AERSP597 Midterm

Ani Perumalla

April 1, 2021

1 Q. #1

[1]:
```
# Import necessary packages
import warnings # Ignore user warnings
import itertools as it # Readable nested for loops
from pathlib import Path # Filepaths
import typing # Argument / output type checking
import numpy as np # N-dim arrays + math
import scipy.linalg as spla # Complex linear algebra
import matplotlib.pyplot as plt # Plots
import matplotlib.figure as figure # Figure documentation
import scipy.signal as spsg # Signal processing
import sympy # Symbolic math + pretty printing

# Logistics
warnings.simplefilter("ignore", UserWarning)
sympy.init_printing()
figs_dir = (Path.cwd() / "figs")
figs_dir.mkdir(parents = True, exist_ok = True)
prob = 1
```

[2]:
```
def etch(sym: str, mat: np.ndarray):
    display(sympy.Eq(sympy.Symbol(sym),
                     sympy.Matrix(mat.round(5)),
                     evaluate = False))
    pass


def d2c(A: np.ndarray, B: np.ndarray,
        dt: float) \
        -> typing.Tuple[np.ndarray, np.ndarray]:
    """Convert discrete linear state space model to continuous linear state␣
↪space model.

    :param np.ndarray A:
    :param np.ndarray B:
    :param float dt: Timestep duration
```

```
    :return: (A_c, B_c) Continuous-time linear state space model
    """
    A_c = spla.logm(A)/dt
    if np.linalg.cond(A - np.eye(*A.shape)) < 1/np.spacing(1):
        B_c = A_c @ spla.inv(A - np.eye(*A.shape)) @ B
    else:
        B_temp = np.zeros(A_c.shape)
        for i in range(200):
            B_temp += (1/((i + 1)*np.math.factorial(i)))*np.linalg.
↪matrix_power(A_c, i)*(dt**(i + 1))
        B_c = B @ spla.inv(B_temp)
    return A_c, B_c


def c2d(A_c: np.ndarray, B_c: np.ndarray,
        dt: float) \
        -> typing.Tuple[np.ndarray, np.ndarray]:
    """Convert continuous linear state space model to discrete linear state␣
↪space model.

    :param np.ndarray A_c:
    :param np.ndarray B_c:
    :param float dt: Timestep duration
    :return: (A, B) Discrete-time linear state space model
    """
    A = spla.expm(A_c*dt)
    if np.linalg.cond(A_c) < 1/np.spacing(1):
        B = (A - np.eye(*A.shape)) @ spla.inv(A_c) @ B_c
    else:
        B_temp = np.zeros(A_c.shape)
        for i in range(200):
            B_temp += (1/((i + 1)*np.math.factorial(i)))*np.linalg.
↪matrix_power(A_c, i)*(dt**(i + 1))
        B = B_temp @ B_c
    return A, B


def sim_ss(A: np.ndarray, B: np.ndarray, C: np.ndarray, D: np.ndarray,
           X_0: np.ndarray, U: np.ndarray,
           nt: int) \
        -> typing.Tuple[np.ndarray, np.ndarray]:
    """Simulate linear state space model via ZOH.

    :param np.ndarray A:
    :param np.ndarray B:
    :param np.ndarray C:
    :param np.ndarray D:
```

```
    :param np.ndarray X_0: Initial state condition
    :param np.ndarray U: Inputs, either impulse or continual
    :param nt: Number of timesteps to simulate
    :return: (X) State vector array over duration; (Z) Observation vector array␣
↪over duration
    """
    assert D.shape == (C @ A @ B).shape
    assert X_0.shape[-2] == A.shape[-1]
    assert U.shape[-2] == B.shape[-1]
    assert A.shape[-2] == B.shape[-2]
    assert C.shape[-2] == D.shape[-2]
    assert A.shape[-1] == C.shape[-1]
    assert B.shape[-1] == D.shape[-1]
    assert (U.shape[-1] == 1) or (U.shape[-1] == nt) or (U.shape[-1] == nt - 1)

    X = np.concatenate([X_0, np.zeros([X_0.shape[-2], nt])], 1)
    Z = np.zeros([C.shape[-2], nt])
    if U.shape[-1] == 1: # Impulse
        X[:, 1] = (A @ X[:, 0]) + (B @ U[:, 0])
        Z[:, 0] = (C @ X[:, 0]) + (D @ U[:, 0])
        for i in range(1, nt):
            X[:, i + 1] = (A @ X[:, i])
            Z[:, i] = (C @ X[:, i])
    else: # Continual
        for i in range(nt):
            X[:, i + 1] = (A @ X[:, i]) + (B @ U[:, i])
            Z[:, i] = (C @ X[:, i]) + (D @ U[:, i])
    return X, Z


def markov_sim(Y: np.ndarray, U: np.ndarray) \
        -> np.ndarray:
    """Obtain observations from Markov parameters and inputs, for zero initial␣
↪conditions

    :param np.ndarray Y: Markov parameter matrix
    :param np.ndarray U: Continual inputs
    :return: (Z) Observation vector array over duration
    :rtype: np.ndarray
    """
    l, m, r = Y.shape
    Y_2_Z = np.zeros([r*l, l])
    Y_2_Z[:r, :] = U
    for i in range(1, l):
        Y_2_Z[r*i:r*(i + 1), :] = np.concatenate([np.zeros([r, i]), U[:, 0:
↪(-i)]], 1)
    Z = np.concatenate(Y, 1) @ Y_2_Z
```

```
    return Z


def ss2markov(A: np.ndarray, B: np.ndarray, C: np.ndarray, D: np.ndarray,
              nt: int) \
        -> np.ndarray:
    """Get Markov parameters from state space model.

    :param np.ndarray A:
    :param np.ndarray B:
    :param np.ndarray C:
    :param np.ndarray D:
    :param nt: Number of Markov parameters to generate (i.e., length of␣
↪simulation)
    :return: (Y) 3D array of Markov parameters
    :rtype: np.ndarray
    """
    assert D.shape == (C @ A @ B).shape
    Y = np.zeros([nt, *D.shape])
    Y[0] = D
    for i in range(1, nt):
        Y[i] = C @ (np.linalg.matrix_power(A, i - 1)) @ B
    return Y


def Hankel(Y: np.ndarray, alpha: int, beta: int, i: int = 0) \
        -> np.ndarray:
    """Hankel matrix.

    :param Y: Markov parameter matrix
    :param alpha: Num. of rows of Markov parameters in Hankel matrix
    :param beta: Num. of columns of Markov parameters in Hankel matrix
    :param i: Start node of Hankel matrix
    :return: Block Hankel matrix.
    :rtype: np.ndarray
    """
    assert (len(Y) - 1) >= (i + alpha + beta - 1)
    m, r = Y.shape[-2:]
    H = np.zeros([alpha*m, beta*r])
    for j in range(beta):
        H[:, (j*r):((j + 1)*r)] = Y[(i + 1 + j):(i + alpha + 1 + j)].
↪reshape([alpha*m, r])
    return H


def era(Y: np.ndarray, alpha: int, beta: int, n: int) \
        -> typing.Tuple[np.ndarray,
```

```python
                        np.ndarray,
                        np.ndarray,
                        np.ndarray,
                        np.ndarray]:
    """Eigensystem Realization Algorithm (ERA).

    :param np.ndarray Y: Markov parameter matrix
    :param int alpha: Num. of rows of Markov parameters in Hankel matrix
    :param int beta: Num. of columns of Markov parameters in Hankel matrix
    :param int n: Order of proposed linear state space system
    :returns: (A, B, C, D) - State space of proposed linear state space system;␣
↪(S) - Singular Values of H(0)
    :rtype: (np.ndarray, np.ndarray, np.ndarray, np.ndarray, np.ndarray, np.
↪ndarray)
    """
    assert (len(Y) - 1) >= (alpha + beta - 1)
    m, r = Y.shape[-2:]
    assert (alpha >= (n/m)) and (beta >= (n/r))
    H_0 = Hankel(Y, alpha, beta, 0)
    print(f"Rank of H(0): {np.linalg.matrix_rank(H_0)}")
    H_1 = Hankel(Y, alpha, beta, 1)
    print(f"Rank of H(1): {np.linalg.matrix_rank(H_1)}")
    U_sim, S, Vh = np.linalg.svd(H_0)
    V = Vh.T
    U_n = U_sim[:, :n]
    V_n = V[:, :n]
    S_n = S[:n]

    E_r = np.concatenate([np.eye(r), np.tile(np.zeros([r, r]), beta - 1)], 1).T
    E_m = np.concatenate([np.eye(m), np.tile(np.zeros([m, m]), alpha - 1)], 1).T
    A = np.diag(S_n**(-1/2)) @ U_n.T @ H_1 @ V_n @ np.diag(S_n**(-1/2))
    B = np.diag(S_n**(1/2)) @ V_n.T @ E_r
    C = E_m.T @ U_n @ np.diag(S_n**(1/2))
    D = Y[0]
    return A, B, C, D, S


def okid(Z: np.ndarray, U: np.ndarray,
         l_0: int,
         alpha: int, beta: int,
         n: int):
    """Observer Kalman Identification Algorithm (OKID).

    :param np.ndarray Z: Observation vector array over duration
    :param np.ndarray U: Continual inputs
    :param int l_0: Order of OKID to execute (i.e., number of Markov parameters␣
↪to generate via OKID)
```

```
    :param int alpha: Num. of rows of Markov parameters in Hankel matrix
    :param int beta: Num. of columns of Markov parameters in Hankel matrix
    :param int n: Number of proposed states to use for ERA
    :return: (Y) Markov parameters
    :rtype: np.ndarray
    """
    r, l_u = U.shape
    m, l  = Z.shape
    assert l == l_u
    V = np.concatenate([U, Z], 0)
    assert (max([alpha + beta, (n/m) + (n/r)]) <= l_0) and (l_0 <= (l - r)/(r +␣
↪m)) # Boundary conditions

    # Form observer
    Y_2_Z = np.zeros([r + (r + m)*l_0, l])
    Y_2_Z[:r, :] = U
    for i in range(1, l_0 + 1):
        Y_2_Z[((i*r) + ((i - 1)*m)):(((i + 1)*r) + (i*m)), :] = np.
↪concatenate([np.zeros([r + m, i]), V[:, 0:(-i)]], 1)
    # Find Observer Markov parameters via least-squares
    Y_obs = Z @ spla.pinv2(Y_2_Z)
    Y_bar_1 = np.array(list(it.chain.from_iterable([Y_obs[:, i:(i + r)]
                                                    for i in range(r, r + (r +␣
↪m)*l_0, r + m)]))).reshape([l_0, m, r])
    Y_bar_2 = -np.array(list(it.chain.from_iterable([Y_obs[:, i:(i + m)]
                                                     for i in range(2*r, r + (r␣
↪+ m)*l_0, r + m)]))).reshape([l_0, m, m])

    # Obtain Markov parameters from Observer Markov parameters
    Y = np.zeros([l_0 + 1, m, r])
    Y[0] = Y_obs[:, :r]
    for k in range(1, l_0 + 1):
        Y[k] = Y_bar_1[k - 1] - \
               np.array([Y_bar_2[i] @ Y[k - (i + 1)]
                         for i in range(k)]).sum(axis = 0)
    # Obtain Observer Gain Markov parameters from Observer Markov parameters
    Y_og = np.zeros([l_0, m, m])
    Y_og[0] = Y_bar_2[0]
    for k in range(1, l_0):
        Y_og[k] = Y_bar_2[k] - \
                  np.array([Y_bar_2[i] @ Y_og[k - (i + 1)]
                            for i in range(k - 1)]).sum(axis = 0)
    return Y, Y_og
```

