RECURRENCE THEOREMS FOR TOPOLOGICAL MARKOV CHAINS

CÉDRIC HO THANH, NATSUKI URABE, AND ICHIRO HASUO

Abstract. In the theory of probabilistic model checking, *recurrence theorems* play crucial roles in reducing an infinitary question to finitary—and thus machine-checkable—ones. They do so specifically by (1) reducing recurrence specifications—which speak about infinite future—to finding bottom strongly connected components (BSCCs), much like the lasso-based algorithm for non-emptiness of Büchi automata, and (2) reducing (quantitative) computation of some probabilities to the (qualitative) graph-theoretic question of strong connectedness. In this paper, we present *infinitary extensions* of those recurrence theorems, ones that apply to Markov chains with infinite (and even continuous) state spaces. The extension calls for careful generalization of the original finitary arguments to *topological* ones. We conduct the extension, imposing Polishness and compactness as key conditions, and using *upper semicontinuity* as an important technical notion in our proofs.

1. Introduction

1.1. Model checking, qualitative and quantitative. The theory of *model checking* [CGP99] is one of the major successes of theoretical computer science. In principle, it features the combination of two different modeling formalisms, namely (inductive) logical specifications and (coinductive) automata-theoretic modeling. Practically, the theory allows one to reduce the logical problem of satisfaction of a temporal formula to an automata-theoretic problem, the latter being handled by efficient algorithms. Model checking is therefore a powerful methodology in *automated formal verification*, whose importance is ever-growing in the modern world. It has resulted in a number of real-world verification tools, including SPIN [Hol19], mCRL2 [GM14, BGK+19], and NuSMV [CCG+02].

The theory of model checking was initially developed in qualitative settings, where target systems are typically nondeterministic, and the satisfaction of a specification is boolean. Its success, however, made the community turn to its *quantitative* extension—extension to settings where continuous values are involved, e.g. transition probabilities, continuous time, satisfaction probabilities, rewards, costs, etc.

It should be noted that a straightforward formalization of quantitative model checking leads to an infinitary problem—dealing with the continuum of real numbers, the set of "configurations" is continuous as well—that is not subject to automated exhaustive search. This is for example the case with model checking of hybrid automata [Hen96, ACH+95], where reachability is undecidable even if a hybrid automaton has a finite set of control states.

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Nevertheless, the community has found a few special classes of quantitative model checking problems that can be effectively decided. Among them is model checking of *timed automata*, where the continuous set of clock values allows finite partitioning by the "region" and "zone" constructions. See e.g. [BY04, And19].

1.2. **Probabilistic model checking.** Another effective setting of quantitative model checking—one of interest in this paper—is *probabilistic model checking*. Here, target systems (such as Markov chains and Markov decision processes) exhibit probabilistic branching, and the satisfaction of (at least some) specifications is given by a real number (such as the satisfaction probability).

In the theory of probabilistic model checking, the crucial fact is that reachability in many types of probabilistic systems can be computed efficiently by linear programming, as long as the systems have finitely many states. Suitable temporal logics have been devised for the purpose of specification. Furthermore, several mathematical results have been developed that allow reduction of the satisfaction of those temporal formulas to computation of reachability probabilities and graph-theoretic properties of the underlying graphs. See [BK08, chapter 10] for a comprehensive account on the theory.

In this paper, we take interest in one of those "reduction" results, namely *recurrence* theorems.

1.3. **A lasso-based algorithm for qualitative model checking.** Before we discuss the probabilistic recurrence results, we briefly discuss a lasso-based algorithm for (qualitative, non-probabilistic) model checking. It sets the right context for interpreting the probabilistic recurrence results discussed in section 1.4.

Recurrence refers to the event that a certain property ϕ becomes true infinitely often over time. In the usual syntax of linear time logic (LTL), recurrence is expressed as $\Box \Diamond \phi$ or $\mathbf{G}\,\mathbf{F}\,\phi$, meaning that at any moment in the future, there is another moment in its future where ϕ holds. Note that recurrence speaks about infinity: the $\Box$ modality refers to the (whole, unbounded) future, and the $\Diamond$ modality poses no finite bound whatsoever.

In classical qualitative model checking, e.g. [Var96], recurrence is checked in the following manner:

(1) a temporal formula, e.g. a recurrence formula $\Box \Diamond \phi$, gets translated into a *Büchi automaton*;
(2) this automaton is combined with a system model, also expressed as an automaton;
(3) the combination (a *product automaton*) is checked for emptiness.

The last problem is decidable by exhaustive search for a *lasso*, i.e. a shape in a graph with a lead and a loop. The finiteness of the relevant automata is essential here, since it justifies the search for a lasso via the infinitary pigeonhole principle. This way, we can reduce the truth of (an infinitary property of) recurrence to (a finite shape of) a lasso.

1.4. **Finitary recurrence in probabilistic model checking.** A counterpart for the above lasso-based reduction in probabilistic model checking is the following results.

Theorem (Finitary weak recurrence [BK08, corollary 10.30])**.** *Let* (X, γ) *be a finite Markov chain, and* $U \subseteq X$ *be such that* $X \vDash \mathbb{P}(\Diamond U) > 0$*, i.e. the probability* *of eventually reaching U is non-zero starting from everywhere in X. Then $X \vDash \mathbb{P}(\Box \Diamond U) = 1$.*

Theorem (Finitary strong recurrence [BK08, corollary 10.30 and 10.33])**.** *Let (X, γ) be a finite Markov chain that is strongly connected, and $U \subseteq X$ non-empty. Then $X \vDash \mathbb{P}(\Box \Diamond U) = 1$.*

Corollary. *Let (X, γ) be a finite Markov chain, $x \in X$ be a state, $Y \subseteq X$ form a bottom strongly connected component (BSCC), and $U \subseteq Y$ be non-empty. If $x \vDash \mathbb{P}(\Diamond Y) = r$ for $r \in [0,1]$, then $x \vDash \mathbb{P}(\Box \Diamond U) = r$.*

Weak recurrence states that, if U is reachable from every state, then it is almost surely visited infinitely often, starting from any state. Strong recurrence states that, if the Markov chain (X, γ) is strongly connected—meaning that every state is reachable from any other state with a non-zero probability—then a nonempty U is almost surely visited infinitely often.

Finally, the corollary is what enables efficient probabilistic model checking. It enables the computation of the probability of a recurrence formula $\Box \Diamond U$, by separating the problem into (1) finding a BSCC Y, and (2) computing the reachability probability to Y. Compared to the qualitative lasso-based algorithm in section 1.3, (1) the BSCC Y corresponds to the loop part of a lasso in section 1.3, and (2) the reachability probability to Y corresponds to the lead part of a lasso.

These results reduce infinite to finite, in the following two ways. Firstly, the infinitary specification of recurrence is reduced to a BSCC, exploiting the finiteness of the state space X, much like in section 1.3. Secondly, the quantitative question of reachability probability is partially reduced to the *qualitative* question of finding a BSCC—note that a BSCC is a qualitative notion that can be found by examining the underlying graph of the Markov chain (where there is an edge if the transition probability is non-zero). The only quantitative question is to compute $\mathbb{P}(\Diamond Y)$, which can be done efficiently by linear programming.

1.5. **Our contribution: infinitary recurrence via topology.** In this paper, we study infinitary extensions of the recurrence results above, that is, when the Markov chain in question can have an infinite (possibly even continuous) state space.

We found that naive generalizations of the finitary statements are not true, see remark 6.11 for a simple counterexample. Moreover, graph-theoretic analysis is no longer possible. For example, if the state space is the unit interval $[0,1]$, and if the transition distribution from any state is the uniform distribution on $[0,1]$, then the probability to transition from any state to any other is 0. Yet this Markov chain is not trivial.

In this paper, we show that suitable topological machineries can lift the finitary results of section 1.4 to the infinitary setting. The usefulness of topology is hardly surprising: topology is a mathematical language about size, neighborhood, and observability, that is used in numerous branches of mathematics. In theoretical computer science specifically, the use of topology to describe infinity in finitary terms is standard [Joh92, Vic96]. In the study of probabilistic systems too, topology have been commonly used [Dob07, DGJP04, DEP02]. Nevertheless, we believe that the topological theory developed in the paper is non-trivial, novel, and of potential practical importance.

The statements that we will prove as the counterpart to section 1.4 are as follows.

Theorem (Topological weak recurrence)**.** *Let* (X, γ) *be a (not necessarily finite) topological Markov chain. If* X *is compact and* $U \subseteq X$ *is an open set such that* $X \vDash \mathbb{P}(\Diamond U) > 0$*, then* $X \vDash \mathbb{P}(\Box \Diamond U) = 1$*.*

Theorem (Topological strong recurrence)**.** *Let* (X, γ) *be a (not necessarily finite) topological Markov chain. If* X *is compact and* (X, γ) *is irreducible, then for all measurable sets* $U \subseteq X$ *with non-empty interior, we have* $X \vDash \mathbb{P}(\Box \Diamond U) = 1$*.*

Corollary. *Let* (X, γ) *be a be a (not necessarily finite) topological Markov chain,* $x \in X$ *be a state,* $Y \subseteq X$ *be an irreducible subchain of* (X, γ)*, and* $U \subseteq Y$ *be a measurable set with non-empty interior. If* $x \vDash \mathbb{P}(\Diamond Y) = r$ *for* $r \in [0, 1]$*, then* $x \vDash \mathbb{P}(\Box \Diamond U) = r$*.*

We note that the statements involve many topological conditions. Some of them come quite naturally. For example, the definition of topological Markov chains requires the underlying state space X to be a *Polish space* [Kec95]. The use of such spaces is standard in the study of probabilistic systems, see e.g. [Dob07, DGJP04, DEP02]. The assumption of *compactness* of X is crucial: it represents "topological finiteness" and allows reasoning that is somewhat similar to the finitary case.

The theory involved in the proof of these theorems is not trivial. We encountered subtle issues about the topology of the Giry monad and the continuity of certain measure extension operators. In fact, said continuity seems to fail in general, which made our efforts much more difficult. Our solution is the use of a suitable compromise, namely *upper semicontinuity* [Bou98, definition IV.6.2.1].

1.6. **Organization of the paper.** We begin by recalling elements of the theory of measurable spaces, Polish spaces, and the Giry monad Δ in section 2. In sections 3 and 4, we introduce two constructions that are foundational to this work, namely the path space $X^{\odot n}$ of a Markov chain (X, γ), and the probability extension operator $\mathrm{ext}_n : \Delta X \longrightarrow \Delta X^{\odot n}$. In section 5, we make use of this topological framework to define a semantics for linear temporal logic (LTL) and probabilistic computation tree logic (PCTL). With all these preparations in place, we develop the topological recurrence theorems in section 6. We conclude in section 7.

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2. Preliminaries

We recall elements of measure theory and descriptive set theory. Refer to [Hal74, Tay06, Kle13, Kec95] for more comprehensive accounts.

