The Compendium

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0.0.0

Perhaps my old age and fearfulness deceive me, but I suspect that the human species—the unique species—is about to be extinguished, but the Library will endure…

Jorge Luis Borges, *The Library of Babel*

Introduction

During a 1985 lecture Paul Erdős stated "You don't have to believe in God, but you should believe in *The Book*." Many colleagues of this brilliant and uncouth mathematician mention his murmurings of *The Book*: God's[1] collection of the most efficient, and thus beautiful proofs for each mathematical theorem.

Though the mathematical community has not gone to the extent of compiling the proofs for every known theorem, there have been attempts. *Proofs from THE BOOK*, the 1998 book by Martin Aigner and Günter Ziegler, a collection of the most elegant proofs from the fields of number theory, geometry, analysis, combinatorics, and graph theory. Recently, Evan Chen—a retired math Olympian and current[2] math PhD at MIT—has created an *exposition* titled The Napkin. The (infinite) Napkin is his attempt to compile the most essential statements one needs to understand the beauty and importance of mathematics. It features overviews of a myriad of different subjects studied by most undergraduate mathematicians. The Napkin provides the curious a non-trivial foundation of their subject of interest; succeeding in honing their mathematical abilities and providing a gateway to more profound investigations.

Additionally, over the past decade, communities of open-source programmers have created multiple chrestomathy websites such as Rosetta Code and the Algorithm archive that aggregate elegant computer programs for different algorithms. Given the Curry-Howard correspondence—a relationship between mathematical proofs and computer programs—I thought that this would be worth mentioning.

All, in all, *that* is what this document tries to be (though with a far more restricted scope and much less precision). I hope to compile every theorem, lemma, corollary, and definition covered over the course of MATH-311. There will be a table of contents which indexes the statements as they appear in **From Calculus to Analysis**[3]. Definitions, in their totality, will be enumerated first and will be subsequently followed by the theorems that we have proved or covered thus far. Any additional propositions that would otherwise be in the compendium will be added at the appropriate time and made note of. Feel free to edit and comment by footnote. This document serves as a palimpsest for our study of real analysis this term.

Finally, this document is meant to be pedagogical. So I will attempt to provide commentary and references to helpful, auxiliary resources when I can. Have fun!

P.S. . . . Do note, that if you want to see the hyper-links in this document, you must download the pdf! Especially in this introduction, I have linked to many extra-curricular sources, which I enjoy visiting or enjoyed having visited. If you have any suggestions, don't be afraid to email me at cainesa21@mail.wlu.edu or alecaines@gmail.com.

[1] If one is given. . .
[2] as of 2020
[3] Register—*for free*—and it is yours.

Versioning

The most critical part of this project is its capacity as an open source document. Given the assumption that most of the my peers are not acquainted with GitHub, a more rudimentary version control system must be developed in order to maintain current and alternative versions of *The Compendium*. Because of the nature of this project, future versions of *The Compendium* will not see great structural changes. Most changes to the document will entail the addition or editing of mathematical statements. Because new definitions are anticipated to be added through the rest of the term (the release date is 4/2/2020), we will avoid a unary versioning system as employed by the TeX typesetting system.

Versioning Schema

I propose the following schema: A major version change will entail a change in structure of the document. A minor version change will entail the correction a previously invalid proof or definition, as well as spelling errors. Patch version changes will entail the addition of new definitions and theorems. If the number of patches explodes in the near future, I will implement a modulo on the patch version and record the multiplicity of patches in an appended digit to the version number.

Submitting Changes

Each individual who intends to edit their copy of *The Compendium* shall make the edits to the LaTeXdocument and resubmit the TeX file to me via email. In the email, make note of

1. The mathematical statements (theorems, proofs, definitions, etc. . .) changed
2. The line numbers at which the changes took place.
3. A brief description (need not be elaborate at all) of the changes.

Given that I believe the changes are appropriate, I will then alter the version of *The Compendium* and its state. I will then release the newest version on Overleaf, for all to access. All version changes will be recorded at after the section titled **Theorems, Lemmas, Corollaries, and Proofs**. Though, do note, no one is stopping you from hosting your own version! If you would like to release your own version of *The Compendium*, then reset the version to 0.0.0, label yourself as the co-author on the title, and LaTeXaway!

Contents

0.1 Definitions

0.1.1 Infinite Decimal

Definition 0.1.1

An **infinite decimal** is an expression of the form $\pm d_0.d_1.d_2\ldots$, where $d_0 \in \mathbb{N}_0$ and $d_k \in \{0,1,2,3,4,5,6,7,8,9\}$ for $k \geq 1$.

Repeating infinite decimal

Definition 0.1.2

An *infinite decimal* is **repeating** if there are $k, m \in \mathbb{N} \ni d_{j+m} = d_j, \forall j \geq k$

Example 0.1.1

$1.67234523452345\ldots := 1.6\overline{72345}$, where $k = 3, m = 4$, and
$j :=$ any arbitrary natural number greater than k.

0.1.2 Finite Decimal

Definition 0.1.3

A **finite decimal** is a decimal of the form: $d_0.d_1d_2\ldots d_n := d_0.d_1d_2\ldots\overline{0}$

0.1.3 Irrational number

Definition 0.1.4

An **irrational number** is a real number that is not rational.

Example 0.1.2

Let $x := 0.101001000100001\ldots$.Note that there is no consistent repeating string of digits in x. So, x is irrational.

0.1.4 Density

Definition 0.1.5

Let A and B be sets. We say A is **dense** in B if any interval centered at a point in B contains at least one point from A. In discrete terms, $\forall b \in B, \forall r > 0, \exists a \in A \ni a \in (b-r, b+r)$[a].

[a] Otherwise written $|a-b| < r$.

0.1.5 Accumulation Point

Definition 0.1.6

Let $D \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. a is an **accumulation point** of D if $\forall \varepsilon > 0, \exists x \in D \ni 0 < |x-a| < \varepsilon$.

Example 0.1.3

Claim: 0 is an accumulation point of $(-1,1)$.
Proof: Let $\varepsilon > 0$ be given. Suppose $\varepsilon \leq 1$. Let $x := \frac{\varepsilon}{2}$. Then, $0 < x = |x-0| < 1$. So, $x \in (-1,1)$. Note, $x = \frac{\varepsilon}{2} < \varepsilon$. So, $0 < |x-0| < \varepsilon$.

$\blacksquare$

Comments: The above proves to be an interesting result because given any $m > 0$, if $0 < r \leq m$, then $\exists x \in D \ni 0 < |x-a| < r$. Consequently, a must be an accumulation point of D.

0.1.6 Convergence

Definition 0.1.7

Let $D \subseteq \mathbb{R}$, $a, L \in \mathbb{R}$, $f : D \to \mathbb{R}$, and a be an accumulation point of D. We say f **converges** to L as x approaches a in D provided $\forall \varepsilon > 0$, $\exists \delta > 0 \ni \forall x \in D, 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.

