Sagnik Nandi

Abstract Algebra

Proof Portfolio Reflection

Proof 1:

I chose this problem because it delves into the non-commutative nature of groups and illustrates how the structure of a group can influence the behavior of its elements. This problem was compelling because it requires a deeper understanding of group theory, particularly the implications of non-commutativity on group elements. As this was one of my first problem set questions and coming from a lower math class, this problem gave me unique insight into what type of thinking I should be doing to manipulate expressions.

While this proof was elementary in nature, coming from one of the first few problems, it still gave me valuable insight into how to view and manipulate interconnected algebraic structures and through their exploration of group structure, operations, and element behavior. In the process of solving this problem, I encountered the challenge of group algebra manipulations, particularly in understanding and proving statements involving non-commutative elements.

Proof 2:

I selected this problem to explore the concept of cyclic subgroups within the general linear group GL(2, R). This problem is significant because it requires verifying subgroup properties and demonstrating cyclicity, which are fundamental aspects of group theory. I specifically remember how my original proof did not gesture to the positive powers of the subgroup. This question specifically taught me how to examine matrix subgroups and some of the manipulation you can do with them to effectively move form N to Z, which was a critical mistake in my original proof. This helped me to exactly understand why cyclic groups behave in the way that they do and gave me a blueprint for future problems involving this style of question. This difficulty underscored the importance of meticulousness in mathematical reasoning and the need to carefully verify each step in a proof, especially when familiar properties (like negatives) do not hold. I also learned to be more patient and methodical in my approach, breaking down the problem into smaller, more manageable parts to gain a clearer understanding.

Proof 3:

I selected this problem because it illustrates a fundamental concept in group theory, the preservation of group structure under isomorphisms. Isomorphisms are crucial in algebra as they provide a way to study groups by examining their structural similarities rather than their specific elements. Understanding how subgroups transform under isomorphisms is essential for grasping more advanced topics we leaned further in the class like quotient groups and group actions. Additionally, this problem requires a clear understanding of subgroup criterion, which is a massively important topic in Algebra.

In solving this problem, I initially made an error by incorrectly considering the possibility that ϕ could be an empty map, which is not valid since an isomorphism is bijective and thus cannot be empty. This mistake highlighted the importance of carefully reviewing the definitions and properties of mathematical objects being discussed. It reinforced the necessity of checking the fundamental properties of the functions and sets involved, especially when dealing with isomorphisms that preserve structure and bijection.

Proof 4:

Interestingly, I found this problem relatively straightforward and did not have to revisit or revise my understanding of the concept. The solution immediately made sense, which was reassuring and indicated that I had a solid grasp of the underlying principles of quotient groups.

Solving this problem reinforced my understanding of the concept of abelian groups and their coset representatives and their preservation under quotient operations (Coset addition and multiplication). From a problem-solving perspective, this experience highlighted the value of being comfortable with core concepts, as it allows for quicker and more confident handling of related problems.

In terms of communication, being able to immediately grasp and articulate the proof helped in explaining the concept clearly and concisely. It was a good exercise in conveying a solution in a straightforward manner without overcomplicating it, which is essential for effective mathematical communication. This experience reaffirmed the importance of a strong foundation in basic concepts, which facilitates tackling more complex topics with ease.

Proof 5:

I found this problem relatively straightforward, as it directly applies to the First Isomorphism Theorem. However, the problem was valuable in reinforcing my understanding of how to construct and manipulate homomorphisms to leverage key theorems effectively. The proof involves defining a natural surjective homomorphism from G/N to G/M and showing that its kernel is M/N. This construction showcases the elegance and utility of the First Isomorphism Theorem, which allows us to relate different quotient structures through homomorphisms.

In terms of communication skills, this problem provided an opportunity to clearly articulate the steps involved in applying the First Isomorphism Theorem. This includes defining the appropriate homomorphism, identifying its image, identifying the kernel, and using these to establish the isomorphism between quotient groups. Overall, this problem solidified my understanding of quotient groups, normal subgroups, and the application of the First Isomorphism Theorem.

Proof 6:

This problem was particularly challenging because it required identifying an appropriate group action, which took considerable time to figure out (until like one day before the submission for this!). The ambiguity in choosing the right subgroup and understanding the nature of the group action meant that I had to think deeply about the structure of the group rather than just focusing on individual elements. The key insight was to use the group action of G on the left cosets of a subgroup of order 3 (which must exist by the Sylow theorems), and then show that this action is transitive and mimics the action on its elements.

Moreover, this problem was a valuable exercise in constructing and analyzing group actions, something that I am still not very comfortable with using. It undoubtedly helped me grasp a better understanding of actions and helped me understand why we choose specific bijections.