2.1. **Measurable spaces.** A *measurable space* $X = (X, \Sigma)$ consists of a set X and a *σ-algebra* Σ on X, which is a collection of subsets of X such that (1) $X \in \Sigma$; (2) Σ is closed under complementation; (3) Σ is closed under countable unions. Elements of Σ are called *measurable* subsets. If (X', Σ') is another measurable space, then a set-map $f : X \longrightarrow X'$ is *measurable* if f^{-1} maps measurable subsets of X' to measurable subsets of X. Let $\mathcal{M}\mathrm{eas}$ be the category of measurable spaces and measurable maps.

A *measure* on X is a map $\mu : \Sigma \longrightarrow [0, +\infty]$ such that (1) $\mu(\varnothing) = 0$; (2) μ is *countably additive*: if $E_1, \ldots \in \Sigma$ are pairwise disjoint, then $\mu\left(\sum_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} \mu(E_n)$. A measure μ is *finite* if $\mu(X) < \infty$. It is a *probability measure* (or *probability distribution*) if $\mu(X) = 1$.

Theorem 2.1 (Carathéodory, [Kle13, theorem 1.41])**.** *Let* $\Gamma \subseteq 2^X$ *be a* ring of sets*, i.e. a set of subsets of* X *such that (1)* $\varnothing \in \Gamma$*; (2)* Γ *is closed under binary unions; (3)* Γ *is closed under subtractions, i.e. if* $A, B \in \Gamma$*, then* $A - B \in \Gamma$*. Let* Σ *be the smallest* σ*-algebra containing* Γ*. If* $\mu : \Gamma \longrightarrow [0, +\infty]$ *is a map satisfying the same conditions as a probability measure, then it extends uniquely into an actual probability measure* $\mu : \Sigma \longrightarrow [0, 1]$*.*

2.2. **Integration.** Let $X = (X, \Sigma)$ be a measurable space and μ be a measure on X. For $E \subseteq X$, let $\chi_E : X \longrightarrow \{0, 1\}$ be the characteristic map of E. A *simple map* is a map $g : X \longrightarrow \mathbb{R}$ of the form $g = \sum_{i=1}^n r_i \chi_{E_i}$, where $r_1, \ldots, r_n \in \mathbb{R} - \{0\}$, and $E_1, \ldots, E_n \in \Sigma$ are pairwise disjoint. The *integral* of such a g is $\int g \, \mathrm{d}\mu := \sum_i r_i \mu(E_i)$. Let now $f : X \longrightarrow \mathbb{R}$ be a measurable positive map, and define

$$\int f \, \mathrm{d}\mu := \sup \left\{ \int g \, \mathrm{d}\mu \mid g \leq f, \ g \text{ simple positive} \right\}$$

Finally, if $f : X \longrightarrow \mathbb{R}$ is a measurable map, write $f = f_+ - f_-$, where $f_+ := \min(0, f)$ and $f_- := \max(0, f)$, and let $\int f \, \mathrm{d}\mu := \int f_+ \, \mathrm{d}\mu - \int f_- \, \mathrm{d}\mu$. If $A \in \Sigma$, we write $\int_A f \, \mathrm{d}\mu$ as a shorthand for $\int \chi_A f \, \mathrm{d}\mu$. In the sequel, it will sometimes be convenient to use the verbose notation $\int_{x \in A} f(x) \mu(\mathrm{d}x)$ for $\int_A f \, \mathrm{d}\mu$.

2.3. **Borel σ-algebra.** We write $\mathcal{T}\mathrm{op}$ for the category of topological spaces and continuous maps. Let $X = (X, \mathcal{T}) \in \mathcal{T}\mathrm{op}$. The *Borel* σ*-algebra* $B(X)$ of X is the σ-algebra generated by $\mathcal{T}$. In the sequel, all topological spaces will be implicitely considered as measurable spaces in this way. Note that continuous maps are measurable.

If μ is a measure on X, then its *support* is

$$\operatorname{supp} \mu := \{ x \in X \mid \forall U \in \mathcal{T}, \ x \in U \implies \mu(U) > 0 \} .$$

It is a closed set since its complement is the union of all open sets of measure 0. In particular, $\operatorname{supp} \mu$ is measurable, and if μ is a probability measure, $\mu(\operatorname{supp} \mu) = 1$.

2.4. **Polish spaces.** A topological space $X = (X, \mathcal{T})$ is a *Polish space* if it is (1) *separable*: it admits a countable dense subset; (2) *completely metrizable*: there exists a complete metric on X that generates $\mathcal{T}$. Let $\mathcal{P}\mathrm{ol}$ be the full subcategory of $\mathcal{T}\mathrm{op}$ spanned by Polish spaces.

Theorem 2.2 ([Kec95, theorem 4.14])**.** *A topological space is a Polish space if and only if it is homeomorphic to a* G_δ*-set (i.e. a countable intersection of open sets) of the* Hilbert cube $[0, 1]^\infty$*. In particular, a* G_δ*-set of a Polish space is itself a Polish space.*[1]

Lemma 2.3 ([Wil70, problem 15.C])**.** *In a metrizable space, every closed set is* G_δ*.*

The following fundamental result is expected, but we were not able to find a complete demonstration in the literature. We include a proof for reference.

Proposition 2.4. *The category* $\mathcal{P}\mathrm{ol}$ *has all countable limits.*

Proof. It is well-known that $\mathcal{P}\mathrm{ol}$ has countable products, see e.g. [Kle13, theorem 14.8]. We now show that it has equalizers. Let $f, g : X \longrightarrow Y$ be two morphisms in $\mathcal{P}\mathrm{ol}$, and let $k : K \longrightarrow X$ be the equalizer of f and g computed in $\mathcal{T}\mathrm{op}$. In other

[1] In fact, this is also a necessary condition, see [Kec95, theorem 3.11].

words, $K = \{x \in X \mid f(x) = g(x)\}$ with the subspace topology. We show that K is a Polish space.

For $r > 0$, consider the infinite product $H_r := [-r, r]^\infty$ with the product topology. It is homeomorphic to the Hilbert cube, and by theorem 2.2, there exists an embedding $i : Y \longrightarrow H_1$. For $j : H_1 \longrightarrow H_2$ the natural inclusion, define $k := jif - jig : X \longrightarrow H_2$, and note that $K = k^{-1}((0, 0, \ldots))$. This show that K is closed in X, and by lemma 2.3 and theorem 2.2 again, it is a Polish space. $\square$

2.5. **The Giry monad.** If X is a topological space, we write $C^b(X)$ (resp. $M^b(X)$) for the set of all bounded continuous (resp. bounded measurable) maps $X \longrightarrow \mathbb{R}$.

Definition 2.5. The *Giry monad* [Gir82, section I.2] $\Delta : \mathcal{P}\mathrm{ol} \longrightarrow \mathcal{P}\mathrm{ol}$ maps a Polish space X to the set of all probability distributions over X, endowed with the coarsest topology $\mathcal{T}_\Delta$ such that for all $f \in C^b(X)$, the map

$$I_f := \int f \,\mathrm{d}(-) : \Delta X \longrightarrow \mathbb{R} \tag{2.6}$$

is continuous. On morphisms, Δ maps $f : X \longrightarrow Y$ to $f_* : \Delta X \longrightarrow \Delta Y$, where for $\mu \in \Delta X$ and $E \in B(Y)$, $f_*\mu(E) := \mu(f^{-1}(E))$.

The monad unit $\delta : X \longrightarrow \Delta X$ maps $x \in X$ to its *Dirac distribution* δ_x, given by $\delta_x(E) = 1$ if $x \in E$ and 0 otherwise.

The monad multiplication $\Delta\Delta X \longrightarrow \Delta X$ is defined naturally but is not needed in this paper—see e.g. [Gir82, theorem 1].

The topology $\mathcal{T}_\Delta$ of ΔX is the *weakest topology with respect to integration of continuous bounded maps*. We shall reuse this formulation in the sequel. It is easy to see that $\mathcal{T}_\Delta$ is generated by sets of the form

$$\beta^{\bowtie p}(f) := \{\mu \in \Delta X \mid I_f(\mu) \bowtie p\},$$

where $\bowtie \in \{>, <\}$, $p \in \mathbb{R}$, and $f \in C^b(X)$. We extend the notation $\beta^{\bowtie p}(f)$ to $\bowtie \in \{\geq, \leq\}$ and any measurable map $f : \Delta X \longrightarrow \mathbb{R}$ in the obvious way. If $f = \chi_E$ for some $E \in B(X)$, then we write $\beta^{\bowtie p}(E)$ as a shorthand for $\beta^{\bowtie p}(\chi_E)$.

Remark 2.7. The Giry monad also has a version for the category $\mathcal{M}\mathrm{eas}$ [Gir82, section I.2], which we denote by $\Delta' : \mathcal{M}\mathrm{eas} \longrightarrow \mathcal{M}\mathrm{eas}$. Here, if $M \in \mathcal{M}\mathrm{eas}$, then the σ-algebra on $\Delta' M$ is the smallest such that for all $f \in M^b(X)$, the map $I_f : \Delta' X \longrightarrow \mathbb{R}$ from (2.6) is measurable. In other words, it is the smallest σ-algebra with respect to integration of measurable bounded maps

With this in mind, and coming back to Polish spaces, one could think that a more natural topology on ΔX than in definition 2.5 would be the weakest w.r.t. the integration of *measurable* (instead of continuous) bounded maps. Unfortunately, this topology, which we shall denote by $\mathcal{T}_{\mathrm{wrong}}$, is usually not desirable. For example the Dirac map $\delta : X \longrightarrow (\Delta X, \mathcal{T}_{\mathrm{wrong}})$ is continuous only when X is discrete. To see this, take $f \in M^b(X)$. Since $f(x) = \int f \,\mathrm{d}\delta_x$ for all $x \in X$, the following diagram commutes.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\delta"'] & & \mathbb{R} \\
& (\Delta X, \mathcal{T}_{\mathrm{wrong}}) \arrow[ur, "I_f"'] &
\end{tikzcd}$$

So assuming that δ is continuous, X has the property that every measurable bounded map $X \longrightarrow \mathbb{R}$ is continuous. In particular, any characteristic map is continuous, forcing X to be discrete...

Still, the following result shows that the Borel σ-algebra of the wrong topology $\mathcal{T}_{\text{wrong}}$, the right topology $\mathcal{T}_\Delta$ of definition 2.5, and the σ-algebra on $\Delta'(X, B(X))$, where $\Delta' : \mathcal{M}\text{eas} \longrightarrow \mathcal{M}\text{eas}$ is the Giry monad on $\mathcal{M}\text{eas}$ [Gir82, section I.2], all coincide. In the sequel, we shall not consider $\mathcal{T}_{\text{wrong}}$ any further.

