Homework #7

Math 222A, 2023F

Instructor: Sung-Jin Oh   Due: Oct. 20th, 11:59pm (PST)

Instruction: The homework should to be submitted to Gradescope as a pdf file. Please work on separate sheets of paper and scan them, or type it up.

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1. Write down an explicit formula for a solution of

$$\begin{cases} \partial_t u - \Delta u + cu = f & \text{in } (0, \infty)_t \times \mathbb{R}^d, \\ u = g & \text{on } \{t = 0\} \times \mathbb{R}^d. \end{cases}$$

Solution. The idea is to make a change of variables to transform this equation into the heat equation. In view of the fact that $\partial_t (e^{ct} u) = e^{ct}(\partial_t u + cu)$ (cf. method of integrating factor) and $\Delta(e^{ct} u) = e^{ct} \Delta u$, the given PDE is equivalent to

$$\begin{cases} (\partial_t - \Delta)(e^{ct} u) = e^{ct} f & \text{in } (0, \infty)_t \times \mathbb{R}^d, \\ e^{ct} u = g & \text{on } \{t = 0\} \times \mathbb{R}^d, \end{cases}$$

where we used $e^{ct} = 1$ when $t = 0$ on the second line. Therefore, we have

$$e^{ct} u = \int_0^t \int \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} e^{cs} f(s, y) \, \mathrm{d}s \mathrm{d}y + \int \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{44}} g(y) \, \mathrm{d}y.$$

Dividing both sides by e^{ct}, we obtain

$$\boxed{\begin{aligned} u(t,x) = &\int_0^t \int \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} e^{-c(t-s)} f(s, y) \, \mathrm{d}s \mathrm{d}y \\ &+ \int \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{44}} e^{-ct} g(y) \, \mathrm{d}y. \end{aligned}}$$

□

2. We say $G(t,x,s,y)$ is a *Green's function* for $(\partial_t - \Delta)$ in $\mathbb{R}_t \times U$ if it solves

$$\begin{cases} (\partial_t - \Delta)G(t,x,s,y) = \delta_0((t,x)-(s,y)) & \text{in } \mathbb{R}_t \times U, \\ G(t,x,s,y) = 0 & \text{on } \mathbb{R}_t \times \partial U, \\ G(t,x,s,y) = 0 & \text{in } (t,x) \in (-\infty, s) \times U). \end{cases}$$

(a) Find a Green's function for $(\partial_t - \Delta)$ in $\mathbb{R}_t \times U$, where U is the half line $(0,\infty)_x$.

(b) Given $f \in C_c^\infty((0,\infty)_t \times (0,\infty)_x)$, write down a solution formula for the problem

$$\begin{cases} (\partial_t - \Delta)u = f & \text{in } \mathbb{R}_t \times (0,\infty)_x, \\ u = 0 & \text{on } \mathbb{R}_t \times \{0\}, \\ u = 0 & \text{in } (t,x) \in (-\infty,0) \times (0,\infty)_x. \end{cases}$$

(c) Given $g \in C_c^\infty((0,\infty)_t)$, write down a solution formula for the problem

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}_t \times (0,\infty)_x, \\ u = g & \text{on } \mathbb{R}_t \times \{0\}, \\ u = 0 & \text{in } (t,x) \in (-\infty,0) \times (0,\infty)_x. \end{cases}$$

[Hint: Consider $v(t,x) = u(t,x) - g(t)$ and reduce it to (b).]

Solution. (a) We use the method of images. Given $(s,y) \in \mathbb{R} \times (0,\infty)$, define

$$\boxed{G(t,x,s,y) = E_+(t-s, x-y) - E_+(t-s, x+y),}$$

where $E_+(t,x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{|x|^2}{4t}}$ is the forward fundamental solution for $\partial_t - \Delta$.

Let us check that G indeed is the desired Green's function. Note that $(\partial_t - \Delta)(E_+(t-s, x+y)) = \delta_0(t-s, x+y)$, which is equal to zero if $x \in U = (0,\infty)$. Moreover, if $x \in \partial U = \{0\}$, then

$$G(t,0,s,y) = E_+(t-s,-y) - E_+(t-s,y) = 0.$$

Finally, $G(t,x,s,y) = 0$ if $t < s$, since both $E_+(t-s, x-y)$ and $E_+(t-s,x+y)$ vanish.

(b) The solution formula is given by

$$\boxed{\begin{aligned} u(t,x) &= \iint_{\mathbb{R}\times(0,\infty)} G(t,x,s,y) f(s,y)\,\mathrm{d}s\mathrm{d}y \\ &= \int_0^t \int_0^\infty \frac{1}{(4\pi(t-s))^{\frac{1}{2}}} \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) f(s,y)\,\mathrm{d}y\mathrm{d}s. \end{aligned}}$$

(c) Following the hint, consider $v(t,x) := u(t,x) - g(t)$. Then v solves the problem

$$\begin{cases} (\partial_t - \Delta) v = -g'(t) & \text{in } \mathbb{R}_t \times (0,\infty)_x, \\ v = 0 & \text{on } \mathbb{R}_t \times \{0\}, \\ v = 0 & \text{in } (t,x) \in (-\infty,0)\times(0,\infty)_x. \end{cases}$$

where the last line follows because $g(0) = 0$ by the compact support assumption. By (b), we have

$$\begin{aligned} u(t,x) &= g(t) + v(t,x) \\ &= g(t) - \iint G(t,x,s,y) g'(s)\,\mathrm{d}y\mathrm{d}s \\ &= \boxed{g(t) - \int_0^t \int_0^\infty \frac{1}{(4\pi(t-s))^{\frac{1}{2}}} \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) g'(s)\,\mathrm{d}y\mathrm{d}s.} \end{aligned}$$

(Note that even though, strictly speaking, (b) assumed that $\operatorname{supp} f$ is compact, it can be checked by hand that the formula is still valid in this case.)