[3]:
```
# Set seed for consistent results
rng = np.random.default_rng(seed = 100)
```

```python
# Simulation dimensions
cases = 3 # Number of cases
n = 2 # Number of states
r = 1 # Number of inputs
m = 2 # Number of measurements
t_max = 50 # Total simulation time
dt = 0.1 # Simulation timestep duration
nt = int(t_max/dt) # Number of simulation timesteps

# Simulation time
train_cutoff = int(20/dt) + 1
t_sim = np.linspace(0, t_max, nt + 1)
t_train = t_sim[:train_cutoff]
t_test = t_sim
nt_train = train_cutoff
nt_test = nt

# Problem parameters
theta_0 = 0.5 # Angular velocity
k = 10 # Spring stiffness
mass = 1 # Point mass

# State space model
A_c = np.array([[0, 1], [theta_0**2 - k/mass, 0]])
B_c = np.array([[0], [1]])
C = np.eye(2)
D = np.array([[0], [1]])
A, B = c2d(A_c, B_c, dt)
eig_A = spla.eig(A_c)[0] # Eigenvalues of true system
etch(f"\lambda", eig_A)
etch(f"\omega_{{n}}", np.abs(eig_A))
etch(f"\zeta", -np.cos(np.angle(eig_A)))
# Note that damping is always positive even when it is displayed as negative.

# True simulation values
X_0_sim = np.zeros([n, 1]) # Zero initial condition
U_sim = np.zeros([cases, r, nt]) # True input vectors
U_sim[0] = rng.normal(0, 0.1, [r, nt]) # True input for case 1
U_sim[1] = spsg.square(2*np.pi*5*t_sim[:-1]) # True input for case 2
U_sim[2] = np.cos(2*np.pi*2*t_sim[:-1]) # True input for case 3
X_sim = np.zeros([cases, n, nt + 1]) # True state vectors
Z_sim = np.zeros([cases, m, nt]) # True observation vectors

# Separation into train and test data
U_train = U_sim[0, :r, :train_cutoff] # Train input vector
U_test = U_sim # Test input vectors
X_train = np.zeros([n, nt_train + 1]) # Train state vector
```

```
X_test = np.zeros([cases, n, nt_test + 1]) # Test state vectors
Z_train = np.zeros([m, nt_train]) # Train observation vector
Z_test = np.zeros([cases, m, nt_test]) # Test observation vectors
V_train = np.zeros([r + m, nt_train]) # Train observation input vectors
V_test = np.zeros([cases, r + m, nt_test]) # Test observation input vectors
```

$$\lambda = \begin{bmatrix} 3.1225i \\ -3.1225i \end{bmatrix}$$

$$\omega_n = \begin{bmatrix} 3.1225 \\ 3.1225 \end{bmatrix}$$

$$\zeta = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$$

[4]:
```
# OKID logistics
order = 50 # Order of OKID algorithm, number of Markov parameters to identify␣
 ↪after the zeroeth
alpha, beta = 15, 20 # Number of block rows and columns in Hankel matrices
n_era = 2 # Number of proposed states
X_0_okid = np.zeros([n_era, 1]) # Zero initial condition

print(f"Min. OKID Order: {max([alpha + beta, (n_era/m) + (n_era/r)]):n}")
print(f"Max. OKID Order: {(nt_train - r)/(r + m):n}")
print(f"Proposed OKID Order: {order:n}")
```

```
Min. OKID Order: 35
Max. OKID Order: 66.6667
Proposed OKID Order: 50
```

We propose to use an OKID order l_0 of 50, since this value falls inside the range of permitted OKID orders.

