(D-1)$-sphere of radius r, the other point, P_1, is uniformly distributed on the surface of that hypersphere. This line segment may intersect with $C \in \mathbb{N}$ unique hyperplanes. This paper studies the probability of the line segment intersecting more than c hyperplanes, $P(C > c|r, D, N, S)$. From there, solutions for crossing less than c hyperplanes and exactly c hyperplanes can be derived. As an example, the classical Buffon's needle problem would be $P(C > 0|r, 2, 1, S)$. Laplace's extension would be represented as $P(C > 0|r, 2, 2, S)$.

We will define the coordinates of line segment using $\vec{x} \in \mathbb{R}^D$ for the location of P_0 and spherical coordinates for the location of P_1 with respect to P_0.

$$y_1 = r\cos\phi_1 \tag{1}$$
$$y_2 = r\sin\phi_1\cos\phi_2 \tag{2}$$
$$\vdots \tag{3}$$
$$y_{D-1} = r\sin\phi_1\dots\sin\phi_{D-2}\cos\phi_{D-1} \tag{4}$$
$$y_D = r\sin\phi_1\dots\sin\phi_{D-2}\sin\phi_{D-1} \tag{5}$$
$$P_1 = \vec{x} + \vec{y} \tag{6}$$
$$\phi_j \in \begin{cases} [0, \pi] & j < D-1 \\ [0, 2\pi] & j = D-1 \end{cases} \tag{7}$$

Translational symmetry of the grid of hyperplanes allows us to consider the domain of P_0 to be $x_i \in [0, S_i]$ as the origin can be moved to any point on the grid. Reflectional symmetry of the grid also allows us to consider the domain of $\vec{y}$ to be a single orthant of the hypersphere. For convenience, we will pick the orthant where $\phi_i \in [0, \pi/2]$.

The rest of the paper is organized as follows. A derivation of the joint probability density function for P_0 and P_1 will be provided in §2. The derivation and validation of the crossing probabilities will be given in §3. Analysis of the limits of the probabilities is explored in §4.

2 Joint Probability Density of the Line Segment

Each coordinate for P_0 can be defined as a uniformly distributed random variable $X_i \sim \mathrm{Uniform}(0, S_i)$. Due to independence, the joint PDF for P_0 is the product $\prod_{i=1}^{D} \frac{1}{S_i}$. By the definition of the problem, the coordinates $\vec{x}$ do not influence the orientation of the line segment defined by $\vec{\phi}$. Typically, the probability density function for the uniform distribution of points on a hypersphere could be determined by calculating the determinant of the Jacobian of the coordinate transformation. However, we will require a more closed-form solution.

Proposition 1. *aoeu*

Proof. aoeu □

3 Headings: first level

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3.1 Headings: second level

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$$\xi_{ij}(t) = P(x_t = i, x_{t+1} = j | y, v, w; \theta) = \frac{\alpha_i(t) a_{ij}^{w_t} \beta_j(t+1) b_j^{v_{t+1}}(y_{t+1})}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i(t) a_{ij}^{w_t} \beta_j(t+1) b_j^{v_{t+1}}(y_{t+1})} \tag{8}$$

3.1.1 Headings: third level

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4 Examples of citations, figures, tables, references

4.1 Citations

Citations use `natbib`. The documentation may be found at

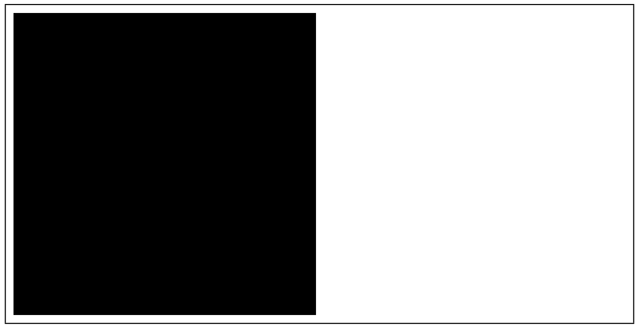

Figure 1: Sample figure caption.

Table 1: Sample table title

Part		
Name	Description	Size (μm)
Dendrite	Input terminal	$\sim$100
Axon	Output terminal	$\sim$10
Soma	Cell body	up to 10^6

`http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf`

Here is an example usage of the two main commands (`citet` and `citep`): Some people thought a thing [Kour and Saabne, 2014a, Hadash et al., 2018] but other people thought something else [Kour and Saabne, 2014b]. Many people have speculated that if we knew exactly why Kour and Saabne [2014b] thought this…

4.2 Figures

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4.3 Tables

See awesome Table 1.

The documentation for `booktabs` ('Publication quality tables in LaTeX') is available from:

`https://www.ctan.org/pkg/booktabs`

4.4 Lists

- Lorem ipsum dolor sit amet
- consectetur adipiscing elit.
- Aliquam dignissim blandit est, in dictum tortor gravida eget. In ac rutrum magna.

[1] Sample of the first footnote.

References

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