Homework #2

Math 222A, 2023F

Instructor: Sung-Jin Oh Due: Sep. 8th, 11:59pm (PST)

Instruction: The homework should to be submitted to Gradescope as a pdf file. Please work on separate sheets of paper and scan them, or type it up.

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1. [Problem 7 in L. C. Evans, *Partial Differential Equations*, 2nd Ed., §3.5] Verify that when Γ is not flat near x_0, the noncharacteristic condition is

$$\boldsymbol{\nu}(x_0) \cdot D_p F(x_0, z_0, p_0) \neq 0.$$

2. [Problem 9 in L. C. Evans, *Partial Differential Equations*, 2nd Ed., §3.5] Consider the problem of minimizing the action $\int_0^t L(\mathbf{w}(s), \dot{\mathbf{w}}(s))\,\mathrm{d}s$ over the class of admissible class

$$\mathcal{A}(t, x) := \{\mathbf{w}(\cdot) \in C^2([0, t]; \mathbb{R}^d : \mathbf{w}(t) = x\}.$$

   (a) Show that a minimizer $\mathbf{x}(\cdot) \in \mathcal{A}$ solves the Euler–Lagrange equations

$$-\frac{\mathrm{d}}{\mathrm{d}s} D_v L(\mathbf{x}(s), \dot{\mathbf{x}}(s)) + D_x L(\mathbf{x}(s), \dot{\mathbf{x}}(s)) = 0$$

   for $0 < s < t$.

   (b) Prove that $D_v L(\mathbf{x}(0), \dot{\mathbf{x}}(0)) = 0$.
   [Hint: Consider variations $h\boldsymbol{\varphi}$ that is nonvanishing at $s = 0$.]

   (c) Suppose now that $\mathbf{x}(\cdot) \in \mathcal{A}$ minimizes the modified action

$$\int_0^t L(\mathbf{w}(s), \dot{\mathbf{w}}(s))\,\mathrm{d}s + g(\mathbf{w}(0)),$$

   for some C^2 function g. Show that $\mathbf{x}$ solves the usual Euler–Lagrange equations and determine the boundary condition at $s = 0$.

3. [Exercises on convex functions] We say that $f : \mathbb{R} \to \mathbb{R}$ is (resp. *strictly*) *convex* if, for all $x \neq y \in \mathbb{R}^d$ and $0 < \tau < 1$,

$$f(\tau x + (1-\tau)y) \leq \tau f(x) + (1-\tau)f(y).$$

   (a) Given any collection $\{f_\alpha : \mathbb{R} \to \mathbb{R}\}_{\alpha \in \mathcal{A}}$ of convex functions such that $\sup_\alpha f_\alpha(x)$ is a finite number for each $x \in \mathbb{R}$, show that $\sup_\alpha f_\alpha(x)$ is convex.

   (b) Show that f is continuous.

   [Hint: Given any three points $x < y < z$, show that the graph of f is contained between the line passing through $(x, f(x))$, $(y, f(y))$ and the line passing through $(y, f(y))$, $(z, f(z))$.]

   (c) We say that m belongs to the *subdifferential* of f at x, and write $m \in \partial f(x)$, if $f(y) \geq m \cdot (y-x) + f(x)$ for all $y \in \mathbb{R}$. Show that f is differentiable at x if and only if ∂f consists of exactly one element (namely, the derivative of f at x).

   [Hint: To prove the "if" part, begin by showing that the left and right limits $\lim_{h\to 0^\pm} \frac{f(x+h)-f(x)}{h}$ always exist, and relate these limits with the subdifferential.]