MULTIDIMENSIONAL EXTENSION OF BUFFON'S NEEDLE PROBLEM

A PREPRINT

Alexander Choi
alexander.e.choi@gmail.com

August 30, 2023

ABSTRACT

Consider a line segment randomly placed on a two-dimensional plane ruled with a set of regularly spaced parallel lines. The classical Buffon's needle problem asks what the probability is that the line segment intersects at least 1 of these lines. This paper extends this problem by considering a line segment randomly placed in $\mathbb{R}^D$ and its probability of intersection with a set of regularly spaced parallel hyperplanes.

Keywords Buffon's needle problem · Geometric Probability

1 Introduction

Given $D \in \mathbb{N}_{>0}$ and $N \in [1, 2, \ldots, D]$, consider a grid on $\mathbb{R}^D$ formed by N orthogonal sets of regularly spaced hyperplanes. Each set of hyperplanes has a potentially unique spacing of S_i. A line segment of length $r \in \mathbb{R}^+$ is randomly located in the space such that one of its end points, P_0, is uniformly distributed across the entire domain. The line segment's orientation is independently distributed such that when considering P_0 as the center of a $(D-1)$-sphere of radius r, the other point, P_1, is uniformly distributed on the surface of that hypersphere. This line segment may intersect with $C \in \mathbb{N}$ unique hyperplanes. This paper studies the probability of the line segment intersecting more than c hyperplanes, $P(C > c|r, D, N, S)$. From there, solutions for crossing less than c hyperplanes and exactly c hyperplanes can be derived. As an example, the classical Buffon's needle problem would be $P(C > 0|r, 2, 1, S)$. Laplace's extension would be represented as $P(C > 0|r, 2, 2, S)$.

We will define the coordinates of line segment using $\vec{x} \in \mathbb{R}^D$ for the location of P_0 and spherical coordinates for the location of P_1 with respect to P_0.

$$y_1 = r \cos \phi_1 \tag{1}$$
$$y_2 = r \sin \phi_1 \cos \phi_2 \tag{2}$$
$$\vdots \tag{3}$$
$$y_{D-1} = r \sin \phi_1 \ldots \sin \phi_{D-2} \cos \phi_{D-1} \tag{4}$$
$$y_D = r \sin \phi_1 \ldots \sin \phi_{D-2} \sin \phi_{D-1} \tag{5}$$
$$P_1 = \vec{x} + \vec{y} \tag{6}$$
$$\phi_j \in \begin{cases} [0, \pi] & j < D-1 \\ [0, 2\pi] & j = D-1 \end{cases} \tag{7}$$

Translational symmetry of the grid of hyperplanes allows us to consider the domain of P_0 to be $x_i \in [0, S_i]$ as the origin can be moved to any point on the grid. Reflectional symmetry of the grid also allows us to consider the domain of $\vec{y}$ to be a single orthant of the hypersphere. For convenience, we will pick the orthant where $\phi_i \in [0, \pi/2]$.

The rest of the paper is organized as follows. A derivation of the joint probability density function for P_0 and P_1 will be provided in §2. The derivation and validation of the crossing probabilities will be given in §3. Analysis of the limits of the probabilities is explored in §4.

2 Joint Probability Density of the Line Segment

Each coordinate for P_0 can be defined as a uniformly distributed random variable $X_i \sim \mathrm{Uniform}(0, S_i)$. Due to independence, the joint PDF for P_0 is the product $\prod_{i=1}^{D} \frac{1}{S_i}$. By the definition of the problem, the coordinates $\vec{x}$ do not influence the orientation of the line segment defined by $\vec{\phi}$. The probability density function for the uniform distribution of points on an orthant of the hypersphere can be determined by calculating the area element in terms of spherical coordinates.

