Mathematics 258 - Harmonic Analysis

Fall 2023

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Problem Set 2

Due: 7 days after we finish Ch.4 or November 1, whichever date is later

In the problems below, when $x \in \mathbb{R}^n \backslash \{0\}$, we denote $x' = \frac{x}{|x|}$.

1. Find the Riesz transform of the Poisson kernel $Q_t^{(j)}(x) = R_j(P_t)(x)$.

2. Let $1 \leqslant p, q \leqslant \infty$. Assuming that T is a linear operator defined on $\mathscr{S}$ that is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ and that T commutes with all translations. Prove that: (i) There exists a unique $K \in \mathscr{S}'(\mathbb{R}^n)$ such that $Tf = K * f$; (ii) T can be extended to a bounded linear operator from $L^{q'}(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$; (iii) $T = 0$ if $p > q$ and (iv) $\widehat{K} \in L^\infty(\mathbb{R}^n)$ if $p = q = 2$.

3. For $0 < \alpha < n$, define the *fractional integral operator* I_α on $\mathscr{S}(\mathbb{R}^n)$ to be

$$\widehat{I_\alpha f}(\xi) = (2\pi|\xi|)^{-\alpha} \widehat{f}(\xi).$$

Prove that I_α is strong (p, q) and weak $(1, \frac{n}{n-\alpha})$ where $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

4. For a C^1 function $f : \mathbb{R}^2 \to \mathbb{C}$ with compact support, prove that

$$\left\| \frac{\partial f}{\partial x_1} \right\|_p + \left\| \frac{\partial f}{\partial x_2} \right\|_p \lesssim_p \left\| \frac{\partial f}{\partial x_1} + \mathrm{i} \frac{\partial f}{\partial x_2} \right\|_p, 1 < p < \infty.$$

5. Assuming the support of $u \in \mathscr{S}'(\mathbb{R}^n)$ is $\{0\}$. Prove that there exists $K \in \mathbb{Z}^+$ and $a_\alpha \in \mathbb{C}$ such that

$$u = \sum_{|\alpha| \leqslant K} a_\alpha \partial^\alpha \delta$$

where δ is the Dirac delta at the origin.

6. (optional) For $f \in \mathscr{S}(\mathbb{R})$, prove that $H(f')(x) = (Hf)'(x)$ and that

$$|H(f')(x)| \lesssim_f |x|^{-2}, \forall |x| \geqslant 1.$$

7. (optional) Find the Hilbert transform of $\chi_{[a,b]}$.

8. (optional) Assuming $f \in \mathscr{S}(\mathbb{R})$. Prove that $Hf \in L^1(\mathbb{R})$ if and only if $\int_{\mathbb{R}} f \mathrm{d}x = 0$.

9. (optional) Find a way to define Hf for $f \in L^\infty$ that is canonical up to the addition of a constant.

10. (optional) For $f(x) = (x^2+1)^{\frac{1}{4}}$, prove $H(f') \in L^\infty$.

11. (optional) Define the translation operator $\tau_h f(x) = f(x+h), h \in \mathbb{R}$, the dilation operator $\delta_\rho f(x) = f(\rho x), \rho > 0$ and the involution operator $\tilde{f}(x) = f(-x)$. For a bounded linear operator T on $L^2(\mathbb{R})$, assuming $\tau_h(Tf) = T(\tau_h f)$, $\delta_\rho(Tf) = T(\delta_\rho f)$

and $\widetilde{Tf} = -T(\tilde{f})$. Prove that $T = CH$ for some constant C. Can you generalize this conclusion to a statement involving the Riesz transforms in $\mathbb{R}^n$?

12. (optional) Let $1 < p < \infty$. Prove that a real-valued function $f \in L^p(\mathbb{R})$ if and only if there exists an analytic function $F(x + \mathrm{i}y)$ on the upper half plane such that

$$\sup_{y>0} \int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^p \mathrm{d}x < \infty$$

and

$$\lim_{y\to 0+} \Re F(x+\mathrm{i}y) = f(x), a.e. x \in \mathbb{R}.$$

13. (optional) Prove that a multiplier T_m is bounded on $L^1(\mathbb{R}^n)$ if and only if $m = \hat{\mu}$ where μ is a bounded Borel measure. Moreover prove that if such μ exists, then the operator norm of T_m on L^1 is equal to the total variation of μ.

14. (optional) Assuming for some $\Omega \in L^1(S^{n-1})$ that is not identically zero, the singular integral $Tf = \text{p.v.}\frac{\Omega(x')}{|x|^n} * f$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, prove that $p = q$.

15. (optional) Let $P_k(x), k \geqslant 1$ be a homogeneous harmonic polynomial of degree k on $\mathbb{R}^n$, prove that

- $(P_k(x)e^{-\pi|x|^2})\hat{}(\xi) = \mathrm{i}^{-k} P_k(\xi) e^{-\pi|\xi|^2}$.
- $\int_{S^{n-1}} P_k(x')\mathrm{d}\sigma(x') = 0$.
- p.v. $\frac{P_k(x)}{|x|^{n+k}}$ has Fourier transform $m(\xi) = \mathrm{i}^{-k}\pi^{\frac{n}{2}} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{n+k}{2})} \frac{P_k(\xi)}{|\xi|^k}$.

16. (optional) Assuming $\phi \geqslant 0$ is in $L^1(\mathbb{R}^n)$ and satisfies that $\phi(rx)$ is decreasing in $r \in (0, \infty)$, $\forall x \in \mathbb{R}^n$. Prove that

$$M_\phi f(x) = \sup_{0<t<\infty} |\phi_t * f(x)|$$

is bounded on $L^p(\mathbb{R}^n), 1 < p < \infty$.

17. (optional) Assuming an even function $\Omega \in L \log L(S^{n-1})$ and has zero average. Prove that the singular integral operator $Tf = \text{p.v.}\frac{\Omega(x')}{|x|^n} * f$ is bounded on $L^p(\mathbb{R}^n), 1 < p < \infty$. Moreover prove that the maximal singular integral operator

$$T^* f(x) = \sup_{0<\varepsilon<R<\infty} \int_{\varepsilon<|y|<R} \frac{\Omega(y')}{|y|^n} f(x-y)\mathrm{d}y$$

is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

18. (optional) Find a function $G \in \mathscr{S}'(\mathbb{R}^n)$ such that $\Delta(G) = \delta$. Here δ is the Dirac delta at the origin.

19. (optional) Let $m(\xi)$ be the Fourier transform of p.v. $\frac{\Omega(x')}{|x|^n}$. Prove that $\Omega(x') \in C^\infty(S^{n-1})$ if and only if $m(\xi) \in C^\infty(\mathbb{R}^n \backslash \{0\})$.