Proposition 2.8 ([Kec95, theorem 17.24], [Dob07, proposition 1.80])**.** *The Borel σ-algebra $B(\Delta X)$ is the smallest σ-algebra w.r.t. the integration of bounded measurable maps. In other words, the following square commutes:*

$$\begin{tikzcd} \mathcal{P}\text{ol} \arrow[r, "\Delta"] \arrow[d] & \mathcal{P}\text{ol} \arrow[d] \\ \mathcal{M}\text{eas} \arrow[r, "\Delta'"] & \mathcal{M}\text{eas} \end{tikzcd}$$

where the vertical arrows are the forgetful functors. Incidentally, $B(\Delta X)$ is also the smallest w.r.t. evaluation of measures on measurable sets.

Theorem 2.9 ("Portmanteau", [Kec95, theorem 17.24], [Dob07, proposition 1.66])**.** *For $\mu, \mu_0, \mu_1, \mu_2, \ldots \in \Delta X$, the following are equivalent:*

(1) the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to μ;
(2) for all $f \in C^b(X)$, we have $\lim_n \int f \,\mathrm{d}\mu_n = \int f \,\mathrm{d}\mu$;
(3) for all open $U \subseteq X$, we have $\liminf_n \mu_n(U) \geq \mu(U)$;
(4) for all closed $F \subseteq X$, we have $\limsup_n \mu_n(F) \leq \mu(F)$.

2.6. **Markov chains.** As we announced earlier, we work on Markov chains in the setting of Polish spaces. The usual definition in terms of stochastic kernels can be simply presented in the following categorical terms.

Definition 2.10 (Markov chain)**.** A *Markov chain* $X = (X, \gamma)$ is the datum of a Polish space X together with a morphism $\gamma : X \longrightarrow \Delta X$ in $\mathcal{P}\text{ol}$. In other words, a Markov chain is simply a Δ-coalgebra. Elements of X are called *states*, and γ is called the *transition kernel*.

For $x, y \in X$ and $E \in B(X)$, we write $\gamma(x, E)$ instead of $\gamma(x)(E)$ and $\gamma(x, y)$ instead of $\gamma(x, \{y\})$.[2]

3. Path spaces

Let $X = (X, \gamma)$ be a Markov chain. In this section, we define a Polish subspace $X^{\odot n} \subseteq X^n$ of *paths of length* n in X, where $n \leq \infty$. Elements of $X^{\odot n}$ are sequences of states that can result from a random walk in X.

In what follows, we study the topological and measurable structures of the path spaces $X^{\odot n}$ (where $n \leq \infty$). We do so because (1) the finitary theory in [BK08, chapter 10] is built similarly on top of path spaces; (2) path spaces are such fundamental constructs that their topological/measurable structures are interesting in themselves. In particular, we will see in proposition 3.8 that the Borel σ-algebra on $X^{\odot\infty}$ is generated by the well-known *cylinder sets* of X, much like in [BK08, definition 10.9] and [Kle13, definition 14.9].

[2]Recall that since X is Hausdorff, every singleton is closed, thus measurable.

3.1. **Finite paths.**

Definition 3.1. We construct the space $X^{\odot n}$ of paths of length n, for $n \in \mathbb{N}$. First, let $X^{\odot 0}$ be the singleton space $*$, and $X^{\odot 1} := X$. If $n = 2$, let $X^{\odot 2} \subseteq X^2$ be the subspace of all pairs (x, y) such that $y \in \operatorname{supp}\gamma(x)$. If $n \geq 3$, let $X^{\odot n}$ be the limit (in $\mathcal{T}\mathrm{op}$) of the following diagram

$$\begin{tikzcd}[column sep=small] X^{\odot 2} \arrow[dr, "p_2"] & & X^{\odot 2} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \cdots \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & X^{\odot 2} \arrow[dl, "p_1"'] \\ & X & & X & & X & \end{tikzcd}$$

where p_1 and p_2 are the projections on the first and second component respectively, and where there are $n-1$ instances of $X^{\odot 2}$.

It is easy to see that $X^{\odot n}$ is the subspace of X^n of sequences $(x_1, \ldots, x_n)$ such that for $1 \leq i < n$, $x_{i+1} \in \operatorname{supp}\gamma(x_i)$. In particular, the topology of $X^{\odot n}$ is spanned by subsets of the form

$$U_1 \odot \cdots \odot U_n := X^{\odot n} \cap (U_1 \times \cdots \times U_n),$$

called *open sequence sets*, where $U_1, \ldots, U_n \in \mathcal{T}$.

Our topological framework is based in Polish spaces, so it is important that path space construction yields Polish spaces as well. We show this in the next two "sanity check" results.

Lemma 3.2. *The subspace $X^{\odot 2} \subseteq X^2$ is Polish.*

Proof. By definition, $(x, y) \in X^{\odot 2}$ if and only if for all open set U, $y \in U$ implies $\gamma(x, U) > 0$. In other words, for all open set U, $g_U(x, y) > 0$, where $g_U(x, y) := 1 - \delta_y(U) + \gamma(x, U)$. This gives $X^{\odot 2} := \bigcap_U g_U^{-1}(0, +\infty)$, and U can even range over a countable topological basis of X. Unfortunately, this is not enough to prove the proposition, as g_U is not continuous in general. The crux of this proof is to replace g_U by some continuous map that allows for a similar characterization of $X^{\odot 2}$.

Choose a metric compatible with the topology of X, and write $B_{x,n}$ (resp. $\bar{B}_{x,n}$) for the open (resp. closed) ball centered around x with radius $\frac{1}{n}$. Let Q be a countable dense subset of X, and recall that $\{B_{q,n} \mid q \in Q,\ n \geq 1\}$ is a topological basis of X. Using the Tietze extension theorem [Mun00, theorem 35.1], choose a continuous map $b_{q,n} : X \longrightarrow \mathbb{R}$ that is 1 on $\bar{B}_{q,2n}$ and 0 on $X - B_{q,n}$. Consider the following continuous map $f_{q,n} : X^2 \longrightarrow \mathbb{R}$:

$$f_{q,n}(x, y) := 1 - b_{q,n}(y) + \int b_{q,n} \, \mathrm{d}\gamma(x).$$

We claim that $y \in \operatorname{supp}\gamma(x)$ if and only if for all $q \in Q$ and $n \geq 1$ we have $f_{q,n}(x, y) > 0$. This implies that, $X^{\odot 2} = \bigcap_{q,n} f_{q,n}^{-1}(0, +\infty)$, which G_δ-set of X^2, and therefore a Polish space by theorem 2.2.

We now prove our claim. If $y \notin \operatorname{supp}\gamma(x)$, then since $\operatorname{supp}\gamma(x)$ is closed, there exists $q \in Q$ and $n \geq 1$ such that $y \in B_{q,2n}$ but $B_{q,n} \cap \operatorname{supp}\gamma(x) = \varnothing$.[3] In particular, $f_{q,n}(x, y) = 0$. Assume now that $y \in \operatorname{supp}\gamma(x)$, and take $q \in Q$ and $n \geq 1$. We distinguish two cases.

(1) If $b_{q,n}(y) = 0$, then $f_{q,n}(x, y) \geq 1$.

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[3] If d is the chosen metric on X and $r := \inf_{z \in \operatorname{supp}\gamma(x)} d(y, z)$ (which is necessarily > 0), choose n such that $\frac{3}{2n} < r$, and $q \in Q \cap B_{y,2n}$.

(2) If $b_{q,n}(y) > 0$, then there exists an open neighborhood U of y and $\varepsilon > 0$ such that $b_{q,n} > \varepsilon$ on U.[4] Then, $f_{q,n}(x,y) \geq \int b_{z,n}\, \mathrm{d}\gamma(x) \geq \varepsilon \times \gamma(x,U) > 0$ since $y \in \operatorname{supp} \gamma(x)$.

In both cases, $f_{q,n}(x,y) > 0$. $\square$

Proposition 3.3. *The path space $X^{\odot n}$ is Polish for all $n \in \mathbb{N}$.*

Proof. The cases $n = 0,1$ are trivial, $n = 2$ is treated by lemma 3.2, and $n \geq 3$ follows from proposition 2.4. $\square$

The next two results provide elementary insights into the Borel σ-algebra of $X^{\odot n}$.

Proposition 3.4. *The Borel σ-algebra $B(X^{\odot n})$ is generated by sets of the form $E_1 \odot \cdots \odot E_n := X^{\odot n} \cap (E_1 \times \cdots \times E_n)$, where $E_1, \ldots, E_n \in B(X)$, called* measurable sequence sets.

Proof. Open sequence sets are by definition measurable sequence sets. Conversely, note that $E_1 \odot \cdots \odot E_n = \bigcap_{i=1}^n X \odot \cdots \odot E_i \odot \cdots \odot X$, and clearly, the sets in this intersection are measurable. $\square$

Proposition 3.5. *Let S_X be the set of all finite unions of measurable sequence sets of $X^{\odot n}$. Then S_X forms a ring of sets.*

Proof. Let us use MSS as an acronym for "measurable sequence set". Trivially, S_X contains $\varnothing$ and is closed under binary unions. We now prove that S_X is closed under subtractions. First, note that the intersection of two MSSs is again an MSS:

$$(E_1 \odot \cdots \odot E_n) \cap (F_1 \odot \cdots \odot F_n) = (E_1 \cap F_1) \odot \cdots \odot (E_n \cap F_n),$$

where $E_i, F_i \in B(X)$. Since elements of S_X are finite unions of MSSs, it follows that S_X is closed under finite intersections. From here, to show that S_X is closed under subtractions, it is enough to show that it is closed under complementation. Observe that for $E_1, \ldots, E_n \in B(X)$, we have

$$X^{\odot n} - (E_1 \odot \cdots \odot E_n) = \bigcup_J E_{1,J} \odot \cdots \odot E_{n,J},$$

where J ranges over the non-empty subsets of $\{1, \ldots, n\}$, and where $E_{i,J} := X - E_i$ if $i \in J$, or E_i otherwise. Therefore, the complement of an MSS is in S_X. Now, let $S \in S_X$, and write it as a finite union of MSSs, say $S = \bigcup_i E_{i,1} \odot \cdots \odot E_{i,n}$. We have $X^{\odot n} - S = \bigcap_i X^{\odot n} - (E_{i,1} \odot \cdots \odot E_{i,n})$, which by the previous observation is a finite intersection of elements of S_X. Since S_X is closed under finite intersections, we conclude that $X^{\odot n} - S \in S_X$. $\square$

3.2. **Infinite paths.** Building on the finitary case, we now define the space $X^{\odot\infty}$ of infinite paths (or infinite random walks) in X. There is an obvious projection $X^{\odot n} \longrightarrow X^{\odot(n-1)}$ that "pops" the last state of a path.

Definition 3.6. Let $X^{\odot\infty}$ be the limit (in $\mathcal{P}$ol, see proposition 2.4) of the following diagram:

$$\cdots \longrightarrow X^{\odot n} \longrightarrow X^{\odot(n-1)} \longrightarrow \cdots \longrightarrow X^{\odot 2} \longrightarrow X.$$

[4] e.g. $\varepsilon = \frac{b_{q,n}(y)}{2}$ and $U = b_{q,n}^{-1}(\varepsilon, +\infty)$.