The Hankel matrix $H(0)$ must be at least rank n. In addition, the sum of the numbers of block rows (α) and columns (β) should be at most the number of estimated Markov parameters (i.e., the OKID order $l_0 = 50$). Moreover, to meet the full rank condition for $H(0)$, $\alpha \geq \frac{n}{m} = 1$ and $\beta \geq \frac{n}{r} = 2$. To meet these criteria, we choose $\alpha = 15$ and $\beta = 20$.

As shown by the SVD plot below, the order of the system appears to be 2, since there is a sharp dropoff in the value of the singular values for $n > 2$. Therefore, we will choose the number of states for the OKID/ERA-proposed system to be 2.

[5]:
```
# OKID System Markov parameters
Y_okid = np.zeros([order + 1, m, r])
# OKID Observer Markov Gain parameters
Y_og_okid = np.zeros([order, m, m])
# OKID state vector, drawn from state space model derived from OKID/ERA
X_okid_train = np.zeros([n_era, nt_train + 1])
X_okid_test = np.zeros([cases, n_era, nt_test + 1])
X_okid_train_obs = np.zeros([n_era, nt_train + 1])
```

```
X_okid_test_obs = np.zeros([cases, n_era, nt_test + 1])
# OKID observations, drawn from state space model derived from OKID/ERA
Z_okid_train = np.zeros([n_era, nt_train])
Z_okid_test = np.zeros([cases, n_era, nt_test])
Z_okid_train_obs = np.zeros([n_era, nt_train])
Z_okid_test_obs = np.zeros([cases, n_era, nt_test])
# Singular values of the Hankel matrix constructed through OKID Markov␣
↪parameters
S_okid = np.zeros([min(alpha*m, beta*r)])
eig_A_okid = np.zeros([n_era], dtype = complex)

# OKID/ERA state space model
A_okid = np.zeros([n_era, n_era])
B_okid = np.zeros([n_era, r])
C_okid = np.zeros([m, n_era])
D_okid = np.zeros([m, r])
G_okid = np.zeros([m, m])
# OKID/ERA state space model augmented with observer
A_okid_obs = np.zeros([n_era, n_era])
B_okid_obs = np.zeros([n_era, r + m])
C_okid_obs = np.zeros([m, n_era])
D_okid_obs = np.zeros([m, r + m])
```

[6]:

```
# Simulation
for i in range(cases):
    X_sim[i], Z_sim[i] = sim_ss(A, B, C, D, X_0 = X_0_sim, U = U_sim[i], nt =␣
↪nt)
    if i == 0:
        # Split between train and test data for case 1
        X_train, Z_train = X_sim[i, :, :train_cutoff], Z_sim[i, :, :
↪train_cutoff]
        # Identify System Markov parameters and Observer Gain Markov parameters
        Y_okid, Y_og_okid = okid(Z_train, U_train,
                                 l_0 = order, alpha = alpha, beta = beta, n =␣
↪n_era)
        # Identify state space model using System Markov parameters for ERA
        A_okid, B_okid, C_okid, D_okid, S_okid = \
            era(Y_okid, alpha = alpha, beta = beta, n = n_era)
        # Construct observability matrix
        O_p_okid = np.array([C_okid @ np.linalg.matrix_power(A_okid, i)
                             for i in range(order)])
        # Find observer gain matrix
        G_okid = spla.pinv2(O_p_okid.reshape([order*m, n_era])) @ Y_og_okid.
↪reshape([order*m, m])
        # Augment state space model with observer
        A_okid_obs = A_okid + G_okid @ C_okid
        B_okid_obs = np.concatenate([B_okid + G_okid @ D_okid, -G_okid], 1)
```

```
        C_okid_obs = C_okid
        D_okid_obs = np.concatenate([D_okid, np.zeros([m, m])], 1)
        V_train = np.concatenate([U_train, Z_train], 0)
        # Simulate OKID realization with "raw" state and OKID realization with␣
↪estimated state
        X_okid_train, Z_okid_train = \
            sim_ss(A_okid, B_okid, C_okid, D_okid,
                   X_0 = X_0_okid, U = U_train, nt = nt_train)
        X_okid_train_obs, Z_okid_train_obs = \
            sim_ss(A_okid_obs, B_okid_obs, C_okid_obs, D_okid_obs,
                   X_0 = X_0_okid, U = V_train, nt = nt_train)
        # Display outputs
        etch(f"A_{{OKID}}", A_okid)
        etch(f"B_{{OKID}}", B_okid)
        etch(f"C_{{OKID}}", C_okid)
        etch(f"D_{{OKID}}", D_okid)
        etch(f"G_{{OKID}}", G_okid)
        # Calculate and display eigenvalues
        eig_A_okid = spla.eig(d2c(A_okid, B_okid, dt)[0])[0] # Eigenvalues of␣
↪identified system
        etch(f"\hat{{\lambda}}", eig_A_okid)
        etch(f"\hat{{\omega}}_{{n}}", np.abs(eig_A_okid))
        etch(f"\hat{{\zeta}}", -np.cos(np.angle(eig_A_okid)))
    X_test[i], Z_test[i] = X_sim[i], Z_sim[i]
    X_okid_test[i], Z_okid_test[i] = \
        sim_ss(A_okid, B_okid, C_okid, D_okid,
               X_0 = X_0_okid, U = U_test[i], nt = nt_test)
    V_test[i] = np.concatenate([U_test[i], Z_test[i]], 0)
    X_okid_test_obs[i], Z_okid_test_obs[i] = \
        sim_ss(A_okid_obs, B_okid_obs, C_okid_obs, D_okid_obs,
               X_0 = X_0_okid, U = V_test[i], nt = nt_test)
```

```
Rank of H(0): 2
Rank of H(1): 2
```

$$A_{OKID} = \begin{bmatrix} 0.95344 & -0.33621 \\ 0.2807 & 0.94985 \end{bmatrix}$$

$$B_{OKID} = \begin{bmatrix} -0.22189 \\ -0.20365 \end{bmatrix}$$

$$C_{OKID} = \begin{bmatrix} 0.06029 & -0.09004 \\ -0.2558 & -0.20438 \end{bmatrix}$$

$$D_{OKID} = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}$$

$$G_{OKID} = \begin{bmatrix} -0.05416 & 0.15911 \\ 0.03786 & 0.1723 \end{bmatrix}$$

$$\hat{\lambda} = \begin{bmatrix} 3.1225i \\ -3.1225i \end{bmatrix}$$

$$\hat{\omega}_n = \begin{bmatrix} 3.1225 \\ 3.1225 \end{bmatrix}$$

$$\hat{\zeta} = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$$

The eigenvalues of the system are accurately identified via OKID. The natural frequencies and damping ratios are essentially exactly identified.

