The Poisson matrix and Kronecker Products

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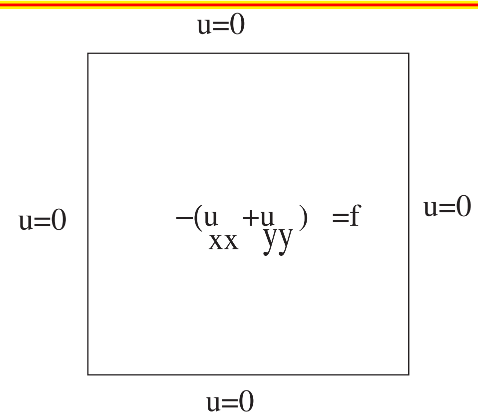
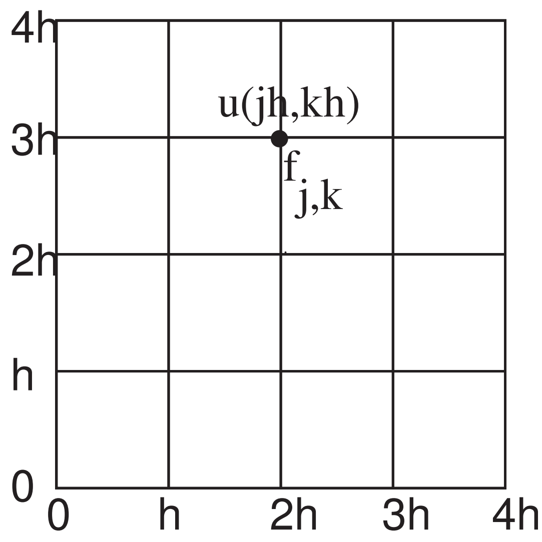
Plan for the day

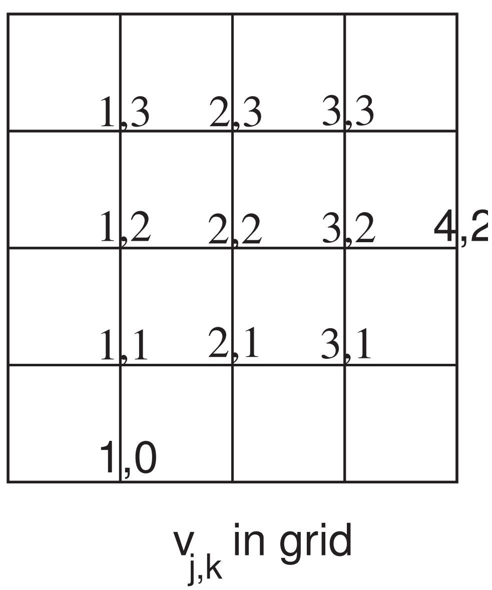
- Numerical solution of the Poisson problem
  - The finite difference scheme
  - Formulation as a matrix equation
  - Formulation in standard form $\boldsymbol{Ax} = \boldsymbol{b}$.
  - The Poisson matrix $\boldsymbol{A}$ is block tridiagonal
- LU factorization of a block tridiagonal matrix
- Solving the Poisson problem, complexity
- Kronecker product
  - Relation to Poisson problem
  - Basic properties
  - Eigenvalues and eigenvectors
- Eigenpairs for the Poisson matrix

Numerical solution of the Poisson Problem



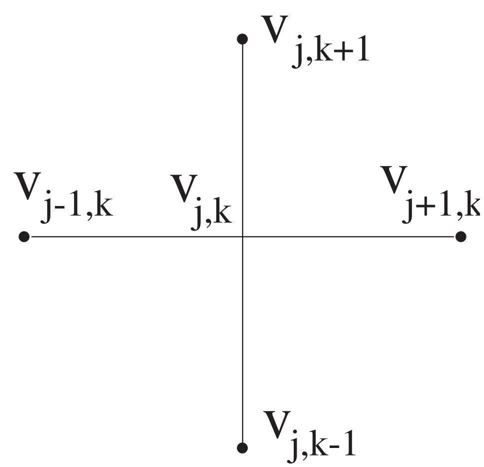
$$m \in \mathbb{N},\ h = 1/(m+1), \quad v_{j,k} \approx u(jh, kh)$$