It is easy to see that $X^{\odot\infty}$ is the subspace of X^∞ of all sequences $(x_i)_{i\in\mathbb{N}} \in X^\infty$ such that for all $i \in \mathbb{N}$ we have $x_{i+1} \in \operatorname{supp}\gamma(x_i)$. In particular, the limit topology is spanned by subsets of the form

$$\begin{aligned}\operatorname{Cyl}(U_1,\dots,U_n) &\coloneqq U_1 \odot \cdots \odot U_n \odot X^{\odot\infty}\\ &= X^{\odot\infty} \cap (U_1 \times \cdots \times U_n \times X^\infty),\end{aligned}$$

called *open cylinder sets*, where $n \in \mathbb{N}$ and $U_1,\dots,U_n \in \mathcal{T}$.

Example 3.7. If X is discrete, then $X^{\odot\infty}$ matches the definition of $\operatorname{Path}(X)$ of [BK08, defintion 3.6]. Further, the limit topology is spanned by sets of the form

$$\operatorname{Cyl}(\{x_0\},\dots,\{x_{n-1}\}) = \left\{(y_i)_{i\in\mathbb{N}} \in X^{\odot\infty} \mid y_j = x_j,\ j < n\right\}$$

where $(x_0,\dots,x_{n-1}) \in X^{\odot n}$ and $n \in \mathbb{N}$, which are precisely the cylinder sets of [BK08, definition 10.9].

Proposition 3.8. *The Borel σ-algebra $B(X^{\odot\infty})$ is generated by sets of the form $\operatorname{Cyl}(E_1,\dots,E_n) \coloneqq E_1 \odot\cdots\odot E_n \odot X^{\odot\infty}$, where $E_1,\dots,E_n \in B(X)$, called* measurable cylinder sets*.*

Proof. Similar to the proof of proposition 3.4. □

The next two results extend the insights of propositions 3.4 and 3.5 regarding the Borel σ-algebra of $X^{\odot n}$ to that of $X^{\odot\infty}$.

Proposition 3.9. *Let C_X be the set of all finite unions of measurable cylinder sets. Then C_X forms a ring of sets.*

Proof. This proof is similar to that of proposition 3.5. Let us use MCS as a acronym for "measurable cylinder set". Trivially, C_X contains $\varnothing$ and is closed under binary unions. We now prove that C_X is closed under subtractions. First, note that the intersection of two MCSs is again an MCS. Indeed, if $E_1,\dots,E_m,F_1,\dots,F_n \in B(X)$, then without loss of generality, we may assume $m \le n$, and

$$\begin{aligned}&\operatorname{Cyl}(E_1,\dots,E_m) \cap \operatorname{Cyl}(F_1,\dots,F_n)\\ &= \operatorname{Cyl}((E_1 \cap F_1),\dots,(E_m \cap F_m),F_{m+1},\dots,F_n).\end{aligned}$$

Since elements of C_X are finite unions of MCSs, it follows that C_X is closed under finite intersections. From here, to show that C_X is closed under subtractions, it is enough to show that it is closed under complementation. Observe that for $E_1,\dots,E_n \in B(X)$, we have

$$X^{\odot\infty} - \operatorname{Cyl}(E_1,\dots,E_m) = \bigcup_J \operatorname{Cyl}(E_{1,J},\dots,E_{m,J}),$$

where J ranges over the non-empty subsets of $\{1,\dots,m\}$, and where $E_{i,J} \coloneqq X - E_i$ if $i \in J$, or E_i otherwise. Therefore, the complement of an MCS is an element of C_X. Now, let $C \in C_X$, and write it as a finite union of MCSs, say $C = \bigcup_i \operatorname{Cyl}(E_{i,1},\dots,E_{i,m_i})$. Then, we have $X^{\odot\infty} - C = \bigcap_i X^{\odot\infty} - \operatorname{Cyl}(E_{i,1},\dots,E_{i,m_i})$, which by the previous observation is a finite intersection of elements of C_X. Since C_X is closed under finite intersections, we conclude that $X^{\odot\infty} - C \in C_X$. □

Lemma 3.10. *Let $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$. If $E \in B(X^{\odot m})$ and $F \in B(X^{\odot n})$, then $E \odot F \in B(X^{\odot(m+n)})$.*

Proof. By definition, we have $X^{\odot(m+n)} \subseteq X^{\odot m} \times X^{\odot n} \subseteq X^{m+n}$. Since $E \times F \in B(X^{\odot m} \times X^{\odot n})$, there exists $K \in B(X^{m+n})$ such that $E \times F = K \cap (X^{\odot m} \times X^{\odot n})$. Finally, $E \odot F = K \cap X^{\odot(m+n)} \in B(X^{\odot(m+n)})$. □

4. Extension of probability measures

Let $X = (X, \gamma)$ be a Markov chain. In this section, we give a sense to the idea of "probability to follow a given path" in X. In formal terms, for $\mu \in \Delta X$ (which acts as an initial distribution), we define a probability distribution $\operatorname{ext}_n \mu$ on $X^{\odot n}$, where $n \leq \infty$.

4.1. **Finitary extension.**

Definition 4.1. We now define the *finitary extension operator* $\operatorname{ext}^\gamma_n : \Delta X \longrightarrow \Delta X^{\odot n}$. Take $\mu \in \Delta X$. First, $\operatorname{ext}^\gamma_0 \mu$ is the unique distribution on the singleton space $X^{\odot 0}$, while $\operatorname{ext}^\gamma_1 \mu \coloneqq \mu$. For $n \geq 2$ and $E_1, \dots, E_n \in B(X)$, the probability $\operatorname{ext}^\gamma_n \mu\,(E_1 \times \cdots \times E_n)$ is given by

$$\int\!\!\cdots\!\!\int_{x_i \in E_i} \mu(\mathrm{d}x_1) \times \gamma(x_1, \mathrm{d}x_2) \times \cdots \times \gamma(x_{n-2}, \mathrm{d}x_{n-1}) \times \gamma(x_{n-1}, E_n).$$

Using Carathéorody's theorem 2.1, this gives rise to a fully fledged probability distribution on X^n. It clearly has support in $X^{\odot n}$, so we consider $\operatorname{ext}^\gamma_n \mu$ as a distribution on $X^{\odot n}$, which completes the definition.

In the sequel, we shall write ext_n instead of $\operatorname{ext}^\gamma_n$ if γ is clear from the context.

Example 4.2. If X is discrete and $E_1, \dots, E_n \in B(X)$ are finite, then

$$\operatorname{ext}_n \mu(E_1 \odot \cdots \odot E_n) = \sum_{x_i \in E_i} \mu(x_1) \times \gamma(x_1, x_2) \times \cdots \times \gamma(x_{n-1}, E_n).$$

This expresses the probability to follow a path that leads from E_1 to E_2 to E_3 to... to E_n, weighted by an initial distribution μ.

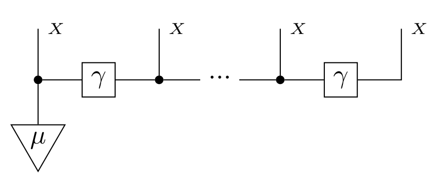
In the infinite case, this intuition still holds. In particular, the following lemma is not surprising:

Lemma 4.3. *For $n \geq 1$ and $E \in B(X^{\odot n})$, we have*

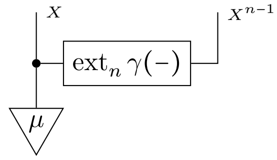
$$\begin{aligned}
\operatorname{ext}_{n+1} \mu(E \odot X) &= \operatorname{ext}_n \mu(E),\\
\operatorname{ext}_{n+1} \mu(X \odot E) &= \int_{x \in X} \mu(\mathrm{d}x) \times (\operatorname{ext}_n \gamma(x)(E)) \,.
\end{aligned}$$

Remark 4.4. The extension $\operatorname{ext}_n \mu$ admits an alternative description in terms of string diagrams, one that can be more familiar to category theorists.

Let $\mathcal{S}\mathrm{toch}$ be the Kleisli category of the Giry monad, also known as the category of *Markov kernels*. A morphism $f : X \longrightarrow Y$ can be understood as a probability distribution on Y parametrized by X. In particular, $\mathcal{S}\mathrm{toch}(*, X) = \Delta X$ as sets. We use the string diagrams of [CJ19] to describe morphisms in $\mathcal{S}\mathrm{toch}$. In this context, for $n \geq 2$, $\operatorname{ext}_n \mu : * \longrightarrow X^n$ can be drawn as



This diagram highlights the same idea as in example 4.2, namely that of sampling a state of X using μ, and then applying transition kernel γ n times, yielding a random walk of length n. Therefore, this combination samples paths of length n, a.k.a. elements of $X^{\odot n}$. Note that $\operatorname{ext}_n \mu$ can also be described as



Here, the pointwise-extended probability kernel $\operatorname{ext}_n \gamma(-)$ walks through n states of X, instead of just one for γ.

The following theorem, and its infinitary counterpart theorem 4.7, provide a crucial (yet loose) control over the way probabilities are extended.

Theorem 4.5. *The finitary probability extension operator is measurable.*

Proof. The cases $n = 0, 1$ are trivial, so we assume $n \geq 2$. Recall from proposition 3.4 that $B(X^{\odot n})$ is generated by measurable sequence sets, so by proposition 2.8, $B(\Delta X^{\odot n})$ is generated by sets of the form $\beta^{\bowtie p}(E_1 \odot \cdots \odot E_n)$, where $\bowtie \in \{>, \geq, \leq, <\}$, $p \in \mathbb{R}$, and $E_1, \ldots, E_n \in B(X)$. We simply show that the preimage of these sets under ext_n is measurable.

If $\mu \in \Delta X$, then by definition, $\operatorname{ext}_n \mu(E_1 \odot \cdots \odot E_n) = I_f(\mu)$, where

$$f(x) \coloneqq \chi_{E_1}(x) \times (\operatorname{ext}_{n-1} \gamma(x)(E_2 \odot \cdots \odot E_n)) .$$

By induction, f is measurable and bounded, so by proposition 2.8 again, $I_f : \Delta X \longrightarrow \mathbb{R}$ is measurable. Finally, the preimage $\operatorname{ext}_n^{-1} \beta^{>p}(E_1 \odot \cdots \odot E_n) = I_f^{-1}(p, +\infty)$ is measurable. The cases $\beta^{\geq p}$, $\beta^{\leq p}$ and $\beta^{<p}$ are completely analogous. □

4.2. **Infinitary extension.**

Definition 4.6. Making use of proposition 3.9, lemma 4.3, and Carathéodory's theorem 2.1, we can define a probability distribution $\operatorname{ext}_\infty^\gamma \mu$ on $X^{\odot \infty}$ by

$$\operatorname{ext}_\infty^\gamma \mu\, (\operatorname{Cyl}(E_1, \ldots, E_n)) \coloneqq \operatorname{ext}_n^\gamma \mu(E_1 \odot \cdots \odot E_n),$$

where $E_1, \ldots, E_n \in B(X)$. This gives rise to the *infinitary extension operator* $\operatorname{ext}_\infty^\gamma : \Delta X \longrightarrow \Delta X^{\odot \infty}$.