Example 0.1.4

Claim: $\lim_{x\to 4} f(x) = 7$.
Proof: Pick $\varepsilon > 0$ and let $\delta - \frac{\varepsilon}{3}$. Then, $\forall x \in \mathbb{R}$,

$$\begin{aligned} |f(x) - 7| &= |3x - 12| \\ &= 3|x-4| \\ &< 3 \cdot \left(\frac{\varepsilon}{3}\right) \\ &= \varepsilon \end{aligned}$$

■

0.1.7 Local Behavior

Definition 0.1.8

Let $D \subseteq \mathbb{R}$ and a an accumulation point in D be given. Additionally, let $f : D \to \mathbb{R}$ where f has **local behavior** near a if $\exists \delta > 0 \ni f(x)$ has the property, $\forall x \in (a-\delta, a+\delta)$.

0.1.8 Restriction

Definition 0.1.9

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. If $E \subseteq D$, we define the **restriction** of f to E to be the function $f\Big|_E$ with domain E and having the same values as f on E. Formally,

$$f\Big|_E : E \to \mathbb{R}, f\Big|_E(x) = f(x), \forall x \in E$$

Note, $f\Big|_E(x)$ is undefined $\forall x \in D \subseteq E$.

0.1.9 Limit at infinity

Definition 0.1.10

$\lim_{x\to\infty} f(x) = L$ is defined when $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}, \forall x \in D, N < x \implies |f(x) - L| < \varepsilon$.

Example 0.1.5

Claim: $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

Proof: Let $\varepsilon > 0$ be given. We want to find N such that $x > N \implies \frac{1}{x^2} < \varepsilon$. Note, $x^2 > \frac{1}{\varepsilon}$. Let $N := \frac{1}{\sqrt{\varepsilon}}$. Therefore, if $x > N \Leftrightarrow x > \frac{1}{\sqrt{\varepsilon}}$, then $x^2 > \frac{1}{\varepsilon}$. So, $|\frac{1}{x^2} - 0| = \frac{1}{x^2} < \varepsilon$.

■

0.1.10 Sequence

Definition 0.1.11

A **sequence** of real numbers is a function $x : \mathbb{N} \to \mathbb{R}$ denoted $(x)_{n=a}^{b}$ where $x_n, \forall n \in [a, b]$ are the element of the sequence at from indices a to b.

0.1.11 Bounded sequence

Definition 0.1.12

A sequence is **bounded** if there is a k such that $|x_n| \leq k, \forall n$.

0.1.12 Null Sequence

Definition 0.1.13

A sequence x_n is **null** if given any $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni \forall n \in \mathbb{N}^{>N}, n \implies |x_n| < \varepsilon$.

Comment:

1. This means that $x_n \to 0$ if x_n is null.
2. This also means $x_n \to p$ if and only if $(x_n - p)$ is null.

Example 0.1.6

Claim: $(1 + (-1)^n)$ is not null.
Proof: Note, if a sequence (x) is not null, then $\exists \varepsilon > 0 \ni \forall N \in \mathbb{N}, \exists n \in \mathbb{N}$ where $n > N$ and $|x_n| \geq \varepsilon$. Let $x_n = 1 + (01)^n, \forall n$. So, $|x_n| = |1 + (-1)^n| = \begin{cases} 0 & \text{when} n \text{ is odd.} \\ 1 & \text{when} n \text{ is even.} \end{cases}$.
Choose $\varepsilon := 1$ and $N \in \mathbb{N}$ be arbitrary. Now, choose $n := 2N$. Then, $n > N$ trivially. Additionally, $|1 + (-1)^n| = |1 + (-1)^{2N}| = 2 > 1$.

■

0.1.13 Continuous function

Definition 0.1.14

Let $D \subseteq \mathbb{R}$. A function$f : D \to \mathbb{R}$ is **continuous** at a point $a \in D$ if $\varepsilon > 0, \exists \delta > 0 \ni \forall x \in D$, $|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$.

Comment: Note, if a is an accumulation point, then $\lim_{x \to a} f(x) = f(a)$. If a is not an accumulation point, but f is continuous, then $|x - a| < \delta \implies x = a$. Such an a is called an **isolated point**.

0.1.14 Removable Discontinuity

Definition 0.1.15

Suppose f is discontinuous at a. Then, f has a **removable discontinuity** at a if there is a function g such that g is continuous at a and $g(x) = f(x), \forall x \neq a$.

Example 0.1.7

Claim: $f(x) := \frac{x^2-1}{x-1}$ has a removable discontinuity at $a = 1$.
Proof: Let $g(x) := x + 1$. Then $g(x) = f(x), \forall x \neq 1$ and g is continuous at $x = 1$. So, f has a removable discontinuity at $x = 1$.

0.1.15 Right-Sided Continuity

Definition 0.1.16

Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. f is **continuous from the right** at a if $\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in D, 0 < |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$.

0.1.16 Left-Sided Continuity

Definition 0.1.17

Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. f is **continuous from the left** at a if $\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in D, -\delta < x - a < 0 \implies |f(x) - f(a)| < \varepsilon$.

0.1.17 Upper Bound

Definition 0.1.18

Let $A \subseteq \mathbb{R}$. A real number $u \in \mathbb{R}$ is an **upper bound** for A if each $a \in A \implies a \leq u$.

0.1.18 Lower Bound

Definition 0.1.19

Let $A \subseteq \mathbb{R}$. A real number $u \in \mathbb{R}$ is a **lower bound** for A if each $a \in A \implies a \geq u$.

0.1.19 Maximum

Definition 0.1.20

Let $A \subseteq \mathbb{R}$. A real number $u \in \mathbb{R}$ is the **maximum** of A if each $u \in A$ implies u is an upper bound for A.

0.1.20 Minimum

Definition 0.1.21

Let $A \subseteq \mathbb{R}$. A real number $u \in \mathbb{R}$ is the **minimum** of A if $u \in A$ implies u is a lower bound for A.

0.1.21 Supremum or Least Upper Bound

Definition 0.1.22

Let $\subseteq \mathbb{R}$ and $u \in$. u is a **supremum** or **least upper bound** if u is an upper bound of A and no number smaller than u is an upper bound.

0.1.22 Infimum or Greatest Lower Bound

Definition 0.1.23

Let $A \subseteq \mathbb{R}$ and $u \in \mathbb{R}$. u is an **infimum** or **greatest lower bound** if u is a lower bound for A and no number larger than u is a lower bound.

0.1.23 Intermediate Value Property

Definition 0.1.24

A subset $A \subseteq \mathbb{R}$ has the **intermediate value property** if $\forall x, y \in A$ and $t \in \mathbb{R}$, $x < t < y \implies t \in A$.

0.1.24 Countable

Definition 0.1.25

A set is called **countable** if its elements can be enumerated by the naturals.

0.1.25 Increasing

Definition 0.1.26

Let I be some interval. A function $f : I \to \mathbb{R}$ is **increasing** if $\forall x, y \in I, x < y \implies f(x) \leq f(y)$.

