CSCI 340, Winter 2025
Math HW02

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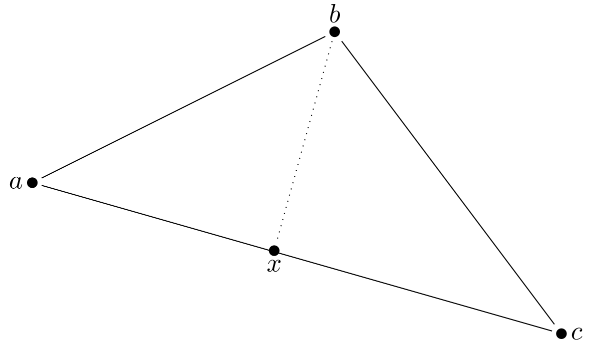
1. In the picture below, a, b, and c are arbitrary points, and the dotted line from b to x is perpendicular to the line from a to c. Give formulas to find each of the distances from a, b, and c to x as a function of the points a, b, and c. Use point subtraction and dot products. Each formula should stand on its own and not depend on the other formulas.



Solution

Here, x is the orthogonal projection of b onto the line passing through a and c. Parameterizing the line, we get any point:

$$x = a + \lambda(c - a) \tag{1}$$

Applying the dot product condition for orthogonal vectors, we get:

$$(x - b) \cdot (c - a) = 0$$

Substituting $x = a + \lambda(c-a)$,

$$(a + \lambda(c-a) - b) \cdot (c-a) = 0$$

$$=> (a - b + \lambda(c-a)) \cdot (c-a) = 0$$

$$=> (a-b) \cdot (c-a) + \lambda(c-a) \cdot (c-a) = 0$$

Solving for λ,

$$\lambda = \frac{(b-a) \cdot (c-a)}{(c-a) \cdot (c-a)}$$

Placing λ in equation (1), we get—

$$x = a + \frac{(b-a) \cdot (c-a)}{(c-a) \cdot (c-a)}(c-a) \tag{2}$$

Distance from a to x

Equation (2) - a:

$$x - a = \frac{(b-a) \cdot (c-a)}{(c-a) \cdot (c-a)}(c-a)$$

Taking its Euclidean norm,

$$\boxed{\|x - a\| = \left|\frac{(b-a) \cdot (c-a)}{(c-a) \cdot (c-a)}\right| \|c-a\|}$$

Distance from c to x

c - Equation (2):

$$c - x = c - \left(a + \frac{(b-a) \cdot (c-a)}{(c-a) \cdot (c-a)}(c-a)\right)$$

$$=> c - x = \left(1 - \frac{(b-a) \cdot (c-a)}{(c-a) \cdot (c-a)}\right)(c-a)$$

Taking its Euclidean norm,

$$\boxed{\|c - x\| = \left|1 - \frac{(b-a) \cdot (c-a)}{(c-a) \cdot (c-a)}\right| \|c-a\|}$$

Distance from b to x

Equation (2) - b:

$$x - b = (a - b) + \frac{(b-a)\cdot(c-a)}{(c-a)\cdot(c-a)}(c-a)$$

Taking its Euclidean norm,

$$\boxed{\|x - b\| = \left\| (b-a) - \frac{(b-a)\cdot(c-a)}{(c-a)\cdot(c-a)}\,(c-a) \right\|}$$

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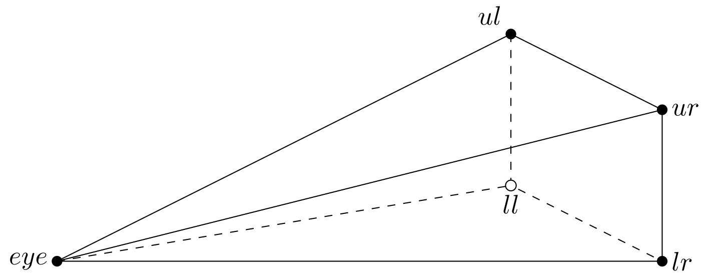
2. Suppose we specify a camera by five points: the eye position and the four corners of the image plane (upper left, upper right, lower left, lower right), as in the figure below (the left side of the view plane is deeper into the picture than the right side).



Given a position in the image plane defined by $percentX$ and $percentY$ (in percent coordinates), and with the origin of the image plane understood as the upper left corner, write an expression giving the vector for a ray from the eye to that point in the view plane. You do not need to normalize the vector.

Solution

Here,

$$World_{\text{top}} = ul + \ percentX\,(ur - ul)$$

$$World_{\text{bottom}} = ll + \ percentX\,(lr - ll)$$

$$WorldCoordinate = World_{\text{top}} + \ percentY\,(World_{\text{bottom}} - World_{\text{top}})$$

Therefore,

$$\text{ray} = WorldCoordinate - eye$$

Thus,

$$\boxed{\text{ray} = (World_{\text{top}} + \ percentY\,(World_{\text{bottom}} - World_{\text{top}})) - eye}$$

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3. Given a camera specified as in the lecture notes, with an eye position, normalized right, up, and foward vectors, and scalars depth, width and height, $\langle p, r, u, f, d, w, h \rangle$, write expressions for each of the four corner points in the camera representation from the previous problem and the center/focus.

Solution

$$\begin{aligned}
(a) \quad & c = p + df \\
(b) \quad & ul = c + \frac{h}{2}u - \frac{w}{2}r \\
(c) \quad & ur = c + \frac{h}{2}u + \frac{w}{2}r \\
(d) \quad & ll = c - \frac{h}{2}u - \frac{w}{2}r \\
(e) \quad & lr = c - \frac{h}{2}u + \frac{w}{2}r
\end{aligned}$$

4. For each of the following implicitly defined quadric surfaces, find fomulas for the coefficients for the quadratic equation, $at^2 + bt + c = 0$ to determine the value of t where a ray defined by $p + tv$ intersects the surface.