Akin to definition 4.1, we shall write $\operatorname{ext}_\infty$ instead of $\operatorname{ext}_\infty^\gamma$ if γ is clear from the context.

Theorem 4.7. *The infinitary probability extension operator is measurable.*

Proof. Recall from proposition 3.8 that $B(X^{\odot \infty})$ is generated by measurable cylinder sets. Thus, by proposition 2.8, the Borel σ-algebra $B(X^{\odot \infty})$ is generated by sets of the form $\beta^{\bowtie p}\,(\operatorname{Cyl}(E_1, \ldots, E_n))$, where $\bowtie \in \{>, \geq, \leq, <\}$, $p \in \mathbb{R}$, and $E_1, \ldots, E_n \in B(X)$. We simply show that the preimage of these sets under $\operatorname{ext}_\infty$ is measurable. We have

$$\begin{aligned}
&\operatorname{ext}_\infty^{-1} \beta^{\bowtie p}\,(\operatorname{Cyl}(E_1, \ldots, E_n)) \\
&= \{\mu \in \Delta X \mid \operatorname{ext}_\infty \mu\,(\operatorname{Cyl}(E_1, \ldots, E_n)) \bowtie p\} \\
&= \{\mu \in \Delta X \mid \operatorname{ext}_n \mu(E_1 \odot \cdots \odot E_n) \bowtie p\} \\
&= \operatorname{ext}_n^{-1} \beta^{\bowtie p}(E_1 \odot \cdots \odot E_n)
\end{aligned}$$

which is measurable by theorem 4.5. □

5. Logic

In this section, we demonstrate how two standard logics for Markov chains can be given a natural semantics in our topological framework.

The first, linear time logic (thereafter LTL) [Pnu77] [BK08, chapter 5] is used to express properties about paths, or *execution traces* of a Markov chain. As such, it appeals to its underlying graph-theoretic nature. A topological Markov chain X can also make sense of the notion of path via the spaces $X^{\odot n}$ and $X^{\odot\infty}$ constructed in section 3. LTL is necessary to define our second logic, probabilistic computation tree logic (PCTL) [HJ94] [BK08, section 10.2], which deals with the probability that, starting at some state, an execution trace satisfies a given LTL formula. It is this syntax that we shall use to formulate our recurrence theorems in section 6.

5.1. **Linear temporal logic.**

Definition 5.1. For $X = (X, \gamma)$ a Markov chain, LTL (for X) is defined by the following grammar:

$$
\begin{array}{llll}
\phi & ::= & E & E \in B(X) \\
& \mid & \top \quad \mid \quad \neg\phi \quad \mid \quad \phi \wedge \phi & \\
& \mid & \bigcirc \phi & \\
& \mid & \phi\, \mathbf{U}^{\leq n}\, \phi & n \leq \infty
\end{array}
$$

The usual logical connectives are defined in terms of $\top$, $\neg$, and $\wedge$. We write $\bigcirc^n \phi$ for $\bigcirc \cdots \bigcirc \phi$ where there are n instances of the "next" modality $\bigcirc$, and $\phi\,\mathbf{U}\,\psi$ instead of $\phi\,\mathbf{U}^{\leq\infty}\,\psi$. The "eventually" and "always" modalities are defined as $\Diamond^{\leq n} \phi \coloneqq \top\,\mathbf{U}^{\leq n}\,\phi$ and $\Box^{\leq n} \phi \coloneqq \neg \Diamond^{\leq n} \neg\phi$ respectively.

Definition 5.2. The *semantics* $[\![\phi]\!] \in B(X^{\odot\infty})$ of an LTL formula ϕ is given as follows:

1. (Atomic predicates) $[\![E]\!] \coloneqq \mathrm{Cyl}(E)$;
2. $[\![\top]\!] \coloneqq X^{\odot\infty}$; $[\![\neg\phi]\!] \coloneqq X^{\odot\infty} - [\![\phi]\!]$; $[\![\phi \wedge \psi]\!] \coloneqq [\![\phi]\!] \cap [\![\psi]\!]$;
3. (Next) $[\![\bigcirc \phi]\!] \coloneqq X \odot [\![\phi]\!]$, which is a measurable set by lemma 3.10;
4. (Until)
$$[\![\phi\,\mathbf{U}^{\leq n}\,\psi]\!] \coloneqq \bigcup_{i=0}^{n} \left([\![\bigcirc^i \psi]\!] \cap \bigcap_{j=0}^{i-1} [\![\bigcirc^j \phi]\!] \right).$$

For $n \leq \infty$, one can check that

$$[\![\Diamond^{\leq n} \phi]\!] = \bigcup_{i=0}^{n} [\![\bigcirc^i \phi]\!], \qquad [\![\Box^{\leq n} \phi]\!] = \bigcap_{i=0}^{n} [\![\bigcirc^i \phi]\!].$$

Given a probability distribution $\mu \in \Delta X$, the *probability* of an LTL formula ϕ is simply $\mu(\phi) \coloneqq \mathrm{ext}_\infty\, \mu([\![\phi]\!])$. If $x \in X$, we write $\gamma(x, \phi)$ instead of $\gamma(x)\,([\![\phi]\!])$. This abuse of notation may seem ambiguous if ϕ is an atomic predicate, but by definition, $\mathrm{ext}_\infty\, \mu([\![E]\!]) = \mathrm{ext}_\infty\, \mu(\mathrm{Cyl}(E)) = \mu(E)$.

The following lemma explores the interplay between the "next" modality $\bigcirc$ and the transition kernel γ. Intuitively, for ϕ a LTL formula, the probability that $\bigcirc \phi$ (or "next, ϕ") happens starting from state x should be the probability that ϕ (or "now, ϕ") happends starting from a successor state of x, weighted by the transition probability $\gamma(x)$. This fact is stated a bit more generally, but the proof in our framework is straightforward.

Lemma 5.3. *For $\mu \in \Delta X$ and ϕ an LTL formula, we have $\mu(\bigcirc \phi) = \int_{x\in X} \mu(\mathrm{d}x)\gamma(x,\phi)$. In particular, for a state $x \in X$, we have $\delta_x(\bigcirc \phi) = \gamma(x,\phi)$.*

Proof. Define two maps $\mu', \mu'' : B(X^{\odot\infty}) \longrightarrow [0,1]$ as $\mu' \coloneqq \operatorname{ext}_\infty \mu(X \odot -)$ and $\mu'' \coloneqq \int_{x\in X} \mu(\mathrm{d}x) \operatorname{ext}_\infty \gamma(x)(-)$. Clearly, μ' and μ'' are probability measure. Further, they agree on measurable cylinder sets, for if $E_1, \ldots, E_n \in B(X)$, then

$$\begin{aligned}
&\mu'\left(\mathrm{Cyl}(E_1,\ldots,E_n)\right)\\
&= \operatorname{ext}_\infty \mu\left(\mathrm{Cyl}(X, E_1, \ldots, X_n)\right)\\
&= \operatorname{ext}_{n+1} \mu\left(X \odot E_1 \odot \cdots \odot X_n\right)\\
&= \int_{x\in X} \mu(\mathrm{d}x) \times \left(\operatorname{ext}_n \gamma(x)(E_1 \odot \cdots \odot E_n)\right) && \spadesuit\\
&= \mu''\left(\mathrm{Cyl}(E_1,\ldots,E_n)\right),
\end{aligned}$$

where $\spadesuit$ follows from lemma 4.3. Therefore, μ' and μ'' agree on the ring C_X of proposition 3.9. By Carathéodory's theorem 2.1, $\mu' = \mu''$. □

5.2. Probabilistic computation tree logic.

Definition 5.4. For $X = (X, \gamma)$ a Markov chain, PCTL (on X) is defined by the following grammar:

$$\begin{array}{lllllll}
\Phi & ::= & E & & & & E \in B(X)\\
 & \mid & \top \quad \mid \quad \neg\Phi \quad \mid \quad \Phi \wedge \Phi & & & & \\
 & \mid & \mathbb{P}(\phi) \bowtie p & & & & \spadesuit
\end{array}$$

where in $\spadesuit$, ϕ is an LTL formula, $p \in [0,1]$, and $\bowtie \in \{>, \geq, \leq, <\}$. The usual logical connectives are defined in terms of $\top$, $\neg$, and $\wedge$.

Definition 5.5. We how define the semantics $[\![\Phi]\!] \in B(X)$ of a PCTL formula Φ. The only non-obvious case is when Φ is of the form $\mathbb{P}(\phi) \bowtie p$. Consider the pointwise-extended kernel $D \coloneqq \operatorname{ext}_\infty \delta_- : X \longrightarrow \Delta X^{\odot\infty}$, that maps $x \in X$ to the infinitary extension of its Dirac distribution δ_x, and let

$$[\![\mathbb{P}(\phi) \bowtie p]\!] \coloneqq D^{-1}\left(\beta^{\bowtie p}\left([\![\phi]\!]\right)\right) = \{x \in X \mid \delta_x(\phi) \bowtie p\}.$$

By theorem 4.7, this is a measurable set. If Φ is an PCTL formula and $Y \subseteq [\![\Phi]\!]$, then we write $Y \vDash \Phi$.

Remark 5.6. In [BK08, definition 10.36], PCTL is defined differently:

$$\begin{array}{lllllll}
\Theta & ::= & E & & & & \\
 & \mid & \top \quad \mid \quad \neg\Theta \quad \mid \quad \Theta \wedge \Theta & & & & \\
 & \mid & \mathbb{P}(\theta) \in J & & & & \spadesuit\\
\theta & ::= & \bigcirc \Theta & & & & \\
 & \mid & \Theta\, \mathbf{U}^{\leq n}\, \Theta & & & & n \leq \infty.
\end{array}$$

where in $\spadesuit$, J is an interval contained in $[0,1]$.[5] In other words, θ is an LTL formula that does not contain any logical connective, and where atomic predicates are of

[5]For algorithmic purposes, it is usually assumed that the bounds of J are rational, but we omit this requirement for simplicity.

the form $[\![\Theta]\!]$, where Θ is a PCTL formula in the present sense (also called *state formula* in [BK08, definition 10.36]).