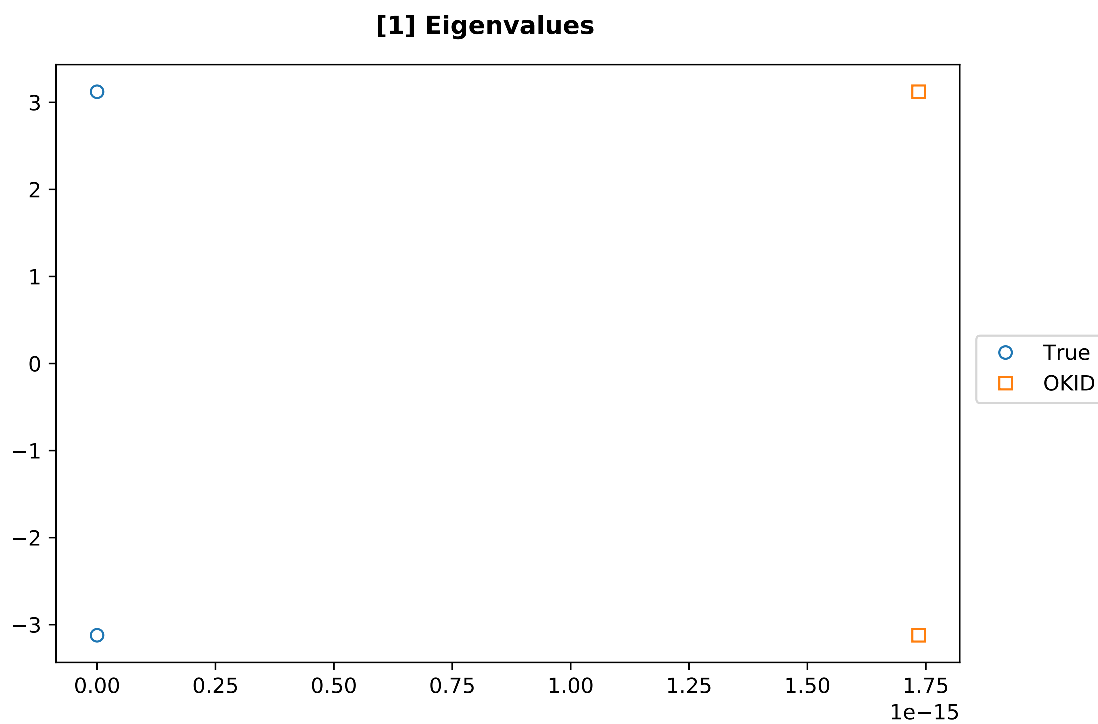
[7]:
```
RMS_train = np.sqrt(np.mean((Z_okid_train - Z_train)**2, axis = 1))
print(f"RMS Error of sim. for system found via OKID for train data:␣
 ↪{RMS_train}")
RMS_test = np.zeros([cases, m])
for i in range(cases):
    RMS_test[i] = np.sqrt(np.mean((Z_okid_test[i] - Z_test[i])**2, axis = 1))
    print(f"RMS Error of sim. for system found via OKID for test data, case {i}:
 ↪ {RMS_test[i]}")
```

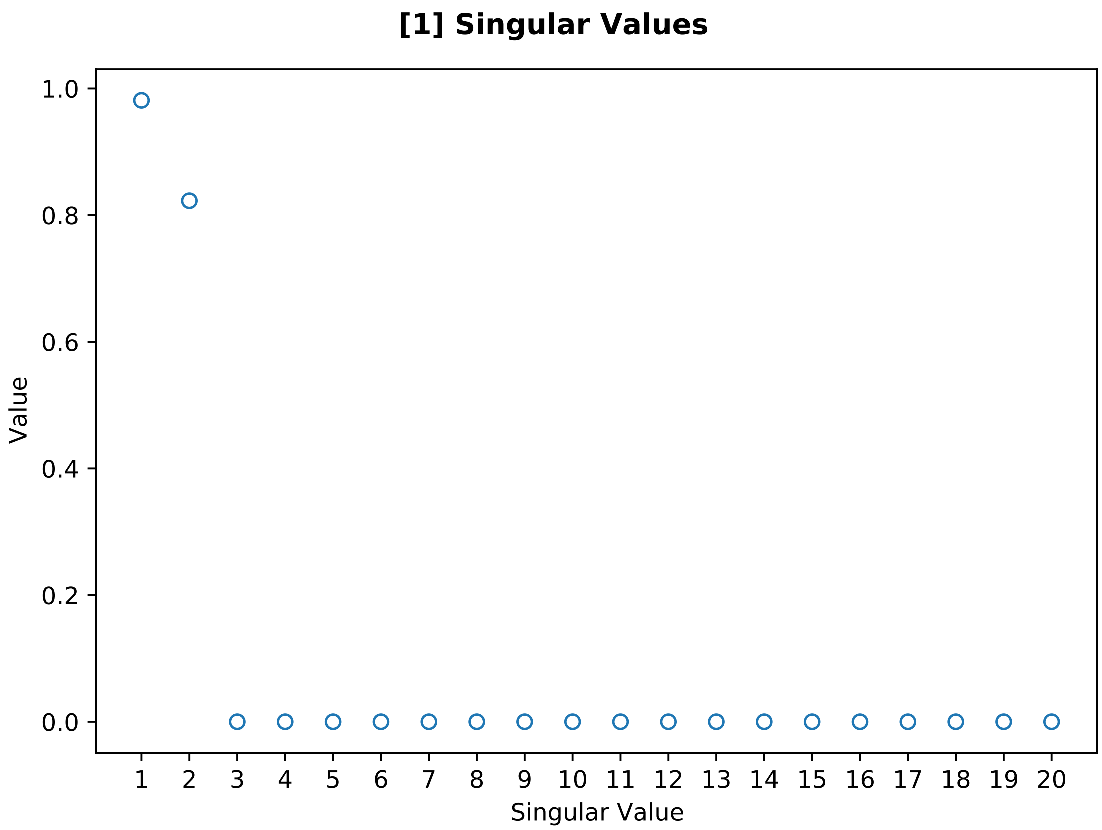
```
RMS Error of sim. for system found via OKID for train data: [5.64018678e-16
1.81879758e-15]
RMS Error of sim. for system found via OKID for test data, case 0:
[1.99758691e-15 6.19485013e-15]
RMS Error of sim. for system found via OKID for test data, case 1:
[6.28794915e-15 1.90445491e-14]
RMS Error of sim. for system found via OKID for test data, case 2:
[6.30791837e-16 2.17401648e-15]
```

[8]:
```
# Eigenvalue plots
fig, ax = plt.subplots(constrained_layout = True) # type:figure.Figure
fig.suptitle(f"[{prob}] Eigenvalues", fontweight = "bold")

ax.plot(np.real(eig_A), np.imag(eig_A),
         "o", mfc = "None")
ax.plot(np.real(eig_A_okid), np.imag(eig_A_okid),
         "s", mfc = "None")

fig.legend(labels = ("True", "OKID"),
           bbox_to_anchor = (1, 0.5), loc = 6)
fig.savefig(figs_dir / f"midterm_{prob}_eigval.pdf",
            bbox_inches = "tight")

# Singular Value plots
fig, ax = plt.subplots(constrained_layout = True) # type:figure.Figure
fig.suptitle(f"[{prob}] Singular Values", fontweight = "bold")

ax.plot(np.linspace(1, len(S_okid), len(S_okid)), S_okid,
```

