MULTIDIMENSIONAL EXTENSION OF BUFFON'S NEEDLE PROBLEM

A PREPRINT

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August 29, 2023

ABSTRACT

Consider a line segment randomly placed on a two-dimensional plane ruled with a set of regularly spaced parallel lines. The classical Buffon's needle problem asks what the probability is that the line segment intersects at least 1 of these lines. This paper extends this problem by considering a line segment randomly placed in $\mathbb{R}^D$ and its probability of intersection with a set of regularly spaced parallel hyperplanes.

Keywords Buffon's needle problem · Geometric Probability

1 Introduction

Given $D \in \mathbb{N}_{>0}$ and $N \in [1, 2, \ldots, D]$, consider a grid on $\mathbb{R}^D$ formed by N orthogonal sets of regularly spaced hyperplanes. Each set of hyperplanes has a potentially unique spacing of S_i. A line segment of length $l \in \mathbb{R}^+$ is randomly located in the space such that one of its end points, P_0, is uniformly distributed across the entire domain. The line segments orientation is distributed such that when considering P_0 as the center of a $(D-1)$-sphere of radius l, the other point, P_1, is uniformly distributed on the surface of that hypersphere. This line segment may intersect with $C \in \mathbb{N}$ unique hyperplanes. This paper studies the probability of the line segment intersecting more than c hyperplanes, $P(C > c|l, D, N, S)$. From there, solutions for crossing less than c hyperplanes and exactly c hyperplanes can be derived.

As an example, the classical Buffon's needle problem would be $P(C > 0|l, 2, 1, S)$. Laplace's extension would be represented as $P(C > 0|l, 2, 2, S)$.

The orientation of the line segment can be represented using spherical coordinates of a $(D-1)$-sphere.

$$
\begin{aligned}
x_1 &= r\cos\phi_1 && (1)\\
x_2 &= r\sin\phi_1\cos\phi_2 && (2)\\
&\vdots && (3)\\
x_{n-1} &= r\sin\phi_1\ldots\sin\phi_{D-2}\cos\phi_{D-1} && (4)\\
x_n &= r\sin\phi_1\ldots\sin\phi_{D-2}\sin\phi_{D-1} && (5)\\
\vec{y} &= \vec{x} + P_0 && (6)
\end{aligned}
$$

There are several symmetries which simplify the problem. Translational symmetry of the grid of hyperplanes allows us to consider the domain of P_0 to be $P_0 \in [0, S_i]^D$ as the origin can be moved to any point on the grid.

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3 Headings: first level

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3.1 Headings: second level

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$$\xi_{ij}(t) = P(x_t = i, x_{t+1} = j | y, v, w; \theta) = \frac{\alpha_i(t) a^{w_t}_{ij} \beta_j(t+1) b^{v_{t+1}}_j (y_{t+1})}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i(t) a^{w_t}_{ij} \beta_j(t+1) b^{v_{t+1}}_j (y_{t+1})} \tag{7}$$

3.1.1 Headings: third level

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4 Examples of citations, figures, tables, references

4.1 Citations

Citations use `natbib`. The documentation may be found at

`http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf`

Here is an example usage of the two main commands (`citet` and `citep`): Some people thought a thing [Kour and Saabne, 2014a, Hadash et al., 2018] but other people thought something else [Kour and Saabne, 2014b]. Many people have speculated that if we knew exactly why Kour and Saabne [2014b] thought this…

4.2 Figures

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[1] Sample of the first footnote.

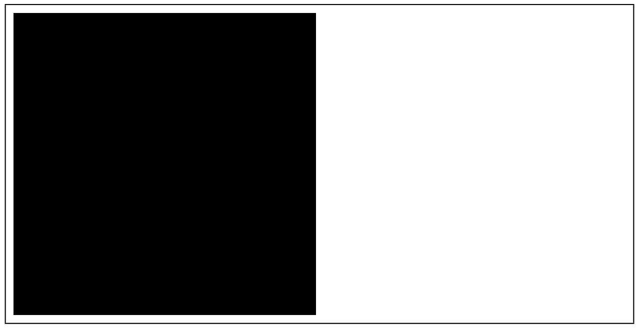

Figure 1: Sample figure caption.

Table 1: Sample table title

Part		
Name	Description	Size (μm)
Dendrite	Input terminal	$\sim$100
Axon	Output terminal	$\sim$10
Soma	Cell body	up to 10^6

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4.3 Tables

See awesome Table 1.

The documentation for `booktabs` ('Publication quality tables in LaTeX') is available from:

`https://www.ctan.org/pkg/booktabs`

4.4 Lists

- Lorem ipsum dolor sit amet
- consectetur adipiscing elit.
- Aliquam dignissim blandit est, in dictum tortor gravida eget. In ac rutrum magna.

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