We can in fact simplify this expression further. Integrating by parts in s and using $g(0) = 0$, we have

$$\begin{aligned} u(t,x) = g(t) &- \lim_{s\to t} g(s) \int_0^\infty \frac{1}{(4\pi(t-s))^{\frac{1}{2}}} \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) \mathrm{d}y \\ &+ \int_0^t \int_0^\infty \partial_s \left(\frac{1}{(4\pi(t-s))^{\frac{1}{2}}} \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) \right) g(s)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

To proceed further, note that

$$\int_0^\infty \frac{1}{(4\pi(t-s))^{\frac{1}{2}}} \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) \mathrm{d}y$$

$$= \int_{-\infty}^{x} \frac{1}{(4\pi(t-s))^{\frac{1}{2}}} e^{-\frac{z^2}{4(t-s)}} \, \mathrm{d}z - \int_{x}^{\infty} \frac{1}{(4\pi(t-s))^{\frac{1}{2}}} e^{-\frac{z^2}{4(t-s)}} \, \mathrm{d}z$$

$$= \int_{-\infty}^{(t-s)^{-\frac{1}{2}}x} \frac{1}{(4\pi)^{\frac{1}{2}}} e^{-\frac{\zeta^2}{4}} \, \mathrm{d}\zeta - \int_{(t-s)^{-\frac{1}{2}}x}^{\infty} \frac{1}{(4\pi)^{\frac{1}{2}}} e^{-\frac{\zeta^2}{4}} \, \mathrm{d}\zeta.$$

The first term converges to 1 as $s \nearrow t$, while the second term converges to 0. Hence,

$$\lim_{s\to t} g(s) \int_0^{\infty} \frac{1}{(4\pi(t-s))^{\frac{1}{2}}} \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) \mathrm{d}y = g(t).$$

Next, note that

$$(-\partial_s - \partial_y^2) \left(\frac{1}{(4\pi(t-s))^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{4(t-s)}} \right) = (\partial_t - \partial_x^2) \left(\frac{1}{(4\pi(t-s))^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{4(t-s)}} \right) = 0,$$

$$(-\partial_s - \partial_y^2) \left(\frac{1}{(4\pi(t-s))^{\frac{1}{2}}} e^{-\frac{(x+y)^2}{4(t-s)}} \right) = (\partial_t - \partial_x^2) \left(\frac{1}{(4\pi(t-s))^{\frac{1}{2}}} e^{-\frac{(x+y)^2}{4(t-s)}} \right) = 0,$$

as long as $s < t$. Therefore, we have

$$\int_0^t \int_0^{\infty} \partial_s \left(\frac{1}{(4\pi(t-s))^{\frac{1}{2}}} \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) \right) g(s) \, \mathrm{d}y \, \mathrm{d}s$$

$$= - \int_0^t \int_0^{\infty} \partial_y^2 \left(\frac{1}{(4\pi(t-s))^{\frac{1}{2}}} \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) \right) \mathrm{d}y g(s) \, \mathrm{d}s$$

$$= - \int_0^t \frac{1}{(4\pi(t-s))^{\frac{1}{2}}} g(s) \left(\partial_y \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) \Big|_{y=0}^{\infty} \right) \mathrm{d}s$$

$$= - \int_0^t \frac{1}{(4\pi(t-s))^{\frac{1}{2}}} g(s) \left(\left(\frac{(x-y)}{2(t-s)} e^{-\frac{(x-y)^2}{4(t-s)}} + \frac{(x+y)}{2(t-s)} e^{-\frac{(x+y)^2}{4(t-s)}} \right) \Big|_{y=0}^{\infty} \right) \mathrm{d}s$$

$$= - \int_0^t \frac{1}{(4\pi(t-s))^{\frac{1}{2}}} g(s) \left(- \frac{x}{(t-s)} e^{-\frac{x^2}{4(t-s)}} \right) \mathrm{d}s$$

$$= \frac{x}{(4\pi)^{\frac{1}{2}}} \int_0^t \frac{1}{(t-s)^{\frac{3}{2}}} e^{-\frac{x^2}{4(t-s)}} g(s) \, \mathrm{d}s.$$

where we used the fundamental theorem of calculus in the third identity. In conclusion,

$$\boxed{u(t,x) = \frac{x}{(4\pi)^{\frac{1}{2}}} \int_0^t \frac{1}{(t-s)^{\frac{3}{2}}} e^{-\frac{x^2}{4(t-s)}} g(s) \, \mathrm{d}s.}$$

$\square$