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1. Jacobi Method

Two assumptions makes by Jacobi Method:

(a) That the system given by has unique solution.

(b) That coefficient matrix A has no zeros on it's main diagonal.

If there is a number are zero in diagonal entries, it's need to changed with others row.

The initial zero iteration: $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \ldots x_n^{(0)})$

The equation of solving Ax = b to obtain xi :

$$
\begin{aligned}
x_1^{k+1} &= \frac{1}{a_{11}}(b_1 - a_{12}x_2^k - a_{13}x_3^k - \ldots - a_{1n}x_n^k) \\
x_2^{k+1} &= \frac{1}{a_{22}}(b_2 - a_{21}x_1^k - a_{23}x_3^k - \ldots - a_{2n}x_n^k) \\
x_3^{k+1} &= \frac{1}{a_{33}}(b_3 - a_{31}x_1^k - a_{32}x_2^k - \ldots - a_{3n}x_n^k) \\
&\vdots \\
x_n^{k+1} &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1^k - a_{n2}x_2^k - \ldots - a_{nn-1}x_{n-1}^k)
\end{aligned}
$$

The Jacobi method in matrix form:

$$
A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}.
$$

$$
A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ -a_{21} & 0 & \cdots & 0 & 0 \\ -a_{31} & -a_{32} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{n,n-1} & 0 \end{pmatrix} - \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ 0 & -a_{22} & -a_{23} & \cdots & -a_{2n} \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad = D - L - U.
$$

$$
x_i^{k+1} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1 \\ j\neq i}}^{n} (-a_{ij}x_j^k) + b1\right] \qquad \text{for } i = 1, 2, 3 \ldots\ldots n
$$

Therefore, the matrix Ax = b can be transformed into (D - L - U) x = b .

Dx = (L + U) x + b

2. Gauss-Seidel Method

Modification of Jacobi method. Gauss-Seidel method require fewer iterations to calculate the same accuracy of the same degree compare to Gauss-Seidel.

$$x_1^{k+1} = \frac{1}{a_{11}}(b_1 - a_{12}x_2^k - a_{13}x_3^k - \ldots - a_{1n}x_n^k)$$
$$x_2^{k+1} = \frac{1}{a_{22}}(b_2 - a_{21}x_1^{k+1} - a_{23}x_3^k - \ldots - a_{2n}x_n^k)$$
$$x_3^{k+1} = \frac{1}{a_{33}}(b_3 - a_{31}x_1^{k+1} - a_{32}x_2^{k+1} - \ldots - a_{3n}x_n^k)$$
$$\vdots$$
$$x_n^{k+1} = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^{k+1} - a_{n2}x_2^{k+1} - \ldots - a_{nn-1}x_{n-1}^{k+1})$$

Similar to Jacobi method but using Xi for i = 1 ~ j -1,

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i=1}\left(a_{ij}x_j^k\right) - \sum_{j=i+1}^{n}\left(a_{ij}x_j^{k+1}\right) + b1\right] \qquad \text{For } i = 1, 2 \ldots n$$

And Gauss-Seidel method in a matrix form:

$$(\mathrm{D} - \mathrm{L})x^{k+1} = \mathrm{U}x^k + \mathrm{b}$$
$$x^{k+1} = (D - L)^{-1}\mathrm{U}x^k + (D - L)^{-1}b$$

3. Convergence of Iterative Method

The rate of convergence of iterative methods determine how fast the error |x^k - x| goes to zero, the number of the iteration increases.

Compare two different method:

Jacobi method:

$$x_n^{k+1} = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^k - a_{n2}x_2^k - \ldots - a_{nn-1}x_{n-1}^k)$$

Gauss-Seidel method:

$$x_n^{k+1} = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^{k+1} - a_{n2}x_2^{k+1} - \ldots - a_{nn-1}x_{n-1}^{k+1})$$

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Gauss-Seidel method when calculate the Xj(k+1), j > 1, it would use the result of Xi(k+1), i < j. In this way, calculating X1 ~ Xn-1 would quickly get closer to exact value. So using Gauss-Seidel method needs less iterations.