Discrete Equation

$$g''(t) \approx \frac{g(t-h) - 2g(t) + g(t+h)}{h^2}$$



From differential equation for $j, k = 1, 2, \ldots, m$

$$(-v_{j-1,k} + 2v_{j,k} - v_{j+1,k}) + (-v_{j,k-1} + 2v_{j,k} - v_{j,k+1}) = h^2 f_{j,k} \qquad (1)$$

From boundary conditions

$$v_{j,k} = 0 \text{ if } j = 0, m+1 \text{ or } k = 0, m+1 \qquad (2)$$

Matrix Equation $\boldsymbol{T}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{T} = h^2\boldsymbol{F}$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

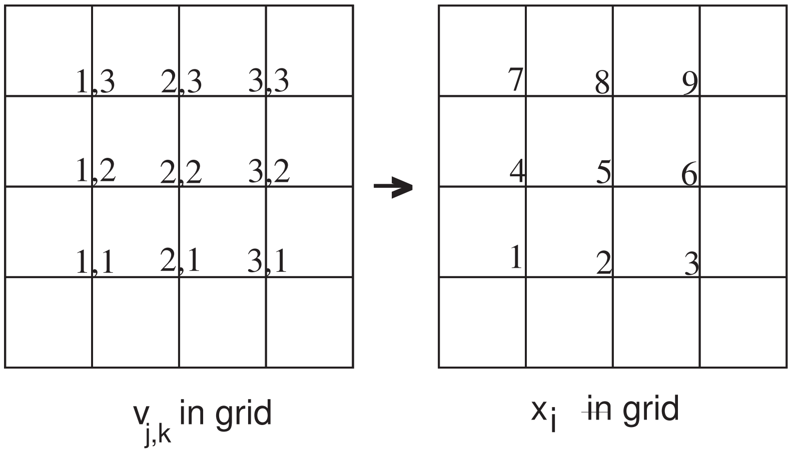
$$\boldsymbol{T} := \boldsymbol{T}_m = \text{tridiag}_m(-1, 2, -1) \in \mathbb{R}^{m,m}$$

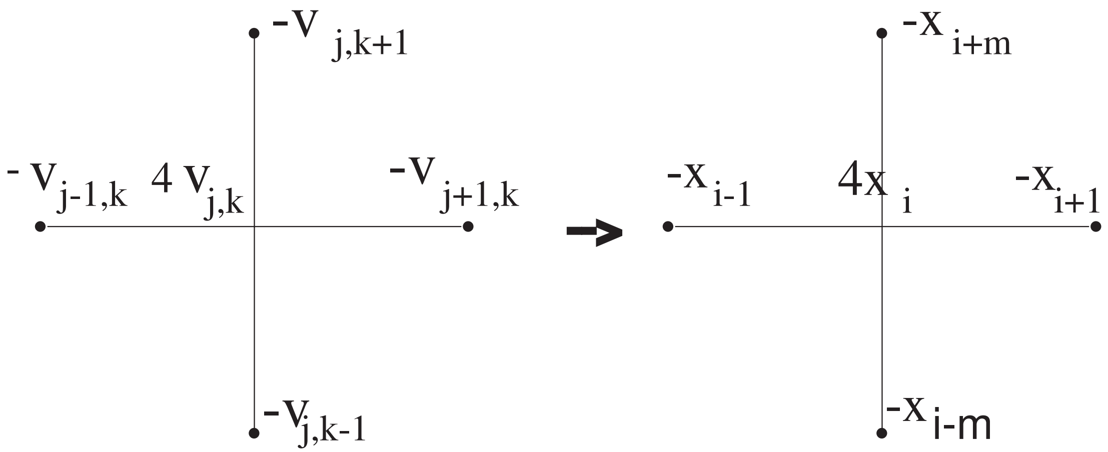
$$\boldsymbol{V} := \begin{bmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & & \vdots \\ v_{m1} & \cdots & v_{mm} \end{bmatrix} \in \mathbb{R}^{m,m} \quad \boldsymbol{F} := \begin{bmatrix} f_{11} & \cdots & f_{1m} \\ \vdots & & \vdots \\ f_{m1} & \cdots & f_{mm} \end{bmatrix} \in \mathbb{R}^{m,m}$$

