Mecânica Quântica Lista I

Affonso Annarumma NUSP 9301753

1) $\underline{|\psi+\phi|^2} = \langle \psi+\phi | \psi+\phi \rangle$

$$\langle \psi | \psi+\phi \rangle + \langle \phi | \psi+\phi \rangle$$

$$\langle \psi | \psi \rangle + \underbrace{\langle \psi | \phi \rangle + \langle \phi | \psi \rangle}_{0} + \langle \phi | \phi \rangle$$

$$= \langle \psi | \psi \rangle + \langle \phi | \phi \rangle = \underline{|\psi|^2 + |\phi|^2}$$

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2) $|\psi+\phi|^2 + |\psi-\phi|^2 = 2|\phi|^2 + 2|\psi|^2$

$$|\psi+\phi|^2 + |\psi-\phi|^2 = \langle \psi+\phi | \psi+\phi \rangle + \langle \psi-\phi | \psi-\phi \rangle$$

$$= (\text{do ex } 1)\ |\psi|^2 + |\phi|^2 + \langle \psi-\phi | \psi-\phi \rangle$$

$$\langle \psi | \psi \rangle + \cancel{\langle \psi | -\phi \rangle}^{0} + \cancel{\langle -\phi | \psi \rangle}^{0} + \langle -\phi | -\phi \rangle$$

$$\langle \psi | \psi \rangle + \langle \phi | \phi \rangle$$

$$|\psi|^2 + |\phi|^2 + |\psi|^2 + |\phi|^2$$

$$= \underline{2|\psi|^2 + 2|\phi|^2}$$

3) $$\int_{-\infty}^{+\infty} \varphi_k^* \cdot \varphi_k \, dx = \int_{-a/2}^{a/2} \frac{1}{\sqrt{a}} e^{-ikx} \cdot \frac{1}{\sqrt{a}} e^{ikx} dx$$

$$= \int_{-a/2}^{a/2} \frac{1}{a} dx = \frac{x}{a} \Big|_{-a/2}^{a/2}$$

$$\frac{a}{2a} - \left(-\frac{a}{2a}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

$$\lim_{a \to \infty} \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx = \lim_{a \to \infty} \frac{1}{a} \left(x \Big|_{-\frac{a}{2}}^{\frac{a}{2}} \right)$$

$$= \lim_{a \to \infty} \frac{a}{a} = 1$$

4) $\langle\psi|\psi\rangle\langle\phi|\phi\rangle \geq |\langle\psi|\phi\rangle|^2$

$$|\gamma\rangle = |\phi\rangle - \left(\frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}\right)|\psi\rangle$$

$$\downarrow$$

(1) $\langle\gamma|\gamma\rangle = \langle\gamma|\phi\rangle - \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}\langle\gamma|\psi\rangle$

(2) $\langle\phi|\gamma\rangle = \langle\phi|\phi\rangle - \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}\langle\phi|\psi\rangle \rightarrow |\langle\psi|\phi\rangle|^2$

$$\langle\psi|\gamma\rangle = \langle\psi|\phi\rangle - \frac{\langle\psi|\phi\rangle}{\cancel{\langle\psi|\psi\rangle}}\cancel{\langle\psi|\psi\rangle}$$

$$\langle\psi|\gamma\rangle = \langle\psi|\phi\rangle - \langle\psi|\phi\rangle = 0 \quad (3)$$

Juntando: (1), (2) e (3)

$$\langle\gamma|\gamma\rangle = \langle\phi|\phi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle} - \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}\langle\gamma|\psi\rangle$$

(2) Já que (2) é totalmente real $\langle\gamma|\phi\rangle = \langle\phi|\gamma\rangle$

Já que $\langle\psi|\gamma\rangle = 0$, $\langle\gamma|\psi\rangle = 0$

$$\langle\gamma|\gamma\rangle = \langle\phi|\phi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle}$$

≥ 0

$$\langle\phi|\phi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle} \geq 0$$

$$\langle\phi|\phi\rangle \geq \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle} \rightarrow \underline{\langle\psi|\psi\rangle\langle\phi|\phi\rangle \geq |\langle\psi|\phi\rangle|^2}$$