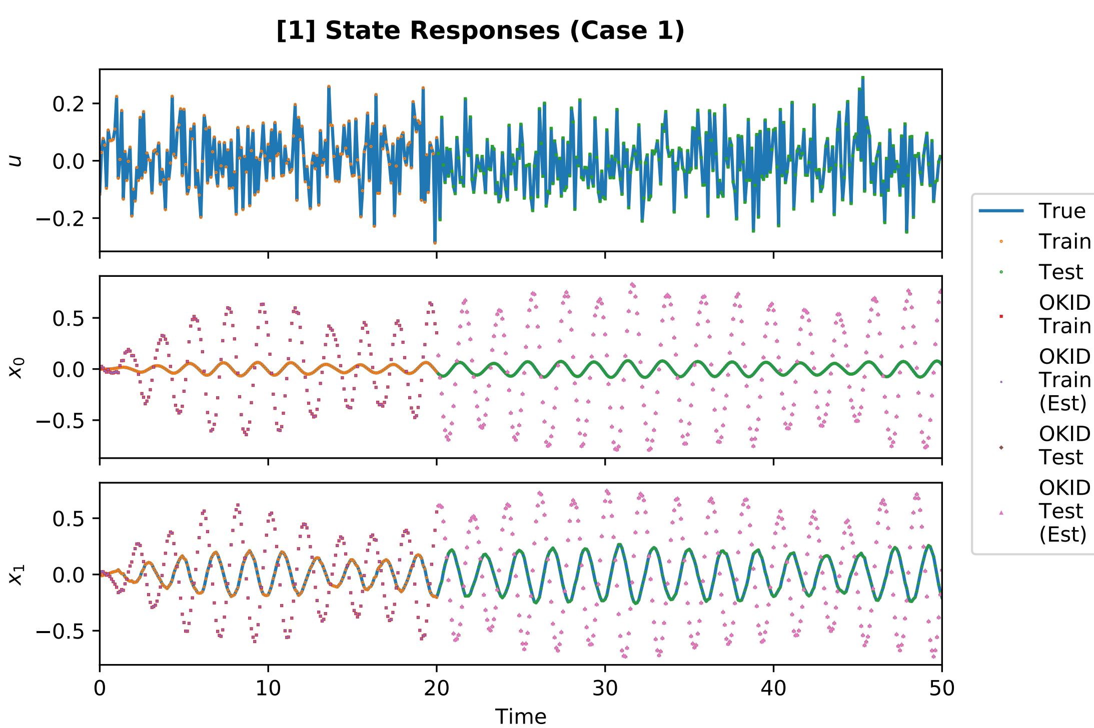
```python
         "o", mfc = "None")
plt.setp(ax, xlabel = f"Singular Value", ylabel = f"Value",
         xticks = np.arange(1, len(S_okid) + 1))

fig.savefig(figs_dir / f"midterm_{prob}_singval.pdf",
            bbox_inches = "tight")

# Response plots
ms = 0.5 # Marker size
for i in range(cases):
    fig, axs = plt.subplots(1 + n, 1,
                            sharex = "col",
                            constrained_layout = True) # type:figure.Figure
    fig.suptitle(f"[{prob}] State Responses (Case {i + 1})",
                 fontweight = "bold")

    if i == 0:
        axs[i].plot(t_sim[:-1], U_sim[i, 0])
        axs[i].plot(t_train, U_train[0],
                    "o", ms = ms, mfc = "None")
        axs[i].plot(t_test[train_cutoff:-1], U_test[i, 0, train_cutoff:],
                    "s", ms = ms, mfc = "None")
        plt.setp(axs[i], ylabel = f"$u$", xlim = [0, t_max])

        for j in range(n):
            axs[j + 1].plot(t_sim, X_sim[i, j])
            axs[j + 1].plot(t_train, X_train[j],
                            "o", ms = ms, mfc = "None")
            axs[j + 1].plot(t_test[train_cutoff:], X_test[i, j, train_cutoff:],
                            "o", ms = ms, mfc = "None")
            axs[j + 1].plot(t_train, X_okid_train[j, :-1],
                            "s", ms = ms, mfc = "None")
            axs[j + 1].plot(t_train, X_okid_train_obs[j, :-1],
                            "*", ms = ms, mfc = "None")
            axs[j + 1].plot(t_test[train_cutoff:], X_okid_test[i, j,␣
↪train_cutoff:],
                            "D", ms = ms, mfc = "None")
            axs[j + 1].plot(t_test[train_cutoff:], X_okid_test_obs[i, j,␣
↪train_cutoff:],
                            "^", ms = ms, mfc = "None")
            plt.setp(axs[j + 1], ylabel = f"$x_{j}$", xlim = [0, t_max])
            if j == 1:
                plt.setp(axs[j + 1], xlabel = f"Time")
        fig.legend(labels = ["_", "_", "_", "True", "Train", "Test",
                             "OKID\nTrain", "OKID\nTrain\n(Est)",
                             "OKID\nTest", "OKID\nTest\n(Est)"],
                   bbox_to_anchor = (1, 0.5), loc = 6)
```

```
    else:
        axs[0].plot(t_sim[:-1], U_sim[i, 0])
        axs[0].plot(t_test[:-1], U_test[i, 0],
                    "o", ms = ms, mfc = "None")
        plt.setp(axs[0], ylabel = f"$u$", xlim = [0, t_max])

        for j in range(n):
            axs[j + 1].plot(t_sim, X_sim[i, j])
            axs[j + 1].plot(t_test, X_test[i, j],
                            "o", ms = ms, mfc = "None")
            axs[j + 1].plot(t_test, X_okid_test[i, j],
                            "D", ms = ms, mfc = "None")
            axs[j + 1].plot(t_test, X_okid_test_obs[i, j],
                            "^", ms = ms, mfc = "None")
            plt.setp(axs[j + 1], ylabel = f"$x_{j}$", xlim = [0, t_max])
            if j == 1:
                plt.setp(axs[j + 1], xlabel = f"Time")
        fig.legend(labels = ["_", "_", "True", "Test",
                             "OKID\nTest", "OKID\nTest\n(Est)"],
                   bbox_to_anchor = (1, 0.5), loc = 6)
    fig.savefig(figs_dir / f"midterm_{prob}_states_case{i + 1}.pdf",
                bbox_inches = "tight")
```