0.1.26 Decreasing

Definition 0.1.27

Let I be some interval. A function $f : I \to \mathbb{R}$ is **decreasing** if $\forall x, y \in I$, $x < y \implies f(x) \geq f(y)$.

0.1.27 Monotone

Definition 0.1.28

A function is **monotone** if it is either increasing or decreasing.

0.1.28 Jump Discontinuity

Definition 0.1.29

A **jump discontinuity** is defined when one-sided limits exist at a point a but are not equal.

0.1.29 Bounded Function

Definition 0.1.30

A function $f : D \to \mathbb{R}$ is **bounded** on $E \subseteq D$ if $\exists M \in \mathbb{R} \ni |f(x)| \leq M, \forall x \in E$.

0.1.30 Compact Interval

Definition 0.1.31

Suppose $a < b$ are real. A **compact interval** I is a closed and bounded interval of the form $I := [a, b]$.

0.1.31 Uniform Continuity

Definition 0.1.32

A function $f : D \to \mathbb{R}$ is **uniformly continuous** on D if given any $\varepsilon > 0$, there is a $\delta > 0 \ni \forall x, y \in D, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

Example 0.1.8

Claim: $f(x) := x^2$ is uniformly continuous on $D := \{x \in \mathbb{R} | |x| < 7\}$.
Proof: Let $\varepsilon > 0$ be given. Note, $|x^2 - y^2| = |(x - y)||(x + y)|$. By the triangle inequality, $|x + y| \leq |x| + |y|$. So, $|(x - y)||(x + y)| < 7 + 7 < 14$. So, let $\delta := \frac{\varepsilon}{14}$. Thus,

$$\begin{aligned} |f(x) - f(y)| &= |x - y||x + y| \\ &= \frac{\varepsilon}{14} \cdot 14 \\ &= 14 \end{aligned}$$

■

0.1.32 Differentiable

Definition 0.1.33

Let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is **differentiable** at $a \in D$ if a is an accumulation point of D with derivative b gives, $\forall \varepsilon > 0 \ni \forall x \in D,) < |x - a| < \delta \implies |\frac{f(x) - f(a)}{x - a} - b| < \varepsilon$.

0.1.33 Right-hand Derivative

Definition 0.1.34

The **right-hand derivative** $(f')^+(a)$ is defined, $(f')^+ := \lim\limits_{x \to a^+} \dfrac{f(x) - f(a)}{x - a}$.

0.1.34 Left-hand Derivative

Definition 0.1.35

The **left-hand derivative** $(f')^{-}(a)$ is defined, $(f')^{-} := \lim_{x \to a^{-}} \dfrac{f(x) - f(a)}{x - a}$.

0.1.35 Local Maximum

Definition 0.1.36

$f(a)$ is a **local maximum** of f if $\exists \delta > 0 \ni x \in (a - \delta, a + \delta) \implies f(x) \geq f(a)$.

0.1.36 Local Minimum

Definition 0.1.37

$f(a)$ is a **local minimum** of f if $\exists \delta > 0 \ni x \in (a - \delta, a + \delta) \implies f(x) \geq f(a)$.

0.2 Theorems, Lemmas, Corollaries, and Proofs

0.2.1 Theorem 1.1.1

Theorem 0.2.1

The set of finite decimals equals the set of infinite decimals terminating in repeating nines.

0.2.2 Theorem 1.1.2

Theorem 0.2.2

An **infinite decimal** is a *rational number* if and only if it is repeating.

Example 0.2.1

Claim: $0.\overline{234} \in \mathbb{Q}$.
Proof: Let $x = 0.\overline{234}$. Then, $1000x = 234.\overline{234}$.

$$1000x = 234 + x$$
$$x = \frac{234}{999}$$

So, $x \in \mathbb{Q}$.

■

Example 0.2.2

Claim: If $\frac{p}{q}$ is rational, then the repeating part of its infinite decimal has a maximum length of $q-1$.
Proof: Without loss of generality, consider $\frac{1}{7} \in \mathbb{Q}$. We want to find the infinite decimal form of $\frac{1}{7}$. Assuming long-divison works, one can discover find that $\frac{1}{7} = 0.\overline{142857}$. Note, there are seven possible elements in the set of remainders, $\mathbb{Z}_7$. If 0 is the remainder, then the decimal terminates. So, the maximum possible length of the repeating decimal for $\frac{1}{7}$ is six.

■

0.2.3 Lemma 1.1.4 (Archimedean property)

Lemma 0.2.1

If $r > 0$, $\exists n \in \mathbb{Z}^{>0} \ni r > \frac{1}{10^N}$.
"*Given any number r, there exists a rational $\frac{1}{10^N}$ smaller than it.*"

Proof: Let r be a decimal $d_0.d_1d_2\cdots > 0$. Since $r \neq 0$, $\exists n \in \mathbb{N} \ni d_n \neq 0$. Note, $\frac{1}{10^n} = 0.00\ldots d_n$ where $d_n = 1$. Additionally, note $r > \frac{1}{10^{n+1}}$.

■

0.2.4 Corollary 1.1.5

Corollary 0.2.1

If $r \geq 0$ and $r < \frac{1}{10^n}, \forall n \in \mathbb{N}$, then $r = 0$.

0.2.5 Theorem 1.1.6 (Density of the Rationals in the Reals)

Theorem 0.2.3

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: Let $r > 0$ and $x \in \mathbb{R}$ be given. Note, x can be expressed as an infinite decimal $x := d_0.d_1d_2\ldots$. By **Lemma 1.1.4**, $\exists n \in \mathbb{Z}^{>0} \ni r > \frac{1}{10^n}$. Let $y, z \in \mathbb{Q} \ni y := d_0.d_1\ldots\overline{0}$ and $z := d_0.d_1\ldots d_n\overline{0}$. Note, if $x > 0$, $y \leq x \leq z$. If $x < 0$, $z < x < y$.
Now, suppose $x > 0$. So, $y \leq x \leq z$. Note, $z - y = \frac{1}{10^n} \Leftrightarrow z = y + \frac{1}{10^n}$.
So, the following,

$$x - r < x \leq y + \frac{1}{10^n} \leq x + \frac{1}{10^n} < r + x$$

Note, $z = y + \frac{1}{10^n}$. So, $x - r < z < x + r$. Additionally, note $z \in \mathbb{Q}$.
Now, let $x < 0$. So, $z \leq x \leq y$. Note, $y = z + \frac{1}{10^n}$. The following,

$$x - r < x \leq z + \frac{1}{10^n} \leq x + \frac{1}{10^n} < r + x$$

So, $x - r < y < x + r$. Note, $y \in \mathbb{Q}$. Finally, let $x = 0$. $0 \in \mathbb{Q}$ and $0 \in (r - x, x + r), \forall r > 0$. So, $\mathbb{Q}$ is dense in $\mathbb{R}$.

■

0.2.6 Theorem 1.1.8 (Density of Irrationals in the Reals)

Theorem 0.2.4

Any open interval contains an irrational number.