$$\text{(1), (2)} \Leftrightarrow \boldsymbol{T}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{T} = h^2\boldsymbol{F}$$

$$\left(\boldsymbol{T}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{T}\right)_{jk} = \sum_{i=1}^{m} \boldsymbol{T}_{ji} v_{ik} + \sum_{i=1}^{m} v_{ji} \boldsymbol{T}_{ik} = \text{ l.h.s in (1)} = h^2(\boldsymbol{F})_{jk}$$

Convert $\boldsymbol{TV} + \boldsymbol{VT} = h^2\boldsymbol{F}$ to $\boldsymbol{Ax} = \boldsymbol{b}$





$$4x_i - x_{i-1} - x_{i+1} - x_{i-m} - x_{i+m} = b_i$$

Linear system $Ax = b$

For $m = 3$ or $n = 9$ we have the following linear system:

$$
\begin{array}{ccc|ccc|ccc}
4x_1 & -x_2 & & -x_4 & & & & & = h^2 f_{11} \\
-x_1 & 4x_2 & -x_3 & & -x_5 & & & & = h^2 f_{21} \\
& -x_2 & 4x_3 & & & -x_6 & & & = h^2 f_{31} \\
\hline
-x_1 & & & 4x_4 & -x_5 & & -x_7 & & = h^2 f_{12} \\
& -x_2 & & -x_4 & 4x_5 & -x_6 & & -x_8 & = h^2 f_{22} \\
& & -x_3 & & -x_5 & 4x_6 & & & -x_9 = h^2 f_{32} \\
\hline
& & & -x_4 & & & 4x_7 & -x_8 & = h^2 f_{13} \\
& & & & -x_5 & & -x_7 & 4x_8 & -x_9 = h^2 f_{23} \\
& & & & & -x_6 & & -x_8 & 4x_9 = h^2 f_{33}
\end{array}
$$

Blockstructure of A

$$A = \left[\begin{array}{ccc|ccc|ccc} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{array}\right]$$

$$A = \left[\begin{array}{c|c|c} T + 2I & -I & 0 \\ \hline -I & T + 2I & -I \\ \hline 0 & -I & T + 2I \end{array}\right]$$

Block tridiagonal

LU-factorization of a tridiagonal matri

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_2 & d_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & d_{n-1} & c_{n-1} \\ & & & a_n & d_n \end{bmatrix} = \begin{bmatrix} 1 & & & \\ l_2 & 1 & & \\ & \ddots & \ddots & \\ & & l_n & 1 \end{bmatrix} \begin{bmatrix} u_1 & c_1 & & \\ & \ddots & \ddots & \\ & & u_{n-1} & c_{n-1} \\ & & & u_n \end{bmatrix}. \quad (3)$$

Note that $\boldsymbol{L}$ has ones on the diagonal, and that $\boldsymbol{A}$ and $\boldsymbol{U}$ have the same super-diagonal.