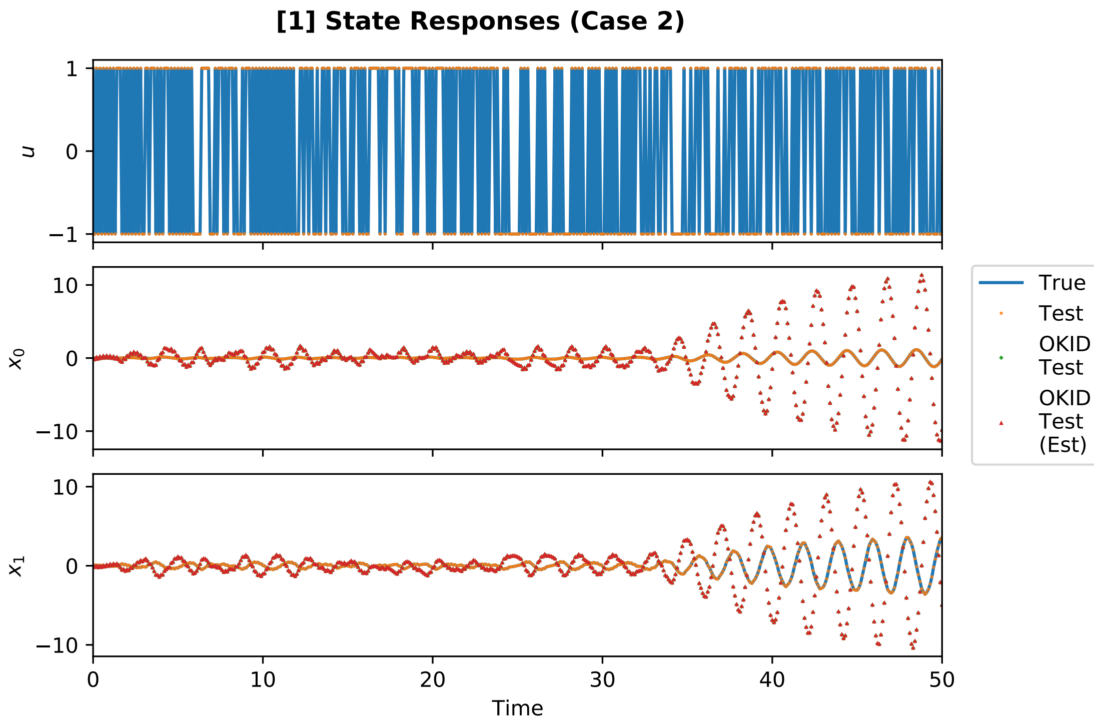


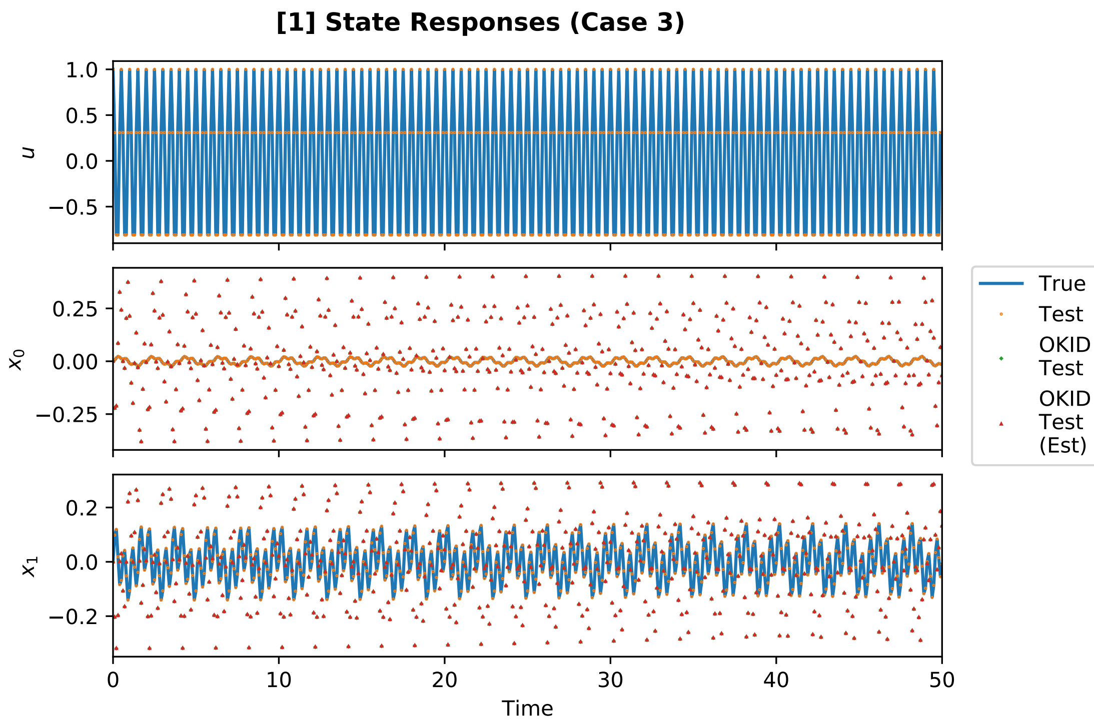


[1] Singular Values
Value
1.0
0.8
0.6
0.4
0.2
0.0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Singular Value



[1] State Responses (Case 1)
u
x0
x1
0.2
0.0
−0.2
0.5
0.0
−0.5
0 10 20 30 40 50
Time
True
Train
Test
OKID Train
OKID Train (Est)
OKID Test
OKID Test (Est)

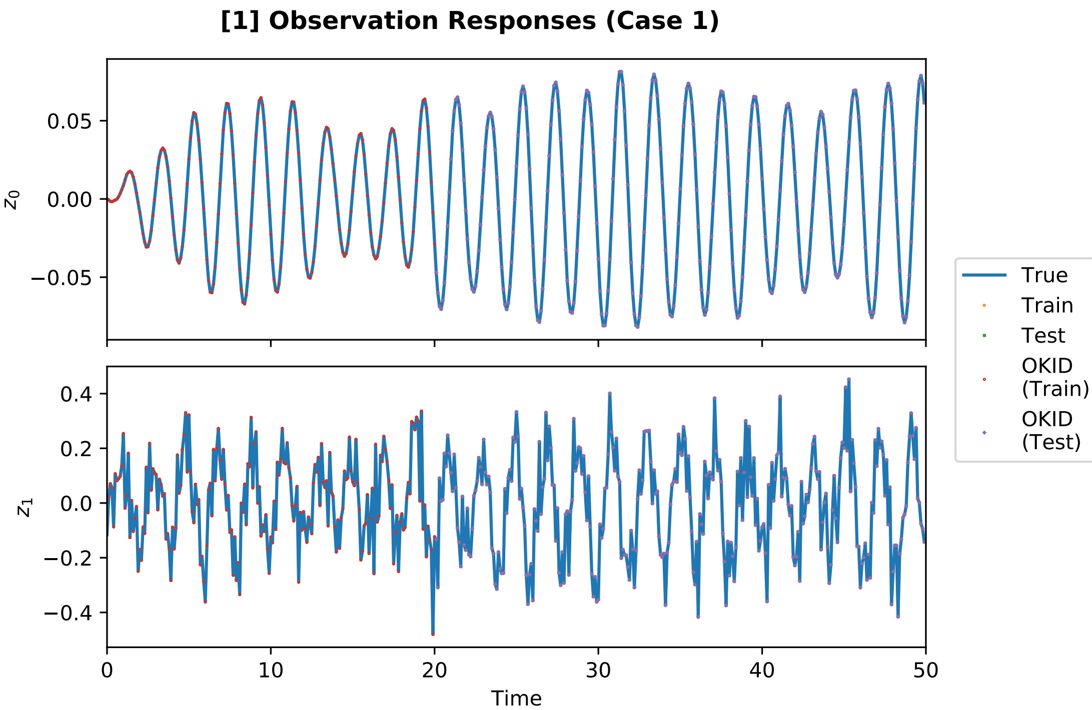

[1] State Responses (Case 2)
u
x0
x1
Time
True
Test
OKID Test
OKID Test (Est)

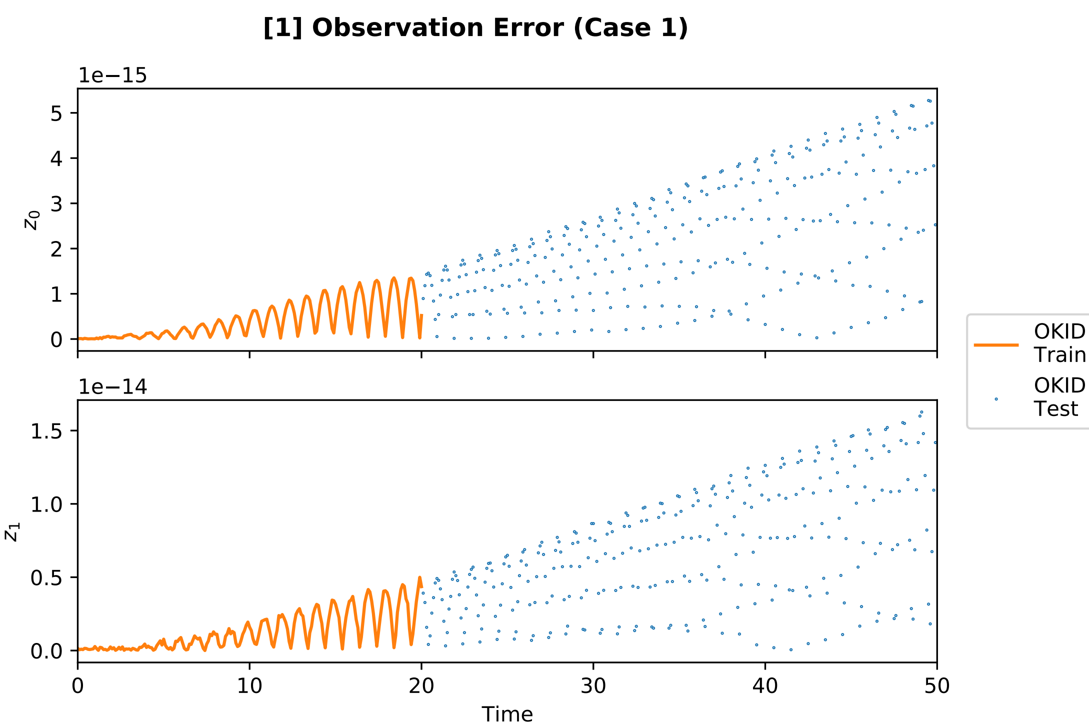

[1] State Responses (Case 3)
u
x0
x1
Time
True
Test
OKID Test
OKID Test (Est)