0.2.7 Theorem 1.3.12 (Local boundedness)

Theorem 0.2.5

Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, and a an accumulation point of D be given. If $\lim_{x\to a} f(x) = L$ exists, then $\exists \delta > 0, \exists M > 0 \ni \forall x \in D, if 0 < |x - a| < \delta$. Consequently, $|f(x)| < M$.

Proof: Since $\lim_{x\to a} f(x) = L$, $\forall \varepsilon > 0$ (and in particular, $\varepsilon = 1$), $\exists \delta > 0 \ni \forall x \in D, 0 < |x - a| \implies |f(x) - L| < 1$. Let $M = 1 + |L|$. Note,

$$\begin{aligned}
|f(x)| &= |f(x) - L + L| \\
&\leq |f(x) - L| + |L| \\
&< 1 + |L| \\
&= M
\end{aligned}$$

■

0.2.8 Lemma 1.4.2 (Constant multiple rule)

[4]

Lemma 0.2.2

Let $\lim_{x\to x_0} f(x) = L$ be given. Then, $\lim_{x\to x_0} (kf)(x) = kL, \forall k \in \mathbb{R}$.

0.2.9 Theorem 1.4.1 (Linearity)

Theorem 0.2.6

Let $D \subseteq \mathbb{R}$, $f, g : D \to \mathbb{R}$, $x_0, a, b, L, M \in \mathbb{R}$ where x_0 is an accumulation point of D. Suppose $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$. Then, $\lim_{x\to x_0} (af(x) + bg(x)) = aL + bM$.

Proof: Let $\varepsilon > 0$ and $k \neq 0$ be given. Then, by **Lemma 1.4.2**,

$$\begin{aligned} |(kf)(x) - kL| &= |kf(x) - kL| \\ &= |k||f(x) - L| \end{aligned}$$

Since $\lim_{x\to x_0} f(x) = L$ is given, $\forall \varepsilon > 0$, in particular $\frac{\varepsilon}{|k|}$, $\exists \delta > 0 \ni \forall x \in D, 0 < |x - a| < \delta \implies |f(x) - L| < \frac{\varepsilon}{|k|}$.

$$\begin{aligned} |k||f(x) - L| &< |k| \cdot \frac{\varepsilon}{|k|} \\ &< \varepsilon \end{aligned}$$

■

0.2.10 Lemma 1.4.4 (Sum Rule)

Lemma 0.2.3

Let $D \subseteq \mathbb{R}$, $f, g : D \to \mathbb{R}$, and $a, L, M \in \mathbb{R}$. Suppose a is an accumulation point of D, $\lim_{x\to a} f(x) = L$, and $\lim_{x\to a} g(x) = M$. Then, $\lim_{x\to a} (f + g)(x) = L + M$.

Proof: Recall, the triangle inequality states

$$|x + y| \leq |x| + |y|$$

[4] Don't ask me why they structured it like this...

So,

$$(f+g)(x)-(L+M) \leq |f(x)-L|+|g(x)-M|$$

Since $\lim_{x\to a} f(x) = L \implies \exists \delta_1 > 0 \ni \forall x \in D, 0 < |x-a| < \delta_1, \implies 0 < |f(x)-L| < \dfrac{\varepsilon}{2}$ (given that $\varepsilon > 0$). Similarly, $\lim_{x\to a} g(x) = M \implies \exists \delta_2 > 0 \ni \forall x \in D, 0 < |x-a| < \delta_2 \implies 0 < |g(x)-M| < \dfrac{\varepsilon}{2}$. Choose $\delta = \min(\delta_1, \delta_2) \ni 0 < |x-a| < \delta$, we have $|(f+g)(x)-(L+M)| \leq |f(x)-L|+|g(x)+M| < \frac{\varepsilon}{2v} + \frac{\varepsilon}{2v} = \varepsilon$

$\blacksquare$

0.2.11 Theorem 1.4.6 (Product Rule)

Theorem 0.2.7

Let $D \subseteq \mathbb{R}$ and $a, L, M \in \mathbb{R}$. Suppose $f, g : D \to \mathbb{R}$ and a is an accumulation point of D. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} (f \cdot g)(x) = L \cdot M$.

Proof: let $\varepsilon > 0$. By the triangle inequality,

$$|(f \cdot g)(x) - (L \cdot M)| = |f(x) \cdot g(x) - L \cdot g(x) + L \cdot g(x) - L \cdot M|$$

Note, $|f(x) \cdot g(x) - L \cdot g(x) + L \cdot g(x) - L \cdot M| \leq |f(x) - L||g(x)| + |L||g(x) - M|$. By the **Local Boundedness Theorem**, $\exists k, \delta > 0 \ni |g(x)| < k$ for $0 < |x - a| < \delta$. So,

$$|f(x) - L||g(x)| + |L||g(x) - M| < k|f(x) + L| + |L||g(x) - M|$$

Since $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, $\exists \delta_2, \delta_3 > 0 \ni |f(x) - L| < \frac{\varepsilon}{2k}$ whenever $0 < |x - a| < \delta_2$ and $|g(x) - M| < \frac{\varepsilon}{2|L|+1}$ whenever $0 < |x - a| < \delta_2$ [a]

[a] $\exists L \ni |L| = 0$

Example 0.2.3

Claim: $\lim_{x \to a} x^n = a^n$.
Proof: Let $n \in \mathbb{N}, a \in \mathbb{R}$.
Base Case: When $n = 1$, $\lim_{x \to a} x^1 = \lim_{x \to a} x = a$. **Inductive Hypothesis:** $\lim_{x \to a} x^n = a^n$. **Inductive Step:** Suppose the **Inductive Hypothesis** holds. Then,

$$\begin{aligned}
\lim_{x \to a} x^{n+1} &= \lim_{x \to a} x^n \cdot x \\
&= \left(\lim_{x \to a} x^n\right)\left(\lim_{x \to a} x\right) \\
&= a^n \cdot a \\
&= a^{n+1}
\end{aligned}$$

■

0.2.12 Theorem 1.4.9 (Quotient Rule)

Theorem 0.2.8

Let $D \subseteq \mathbb{R}$, $f, g : D \to \mathbb{R}$ and $a, L, M \in \mathbb{R}$. Let a be an accumulation point of D. If $M \neq 0$ and $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then $\lim\limits_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{L}{M}$.

Proof: By exercise 1.3.15[a] in **From Calculus to Analysis** it is shown that $\forall x \in D$ near a, $g(x) > \frac{|M|}{2}$. Specifically, $g(x) \neq 0$ near a. So, $\frac{1}{g(x)} \neq 0$ near a. Let $\varepsilon > 0$. Observe, $|\frac{1}{g(x)} - \frac{1}{M}| = |\frac{M-g(x)}{Mg(x)}| = \frac{1}{|M||g(x)|} \cdot |M - g(x)|$. Since $|g(x)| > \frac{|M|}{2}$, $\frac{1}{g(x)} < \frac{2}{|M|}$, by a generalization of the local positivity theorem. Let $\delta > 0 \ni |M - g(x)| < \frac{M|^2\varepsilon}{2}$, if $0 < |x - a| < \delta$. Then ,$|\frac{1}{g(x)} - \frac{1}{M}| - \frac{1}{|M||g(x)|}|M - g(x)| < \frac{1}{|M|} \cdot \frac{2}{|M|} \cdot \frac{\varepsilon|M|^2}{2} = \varepsilon$.