By equating entries in (3) we find

$$d_1 = u_1, \quad a_k = l_k u_{k-1}, \quad d_k = l_k c_{k-1} + u_k, \quad k = 2, 3, \ldots, n.$$

The Algorithm

- $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ (LU-factorization)
- $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$ (forward substitution)
- $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$ (backward substituion)
- $u_1 = d_1, \quad l_k = \frac{a_k}{u_{k-1}}, \quad u_k = d_k - l_k c_{k-1}, \quad k = 2, 3, \ldots, n.$
- $y_1 = b_1, \qquad y_k = b_k - l_k y_{k-1}, \qquad k = 2, 3, \ldots, n,$
- $x_n = y_n/d_n, \quad x_k = (y_k - c_k x_{k+1})/u_k, \quad k = n-1, \ldots, 2, 1.$
- This process is well defined if $\boldsymbol{A}$ is strictly diagonally dominant.
- The number of arithmetic operations (flops) is $O(n)$.

LU of Block Tridiagonal Matrix

$$\begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{C}_1 & & & \\ \boldsymbol{A}_2 & \boldsymbol{D}_2 & \boldsymbol{C}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \boldsymbol{A}_{m-1} & \boldsymbol{D}_{m-1} & \boldsymbol{C}_{m-1} \\ & & & \boldsymbol{A}_m & \boldsymbol{D}_m \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \boldsymbol{L}_2 & 1 & & \\ & \ddots & \ddots & \\ & & \boldsymbol{L}_m & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{C}_1 & & \\ & \ddots & \ddots & \\ & & \boldsymbol{U}_{m-1} & \boldsymbol{C}_{m-1} \\ & & & \boldsymbol{U}_m \end{bmatrix}.$$

$$\boldsymbol{U}_1 = \boldsymbol{D}_1, \quad \boldsymbol{L}_k = \boldsymbol{A}_k \boldsymbol{U}_{k-1}^{-1}, \quad \boldsymbol{U}_k = \boldsymbol{D}_k - \boldsymbol{L}_k \boldsymbol{C}_{k-1}, \quad k = 2, 3, \ldots, m. \quad (4)$$

To solve the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ we partition $\boldsymbol{b}$ conformally with $\boldsymbol{A}$ in the form $\boldsymbol{b}^T = [\boldsymbol{b}_1^T, \ldots, \boldsymbol{b}_m^T]$. The formulas for solving $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$ and $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$ are as follows:

$$\begin{aligned} \boldsymbol{y}_1 &= \boldsymbol{b}_1, & \boldsymbol{y}_k &= \boldsymbol{b}_k - \boldsymbol{L}_k \boldsymbol{y}_{k-1}, & k &= 2, 3, \ldots, m, \\ \boldsymbol{x}_m &= \boldsymbol{U}_m^{-1} \boldsymbol{y}_m, & \boldsymbol{x}_k &= \boldsymbol{U}_k^{-1}(\boldsymbol{y}_k - \boldsymbol{C}_k \boldsymbol{x}_{k+1}), & k &= m-1, \ldots, 2, 1. \end{aligned} \quad (5)$$

The solution is then $\boldsymbol{x}^T = [\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_m^T]$.

Poisson Problem

- The discrete Poisson problem can be solved writing it as a block tridiagonal matrix.
- We can also use Cholesky factorization of a band matrix.
- The matrix has bandwidth $d = \sqrt{n}$.
- Cholesky requires $O(2nd^2) = O(2n^2)$ flops.
- The work using block tridiagonal solver is also $O(n^2)$.
- Need better methods (next time).

Kronecker Product

Definition 1. *For any positive integers p, q, r, s we define the Kronecker product of two matrices $\boldsymbol{A} \in \mathbb{R}^{p,q}$ and $\boldsymbol{B} \in \mathbb{R}^{r,s}$ as a matrix $\boldsymbol{C} \in \mathbb{R}^{pr,qs}$ given in block form as*

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{A}b_{1,1} & \boldsymbol{A}b_{1,2} & \cdots & \boldsymbol{A}b_{1,s} \\ \boldsymbol{A}b_{2,1} & \boldsymbol{A}b_{2,2} & \cdots & \boldsymbol{A}b_{2,s} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{A}b_{r,1} & \boldsymbol{A}b_{r,2} & \cdots & \boldsymbol{A}b_{r,s} \end{bmatrix}.$$

We denote the Kronecker product of $\boldsymbol{A}$ and $\boldsymbol{B}$ by $\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B}$.

