Mathematics 258 - Harmonic Analysis

Fall 2023

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Problem Set 1

Due: 7 days after we finish Ch.2 or October 4, whichever date is later

1. Assuming a function f on $\mathbb{R}^n$ satisfies $|f(x)| \leqslant \frac{C}{(1+|x|)^{n+\delta}}$ and $|\hat{f}(\xi)| \leqslant \frac{C}{(1+|\xi|)^{n+\delta}}$ with constants $C, \delta > 0$. Prove the *Poisson summation formula*:

$$\sum_{k \in \mathbb{Z}^n} f(k) = \sum_{m \in \mathbb{Z}^n} \hat{f}(m).$$

2. Assuming $f \in L^1(\mathbb{R}^n)$ is continuous at 0 and $\hat{f} \geqslant 0$. Prove that

$$\hat{f} \in L^1(\mathbb{R}^n).$$

3. Prove the Riesz-Thorin interpolation theorem.

4. Assuming $\phi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = 1$. Let $\psi(x) = \operatorname{esssup}_{|y| \geqslant |x|} |\phi(y)|$ and assume further $\psi \in L^1(\mathbb{R}^n)$. For a Lebesgue point x_0 of $f(x) \in L^p(\mathbb{R}^n) (p > 1)$, prove that

$$\lim_{t \to 0+} \phi_t * f(x_0) = f(x_0).$$

5. Let $P(x)$ be a trigonometric polynomial of degree N on $\mathbb{T}$, prove that

$$\|P'\|_\infty \lesssim N \|P\|_\infty.$$

6. (optional) Let f be a function of bounded variation on $\mathbb{T}$. Prove that

$$\hat{f}(k) = O(\frac{1}{|k|}).$$

7. (optional) For a given $p \in (2, \infty)$, find a function $f \in L^p$ such that $\hat{f}$ is not in L^1_{loc}.

8. (optional) Assuming $f \in L^2(\mathbb{R})$ such that $xf(x) \in L^2(\mathbb{R})$ and $\xi \hat{f}(\xi) \in L^2(\mathbb{R})$, define

$$\Delta(f) = \frac{1}{\|f\|_2} \left(\int_{\mathbb{R}} (x - \bar{x})^2 |f(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}}$$

where

$$\bar{x} = \frac{1}{\|f\|_2^2} \int_{\mathbb{R}} x |f(x)|^2 \mathrm{d}x$$

and define $\Delta(\hat{f})$ similarly. Prove the *Heisenberg uncertainty principle*:

$$4\pi \Delta_f \Delta_{\hat{f}} \geqslant 1$$

and prove that the equality holds if and only if

$$f(x) = ce^{\mathrm{i}ax}e^{-\frac{(x-b)^2}{s}}, a, b, c \in \mathbb{R}, s > 0.$$

9. (optional) Define

$$Hf(x) = \frac{1}{x}\int_0^x f(y)\mathrm{d}y, x \in (0, \infty).$$

Prove that

$$\|Hf\|_{L^p(0,\infty)} \lesssim_p \|f\|_{L^p(0,\infty)}, 1 < p \leqslant \infty.$$

10. (optional) Use the Vitali covering lemma to prove that the Hardy-Littlewood maximal function is weak-type $(1,1)$.

11. (optional) Let $B \subseteq \mathbb{R}^n$ be a ball and assume $\mathrm{supp} f \subseteq B$ with $f \in L^1(B)$. Prove that $Mf \in L^1(B)$ if and only is $f \log^+ f \in L^1(B)$.