Mid-term Exam

This assignment requires the implementation of analytical concepts studied in class. You can use any programming language and make use of built-in functions. However, you need to numerically implement all the steps of the problem - you can't bypass it with a built-in function. You are required to provide the programming files created to complete this assignment as well as a detailed report that explain your work and results. This assignment is to be completed individually without any kind of assistance. You are allowed to refer to your class-notes and textbook. Copying of solutions, computer programs and reports will be considered a violation of the University honor code.

The objective of this exam is to use a subspace method known as the Observer Kalman IDentification (OKID) to identify a linear time invariant (LTI) dynamical system from time history of input-output (I/O) data. In this respect, let us consider the dynamics of a point mass in a rotating tube as shown in the schematic of Fig 1.

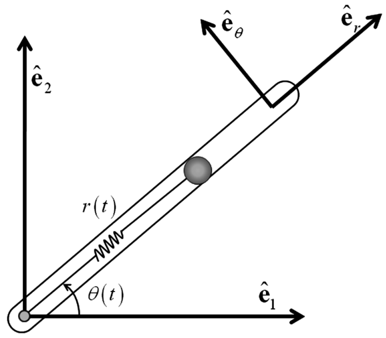

Figure 1: Schematic depicting the point mass in a rotating tube.

Dynamics of such a point mass is governed by a second order differential equation given by

$$\delta\ddot{r}(t) = \left(\dot{\theta}^2(t) - \frac{k}{m}\right)\delta r(t) + u(t) + l\dot{\theta}^2(t) \tag{1}$$

where the new variable $\delta r(t) = r(t) - l$ has been introduced, together with the definition of l, as the free length of the spring (when no force is applied on it, i.e., Hooke's Law applies as $F_s = -k\delta r$). The function $u(t)$ is the radial control force applied on the point mass, m and k represents the spring stiffness. The value of the mass and spring constant are $m = 1$ and $k = 10$, respectively. The angular velocity of the rotating tube $\dot{\theta}(t)$, is assumed to be constant:

$$\dot{\theta}(t) = \dot{\theta}_0 = \text{cst} = 0.5 \text{ rad/s}. \tag{2}$$

Choosing the origin of the coordinate system at the position $r_0 = l$ (without loss of generality), the second order differential equation is given by

$$\delta\ddot{r}(t) = \left(\dot{\theta}_0^2 - \frac{k}{m}\right)\delta r(t) + u(t) \tag{3}$$

The equations of motion in the first order state space form $(x_1(t) = \delta r(t), x_2(t) = \delta\dot{r}(t))$ can be written as:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ \dot{\theta}_0^2 - \dfrac{k}{m} & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) = A_c\boldsymbol{x}(t) + B_c u(t), \tag{4}$$

together with the measurement equations

$$\boldsymbol{y}(t) = \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) = C\boldsymbol{x}(t) + Du(t). \tag{5}$$

Equations (4) and (5) are used for simulation purposes in order to record I/O data. Following the flowchart in Fig. 2, your goal is to implement your own version of OKID/ERA and all the necessary subroutines in order to identify the discrete time system matrices $\{A, B, C, D\}$ such that

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k, \tag{6}$$

$$\boldsymbol{y}_k = C\boldsymbol{x}_k + Du_k. \tag{7}$$

Your tasks are as follows:

1. **Identification with no noise and zero initial condition**

   (a) **Simulation of I/O data**
   The first task is to acquire input/output data for training and testing purposes.

   i. Integrate the equations of motion given by (4) and (5) for 50 seconds assuming zero initial condition and input $u(t)$ to be zero-mean Gaussian white noise with standard deviation of $\sigma_0 = 0.1$. Save the propagated true states, control input and measurement data at a sampling frequency of 10 Hz. The first 20 seconds of this simulation will constitute the training data set for identification purpose while the rest of the 30 seconds of data will constitute the testing data set.

   ii. Similarly, perform two other simulations for 50 seconds at 10 Hz with:
   - $u(t)$ as a square wave of magnitude 1 and frequency 5 Hz,
   - $u(t)$ as a cosine wave of magnitude 1 and frequency 2 Hz.

   These two simulations will also be part of the testing data set.

   (b) **OKID**
   What are possible values for the observer order, i.e. p? Pick a suitable value of p and implement OKID to estimate the observer and system Markov parameters as well as the observer gain from the training data set.

   (c) **ERA**
   What are admissible sizes for the Hankel matrices? Pick a suitable size and implement ERA to estimate the discrete-time system matrices $\{A, B, C, D\}$ from the training data set. Show the SVD plot and clearly explain your choice for the order of the system.

   (d) **Verification**
   Verify that your identified model accurately reproduces I/O data for the training as well as the testing data sets. Compare output errors and compute the root mean squared error (RMSE). Use the observer gain matrix to estimate the state and compare the estimated states with the true state values. Comment on your findings.

   (e) **Modal analysis**
   Compare the eigenvalues of true and identified system matrices. Estimate the identified natural frequencies and damping coefficients and compare them with their true values.

2. **Identification with noise**
   Let us corrupt the true measurement signal $\boldsymbol{y}_k$ with a zero mean Gaussian noise $\boldsymbol{\nu}$ of standard deviation $\sigma = \eta I_2$,

$$\tilde{\boldsymbol{y}}_k = \tilde{\boldsymbol{y}}(t_k) = \boldsymbol{y}(t_k) + \boldsymbol{\nu}(t_k), \forall\ k. \tag{8}$$

   Repeat Part 1) using the corrupted signal $\tilde{\boldsymbol{y}}$ for $\eta = 0.001, 0.01, 0.1, 0.5$. Compare your results with what you have calculated in Part 1 and discuss the effect of noise during the different phases of the identification process.