Lastly, this question was a problem that I have repeatedly missed/mischaracterized as I did not understand the requirements of the problems, nor the tools required to solve it (I used the relation representation the first time I did the proof). I am happy that there was something so challenging and that it did force me to think outside the box.

Proof 7:

Breaking down the proof into verifying each part of the ideal definition helped in organizing thoughts and ensuring that no step was overlooked. This approach is essential in mathematics, where rigor and attention to detail are crucial for constructing valid proofs.

From a communication perspective, the problem was a good exercise in clearly articulating the reasoning behind each step, particularly in explaining why certain elements belong to the annihilator and how the ideal properties are satisfied. This reinforced the need for precision and clarity in mathematical writing, ensuring that the argument is accessible and logically coherent.

This question also took me back to the section of the textbook regarding stabilizers and normalizers in rings instead of groups. (While I do not want to admit it this was one of the parts of the book I was very confident about so I did not read it very thoroughly, however this question making me revisit the section helped me solve the proof as the text book examples gave direct insight on how to solve this question.)

Proof 8:

The Chinese Remainder Theorem is a powerful tool that not only simplifies computations in number theory but also provides deep insights into the structure of rings and modules. Understanding this theorem in the context of rings, especially how the isomorphism works between different quotient structures was a key part to solving this problem. This problem was particularly challenging, especially in proving the sujectivity of the map involved.

Initially, my proof was incorrect, which required me to revisit the concept and correct my approach. This experience underscored the importance of precision and attention to detail in mathematical proofs as I completely misrepresented R as having a 1, particularly when establishing the onto-proof. Proving surjectivity, in this case, required carefully constructing the right map and verifying that every element in the quotient ring was mapped by some representative. I also learned about partial/quotient maps from researching this question which proved to be an immensely useful tool in solving this problem.

Proof 9:

Solving this problem required a thorough understanding of both the ascending chain condition and the concept of finitely generated ideals. Both of these were new concepts to me when I was starting on solving this problem so it required a lot of time to grapple with the ideas represented in the problem.

This problem deepened my understanding of the relationships between different conditions on ideals and the structure of rings. It reinforced the importance of Noetherian properties in ensuring manageable and well-behaved ideal structures within rings.

This question along with Proof 8, were arguably the most difficult math questions/problems I have seen in a long time, and the prospect of solving them was both interesting but also very difficult. I think that one of the key takeaways from these 2 problems as a whole is that sometimes you need to stop and look at the issue form a different perspective.

Q1:

The problems I included are deeply interconnected through their exploration of core concepts in abstract algebra, particularly group theory and ring theory These topics are related not just by their common theme of algebraic structures but also by their emphasis on understanding the behavior of these structures under different operations and establishing basic characteristics.

The collective tasks highlight the course's main focus, which is the study of algebraic structures and their characteristics. These proofs show how to analyze and establish the basic features of these structures. Group theory and ring theory both study sets furnished with operations that obey specific axioms. They also emphasize how crucial it is to comprehend the connections between various algebraic structures, like as rings and ideals or groups and subgroups, and how functions like homomorphisms and isomorphisms may change these structures while maintaining their fundamental characteristics.

Q2:

Throughout the process of tackling these problems, I've experienced significant growth as both a problem-solver and a mathematical communicator. Engaging with challenging problems has enhanced my ability to think critically and approach problems systematically. This growth has been driven by a few key factors.

First, the diverse nature of the problems required me to adopt various strategies and methods. Initially, I tended to approach problems in a linear and formulaic manner, relying heavily on rote memorization and standard methods which I had learned from previous math classes. However, as I encountered more complex and nuanced problems, I realized the importance of flexible thinking and creativity. This pushed me to explore multiple perspectives and solutions, often leading me to discover more efficient or elegant approaches. For example, I learned to appreciate the power of visualization and diagrams in understanding and solving problems, which is a departure from my previous reliance on purely algebraic mental methods. One of the first times I used this was when visualizing the first theorem of isomorphisms in class or graphically illustrating why the second theorem of isomorphisms is held while writing the proof of it.

I also grew as a mathematical communicator through the course of the class. I recall one day when I was solving quotient fields using an irreducible polynomial. I recall using e and the polar complex form of numbers to solve a simple question. Professor Hunter showed how it was much simpler and cleaner using no outside knowledge and only applying the specific tools given could in essence build up to these much more complex ideas without ever directly dealing with those concepts. I suppose in brief this class has taught me how to be both methodical and also resourceful, but most importantly it showed me that the tools that we are shown in mathematics come from heavy mathematical machinery that can sometimes obfuscate the nature of a problem. (Don’t use machine guns to duck hunt is I guess is my biggest takeaway).