■

[a] The proof of local positivity

0.2.13 Theorem 1.4.14 (Composition Rule)

Theorem 0.2.9

Let $A, B, C \subseteq \mathbb{R}$. Suppose $f : A \to B, g : B \to C$, a is an accumulation point of A, b is an accumulation point of B, and $f(x) \neq b$ when $x \neq a$ is *close* to a. If $\lim\limits_{x\to a} f(x) = b$ and $\lim\limits_{x\to b} g(x) = c$, then $\lim\limits_{x\to a} (g \circ f)(x) = c$.

Proof: Let $\varepsilon > 0$ be given. Since $\lim\limits_{y\to b} g(y) = c$, $\exists \delta_1 > 0 \ni 0 < |y - b| < \delta_1 \implies |g(y) - c| < \varepsilon$. Similarly, since $\lim_{x\to a} f(x) = b, \exists \delta_2 > 0 \ni 0 < |x - a| < \delta_2 \implies |f(x) - b| < \varepsilon = \delta_1$.
Now, let $\gamma > 0 \ni 0 < |x - a| < \delta \implies f(x) \neq b, \forall x$[a]. Let $\delta := \min(\gamma, \delta_2) \ni 0 < |x - a| < \gamma$. So, $0 < |x - a| < \gamma \implies 0 < |f(x) - b| < \delta_1 \implies |(g \circ f)(x) - c| < \varepsilon$.

■

[a] Note, that $f(x) \neq b$ is assumed.

0.2.14 Theorem 1.4.16 (Squeeze Theorem)

Theorem 0.2.10

Let $f, g, h : D \to \mathbb{R}$ and Suppose $f(x) \leq h(x) \leq g(x), \forall x \neq a$ near a, some accumulation point of D. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} h(x) = L$.

Proof: Let $\varepsilon > 0$ be given. Let $\delta_f > 0 \ni 0 < |x-a| < \delta_f \implies |f(x) - L| < \varepsilon$. Similarly, for $\delta_g > 0 \ni 0 < |x-a| < \delta_g \implies |g(x) - L| < \varepsilon$. Note,

$$|f(x) - L| < \varepsilon \implies -\varepsilon < f(x) - L < \varepsilon$$

and

$$|g(x) - L| < \varepsilon \implies -\varepsilon < g(x) - L < \varepsilon$$

So,

$$-\varepsilon < f(x) - L \leq h(x) - L \leq g(x) - L < \varepsilon$$

when $0 < |x-a| < \delta, \delta := \min(\delta_g, \delta_f)$. Then,

$$0 < |x-a| < \delta \implies -\varepsilon < f(x) - L < \varepsilon \text{ and } -\varepsilon < g(x) - L < \varepsilon$$

$$\implies -\varepsilon < f(x) - L \leq h(x) - L \leq g(x) - L < \varepsilon.$$

■

0.2.15 Theorem 1.4.19

Theorem 0.2.11

Suppose $D, D_1, D_2 \subseteq \mathbb{R}$, $D := D_1 \cup D_2$, $f : D \to \mathbb{R}$, and a is an accumulation point of D_1 and D_2. The following,

(1) $\lim_{x \to a} f(x) = L \implies \lim_{x \to a} f\Big|_{D_1}(x) = L$ and $\lim_{x \to a} f\Big|_{D_2}(x) = L$

(2) $\lim_{x \to a} f\Big|_{D_1}(x) - L$ and $\lim_{x \to a} f\Big|_{D_2}(x) = L \implies \lim_{x \to a} f(x) = L.$

0.2.16 Corollary 1.4.20

Corollary 0.2.2

Let $D := D_1 \cup D_2$, $f : D \to \mathbb{R}$ and a be an accumulation point of D_1 and D_2. If f is a function, then $\lim_{x \to a} f(x)$ exists $\Leftrightarrow \lim_{x \to a} f\Big|_{D_1}(x) = \lim_{x \to a} f\Big|_{D_2}(x)$ both exist.

Example 0.2.4

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ Let $a \in \mathbb{R}$. We know that a is an accumulation point of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$.[a] By this, note $\lim_{x \to a} f\Big|_{\mathbb{Q}}(x) = 1$ and $\lim_{x \to a} f\Big|_{\mathbb{R} \subseteq \mathbb{Q}}(x) = 0$. So, $\lim_{x \to a} f(x)$ does not exist as either of its component limits[b] are not equal. So, this function does not converge anywhere!

[a] By the density of rationals and irrationals in the reals

[b] Those above.

0.2.17 Theorem (Convergence of Bounded Sequences)

Theorem 0.2.12

A convergent sequence is bounded[a] [b].

[a] where x is a sequence and $p \in \mathbb{R}$, convergence of a sequences is denoted $x_n \to p$

[b] additionally, the complete proof was omitted from class. I may supply the full proof at a later date. Really trying not to cop out here. . .

0.2.18 Theorem (Threshold of Convergent Sequences)

Theorem 0.2.13

Suppose $x_n \to p$ and $p \neq 0$. $\exists N \in \mathbb{N} \ni n > N \implies |x_n| > |\frac{|p|}{2}|$.

0.2.19 Theorem (Properties of Convergent Sequences)

Theorem 0.2.14

Suppose sequences $a_n \to a$, $b_n \to b$, and $a, b, k \in \mathbb{R}$. Then,

(1) $ka_n \to ka$.

(2) $a_n + b_n \to a + b$.

(3) $a_n b_n \to ab$.

(4) If $b \neq 0$, $\frac{a_n}{b_n} \to \frac{a}{b}$.

(5) If $a_n \leq b_n \leq c_n, \forall n$ and $a_n, c_n \to a$, then $b_n \to a$.

0.2.20 Theorem 2.1.1 (Continuity of Composed Functions)

Theorem 0.2.15

If f is continuous at a and g is continuous at $b := f(a)$, then $g \circ f$ is continuous at a.

Proof: Let $\varepsilon > 0$ be given. Since g is continuous at b, $\exists \delta_g \ni \forall y \in D, |y - b| < \delta_g \implies |g(y) - g(a)| < \varepsilon$. Since f is continuous at a, $\exists \delta_f > 0 \ni \forall x \in D, |x - a| < \delta_f \implies |f(x) - f(a)| < \delta_g$. Note, $f(x) \in D$ and $b = f(a)$. So, $|f(x) - f(a)| < \delta_g \Leftrightarrow |y - b| < \delta_g$. Choose $\delta = \delta_f$. So, $|x - a| < \delta \implies |(g \circ f)(x) - (g \circ f)(a)| < \varepsilon$.