- Defined for rectangular matrices of any dimension
- $\#$ of rows(columns) = product of $\#$ of rows(columns) in $\boldsymbol{A}$ and $\boldsymbol{B}$

Kronecker Product Example

$$\boldsymbol{T} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T} \otimes \boldsymbol{I} = \begin{bmatrix} \boldsymbol{T} & 0 \\ 0 & \boldsymbol{T} \end{bmatrix} = \left[\begin{array}{cc|cc} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$$\boldsymbol{I} \otimes \boldsymbol{T} = \begin{bmatrix} 2\boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & 2\boldsymbol{I} \end{bmatrix} = \left[\begin{array}{cc|cc} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ \hline -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{array}\right]$$

Poisson Matrix $= \boldsymbol{T} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{T}$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{T} & & \\ & \boldsymbol{T} & \\ & & \boldsymbol{T} \end{bmatrix} + \begin{bmatrix} 2\boldsymbol{I} & -\boldsymbol{I} & 0 \\ -\boldsymbol{I} & 2\boldsymbol{I} & -\boldsymbol{I} \\ 0 & -\boldsymbol{I} & 2\boldsymbol{I} \end{bmatrix} = \boldsymbol{T} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{T}$$

Definition 2. *Let for positive integers r, s, k, $\boldsymbol{A} \in \mathbb{R}^{r,r}$, $\boldsymbol{B} \in \mathbb{R}^{s,s}$ and $\boldsymbol{I}_k$ be the identity matrix of order k. The sum $\boldsymbol{A} \otimes \boldsymbol{I}_s + \boldsymbol{I}_r \otimes \boldsymbol{B}$ is known as the Kronecker sum of $\boldsymbol{A}$ and $\boldsymbol{B}$.*

The Poisson matrix is the Kronecker sum of $\boldsymbol{T}$ with itself.

Matrix equation $\leftrightarrow$ Kronecker

Given $\boldsymbol{A} \in \mathbb{R}^{r,r}$, $\boldsymbol{B} \in \mathbb{R}^{s,s}$, $\boldsymbol{F} \in \mathbb{R}^{r,s}$. Find $\boldsymbol{V} \in \mathbb{R}^{r,s}$ such that

$$\boldsymbol{A}\boldsymbol{V}\boldsymbol{B}^T = \boldsymbol{F}$$

For $\boldsymbol{B} \in \mathbb{R}^{m,n}$ define

$$\text{vec}(\boldsymbol{B}) := \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_n \end{bmatrix} \in \mathbb{R}^{mn}, \quad \boldsymbol{b}_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} \quad j\text{th column}$$

Lemma 1.

$$\boldsymbol{A}\boldsymbol{V}\boldsymbol{B}^T = \boldsymbol{F} \quad \Leftrightarrow \quad (\boldsymbol{A} \otimes \boldsymbol{B})\,\mathrm{vec}(\boldsymbol{V}) = \mathrm{vec}(\boldsymbol{F}) \tag{6}$$

$$\boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{B}^T = \boldsymbol{F} \quad \Leftrightarrow \quad (\boldsymbol{A} \otimes \boldsymbol{I}_s + \boldsymbol{I}_r \otimes \boldsymbol{B})\,\mathrm{vec}(\boldsymbol{V}) = \mathrm{vec}(\boldsymbol{F}). \tag{7}$$