Solution

Given Elliptic Paraboloid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z \ = 0$$

Points on the line generated by:

$$p + tv = (p_x, p_y, p_z) + t(v_x, v_y, v_z)$$

Pluggin it in:

$$\frac{(p_x + tv_x)^2}{a^2} + \frac{(p_y + tv_y)^2}{b^2} - (p_z + tv_z) = 0$$

$$\Rightarrow \frac{p_x^2 + 2tp_xv_x + t^2v_x^2}{a^2} + \frac{p_y^2 + 2tp_yv_y + t^2v_y^2}{b^2} - (p_z + tv_z) = 0$$

$$\Rightarrow \left(\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2}\right)t^2 + \left(\frac{2p_xv_x}{a^2} + \frac{2p_yv_y}{b^2} - v_z\right)t + \left(\frac{p_x^2}{a^2} + \frac{p_y^2}{b^2} - p_z\right) = 0$$

Therefore,

$$\boxed{\begin{aligned} a &= \frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} \\ b &= \frac{2\,p_x\,v_x}{a^2} + \frac{2\,p_y\,v_y}{b^2} - v_z \\ c &= \frac{p_x^2}{a^2} + \frac{p_y^2}{b^2} - p_z \end{aligned}}$$

Solution

Given the hyperbolic paraboloid equation,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$$

Points on the line generated by:

$$p + tv = (p_x, p_y, p_z) + t(v_x, v_y, v_z)$$

Plugging it in:

$$\frac{(p_x + tv_x)^2}{a^2} - \frac{(p_y + tv_y)^2}{b^2} - (p_z + tv_z) = 0$$

$$\Rightarrow \frac{p_x^2 + 2tp_xv_x + t^2v_x^2}{a^2} - \frac{p_y^2 + 2tp_yv_y + t^2v_y^2}{b^2} - (p_z + tv_z) = 0$$

$$\Rightarrow \left(\frac{v_x^2}{a^2} - \frac{v_y^2}{b^2}\right)t^2 + \left(\frac{2p_xv_x}{a^2} - \frac{2p_yv_y}{b^2} - v_z\right)t + \left(\frac{p_x^2}{a^2} - \frac{p_y^2}{b^2} - p_z\right) = 0$$

Therefore,

$$\boxed{\begin{aligned} a &= \frac{v_x^2}{a^2} - \frac{v_y^2}{b^2} \\ b &= \frac{2\,p_x\,v_x}{a^2} - \frac{2\,p_y\,v_y}{b^2} - v_z \\ c &= \frac{p_x^2}{a^2} - \frac{p_y^2}{b^2} - p_z \end{aligned}}$$

Solution

Elliptic hyperboloid of one sheet,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\Rightarrow \frac{(p_x + t\,v_x)^2}{a^2} + \frac{(p_y + t\,v_y)^2}{b^2} - \frac{(p_z + t\,v_z)^2}{c^2} = 1$$

$$\Rightarrow \left(\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} - \frac{v_z^2}{c^2}\right) t^2 + \left(\frac{2p_xv_x}{a^2} + \frac{2p_yv_y}{b^2} - \frac{2p_zv_z}{c^2}\right) t + \left(\frac{p_x^2}{a^2} + \frac{p_y^2}{b^2} - \frac{p_z^2}{c^2} - 1\right) = 0$$

Therefore,

$$\boxed{\begin{aligned} a &= \frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} - \frac{v_z^2}{c^2} \\ b &= \frac{2\,p_x\,v_x}{a^2} + \frac{2\,p_y\,v_y}{b^2} - \frac{2\,p_z\,v_z}{c^2} \\ c &= \frac{p_x^2}{a^2} + \frac{p_y^2}{b^2} - \frac{p_z^2}{c^2} - 1 \end{aligned}}$$

Solution

Elliptic hyperboloid of two sheets,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

$$\Rightarrow \frac{(p_x + t\, v_x)^2}{a^2} + \frac{(p_y + t\, v_y)^2}{b^2} - \frac{(p_z + t\, v_z)^2}{c^2} = -1$$

$$\Rightarrow \left(\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} - \frac{v_z^2}{c^2}\right) t^2 + \left(\frac{2p_x v_x}{a^2} + \frac{2p_y v_y}{b^2} - \frac{2p_z v_z}{c^2}\right) t + \left(\frac{p_x^2}{a^2} + \frac{p_y^2}{b^2} - \frac{p_z^2}{c^2} + 1\right) = 0$$

Therefore,

$$\boxed{\begin{aligned} a &= \frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} - \frac{v_z^2}{c^2} \\ b &= \frac{2\, p_x\, v_x}{a^2} + \frac{2\, p_y\, v_y}{b^2} - \frac{2\, p_z\, v_z}{c^2} \\ c &= \frac{p_x^2}{a^2} + \frac{p_y^2}{b^2} - \frac{p_z^2}{c^2} + 1 \end{aligned}}$$

Find a formula for a vector normal to each of the following surfaces, given a point (x, y, z) on the surface.

Solution: Normal Vectors

For each implicitly defined surface $f(x, y, z) = 0$, the normal vector is given by:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

(a) Elliptic Paraboloid

$$\boxed{\nabla f = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, -1\right)}$$

(b) Hyperbolic Paraboloid

$$\boxed{\nabla f = \left(\frac{2x}{a^2}, -\frac{2y}{b^2}, -1\right)}$$

(c) Elliptic Hyperboloid of One Sheet

$$\boxed{\nabla f = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, -\frac{2z}{c^2}\right)}$$

(d) Elliptic Hyperboloid of Two Sheets

$$\boxed{\nabla f = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, -\frac{2z}{c^2}\right)}$$