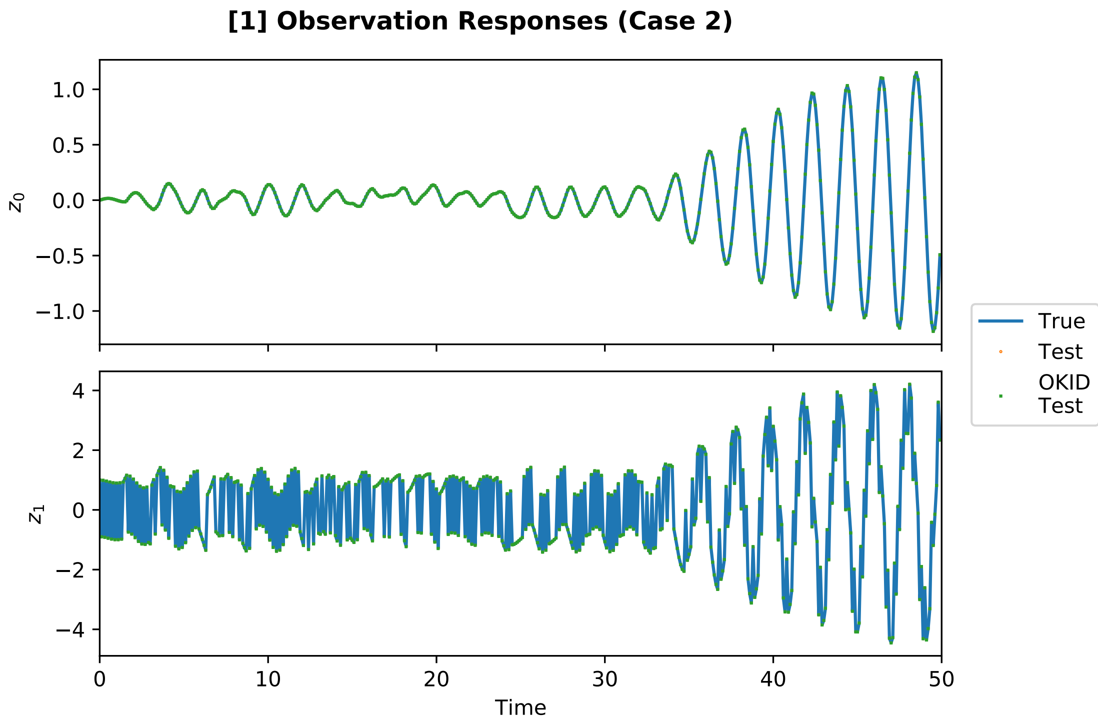
The improvement of the observer to the state estimate is negligible, as the "raw" state is already a perfect realization of the system state that reproduces the test outputs at each sample time.

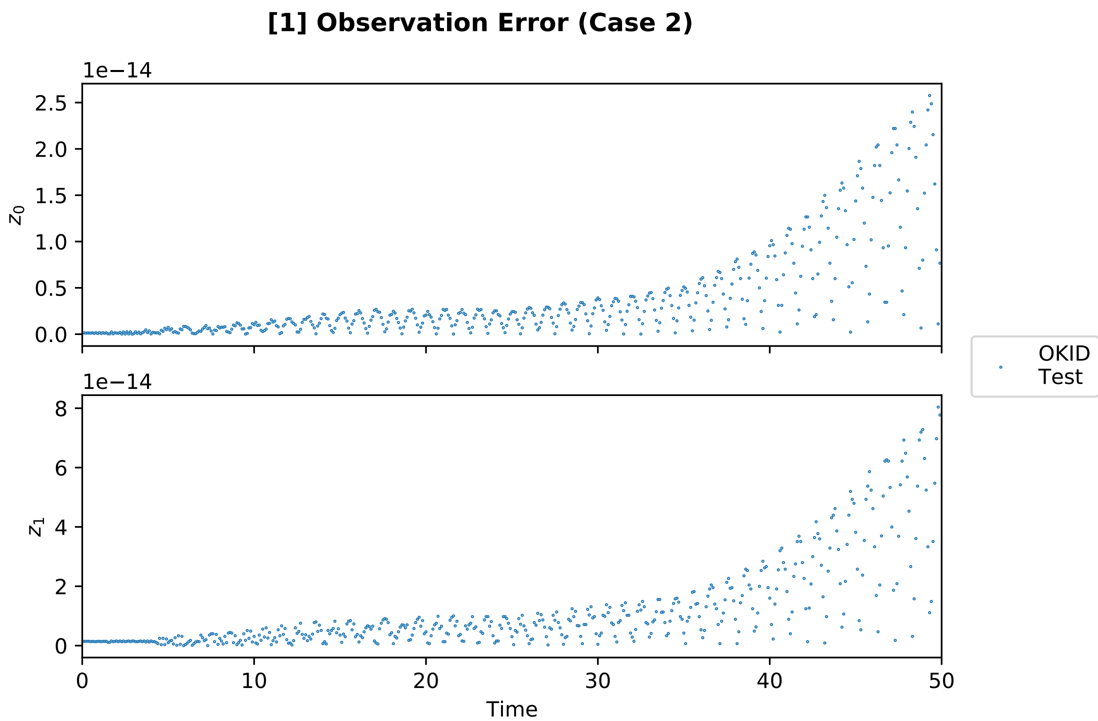
[9]:
```python
# Observation plots
for i in range(cases):
    # Raw observations
    fig, axs = plt.subplots(m, 1,
                            sharex = "col",
                            constrained_layout = True) # type:figure.Figure
    fig.suptitle(f"[{prob}] Observation Responses (Case {i + 1})",
                 fontweight = "bold")
    if i == 0:
        for j in range(m):
            axs[j].plot(t_sim[:-1], Z_sim[i, j])
            axs[j].plot(t_train, Z_train[j],
                        "o", ms = ms, mfc = "None")
            axs[j].plot(t_test[train_cutoff:-1], Z_test[i, j, train_cutoff:],
                        "s", ms = ms, mfc = "None")
            axs[j].plot(t_train, Z_okid_train[j],
                        "o", ms = ms, mfc = "None")
            axs[j].plot(t_test[train_cutoff:-1], Z_okid_test[i, j, train_cutoff:
↪],
                        "D", ms = ms, mfc = "None")
            plt.setp(axs[j], ylabel = f"$z_{j}$",
                     xlim = [0, t_max])
            if j == (m - 1):
                plt.setp(axs[j], xlabel = f"Time")
        fig.legend(labels = ["True", "Train", "Test",
                             "OKID\n(Train)", "OKID\n(Test)"],
                   bbox_to_anchor = (1, 0.5), loc = 6)
    else:
        for j in range(m):
            axs[j].plot(t_sim[:-1], Z_sim[i, j])
            axs[j].plot(t_test[:-1], Z_test[i, j],
                        "o", ms = ms, mfc = "None")
            axs[j].plot(t_test[:-1], Z_okid_test[i, j],
                        "s", ms = ms, mfc = "None")
            plt.setp(axs[j], ylabel = f"$z_{j}$",
                     xlim = [0, t_max])
            if j == (m - 1):
                plt.setp(axs[j], xlabel = f"Time")
        fig.legend(labels = ["True", "Test", "OKID\nTest"],
                   bbox_to_anchor = (1, 0.5), loc = 6)
    fig.savefig(figs_dir / f"midterm_{prob}_obs_case{i + 1}.pdf",
                bbox_inches = "tight")
```