Proof

We partition $\boldsymbol{V}$, $\boldsymbol{F}$, and $\boldsymbol{B}^T$ by columns as $\boldsymbol{V} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s)$, $\boldsymbol{F} = (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s)$ and $\boldsymbol{B}^T = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_s)$. Then we have

$$
\begin{aligned}
&(\boldsymbol{A} \otimes \boldsymbol{B}) \operatorname{vec}(\boldsymbol{V}) = \operatorname{vec}(\boldsymbol{F}) \\
&\qquad \Leftrightarrow \quad \begin{bmatrix} \boldsymbol{A} b_{11} & \cdots & \boldsymbol{A} b_{1s} \\ \vdots & & \vdots \\ \boldsymbol{A} b_{s1} & \cdots & \boldsymbol{A} b_{ss} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_s \end{bmatrix} \\
&\qquad \Leftrightarrow \quad \boldsymbol{A} \sum_j b_{ij} v_j = f_i, \; i = 1, \ldots, m \\
&\qquad \Leftrightarrow \quad [\boldsymbol{A}\boldsymbol{V}\boldsymbol{b}_1, \ldots, \boldsymbol{A}\boldsymbol{V}\boldsymbol{b}_s] = \boldsymbol{F} \quad \Leftrightarrow \quad \boldsymbol{A}\boldsymbol{V}\boldsymbol{B}^T = \boldsymbol{F}.
\end{aligned}
$$

This proves (6)

Proof Continued

$$
\begin{aligned}
&(\boldsymbol{A} \otimes \boldsymbol{I}_s + \boldsymbol{I}_r \otimes \boldsymbol{B}) \operatorname{vec}(\boldsymbol{V}) = \operatorname{vec}(\boldsymbol{F}) \\
&\Leftrightarrow \quad (\boldsymbol{A}\boldsymbol{V}\boldsymbol{I}_s^T + \boldsymbol{I}_r\boldsymbol{V}\boldsymbol{B}^T) = \boldsymbol{F} \quad \Leftrightarrow \quad \boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{B}^T = \boldsymbol{F}.
\end{aligned}
$$

$\square$

This gives a slick way to derive the Poisson matrix. Recall

$$
\boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{B}^T = \boldsymbol{F} \quad \Leftrightarrow \quad (\boldsymbol{A} \otimes \boldsymbol{I}_s + \boldsymbol{I}_r \otimes \boldsymbol{B}) \operatorname{vec}(\boldsymbol{V}) = \operatorname{vec}(\boldsymbol{F})
$$

Therefore,

$$
\boldsymbol{T}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{T} = h^2\boldsymbol{F} \Leftrightarrow (\boldsymbol{T} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{T}) \operatorname{vec}(\boldsymbol{V}) = h^2 \operatorname{vec}(\boldsymbol{F})
$$

Mixed Product Rule

$$(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = (\boldsymbol{AC}) \otimes (\boldsymbol{BD})$$

Proof If $\boldsymbol{B} \in \mathbb{R}^{r,t}$ and $\boldsymbol{D} \in \mathbb{R}^{t,s}$ for some integers r, t then

$$\begin{bmatrix} \boldsymbol{A}b_{1,1} & \cdots & \boldsymbol{A}b_{1,t} \\ \vdots & & \vdots \\ \boldsymbol{A}b_{r,1} & \cdots & \boldsymbol{A}b_{r,t} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}d_{1,1} & \cdots & \boldsymbol{C}d_{1,s} \\ \vdots & & \vdots \\ \boldsymbol{C}d_{t,1} & \cdots & \boldsymbol{C}d_{t,s} \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}_{1,1} & \cdots & \boldsymbol{E}_{1,s} \\ \vdots & & \vdots \\ \boldsymbol{E}_{r,1} & \cdots & \boldsymbol{E}_{r,s} \end{bmatrix},$$

where for all i, j

$$\boldsymbol{E}_{i,j} = \sum_{k=1}^{t} b_{i,k} d_{k,j} \boldsymbol{AC} = (\boldsymbol{AC})(\boldsymbol{BD})_{i,j} = ((\boldsymbol{AC}) \otimes (\boldsymbol{BD}))_{i,j}.$$

where in the last formula i, j refers to the ij-block in the Kronecker product. $\square$

Properties of Kronecker Products

The usual arithmetic rules hold. Note however that

- $(\boldsymbol{A} \otimes \boldsymbol{B})^T = \boldsymbol{A}^T \otimes \boldsymbol{B}^T$
- $(\boldsymbol{A} \otimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1}$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsingular
- $\boldsymbol{A} \otimes \boldsymbol{B} \neq \boldsymbol{B} \otimes \boldsymbol{A}$
- $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = (\boldsymbol{AC}) \otimes (\boldsymbol{BD})$ (mixed product rule)

The mixed product rule is valid for any matrices as long as the products $\boldsymbol{AC}$ and $\boldsymbol{BD}$ are defined.