Proposition 1. *In spherical coordinates, the probability density function for a uniform distribution on an orthant of a hypersphere is* $\frac{2^D}{A_{D-1}} \prod_{i=1}^{D-1} r \sin^{D-1-i} \phi_i$ *where* A_{D-1} *is the surface area of a* $(D-1)$*-sphere.*

Proof. The area element of an $(D-1)$-sphere of radius r can be expressed as

$$d\Omega = \left(\prod_{i=1}^{D-1} r \sin^{D-1-i} \phi_i \right) d\phi_1 \ldots d\phi_{D-1}$$

The probability that a point lies in this differential element can be expressed as follows.

$$f_\Omega(\Omega) d\Omega = f_{\vec{\phi}}(\phi_1, \ldots, \phi_{D-1}) d\phi_1 \ldots d\phi_{D-1}$$

The points are uniformly distributed over the surface of an orthant of the hypersphere implying that $f_\Omega(\Omega) = \frac{2^D}{A_{D-1}}$. Combining this with the above equations results in the following.

$$\frac{2^D}{A_{D-1}} \prod_{i=1}^{D-1} r \sin^{D-1-i} \phi_i = f_{\vec{\phi}}(\phi_1, \ldots, \phi_{D-1})$$

□

3 Headings: first level

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula. See Section 3.

3.1 Headings: second level

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.

$$\xi_{ij}(t) = P(x_t = i, x_{t+1} = j | y, v, w; \theta) = \frac{\alpha_i(t) a_{ij}^{w_t} \beta_j(t+1) b_j^{v_{t+1}}(y_{t+1})}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i(t) a_{ij}^{w_t} \beta_j(t+1) b_j^{v_{t+1}}(y_{t+1})} \tag{8}$$

3.1.1 Headings: third level

Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.

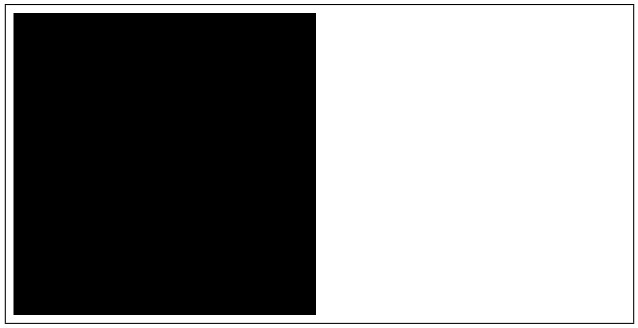

Figure 1: Sample figure caption.

Paragraph Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.

4 Examples of citations, figures, tables, references

4.1 Citations

Citations use `natbib`. The documentation may be found at

`http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf`

Here is an example usage of the two main commands (`citet` and `citep`): Some people thought a thing [Kour and Saabne, 2014a, Hadash et al., 2018] but other people thought something else [Kour and Saabne, 2014b]. Many people have speculated that if we knew exactly why Kour and Saabne [2014b] thought this...

4.2 Figures

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi. See Figure 1. Here is how you add footnotes. [1] Sed feugiat. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Ut pellentesque augue sed urna. Vestibulum diam eros, fringilla et, consectetuer eu, nonummy id, sapien. Nullam at lectus. In sagittis ultrices mauris. Curabitur malesuada erat sit amet massa. Fusce blandit. Aliquam erat volutpat. Aliquam euismod. Aenean vel lectus. Nunc imperdiet justo nec dolor.

4.3 Tables

See awesome Table 1.

The documentation for `booktabs` ('Publication quality tables in LaTeX') is available from:

`https://www.ctan.org/pkg/booktabs`

4.4 Lists

- Lorem ipsum dolor sit amet
- consectetur adipiscing elit.
- Aliquam dignissim blandit est, in dictum tortor gravida eget. In ac rutrum magna.

[1]Sample of the first footnote.

Table 1: Sample table title

Part		
Name	Description	Size (μm)
Dendrite	Input terminal	$\sim$100
Axon	Output terminal	$\sim$10
Soma	Cell body	up to 10^6

References

George Kour and Raid Saabne. Real-time segmentation of on-line handwritten arabic script. In *Frontiers in Handwriting Recognition (ICFHR), 2014 14th International Conference on*, pages 417–422. IEEE, 2014a.

Guy Hadash, Einat Kermany, Boaz Carmeli, Ofer Lavi, George Kour, and Alon Jacovi. Estimate and replace: A novel approach to integrating deep neural networks with existing applications. *arXiv preprint arXiv:1804.09028*, 2018.

George Kour and Raid Saabne. Fast classification of handwritten on-line arabic characters. In *Soft Computing and Pattern Recognition (SoCPaR), 2014 6th International Conference of*, pages 312–318. IEEE, 2014b. doi:10.1109/SOCPAR.2014.7008025.