Clearly, this "standard" PCTL (thereafter sPCTL) is a subset of our PCTL (thereafter simply PCTL). Conversely, formulas in PCTL can be translated to sPCTL using the $\overline{(-)} : \Phi \longmapsto \Theta$ and $\widetilde{(-)} : \phi \longmapsto \theta$ operators defined as follows (for the non-trivial cases):

$$\overline{\mathbb{P}(\phi) > p} \;:=\; \mathbb{P}(\widetilde{\phi}) \in (p, 1] \text{ and likewise for } \geq, \leq, <,$$
$$\widetilde{E} \;:=\; \top\, \mathbf{U}^{\leq 0}\, E, \qquad \widetilde{\top} \;:=\; \widetilde{X},$$
$$\widetilde{\neg\phi} \;:=\; \widetilde{X - [\![\phi]\!]}, \qquad \widetilde{\phi \wedge \phi'} \;:=\; \widetilde{[\![\phi]\!] \cap [\![\phi']\!]}.$$

Note that this translation assumes that every measurable set $E \subseteq X$ corresponds to an atomic predicate in sPCTL.

6. Recurrence

Let $X = (X, \gamma)$ be a Markov chain. The goal of this section is to place conditions on X and $E \in B(X)$ under which the following statements hold: $X \vDash \mathbb{P}(\Diamond\, E) > 0$ (reachability), and $X \vDash \mathbb{P}(\Box \Diamond\, E) = 1$ (recurrence). Reachability results serve as a stepping stone to establish recurrence, which is the subject of our main results: theorems 6.10 and 6.20.

6.1. **Preliminaries on recurring properties.** We start off with a few fairly intuitive observations. The first considerably simplifies the proof of statements of the form "event E almost surely eventually happens", i.e. $X \vDash \mathbb{P}(\Diamond\, E) = 1$, for some $E \in B(X)$. The intuition is that if starting from any state, "E probably happens soon", and that this probability and "soonness" are *globally bounded*, then surely, avoiding E forever is impossible.

Lemma 6.1. *Let $E \in B(X)$, $k \geq 1$, and $r > 0$ be such that $X \vDash \mathbb{P}(\Diamond^{\leq k} E) > r$. Then $X \vDash \mathbb{P}(\Diamond\, E) = 1$.*

Proof. By assumption, for all $x \in X$ we have $\delta_x(\Box^{\leq k} \neg E) < 1 - r$. In plain words, starting from any state x, the probability of staying out of E for the next k steps is $< 1 - r$. So by taking increasingly many successive k-steps walks, the probability of never reaching E should shrink to 0. Formally,

$$
\begin{aligned}
1 - \delta_x(\Diamond E) = \delta_x(\Box \neg E) = \lim_{n\to\infty} \delta_x(\mathrm{Cyl}(\underbrace{X - E, \dots, X - E}_{nk}))& \\
= \lim_{n\to\infty} \chi_{X-E}(x) \int\!\!\cdots\!\!\int_{y_i \in X-E} \Big(& \\
\underbrace{\gamma(x, \mathrm{d}y_2) \times \cdots \times \gamma(y_{k-1}, \mathrm{d}y_k)}_{<\delta_x(\Box^{\le k} \neg E)}& \\
\times \underbrace{\gamma(y_k, \mathrm{d}y_{k+1}) \times \cdots \times \gamma(y_{2k-1}, \mathrm{d}y_{2k})}_{<\delta_{y_k}(\Box^{\le k} \neg E)}& \\
\times \cdots \times \underbrace{\cdots \times \gamma(y_{nk-1}, X_E)}_{<\delta_{y_{(n-1)k}}(\Box^{\le k} \neg E)} \Big)& \\
\le \lim_{n\to\infty} (1 - r)^n& \\
= 0.&
\end{aligned}
$$

□

Next, if an event E eventually happens starting from any state, then surely, it must happen infinitely often. Indeed, once a run has reached E, then by assumption, E must eventually happen again. The following lemma formalizes this.

Lemma 6.2. *Let $E \in B(X)$ be such that $X \vDash \mathbb{P}(\Diamond E) = 1$. Then $X \vDash \mathbb{P}(\Box \Diamond E) = 1$.*

Proof. Recall from section 5.1 that $[\![\Box \Diamond E]\!] = \bigcap_{n\in\mathbb{N}} [\![\bigcirc^n \Diamond E]\!]$. For $x \in X$, we have $\delta_x(\Diamond E) = 1$ by assumption, and if $n \ge 1$,

$$
\begin{aligned}
\delta_x(\bigcirc^n \Diamond E) &= \int\!\!\cdots\!\!\int_{y_i} \gamma(x, \mathrm{d}y_1) \times \cdots \times \gamma(y_{n-1}, \mathrm{d}y_n) \times \underbrace{\gamma(y_n, \Diamond E)}_{=1} \\
&= \int\!\!\cdots\!\!\int_{y_i} \gamma(x, \mathrm{d}y_1) \times \cdots \times \gamma(y_{n-1}, \mathrm{d}y_n) \times \gamma(y_n, X) \\
&= \delta_x(\bigcirc^n X) = 1.
\end{aligned}
$$

□

6.2. **Upper semicontinuity and some semantics sets.** The purpose of the next few technical results is to prove that the semantics of PCTL formulas of the form $\mathbb{P}(\Diamond^{\le k} U) \le r$ is closed, where U is an open set. As we will see later, this is an important step towards bounding reachability probabilities, i.e. the value of $\delta_x(\Diamond U)$ for $x \in X$, which is central to our recurrence theorems 6.10 and 6.20. The overall plan is to express the semantics set $[\![\mathbb{P}(\Diamond^{\le k} U) \le r]\!]$ as the preimage of some convenient function $X \longrightarrow \mathbb{R}$ that we define in lemma 6.6.

In remark 2.7, we briefly discussed the subtle topological properties of ΔX. The core difficulty we encounter is that for $f \in M^b(X)$, the integration operator $I_f : \Delta X \longrightarrow \mathbb{R}$ is not continuous in general. Nonetheness, certain properties of f can be reflected in I_f.

Definition 6.3 ([Bou98, definition IV.6.2.1])**.** For K a topological space, a set-map $f : K \longrightarrow \mathbb{R}$ is *upper semicontinous* (USC for short) if for all sequence $(x_n)_{n\in\mathbb{N}}$ of elements of K converging to x, we have $\limsup_n f(x_n) \le f(x)$. Equivalently, for all

$r \in \mathbb{R}$, the preimage $f^{-1}[r, +\infty)$ is closed (this is the dual[6] of [Bou98, proposition IV.6.2.1]).

Lemma 6.4 ([Bou98, proposition IV.6.2.2])**.** *The product of two positive USC maps $K \longrightarrow \mathbb{R}$ is also USC.*

Let $f : X \longrightarrow \mathbb{R}$ and consider the integration operator $I_f : \Delta X \longrightarrow \mathbb{R}$. By definition 2.5, if f is continuous, then so is I_f. By proposition 2.8, if f is measurable, then so is I_f. For our purpose, measurability of I_f is too weak, and continuity is too restrictive on f (see remark 2.7). The following key result asserts that upper semicontinuity is the compromise we need.

Theorem 6.5 ([DE97, Theorem A.3.12])**.** *If $f : X \longrightarrow \mathbb{R}$ is a bounded positive and USC map, then $I_f : \Delta X \longrightarrow \mathbb{R}$ (see definition 2.5) is USC as well. In particular, the expression $\int f \, \mathrm{d}\gamma(x)$ is USC in x.*

We saw in definition 6.3 that for f a USC map, certain preimage sets of f are closed. The goal of the next two results is to express the semantics set $\llbracket \mathbb{P}(\Diamond^{\leq k} U) \leq r \rrbracket$ as the preimage of some desirable USC f.

Lemma 6.6. *Let $F \subseteq X$ be closed, $k \geq 1$, and consider the map $\Gamma_{F,k} : X \longrightarrow \mathbb{R}$ mapping $x \in X$ to $\operatorname{ext}_k \gamma(x)(F^{\odot k})$.*[7] *Then $\Gamma_{F,k}$ is USC.*

Proof. We proceed by induction. If $k = 1$, then $\Gamma_{F,k}(x) = \gamma(x, F)$. Take $r \in \mathbb{R}$, write $A_r \coloneqq \Gamma_{F,1}^{-1}[r, +\infty)$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of A_r that converges to some $x \in X$. Recall that by continuity of γ, $(\gamma(x_n))_{n\in\mathbb{N}}$ is a sequence of probability measures that converges to $\gamma(x)$. By the Portmanteau theorem 2.9, $\gamma(x, F) \geq \limsup_n \gamma(x_n, F) \geq r$, so that $x \in A_r$. This shows that A_r is closed for all $r \in \mathbb{R}$, and thus that $\Gamma_{F,1} = \gamma(-, F)$ is USC.

Assume now that $k \geq 2$, and note that

$$\begin{aligned} \Gamma_{F,k}(x) &= \operatorname{ext}_k \gamma(x)(F^{\odot k}) \\ &= \int_{y \in F} \gamma(x, \mathrm{d}y) \times \left(\operatorname{ext}_{k-1} \gamma(y)(F^{\odot k-1})\right) \\ &= \int \chi_F \times \Gamma_{F,k-1} \, \mathrm{d}\gamma(x). \end{aligned}$$

Since F is closed, χ_F is USC, and by induction, so is $\Gamma_{F,k-1}$. Using lemma 6.4 and theorem 6.5, we conclude that $\Gamma_{F,k}$ is USC. ☐

Lemma 6.7. *For $U \subseteq X$ open, $k \in \mathbb{N}$, and $r > 0$, the set $\llbracket \mathbb{P}(\Diamond^{\leq k} U) \leq r \rrbracket$ is closed.*

Proof. We write R_k as a shorthand for $\llbracket \mathbb{P}(\Diamond^{\leq k} U) \leq r \rrbracket$. Note that

$$R_k = \llbracket \mathbb{P}(\Box^{\leq k} F) \geq 1 - r \rrbracket = \left\{ x \in X \mid \delta_x(\Box^{\leq k} F) \geq 1 - r \right\},$$

where $F \coloneqq X - U$. In order to show that R_k is closed, we show that $\delta_x(\Box^{\leq k} F)$ is USC (in x). We proceed by induction on k.

(1) If $k = 0$, then $\delta_x(\Box^{\leq 0} F) = \delta_x(F) = \chi_F(x)$, which is USC since F is closed.

[6][Bou98, section IV.6.2] mostly deals with lower semicontinuous (LSC) functions. However, properties of a USC f can be transposed from "dual" properties of $-f$ which is LSC, or even $(\max f) - f$ if f is bounded.

[7]In details, this is the evaluation of $\operatorname{ext}_k \gamma(x) \in \Delta X^{\odot k}$ (the k-ary extension of the transition distribution $\gamma(x) \in \Delta X$) at the measurable sequence set $F^{\odot k} \subseteq X^{\odot k}$.

(2) If $k = 1$, then we have $\delta_x(\Box^{\leq 1} F) = \mathrm{ext}_2\, \delta_x\,(F \odot F) = \chi_F(x)\gamma(x, F)$, which is USC by lemmas 6.4 and 6.6.

