Homework #1

Math 222A, 2023F

Instructor: Sung-Jin Oh — Due: Sep. 1st, 11:59pm (PST)

Instruction: The homework should to be submitted to Gradescope as a pdf file. Please work on separate sheets of paper and scan them, or type it up.

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1. Let $f : \mathbb{R}^{1+1} \to \mathbb{R}$ and $u_0 : \mathbb{R} \to \mathbb{R}$ be smooth functions. Solve, using the method of characteristics,
$$\begin{cases} \partial_t u + t\partial_x u = f \text{ in } \mathbb{R}^{1+1}, \\ u(0,x) = u_0(x). \end{cases}$$

2. [Problem 5 in L. C. Evans, *Partial Differential Equations*, 2nd Ed., §3.5] Solve, using the method of characteristics, the following BVPs.
   (a) $x^1 u_{x^1} + x^2 u_{x^2} = 2u$, $u(x^1, 1) = g(x^1)$.
   (b) $x^1 u_{x^1} + 2x^2 u_{x^2} + u_{x^3} = 3u$, $u(x^1, x^2, 0) = g(x^1, x^2)$.
   (c) $uu_{x^1} + u_{x^2} = 1$, $u(x^1, x^1) = \frac{1}{2}x^1$.

3. [Problem 6 in L. C. Evans, *Partial Differential Equations*, 2nd Ed., §3.5] Given a smooth vector field $\mathbf{b}$ on $\mathbb{R}^d$, let $\mathbf{x}(s) = \mathbf{x}(s, x, t)$ solve the ODE
$$\begin{cases} \dot{\mathbf{x}} = \mathbf{b}(\mathbf{x}) \quad (s \in \mathbb{R}) \\ \mathbf{x}(t) = x. \end{cases}$$
   (a) Define the Jacobian $J(s, x, t) := \det D_x \mathbf{x}(s, x, t)$ and derive *Euler's formula*
$$J_s = (\operatorname{div} \mathbf{b}(\mathbf{x}))J.$$
   (b) Demonstrate that u solves
$$\begin{cases} u_t + \operatorname{div}(u\mathbf{b}) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^d \times \{t = 0\}, \end{cases}$$
   and derive the representation formula
$$u(x,t) := g(\mathbf{x}(0, x, t))J(0, x, t).$$
   (Hint: Show $\partial_s(u(\mathbf{x}, s)J) = 0$.)