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Optimally Connected Pairs in Weighted Complete Undirected Graphs

Definitions:

I. Consider a Weighted Complete Undirected graph (WCU graph). If node k is in the WCU graph, of all the nodes connected to node k, the node min(k) is the closest to k; that is, the edge(k, min(k)) has the smallest weight of all the edges connected to node k. Every node within a WCU graph has one or more min(node).

Therefore, we define *min(k)* as the closest node to k.

II. In the case of a WCU graph, if there exists two nodes, a and b, where:

min(a)=b and min(b)=a

Then we define these nodes as an *Optimally Connected Pair* (OCP). An OCP contains two nodes, of which are closer to each other than any other nodes within the graph.

Theorem: For every WCU graph, there exists at least one OCP comprised of nodes within the WCU graph.

Proof: To prove this, we will use contradiction.

Consider a WCU graph G such that:

a. $|G| = n$

b. $G = \{g_1, g_2, g_3, \ldots, g_m, g_{m+1}, \ldots, g_{n-1}, g_n\}$

c. G has no OCP's.

If there are no OCP's in WCU graph G, then:

$\min(g_1) = g_2 \quad \min(g_2) = g_3 \quad \ldots \quad \min(g_{n-1}) = g_n \quad \min(g_n) = g_m$

However, if the aforementioned is true, then:

$|(g_1, g_2)| > |(g_2, g_3)| > \ldots > |(g_{n-1}, g_n)| > |(g_n, g_m)| > |(g_m, g_{m+1})|$

Simplified,

$|(g_m, g_{m+1})| > |(g_n, g_m)| > |(g_m, g_{m+1})|$ (Not Possible)

So, a WCU cannot exist without an OCP.