Eigenvalues and Eigenvectors

The Kronecker product of two vectors $\boldsymbol{u} \in \mathbb{R}^p$ and $\boldsymbol{v} \in \mathbb{R}^r$ is a vector $\boldsymbol{u} \otimes \boldsymbol{v} \in \mathbb{R}^{pr}$ given by $\boldsymbol{u} \otimes \boldsymbol{v} = \left[\boldsymbol{u}^T v_1, \ldots, \boldsymbol{u}^T v_r\right]^T$ Suppose now $\boldsymbol{A} \in \mathbb{R}^{r,r}$ and $\boldsymbol{B} \in \mathbb{R}^{s,s}$ and

$$\boldsymbol{A}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i, \quad i = 1, \ldots, r, \quad \boldsymbol{B}\boldsymbol{v}_j = \mu_j \boldsymbol{v}_j, \quad j = 1, \ldots, s,$$

then for $i = 1, \ldots, r, \quad j = 1, \ldots, s$

$$(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{u}_i \otimes \boldsymbol{v}_j) = \lambda_i \mu_j (\boldsymbol{u}_i \otimes \boldsymbol{v}_j), \tag{8}$$

$$(\boldsymbol{A} \otimes \boldsymbol{I}_s + \boldsymbol{I}_r \otimes \boldsymbol{B})(\boldsymbol{u}_i \otimes \boldsymbol{v}_j) = (\lambda_i + \mu_j)(\boldsymbol{u}_i \otimes \boldsymbol{v}_j). \tag{9}$$

Thus the eigenvalues of a Kronecker product(sum) are the products (sums) of the eigenvalues of the factors. The eigenvectors of a Kronecker product(sum) are the products of the eigenvectors of the factors.

Proof of Eigen-formulae

This follows directly from the mixed product rule. For (8)

$$(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{u}_i \otimes \boldsymbol{v}_j) = (\boldsymbol{A}\boldsymbol{u}_i) \otimes (\boldsymbol{B}\boldsymbol{v}_j) = (\lambda_i \boldsymbol{u}_i) \otimes (\mu_j \boldsymbol{v}_j) = (\lambda_i \mu_j)(\boldsymbol{u}_i \otimes \boldsymbol{v}_j).$$

From (8)

$$(\boldsymbol{A} \otimes \boldsymbol{I}_s)(\boldsymbol{u}_i \otimes \boldsymbol{v}_j) = \lambda_i(\boldsymbol{u}_i \otimes \boldsymbol{v}_j), \quad \text{and} \quad (\boldsymbol{I}_r \otimes \boldsymbol{B})(\boldsymbol{u}_i \otimes \boldsymbol{v}_j) = \mu_j(\boldsymbol{u}_i \otimes \boldsymbol{v}_j)$$

The result now follows by summing these relations. □

The Poisson Matrix $\boldsymbol{A} = \boldsymbol{T} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{T}$

Recall eigenvalues and eigenvectors of the $\boldsymbol{T}$ matrix. Let $h = 1/(m+1)$.