(3) Assume now that $k \geq 2$. We have

$$\begin{aligned}
&\delta_x(\Box^{\leq k} F)\\
&= \mathrm{ext}_{k+1}\, \delta_x(F^{\odot k+1})\\
&= \chi_F(x) \int_{y \in F} \gamma(x, \mathrm{d}y) \times \left(\mathrm{ext}_{k-1}\, \gamma(y)(F^{\odot k-1})\right)\\
&= \chi_F(x) \int \chi_F \times \Gamma_{F,k-1}\, \mathrm{d}\gamma(x).
\end{aligned}$$

By lemmas 6.4 and 6.6 and theorem 6.5, this is USC in x. □

6.3. **Weak recurrence.** In what follows, we impose compactness on the state spaces of Markov chains. Here we view compactness as "topological finiteness"; using the property, the essence of some arguments in the finitary setting (such as in [BK08]) can be carried over to our current infinitary and topological setting.

In terms of the dynamics of Makov chains, compactness can be understood as a condition that the underlying topology "does not allow for escapes", that is to say, does not allow for paths to "stray" forever (see remark 6.11). This collective containment property that execution traces share is what allows us to deduce recurrence theorems, namely that some events must happen with a certain positive probability.

In lemma 6.7, we showed that semantics sets of the form $\llbracket \mathbb{P}(\Diamond^{\leq k} U) \leq r \rrbracket$ are closed. This allows us to leverage certain interesting properties of compact spaces:

Lemma 6.8. *Let K be a compact topological space. Then for every descending chain $F_0 \supseteq F_1 \supseteq \cdots$ of non-empty closed sets, the intersection $\bigcap_n F_n$ is not empty.*

Proposition 6.9. *Assume that X is compact, and let $U \in \mathfrak{T}$ be such that $X \vDash \mathbb{P}(\Diamond U) > 0$. Then $X \vDash \mathbb{P}(\Diamond U) = 1$.*

Proof. By lemma 6.7, the set $R_{k,n} \coloneqq \llbracket \mathbb{P}(\Diamond^{\leq k} U) \leq \frac{1}{n} \rrbracket$ is closed. Towards a contradiction, assume that $R_{k,n} \neq \varnothing$ for all k and n. Note that $R_{k,n} \supseteq R_{k+1,n}, R_{k,n+1}$. By assumption, X is compact, so by lemma 6.8 (applied twice), the intersection $\bigcap_k \bigcap_n R_{k,n}$ is not empty. However,

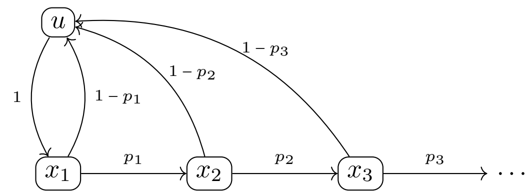
$$\begin{aligned}
\bigcap_k \bigcap_n R_{k,n} &= \bigcap_k \bigcap_n \left\llbracket \mathbb{P}(\Diamond^{\leq k} U) \leq \frac{1}{n} \right\rrbracket\\
&= \bigcap_k \llbracket \mathbb{P}(\Diamond^{\leq k} U) = 0 \rrbracket\\
&= \llbracket \mathbb{P}(\Diamond U) = 0 \rrbracket,
\end{aligned}$$

which contradicts the assumption that $X \vDash \mathbb{P}(\Diamond U) > 0$. Therefore, there exists k and n such that $R_{k,n} = \varnothing$. In particular, $X \vDash \mathbb{P}(\Diamond^{\leq k} U) > \frac{1}{n}$, and lemma 6.1 applies. □

We now arrive at our first recurrence theorem, which is a simple combination of proposition 6.9 and lemma 6.2.

Theorem 6.10 (Topological weak recurrence)**.** *Assume that X is compact, and let $U \in \mathfrak{T}$ be such that $X \vDash \mathbb{P}(\Diamond U) > 0$. Then $X \vDash \mathbb{P}(\Box \Diamond U) = 1$.*

Remark 6.11. Unfortunately, the compactness condition of proposition 6.9 (and thus of theorem 6.10) cannot be dropped. Indeed, consider the discrete space $X := \{u, x_1, x_2, \ldots\}$ with the transition kernel given by $\gamma(u, x_1) := 1$ and $\gamma(x_i, x_{i+1}) = 1 - \gamma(x_i, u) := p_i$, with $p_i := 1 - \frac{1}{(i+1)^2}$. Graphically,



Clearly, this Markov chain is irreducible (every state is reachable from every other), but we have

$$\delta_{x_1}(\Diamond\{u\}) = 1 - \mathrm{ext}_\infty\, \delta_{x_1}(\{(x_1, x_2, \ldots)\}) = 1 - \prod_{i=1}^{\infty} p_i = \frac{1}{2}.$$

Additionally, it is easy to see that $\delta_{x_n}(\Diamond\{u\}) < \delta_{x_1}(\Diamond\{u\})$ for all $n \geq 2$.

6.4. **Subchains and irreducibility.** We argued that compact Markov chains are the suitable generalization of finite ones. Another important notion is that of *strong connectedness*, whereby every state is reachable from every other. Consequently, no state can be removed from the chain, i.e. the chain is irreducible:

Definition 6.12. A *subchain* (or *subcoalgebra*) of X is a Polish subspace $Y \subseteq X$ such that for all $y \in Y$, $\gamma(y)$ has support in Y.

We say that X is *irreducible* if it is non-empty and does not admit a proper subchain. In particular, for every $x \in X$, there exist $y \in X$ such that $x \in \operatorname{supp} \gamma(y)$.[8]

Remark 6.13. Classically, such a subspace Y is also called *closed*, as in closed with respect to the coalgebra structure γ [Num84, definition 2.1] [PGS00, section 2.1] [Bré20, definition 2.3.2]. Unfortunately, Y is not necessarily closed in the topological sense. In fact, it may not even be measurable. Indeed, consider $X := \mathbb{R}$ with the Euclidean topology, and $\gamma(x) := \delta_x$ for all $x \in X$. In this case, every subset of X is a subchain. In particular, Vitali sets [Her06, section 5.7] are subchains despite being non-measurable.

However, measurability can be guaranteed if Y is an irreducible subchain, see proposition 6.15.

Lemma 6.14. *If $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of elements of ΔX that converges to μ, then* $\operatorname{supp} \mu \subseteq \liminf_n \operatorname{supp} \mu_n$.

Proof. Follows from the Portmanteau theorem 2.9. □

Proposition 6.15. *If Y is an irreducible subchain of X, then it is an F_σ-set (i.e. a countable union of closed sets).*

Proof. By irreducibility, $Y = \bigcup_{y \in Y} \operatorname{supp} \gamma(y)$. Let $Q \subseteq Y$ be countable dense subset. By definition, any state $y \in Y$ is the limit of some sequence $(q_n)_{n \in \mathbb{N}}$ of elements of Q. By lemma 6.14, $\operatorname{supp} \gamma(y) \subseteq \liminf_n \operatorname{supp} \gamma(q_n)$. Therefore, $Y = \bigcup_{y \in Y} \operatorname{supp} \gamma(y) = \bigcup_{q \in Q} \operatorname{supp} \gamma(q)$. □

[8] This condition is not sufficient, however.

Remark 6.16. Unfortunately, irreducibility of Y cannot be leveraged to obtain any basic topological property, as the few following counterexamples show.

(1) Y is not closed in general. Let the state space be $X \coloneqq [0, +\infty)$ with the Euclidean topology, and $\gamma(x) \coloneqq \delta_0$ if $x = 0$, or the uniform distribution on $\left[\frac{x}{2}, \frac{3x}{2}\right]$ if $x > 0$. Then $Y \coloneqq (0, +\infty)$ is an non-closed irreducible subchain. In particular, Y is not compact.
(2) Y is not open in general. Like in remark 6.13, consider $X \coloneqq \mathbb{R}$ with the Euclidean topology, and $\gamma(x) \coloneqq \delta_x$ for all $x \in X$. Any singleton is a non-open irreducible subchain.
(3) Y is not connected in general. Consider $X \coloneqq \{x_0, x_1\}$ with the discrete topology, and $\gamma(x_i) \coloneqq \delta_{x_{1-i}}$. Then X is irreducible but not connected.

6.5. **Strong recurrence.**

Definition 6.17. We say that a subchain $Y \subseteq X$ has the *reachability property* (in X) if for all non-empty open subset $U \subseteq X$, we have $Y \vDash \mathbb{P}(\lozenge U) > 0$.

Proposition 6.18. *Every dense irreducible subchain of X has the reachability property. Conversely, if X has the reachability property (in itself), then every irreducible subchain is dense.*

Proof.
(1) Take $U \in \mathcal{T}$ non-empty, and write $R_U \coloneqq Y \cap [\![\mathbb{P}(\bigcirc \lozenge U) > 0]\!]$. Since $R_U \subseteq Y \cap [\![\mathbb{P}(\lozenge U) > 0]\!]$, it is enough to show that $R_U = Y$. Towards a contradiction, assume that $R_U \neq Y$. First, we show that $R_U \neq \varnothing$. Since Y is dense, $Y \cap U \neq \varnothing$, and let $x \in Y \cap U$. By irreducibility, there exist $y \in Y$ such that $x \in \operatorname{supp} \gamma(y)$. In particular, $\gamma(y, U) > 0$, so $y \in R_U$. Consequently, $Y - R_U = Y \cap [\![\mathbb{P}(\bigcirc \lozenge U) = 0]\!]$ is a proper subset of Y. We now show that it is a proper subchain. If not, then there exists $x \in Y - R_U$ such that $\gamma(x, R_U) > 0$. Thus, $\delta_x(\bigcirc \lozenge U) \geq \delta_x(\bigcirc \bigcirc \lozenge U) > 0$, a contradiction with the fact that $x \in Y - R_U$. We arrive at the conclusion that $Y - R_U$ is a proper subchain of Y, which contradicts the irreducibility of Y. Therefore, $R_U = Y$.
(2) Let $Y \subseteq X$ be an irreducible subchain, and let U be the interior of $X - Y$. If $U \neq \varnothing$, then by assumption, $Y \vDash \mathbb{P}(\lozenge U) > 0$. This means that there exists an element $y \in Y$ such that $\gamma(y)$ does not have all its support in Y, a contradiction. Therefore $U = \varnothing$ and Y is dense. ☐

Remark 6.19. Unfortunately, a Markov chain X that has the reachability property is not necessarily irreducible. For example, consider $X \coloneqq [0,1]_A + (0,1]_B$, where the subscripts are purely decorative, and each component has the Euclidean topology. If $x \in [0,1]$, let x_A be the corresponding element in the A component, and likewise for x_B (if $x > 0$). The transition kernel $\gamma : X \longrightarrow \Delta X$ is given as follows: $\gamma(x_A)$ is the uniform distribution on $(0,1]_B$, while $\gamma(x_B) = \delta_{x_A}$. This Markov chain clearly has the reachability property, but it is not irreducible, for $Y \coloneqq (0,1]_A + (0,1]_B$ is a proper subchain. Still, as predicted by proposition 6.18, Y is dense.