■

Comment: If a is an accumulation point in D, then continuity can be re-expressed as

$$\lim_{x \to a} f(x) = f(a)$$

and given that the limit of f meets the following criteria:

(1) Given that a is indeed an accumulation point of $\mathbb{R}$.

(2) Given that both one-sided limits of f exist.

(3) Given that both one-sided limits of f are equivalent.

0.2.21 Theorem 3.1.5 (Order Completeness)

Theorem 0.2.16

Let $A \subseteq \mathbb{R}$ and $A \neq \emptyset$. If A has an upper bound, then A has a least upper bound.

0.2.22 Theorem 3.1.8 (Greatest Lower Bound Theorem)

Theorem 0.2.17

Let $A \subseteq \mathbb{R}$. If A has a greatest lower bound.

0.2.23 Theorem 3.2.1 (Interval Theorem)

Theorem 0.2.18

Let $A \subseteq \mathbb{R}$ and $A \neq \emptyset$. Then, A has the intermediate value property if and only if it A is an interval.

Proof: ($\Leftarrow$) Suppose $A \subseteq \mathbb{R} \ni A = \emptyset$ has the intermediate value property. Let a $= \inf A$ and $b = \sup A$ such that a and b are both finite. Then, a is a lower bound and b is an upper bound. So $A \subseteq [a, b]$. Let $t \in \mathbb{R} \ni a < t < b$. Since $a < t$ and $a = \inf A$, t is not a lower bound for A. Thus, $\exists x \in A \ni \inf A = a \leq x < t$. Similarly, $\exists y \in A \ni t < y \leq b = \sup A$. Thus, by the intermediate value property, since $x, y \in A$ and $x < t < y$, we have $t \in A$. Therefore, $(a, b) \subseteq A$.

$\blacksquare$

0.2.24 Theorem 3.3.2 (Nested Interval Theorem)

Theorem 0.2.19

Suppose the closed intervals $([a_n, b_n])_{n\in\mathbb{N}}$ are nested and the sequence of lengths $(b_n - a_n)$ is null[a]. Then, $\exists x \in \mathbb{R} \ni \cap_{n=1}^{\infty}[a_n, b_n] = \{x\}$.

Proof: Suppose the closed intervals $([a_n, b_n])_{n\in\mathbb{N}}$ are nested and $(b_n - a_n)$ is null. Then,

$$a_i \leq a_{i+1} \leq \cdots \leq a_j \leq b_j \leq \cdots \leq b_{j+1} \leq b_j$$

where $i < j, \forall i, j \in \mathbb{N}$. So, $a_i \leq a_j \leq b_j \leq b_i \implies a_i \leq b_j, a_i \leq a_j$. Then, $\forall m, n \in \mathbb{N}$, $a_m \leq b_n$. Let $A := \{a_m | m \in \mathbb{N}\}, B := \{b_n | n \in \mathbb{N}\}$. Then, $\forall a \in A$ and $\forall b \in B$, $a \leq b$.If $a := \sup A$ and $b := \inf B$, $a \leq b$. If $a < b$, then $0 < b - a \leq b_n - a_n, \forall n \in \mathbb{N}$. Note, if $\forall a_n, b_n > 0$, then $(b_n - a_n) \not\to 0$. This contradicts $(b_n - a_n)$ being null. So, $a \not< b$ by contradiction. So, $a = \sup A = \inf B = b$.
Note, $\forall n \in \mathbb{N}, a_n \leq a$. Similarly, $b_n \geq b, \forall n \in \mathbb{N}$. So, $a_n \leq a = b \leq b_n$. So, $a_n \in [a_n, b_n], \forall n \in \mathbb{N}$. So, $\{a\} \in \cup_{n=1}^{\infty}[a_n, b_n]$.
Now, let $x \in \cup_{n=1}^{\infty}[a_n, b_n]$ .Then, $x \in [a_n, b_n], \forall n \in \mathbb{N}$. So, $a_n \leq x \leq b_n$. So, x is an pper bound] for the set A and a ower bound] for the set B. So, $a \leq x \leq b$. But, $a = b$. So, $x = a = b$. Thus, $x \in \{a\}$. So, $\cup_{n=1}^{\infty}[a_n, b_n] \subseteq \{a\}$.

■

[a]That the length of the intervals approaches 0

0.2.25 Corollary 3.3.4 (Binary Nested Interval Theorem)

Corollary 0.2.3

Let aMb be real numbers. Let $a_0 := a$, $b_0 := b$. Then, $\forall n \in \mathbb{N}_0$, $c_n := \frac{a_n + b_n}{2}$. Now, suppose either $a_{n+1} = a_n$ and $b_{n+1} = c_n$ or $a_{n+1} - c_n$ and $b_{n+1} = b_n$. Then, $\cap_{n=0}^{\infty}[a_n, b_n] = \{x\}, \exists x \in \mathbb{R}$.

0.2.26 Theorem 3.5.1 (Roots of an Exponential)

Theorem 0.2.20

Let $x \in \mathbb{R}^{>0}$, $k \in \mathbb{N}$. Then, $\exists y \in \mathbb{R}^{>0} \ni y^k = x$. So, $x^{\frac{1}{k}} = y$.

0.2.27 Theorem 3.5.4 (Irrationality of the Square Root of Two)

Theorem 0.2.21

Suppose $\sqrt{2}$ is rational. So, $\exists p, q \in \mathbb{N} \ni \sqrt{2} - \frac{p}{2}$. Suppose p and q have no common factors. Then, $2 = \frac{p^2}{q^2} \implies 2q^2 = p^2$. So, p^2 and furthermore, p is even. Since p is even, $p = 2k, \exists k \in \mathbb{N}$. So, $2q^2 = (2k)^2 = 4k^2$. So, $q^2 = 2k^2$ and q is even. So, p and q have a common factor, 2. But p and q have no common factors! So, by contradiction, $\sqrt{2}$ is irrational.

■

0.2.28 Lemma (Finite Union of Countable Sets)

Lemma 0.2.4

The union of two countable sets is a countable set.

Proof: Let A and B be countable. Let $a_1, a_2, a_3, \dots \in A$be an enumeration of A and $b_1, b_2, b_3, \dots \in B$ be an enumeration of B. Note, $a_1, b_1, a_2, b_2, a_3, b_3, \dots$ is an enumeration in $A \cup B$. So, $A \cup B$ a finite union of countable sets.

■

0.2.29 Lemma (Infinite Union of Finite Sets)

Lemma 0.2.5

If $A_k, \forall k \in \mathbb{N}$ is finite, then $\cup_{k=1}^{\infty} A_k$ is countable.

0.2.30 Theorem 4.1.5 (Infinite Union of Countable Sets)

Theorem 0.2.22

Let $A_k := \{a_{k,1}, a_{k,2}, a_{k,3}, \dots\}$ and let $B_n := \{a_{ij} | i + j = n\}$. Since, B_n has $n - 1$ elements, it is finite and countable. Consequently, $\cup_{k=1}^{\infty} A_k$ is countable.