1. We know that $\boldsymbol{T}\boldsymbol{s}_j = \lambda_j \boldsymbol{s}_j$ for $j = 1, \ldots, m$, where

$$\boldsymbol{s}_j = (\sin(j\pi h), \sin(2j\pi h), \ldots, \sin(mj\pi h))^T, \tag{10}$$

$$\lambda_j = 2 - 2\cos(j\pi h) = 4\sin^2(\frac{j\pi h}{2}). \tag{11}$$

2. The eigenvalues are distinct and the eigenvectors are orthogonal

$$\boldsymbol{s}_j^T \boldsymbol{s}_k = \frac{1}{2h}\delta_{j,k}, \quad j, k = 1, \ldots, m. \tag{12}$$

Eigenvalues/-vectors $\boldsymbol{A} = \boldsymbol{T} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{T}$

1. We have $\boldsymbol{A}\boldsymbol{x}_{j,k} = \lambda_{j,k}\boldsymbol{x}_{j,k}$ for $j,k = 1,\dots,m$, where

$$\boldsymbol{x}_{j,k} = \boldsymbol{s}_j \otimes \boldsymbol{s}_k, \tag{13}$$

$$\boldsymbol{s}_j = (\sin(j\pi h), \sin(2j\pi h), \dots, \sin(mj\pi h))^T, \tag{14}$$

$$\lambda_{j,k} = 4\sin^2(\frac{j\pi h}{2}) + 4\sin^2(\frac{k\pi h}{2}). \tag{15}$$

2. The eigenvectors are orthogonal

$$\boldsymbol{x}_{j,k}^T\boldsymbol{x}_{p,q} = \frac{1}{4h^2}\delta_{j,p}\delta_{k,q}, \quad j,k,p,q = 1,\dots,m. \tag{16}$$

3. $\boldsymbol{A}$ is symmetric $\boldsymbol{A}^T = (\boldsymbol{T} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{T})^T = \boldsymbol{T}^T \otimes \boldsymbol{I}^T + \boldsymbol{I}^T \otimes \boldsymbol{T}^T = \boldsymbol{A}$
4. $\boldsymbol{A}$ is positive definite (positive eigenvalues)

$\boldsymbol{A} = \boldsymbol{T} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{T}$ for $m = 2$

$$\boldsymbol{s}_1 = \begin{bmatrix} \sin(\frac{\pi}{3}) \\ \sin(\frac{2\pi}{3}) \end{bmatrix} = \frac{\sqrt{3}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{s}_2 = \begin{bmatrix} \sin(\frac{2\pi}{3}) \\ \sin(\frac{4\pi}{3}) \end{bmatrix} = \frac{\sqrt{3}}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_1 = 4\sin^2\left(\frac{\pi}{6}\right) = 1, \quad \lambda_2 = 4\sin^2\left(\frac{2\pi}{6}\right) = 3$$

$$\boldsymbol{A}\boldsymbol{x}_{ij} = \mu_{ij}\boldsymbol{x}_{ij},\ \mu_{ij} = \lambda_i + \lambda_j,\ \boldsymbol{x}_{ij} = \boldsymbol{s}_i \otimes \boldsymbol{s}_j \quad i,j = 1,2$$

$$\mu_{11} = 2,\ \mu_{12} = \mu_{21} = 4,\ \mu_{22} = 6.$$

$$\begin{aligned} \boldsymbol{s}_1 \otimes \boldsymbol{s}_1 &= \frac{3}{4}[1,1,1,1]^T, & \boldsymbol{s}_1 \otimes \boldsymbol{s}_2 &= \frac{3}{4}[1,1,-1,-1]^T, \\ \boldsymbol{s}_2 \otimes \boldsymbol{s}_1 &= \frac{3}{4}[1,-1,1,-1]^T, & \boldsymbol{s}_2 \otimes \boldsymbol{s}_2 &= \frac{3}{4}[1,-1,-1,1]^T \end{aligned}$$

$$\boldsymbol{x}_{j,k}^T \boldsymbol{x}_{p,q} = \frac{9}{4}\delta_{j,p}\delta_{k,q}, \quad j,k,p,q = 1,2.$$

Check

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

etc.

Summary

- Studied the Poisson matrix A
- Shown that A can be written as a Kronecker sum
- studied properties of general Kronecker products and Kronecker sums
- used this to derive properties of A