Theorem 6.20 (Topological strong recurrence)**.** *If X is a compact and irreducible Markov chain, and $U \in \mathcal{T}$ is non-empty, then $X \vDash \mathbb{P}(\square \lozenge U) = 1$.*

Proof. Follows from the weak recurrence theorem 6.10 and proposition 6.18. ☐

Using the above results, it is easy to derive the following corollaries, which are most important when it comes to the application to model checking.

Corollary 6.21. *If X is compact and irreducible, then for all $E \in B(X)$ with non-empty interior, $X \vDash \mathbb{P}(\Box \Diamond E) = 1$.*

Corollary 6.22. *Let $Y \subseteq X$ be an irreducible subchain, $x \in X$, and $E \in B(Y)$ have a non-empty interior. We have $\delta_x(\Box \Diamond E) = \delta_x(\Diamond Y)$.*[9]

7. Conclusions and future work

We developed a mathematical theory for infinitary probabilistic model checking, focusing on the recurrence theorems in the existing finitary theory and extending them to the infinitary setting. This endeavor was enabled by the use of a suitable topological machinery. In the end, we formulated and proved topological recurrence theorems, which reduce different aspects of infinitary probabilistic model checking to problems that are easier to solve, such as irreducibility, which is qualitative rather than quantitative.

For future works, we shall lift our current theory from Polish spaces to the more general framework of analytic spaces, as is done in [DEP02, DGJP04].

Another important direction is to build a bridge from outside theoretical computer science to the theory of Markov chains. The study of Markov chains in theoretical computer science has been centered around probabilistic model checking and thus on finitary cases. The current paper makes a topological step towards the study of infinite Markov chains, and it is natural to pursue the connection to other fields, where their interests are often in stationary distributions, ergodicity, and other dynamical properties, see e.g. [Chu60, Num84].

Finally, the application of the current results to practical formal verification is also a relevant topic for future works. Probabilistic verification of continuous systems is studied energetically in the community of hybrid systems, see e.g. [MMS20, FCX+20]. Relationship to these works should be investigated. Another community to which the present paper is relevant is that of probabilistic programming languages. In particular, we shall look into semantical works such as [VKS19].

References

[ACH+95] Rajeev Alur, Costas A. Courcoubetis, Nicolas Halbwachs, Thomas A.A. Henzinger, Pei-Hsin Ho, Xavier Nicollin, Alfredo M. Olivero, Joseph Sifakis, and Sergio Yovine. The algorithmic analysis of hybrid systems. *Theoretical Computer Science*, 138(1):3–34, February 1995.

[And19] Étienne André. What's decidable about parametric timed automata? *International Journal on Software Tools for Technology Transfer*, 21(2):203–219, April 2019.

[BGK+19] Olav Bunte, Jan Friso Groote, Jeroen J. A. Keiren, Maurice Laveaux, Thomas Neele, Erik P. de Vink, Wieger Wesselink, Anton Wijs, and Tim A. C. Willemse. The mCRL2 Toolset for Analysing Concurrent Systems: Improvements in Expressivity and Usability. In Tomáš Vojnar and Lijun Zhang, editors, *Tools and Algorithms for the Construction and Analysis of Systems*, volume 11428, pages 21–39. Springer International Publishing, Cham, 2019.

[BK08] Christel Baier and Joost-Pieter Katoen. *Principles of Model Checking*. The MIT Press, Cambridge, Mass, 2008.

[Bou98] Nicolas Bourbaki. *Elements of Mathematics. Chapters 1/4: 3. General Topology Chapters 1 - 4*. Springer, Berlin Heidelberg, softcover ed., [nachdr.] - [1998] edition, 1998.

[Bré20] Pierre Brémaud. *Markov Chains: Gibbs Fields, Monte Carlo Simulation and Queues*. Number volume 31 in Texts in Applied Mathematics. Springer, Cham, second edition edition, 2020.

[9] Recall that by proposition 6.15, Y and E are measurable in X.

[BY04] Johan Bengtsson and Wang Yi. Timed Automata: Semantics, Algorithms and Tools. In Takeo Kanade, Josef Kittler, Jon M. Kleinberg, Friedemann Mattern, John C. Mitchell, Moni Naor, Oscar Nierstrasz, C. Pandu Rangan, Bernhard Steffen, Madhu Sudan, Demetri Terzopoulos, Dough Tygar, Moshe Y. Vardi, Gerhard Weikum, Jörg Desel, Wolfgang Reisig, and Grzegorz Rozenberg, editors, *Lectures on Concurrency and Petri Nets*, volume 3098, pages 87–124. Springer Berlin Heidelberg, Berlin, Heidelberg, 2004.

[CCG+02] Alessandro Cimatti, Edmund Clarke, Enrico Giunchiglia, Fausto Giunchiglia, Marco Pistore, Marco Roveri, Roberto Sebastiani, and Armando Tacchella. NuSMV 2: An OpenSource Tool for Symbolic Model Checking. In Gerhard Goos, Juris Hartmanis, Jan van Leeuwen, Ed Brinksma, and Kim Guldstrand Larsen, editors, *Computer Aided Verification*, volume 2404, pages 359–364. Springer Berlin Heidelberg, Berlin, Heidelberg, 2002.

[CGP99] Edmund M. Clarke, Orna Grumberg, and Doron A. Peled. *Model Checking*. MIT Press, Cambridge, Mass, 1999.

[Chu60] Kai Lai Chung. *Markov Chains with Stationary Transition Probabilities*. Springer, Berlin, 1960.

[CJ19] Kenta Cho and Bart Jacobs. Disintegration and Bayesian inversion via string diagrams. *Mathematical Structures in Computer Science*, 29(7):938–971, August 2019.

[DE97] Paul Dupuis and Richard S. Ellis. *A Weak Convergence Approach to the Theory of Large Deviations: Dupuis/A Weak*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., Hoboken, NJ, USA, February 1997.

[DEP02] Josée Desharnais, Abbas Edalat, and Prakash Panangaden. Bisimulation for Labelled Markov Processes. *Information and Computation*, 179(2):163–193, December 2002.

[DGJP04] Josée Desharnais, Vineet Gupta, Radha Jagadeesan, and Prakash Panangaden. Metrics for labelled Markov processes. *Theoretical Computer Science*, 318(3):323–354, June 2004.

[Dob07] Ernst-Erich Doberkat. *Stochastic Relations: Foundations for Markov Transition Systems*. Chapman & Hall/CRC Studies in Informatics Series. Chapman & Hall/CRC, Boca Raton, FL, 2007.

[FCX+20] Shenghua Feng, Mingshuai Chen, Bai Xue, Sriram Sankaranarayanan, and Naijun Zhan. Unbounded-Time Safety Verification of Stochastic Differential Dynamics. In Shuvendu K. Lahiri and Chao Wang, editors, *Computer Aided Verification*, volume 12225, pages 327–348. Springer International Publishing, Cham, 2020.

[Gir82] Michèle Giry. A categorical approach to probability theory. In A. Dold, B. Eckmann, and B. Banaschewski, editors, *Categorical Aspects of Topology and Analysis*, volume 915, pages 68–85. Springer Berlin Heidelberg, Berlin, Heidelberg, 1982.

[GM14] J. F. Groote and Mohammad Reza Mousavi. *Modeling and Analysis of Communicating Systems*. The MIT Press, Cambridge, Massachusetts, 2014.

[Hal74] Paul R. Halmos. *Measure Theory*. Number 18 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1974.

[Hen96] T.A. Henzinger. The theory of hybrid automata. In *Proceedings 11th Annual IEEE Symposium on Logic in Computer Science*, pages 278–292, New Brunswick, NJ, USA, 1996. IEEE Comput. Soc. Press.

[Her06] Horst Herrlich. *Axiom of Choice*. Number 1876 in Lecture Notes in Mathematics. Springer, Berlin, 2006.

[HJ94] Hans Hansson and Bengt Jonsson. A logic for reasoning about time and reliability. *Formal Aspects of Computing*, 6(5):512–535, September 1994.

[Hol19] Gerard J. Holzmann. Spin, 2019.

[Joh92] Peter T. Johnstone. *Stone Spaces*. Number 3 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, Cambridge, paperback ed., repr edition, 1992.

[Kec95] Alexander S. Kechris. *Classical Descriptive Set Theory*. Number 156 in Graduate Texts in Mathematics. Springer, New York Berlin Heidelberg, 1995.

[Kle13] Achim Klenke. *Probability Theory: A Comprehensive Course*. Universitext. Springer, New York, 2013.

[MMS20] Rupak Majumdar, Kaushik Mallik, and Sadegh Soudjani. Symbolic controller synthesis for Büchi specifications on stochastic systems. In *Proceedings of the 23rd International*

Conference on Hybrid Systems: Computation and Control, pages 1–11, Sydney New South Wales Australia, April 2020. ACM.

[Mun00] James R. Munkres. *Topology*. Prentice Hall, Inc, Upper Saddle River, NJ, 2nd ed edition, 2000.

[Num84] Esa Nummelin. *General Irreducible Markov Chains and Non-Negative Operators*. Cambridge University Press, first edition, October 1984.

[PGS00] H. Peter Gumm and Tobias Schröder. Coalgebraic structure from weak limit preserving functors. *Electronic Notes in Theoretical Computer Science*, 33:111–131, 2000.

[Pnu77] Amir Pnueli. The temporal logic of programs. In *18th Annual Symposium on Foundations of Computer Science (Sfcs 1977)*, pages 46–57, Providence, RI, USA, September 1977. IEEE.

[Tay06] Michael E. Taylor. *Measure Theory and Integration*. Number v. 76 in Graduate Studies in Mathematics. American Mathematical Society, Providence, R.I, 2006.

[Var96] Moshe Y. Vardi. An automata-theoretic approach to linear temporal logic. In Gerhard Goos, Juris Hartmanis, Jan Leeuwen, Faron Moller, and Graham Birtwistle, editors, *Logics for Concurrency*, volume 1043, pages 238–266. Springer Berlin Heidelberg, Berlin, Heidelberg, 1996.

[Vic96] Steven Vickers. *Topology via Logic*. Number 5 in Cambridge Tracts in Theoretical Computer Science. Cambridge Univ. Press, Cambridge, 1. paperback ed edition, 1996.

[VKS19] Matthijs Vákár, Ohad Kammar, and Sam Staton. A domain theory for statistical probabilistic programming. *Proceedings of the ACM on Programming Languages*, 3(POPL):1–29, January 2019.

[Wil70] Stephen Willard. *General Topology*. Addison-Wesley Series in Mathematics. Addison-Wesley, Reading/Mass., 1970.

Email address: `hothanh@nii.ac.jp`

Email address: `urabe@nii.ac.jp`

Email address: `hasuo@nii.ac.jp`

National Institute for Informatics, 2-1-2 Hitotsubashi, Tokyo 101-8430, Japan