■

0.2.31 Theorem 4.1.7 (Cantor)

Theorem 0.2.23

The interval $[0,1]$ is not countable.[a]

Proof: Suppose $[0,1]$ is countable. Let $\{y_k | k \in \mathbb{N}\} := [0,1]$[b] Suppose $x := 0.d_1 d_2 \dots$ where

$$d_k := \begin{cases} 7, & k^{\text{th}} \text{ digit of } y_k \leq 4 \\ 2, & k^{\text{th}} \text{ digit of } y_k \geq 5 \end{cases}$$

For any k, $x_k \neq y_k$ since they have different k^{th} digits.

■

[a] Really sorry about lack of order in the theorem counting here. . . I need to create a template for propositions, which is what this statement really is

[b] Very weird notation. . .

0.2.32 Theorem 4.2.1

Theorem 0.2.24

Let A and B be sets. There is an injective map $f : A \to B$ if and only if there is a surjective map $g : B \to A$.

0.2.33 Theorem (Cantor-Bernstein-Schröder Theorem)

Theorem 0.2.25

If $|A| \geq |B|$ and $|A| \leq |B|$, then $|A| = |B|$.

Example 0.2.5

Claim: The Cantor Set is uncountable. **Proof:** Let the Cantor function $f : C \to [0,1]$, $f(d_0.d_1 \dots) := \sum_{k=0}^{n} \frac{d_k}{n}$.

Claim: f is not injective. Let $x \in [0,1]$ be expressed $x := 0._2 d_1 d_2 \dots$. Since $d_j \in \{0,2\}, \forall j, y \in C$ and $f(y) = x$. So, $f : C \to [0,1]$ is a surjection. By Theorem 4.2.1, there is an injection $g : [0,1] \to C$.

Note, $h : C \to [0,1], h(c) = c, \forall c \in C$ is a satisfactory injection as $C \subseteq [0,1]$. So, by

Cantor-Bernstein-Schröder Theorem $|C| = |[0,1]|$ and thus f is a bijection. So, C is uncountable.

■

0.2.34 Theorem (Jump Discontinuity Theorem)

Theorem 0.2.26

Let I be an interval and suppose $ff : I \to \mathbb{R}$ is monotone. Let $a \in I$. Then, f is discontinuous at a if and only if f has a jump discontinuity at a.

0.2.35 Corollary 5.1.4

Corollary 0.2.4

A monotone function has a countable or empty set of discontinuities.

Proof: Without loss of generality, consider f to be increasing. Let A be the set of discontinuities of f. If $a \in A$, then $\lim_{x \to a^-} f(x) < \lim_{x \to a^+} f(x)$ since f is increasing and discontinuous at a. Consider the open interval,

$$I_0 := (\lim_{x \to a^-} f(x), \lim_{x \to a^+} f(x))$$

. If $\exists a' \in A \ni a' < a$, then

$$\lim_{x \to a^-} f(x) \leq f(\frac{a + a'}{2}) \leq \lim_{x \to a^-} f(x)$$

■

0.2.36 Theorem 5.1.6 (Continuity Theorem for Monotone Functions)

Theorem 0.2.27

A monotone function defined on an interval is continuous if and only if its range is an interval.

0.2.37 Corollary 5.1.8

Corollary 0.2.5

A monotone function defined on an interval is continuous if and only if its range has the intermediate value property.

0.2.38 Corollary 5.1.9

Corollary 0.2.6

The inverse of a strictly increasing function is strictly increasing.

0.2.39 Corollary 5.1.10

Corollary 0.2.7

The inverse of a strictly monotone, continuous function defined on an interval is continuous.

Proof: Let I be an interval and suppose $f : I \to \mathbb{R}$ is strictly increasing (without loss of generality) and continuous. By Corollary 5.1.9, f^{-1} is strictly increasing. By Theorem 5.1.6, range f is an interval, J. So, $f^{-1} : J \to I$. Therefore, f^{-1} is continuous.

∎

0.2.40 Theorem 5.2.1 (Bolzano's Intermediate Value Theorem)

Theorem 0.2.28

Let I be an interval and $f : I \to \mathbb{R}$ be a continuous function. If there are points $a, b \in I \ni f(a) < 0$ and $f(b) > 0$, then there is a point $c \in (a, b) \ni f(c) = 0$.

Proof: Without loss of generality, $a < b$. Let $a_0 := a$ and $b_0 := b$. Then, $f(a_0) < 0$ and $f(b_0) > 0$. Let $c_0 := \frac{a_0+b_0}{2}$. If $f(c_0) = 0 \implies c = c_0$, then there exists an element in between a and b that equals 0. If $f(c_0) < 0$, let $a_1 := c_0$ and $b_1 := b_0$. If $f(c_0) > 0$, let $a_1 := a_0$ and $b_1 := c_0$. In either case, $f(a_1) < 0$ and $f(b_1) > 0$. So, sequences (a_n) and (b_n) with $f(a_n) < 0$ and $f(b_n) > 0, \forall n \in \mathbb{N}_0$. For $c_n := \frac{a_n+b_n}{2}, a_{n+1} := a_n$ and $b_{n+1} := c_n$ or $a_{n+1} := c_n$ and $b_{n+1} := b_n$. Thus, $([a_n, b_n])$ are nested and $\cup_{n=0}^{\infty}[a_n, b_n] = \{c\}, \exists c \in \mathbb{R}$. Since $f(a_n) < 0, \forall n \in \mathbb{N}, f(c) = \lim_{n\to\infty} a_n \leq 0$. Similarly, $f(b_n) < 0 \implies f(c) = \lim_{n\to\infty} b_n \geq 0, \forall n \in \mathbb{N}$. Therefore, $f(c) = 0$.

■

0.2.41 Theorem 5.2.2 (Intermediate Value Theorem)

Theorem 0.2.29

If a real-valued function f is continuous on some interval I, then the image $f(I)$ has the intermediate value property.

Proof: Let $a, b \in I \ni a < b$ and y_0 be in between $f(a)$ and $f(b)$. If $f(a) < f(b)$, apply the intermediate value theorem to $g(x) := f(x) - y_0$. If $f(b) < f(a)$, apply the intermediate value theorem to $g(x) := y_0 - f(x)$.

■

0.2.42 Corollary 5.2.3

Corollary 0.2.8

If a real-valued function f is continuous on some interval I ,then $f(I)$ is an interval.

0.2.43 Theorem 5.5.3 (Global Boundedness Theorem)

Theorem 0.2.30

Let I be a compact interval. If $f : I \to \mathbb{R}$ is continuous, then f is bounded on I.