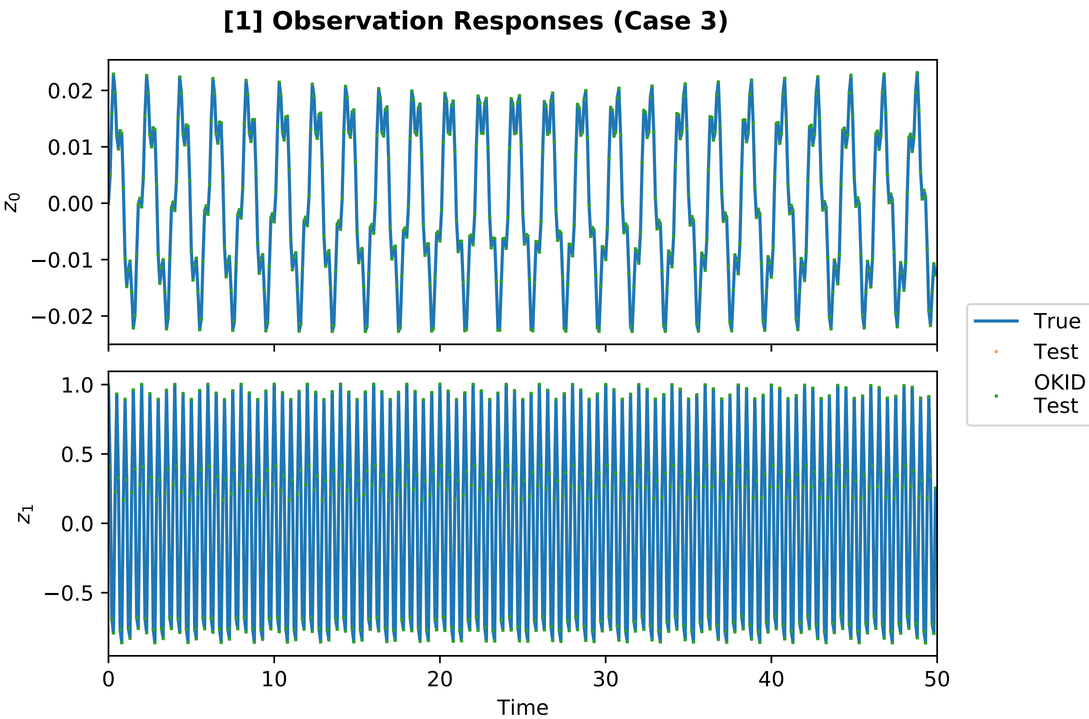
```python
# Observation error
fig, axs = plt.subplots(m, 1,
                        sharex = "col",
                        constrained_layout = True) # type:figure.Figure
fig.suptitle(f"[{prob}] Observation Error (Case {i + 1})",
             fontweight = "bold")
if i == 0:
    for j in range(m):
        axs[j].plot(t_train, np.abs(Z_okid_train[j] - Z_train[j]),
                    c = "C1")
        axs[j].plot(t_test[train_cutoff:-1], np.abs(Z_okid_test[i, j,␣
↪train_cutoff:] - Z_test[i, j, train_cutoff:]),
                    "o", ms = ms, mfc = "None", c = "C0")
        plt.setp(axs[j], ylabel = f"$z_{j}$",
                 xlim = [0, t_max])
        if j == (m - 1):
            plt.setp(axs[j], xlabel = f"Time")
    fig.legend(labels = ["OKID\nTrain", "OKID\nTest"],
               bbox_to_anchor = (1, 0.5), loc = 6)
else:
    for j in range(m):
        axs[j].plot(t_test[:-1], np.abs(Z_okid_test[i, j] - Z_test[i, j]),
                    "o", ms = ms, mfc = "None")
        plt.setp(axs[j], ylabel = f"$z_{j}$",
                 xlim = [0, t_max])
        if j == (m - 1):
            plt.setp(axs[j], xlabel = f"Time")
    fig.legend(labels = ["OKID\nTest"],
               bbox_to_anchor = (1, 0.5), loc = 6)
fig.savefig(figs_dir / f"midterm_{prob}_obs-error_case{i + 1}.pdf",
            bbox_inches = "tight")
```

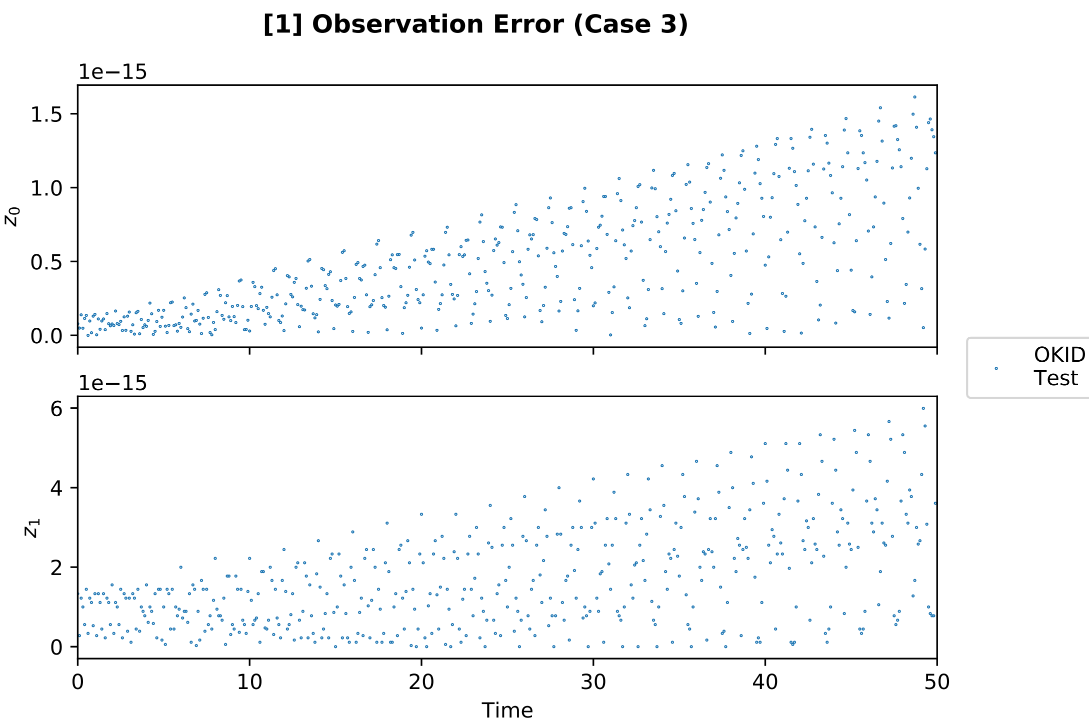

[1] Observation Responses (Case 1)
z0
z1
Time
True
Train
Test
OKID (Train)
OKID (Test)
0.05
0.00
−0.05
0.4
0.2
0.0
−0.2
−0.4
0
10
20
30
40
50


[1] Observation Error (Case 1)
1e−15
1e−14
z0
z1
Time
OKID Train
OKID Test
5
4
3
2
1
0
1.5
1.0
0.5
0.0
0
10
20
30
40
50


[1] Observation Responses (Case 2)
z_0
z_1
Time
True
Test
OKID
Test


[1] Observation Error (Case 2)
1e−14
z_0
z_1
Time
OKID
Test


[1] Observation Responses (Case 3)
z_0
z_1
Time
True
Test
OKID Test


[1] Observation Error (Case 3)
1e−15
z_0
z_1
Time
OKID Test

The observation sequence is reproduced essentially flawlessly for each case.