3. **Identification with nonzero initial condition**
   The simulations in Part 1.(a) are now performed with a nonzero initial condition. For each of the

three simulations, $x_1(0)$ and $x_2(0)$ are both uniformly chosen between -1 and 1. Using the observer formulation for a given $p > 0$, one can write

$$\bar{\boldsymbol{y}} = C\bar{A}^p\bar{\boldsymbol{x}} + \bar{Y}\bar{V} \tag{9}$$

with

$$\bar{\boldsymbol{y}} = \begin{bmatrix}\boldsymbol{y}_p & \boldsymbol{y}_{p+1} & \cdots & \boldsymbol{y}_{l-1}\end{bmatrix} \tag{10}$$

$$\bar{\boldsymbol{x}} = \begin{bmatrix}\boldsymbol{x}_0 & \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_{l-p-2}\end{bmatrix} \tag{11}$$

$$\bar{Y} = \begin{bmatrix}D & C\bar{B} & C\bar{A}\bar{B} & \cdots & C\bar{A}^{p-1}\bar{B}\end{bmatrix} \tag{12}$$

$$\bar{V} = \begin{bmatrix} u_p & u_{p+1} & \cdots & u_{l-1} \\ \boldsymbol{v}_{p-1} & \boldsymbol{v}_p & \cdots & \boldsymbol{v}_{l-2} \\ \boldsymbol{v}_{p-2} & \boldsymbol{v}_{p-1} & \cdots & \boldsymbol{v}_{l-3} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{v}_0 & \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_{l-p-1} \end{bmatrix} \tag{13}$$

Note that the first term in equation (9) represents the effect of the preceding $p-1$ time steps. For the case where $\bar{A}^p$ is sufficiently small and all the states in $\bar{\boldsymbol{x}}$ are bounded, equation (9) can be approximated by neglecting the first term in the right-hand side,

$$\bar{\boldsymbol{y}} = \bar{Y}\bar{V}. \tag{14}$$

For nonzero unknown initial condition, equation 14 must be used to make sure that the transient effect due to initial conditions has died out.

(a) Repeat Part 1.(a)-(c) using a nonzero initial condition for the simulations. Appropriately modify the OKID algorithm to take into account the approximation shown in equation 14.

(b) One can write the following weighting sequence description at time k while using identified system matrices $\{A, B, C, D\}$:

$$\boldsymbol{y}_k = CA^k\boldsymbol{x}_0 + \sum_{i=1}^{k} CA^{i-1}Bu_{k-i} + Du_k. \tag{15}$$

Use the least-squares procedure to obtain an estimation of the initial condition $\hat{\boldsymbol{x}}_0$. Comment on your findings.

(c) Complete Parts 1.(d)-(e) using the identified initial conditions.

4. **Effect of design parameters**
Conduct a brief but *detailed* study on how design parameters of OKID and ERA algorithms effect the identification process (at least for Part 1). Specifically, analyze the effect of the choice of (but not limited to):

- the observer order (try at least three different values),
- the size of the Hankel matrices,
- data acquisition frequency.

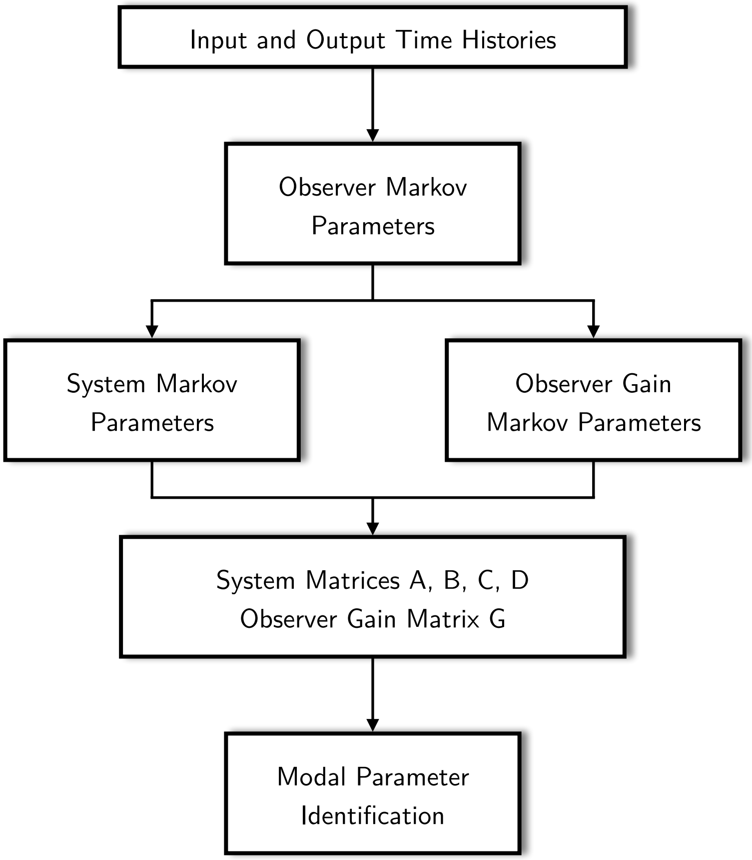

Figure 2: Flowchart for the OKID