5 $|u\rangle = \cos(\theta)|i\rangle + \sin(\theta)|j\rangle$

$|m\rangle = \operatorname{sen}(\theta)|i\rangle + \cos(\theta)|j\rangle$

$|\psi\rangle = a|i\rangle + b|j\rangle$ e $|\psi\rangle = a$

$\langle u|u\rangle = (\cos(\theta)\langle i| + \operatorname{sen}(\theta)\langle j|)$
$\cdot(\cos(\theta)|i\rangle + \operatorname{sen}(\theta)|j\rangle)$

ortonormal

// Já que $\langle i|j\rangle = 0$

$\cos^2(\theta)\langle i|i\rangle + \operatorname{sen}^2(\theta)\langle j|j\rangle = 1$

$\langle m|m\rangle = \operatorname{sen}^2(\theta)\langle i|i\rangle + \cos^2(\theta)\langle j|j\rangle = 1$

$\langle u|m\rangle = (\cos\theta)\langle i| + \operatorname{sen}\theta\langle j|)$
$\cdot(-\operatorname{sen}(\theta)|i\rangle + \cos(\theta)|j\rangle)$

$= -\cos(\theta)\operatorname{sen}(\theta)\langle i|i\rangle - \cancel{\operatorname{sen}^2(\theta)\langle j|i\rangle}^{0}$
$+ \operatorname{sen}(\theta)\cos(\theta)\langle i|j\rangle + \cancel{\cos^2(\theta)\langle i|j\rangle}^{0}$

$\langle u|m\rangle = \operatorname{sen}\theta\cos\theta - \operatorname{sen}\theta\cos(\theta) = 0$

e $\langle u|m\rangle^* = \langle m|u\rangle = 0$ já que é um número real

e portanto $|u\rangle$ e $|m\rangle$ são ortonormais

6. Para $|\omega\rangle$ ser uma base ortonormal

$$\langle \omega_i | \omega_j \rangle = \delta_{ij}$$

com $|\omega_i\rangle = A e^{i\omega_i x}$

$|\omega_j\rangle = A e^{i\omega_j x}$

$$\langle \omega_i | \omega_j \rangle = \int_{-\infty}^{\infty} \omega_i^{*} \omega_j \, dx$$

$$\langle \omega_i | \omega_j \rangle = \int_{-\pi}^{\pi} A e^{-i\omega_i x} A e^{i\omega_j x} \, dx$$

$$\langle \omega_i | \omega_j \rangle = \int_{-\pi}^{\pi} A^2 e^{-i(\omega_i - \omega_j)x} \, dx$$

$$\langle \omega_i | \omega_j \rangle = A^2 \left. \frac{e^{-i(\omega_i - \omega_j)x}}{-i(\omega_i - \omega_j)} \right|_{-\pi}^{\pi}$$

$$\langle \omega_i | \omega_j \rangle = \frac{A^2}{-i(\omega_i - \omega_j)} \underbrace{\left(e^{-i(\omega_i - \omega_j)\pi} - e^{i(\omega_i - \omega_j)\pi} \right)}$$

$$\langle \omega_i | \omega_j \rangle \left(\frac{2}{2} \right) \cdot \frac{A^2}{-i(\omega_i - \omega_j)} \cdot \ldots$$

$$\langle \omega_i | \omega_j \rangle = \frac{2A^2}{-2i(\omega_i - \omega_j)} \cdot \ldots$$

$$\langle \omega_i | \omega_j \rangle = \frac{2A^2}{\omega_i - \omega_j} \cdot \operatorname{sen}\left((\omega_i - \omega_j)\pi \right)$$

Se $|\omega\rangle$ forma una base de nuevo interés

$\langle \omega_i | \omega_j \rangle = 0$ se $i \neq j$

caso contrario, $i = j$

$$\lim_{\omega_i \to \omega_j} 2A^2 \frac{\operatorname{sen}[(\omega_i - \omega_j)\pi] \cdot \pi}{(\omega_i - \omega_j)\pi} = 2A^2\pi$$

$$\downarrow$$

$$\frac{\operatorname{sen} 0}{0}$$

O que significa que, se $i = j$

$$\langle \omega_i | \omega_j \rangle = 2\pi A^2$$

para ser ortonormal

$$\langle \omega_i | \omega_j \rangle = 1 = 2\pi A^2 \rightarrow \boxed{A = \frac{1}{\sqrt{2\pi}}}$$