Proof: By the ntemediate value theorem], $f(I)$ is an interval. Suppose f is unbounded on I. Then, $\exists x \in E, \forall M \in \mathbb{R} \ni |f(x)| \geq M$.
Bisect I. Since f is unbounded on I, it is unbounded on at least one of the halves of I. Bisect the unbounded half of I. Repeat this argument to obtain a sequence of nested intervals $I_{n+1} \subseteq I_n, \forall n \in \mathbb{N} \ni f$ is unbounded on each I_n. By the nested interval theorem , $\cap_{n=0}^{\infty} I_n = \{x_0\}$ for $\exists x_9 \in \mathbb{R}$. Note, f is continuous on I and specifically at x_0. So, by local boundedness, $\exists \delta > 0 \ni f$ is bounded on $(x_0 - \delta, x_0 + \delta)$. By **Example 3.3.3** in **From Calculus to Analysis**, $\exists n \in \mathbb{N} \ni I_n \subseteq (x_0 - \delta, \delta + x_0)$. So, I_n is bounded by contradiction.

0.2.44 Theorem 5.3.5 (Extreme Value Theorem)

Theorem 0.2.31

Let I be a compact interval. If $f : I \to \mathbb{R}$ is continuous on I, then there are x_{min} and x_{max}[a] such that $f(x_{\text{min}}(x) \leq f(x) \leq f(x_{\text{max}}(x), \forall x \in I$.

Proof: Let $M := \sup f(I)$. Since $f$is bounded, by the global boundedness theorem, $m \in \mathbb{R}$. Now, bisect I. We know that at least one-half of I gives $\sup f(I) = M$. Choose this half. Repeat this argument to obtain a sequence of nested intervals. $I_n \ni \sup f(I_n - M$. So, by the nested interval theorem, $\cup_{n=0}^{\infty} I_n - \{x_0\}$ for $\exists x_0 \in \mathbb{R}$. Since $M = \sup f(I)$, $f(x) \leq M, \forall x \in I$. So, $f(x_0) \leq M$. Suppose $f(x_0) < M$. Let $\varepsilon := \frac{1}{2}\bigg(M - f(x_0)\bigg) > 0$. Since f is continuous, $\exists \delta > 0 \ni \forall x \in I, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon = \frac{1}{2}\bigg(M - f(x_0)\bigg)$. Viz., $\exists x \in (x_0 - \delta, x_0 + \delta) \ni f(x) < f(x_0) + \varepsilon$. So, $f(x) < \frac{1}{2}M + \frac{1}{2}f(x_0) < \frac{1}{2}M + \frac{1}{2}M = M$. Choose $N \ni I_n \subseteq (x_0 - \delta, x_0 + \delta)$. Then, $M = \sup\{f(x)|x \in I_n\} \leq \sup\{f(x)|x \in (x_0 - \delta, x_0 + \delta)\} < f(x_0) + \varepsilon = M$ by contradiction. The argument for x_{min} follows similarly.

■

[a] Otherwise denoted as $\arg\min I$ and $\arg\max I$ respectively.

0.2.45 Corollary 5.3.6

Corollary 0.2.9

If $f : [a, b] \to \mathbb{R}$ is continuous and $x_{\text{min}}, x_{\text{max}} \in [a, b] \ni f(x_{\text{min}}) \leq f(x) \leq f(x_{\text{max}}), \forall x \in [a, b]$ then $f([a, b]) = [f(x_{\text{min}}), f(x_{\text{max}})$.

0.2.46 Lemma 5.4.5

Lemma 0.2.6

Let $f : [0,1] \to \mathbb{R}$. Suppose f is uniformly continuous on $I_1 := [0, \frac{1}{2}], I_2 := [\frac{1}{2}, 1]$, and $I_3 = [\frac{1}{4}, \frac{3}{4}]$. Then, f is uniformly continuous on $[0,1]$.

Proof: Let $\varepsilon > 0$ be given. For each j, let $\exists \delta_j > 0$ such that uniform continuity is satisfied for ε on I_j. Let $\delta := \min \delta_1, \delta_2, \delta_3, \frac{1}{2}$. Then, if $x, y \in [0,1]$ with $|x-y| < \delta \leq \frac{1}{2}$. So, $x, y \in I, \exists! j$. Thus, $|x-y| < \delta \leq \delta_j \implies |f(x) - f(y)| < \varepsilon$.

■

0.2.47 Theorem 5.4.6 (Uniform Continuity Theorem)

Theorem 0.2.32

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then, f is uniformly continuous on $[a,b]$.[a]

Proof: Let $f : [a.b] \to \mathbb{R}$ be continuous on $[a,b]$. Suppose f is not uniformly continuous on $[a,b]$. Then, $\exists \varepsilon > 0, \forall \delta > 0, \exists x, y \in [a,b] \ni |x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$. Let $\varepsilon_0 > 0$ be such that there is no $\delta > 0$ satisfying uniform continuity on $[a,b]$. By Lemma 5.4.5, there is no $\delta > 0$ satisfying uniform continuity for ε_0 on at least one of the three subintervals ,$[a, \frac{a+b}{2}], [\frac{a+b}{2}, b], [\frac{3a+b}{4}, \frac{a+3b}{4}]$. Let $a_1 := a$ and $b_1 := b$. Repeat the argument so $\forall n \in \mathbb{N}$, $[a_n n + 1, b_{n+1}] \subseteq [a_n, b_n]$ and $b_{n+1} - a_{n+1} = \frac{|b_n - a_n|}{2}$ and there is no δ satisfying uniform continuity for ε_0 on $[a_n, b_n]$. Note, $[a_n, b_n]$ is nested $\forall n \in \mathbb{N}$. Additionally, $b_n - a_n = \frac{(b-a)}{2^{n-1}}$ is null. By the Nested Interval Theorem, $\exists x_0 \in \mathbb{R} \ni \{x_0\} = \cap_{n=1}^{\infty} [a_n, b_n]$. So, f is continuous at x_0. Thus, $\exists \delta_0 > 0, \forall x \in [a,b], |x - x_0| < \delta_0 \implies |f(x) - f(x_0)| < \frac{\varepsilon_0}{2}$. Let N be such that $[a_N, b_N] \in (x_0 - \delta_0, x_0 + \delta_0)$. So, $\forall x, y \in [a_N, b_N], |x - x_0| < \delta_0$ and $|y - x_0| < \delta_0$. Thus, $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$ and $|f(y) - f(x_0)| < \frac{\varepsilon}{2}$. So, $|f(x) - f(y)| \leq |f(x_0) - f(y)| + |f(x) + f(x_0)| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0$.

So, $x, y \in [a_N, b_N]$ satisfy uniform continuity for ε_0.[b] So. f is uniformly continuous on $[a,b]$.

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[a] If f is continuous on a compact interval, it is uniformly continuous

[b] This argument shows that any $\delta > 0$ satisfies uniform continuity for ε on $[a_N, b_N]$. But, there is supposedly no δ_0 that satisfies uniform continuity for ε_0 on $[a_N, b_N]$

0.3 Version History

0.3.1 v0.0.0 (4/2/2020)

Initial release of *The Compendium*.