# AERSP597 Midterm

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## 1 Q. #4

[1]:
```python
# Import all the functions used in part 1
from era_okid_tools import *

# Logistics
warnings.simplefilter("ignore", UserWarning)
sympy.init_printing()
figs_dir = (Path.cwd() / "figs")
figs_dir.mkdir(parents = True, exist_ok = True)
prob = "4-3"

# Set seed for consistent results
rng = np.random.default_rng(seed = 100)

# Simulation dimensions
cases = 3 # Number of cases
n = 2 # Number of states
r = 1 # Number of inputs
m = 2 # Number of measurements
t_max = 50 # Total simulation time
dt_sim = 0.1 # Simulation timestep duration
nt_sim = int(t_max/dt_sim) # Number of simulation timesteps
dt_sample = 0.2 # Sample timestep duration
nt_sample = int(t_max/dt_sample) # Number of sample timesteps
interval = int(dt_sample/dt_sim) # Sample index interval

# Simulation time
train_cutoff = int(20/dt_sample) + 1
t_sim = np.linspace(0, t_max, nt_sim + 1)
t_sample = np.linspace(0, t_max, nt_sample + 1)
t_train = t_sample[:train_cutoff]
t_test = t_sample
nt_train = train_cutoff
nt_test = nt_sample

# Problem parameters
```

```python
theta_0 = 0.5 # Angular velocity
k = 10 # Spring stiffness
mass = 1 # Point mass

# State space model
A_c = np.array([[0, 1], [theta_0**2 - k/mass, 0]])
B_c = np.array([[0], [1]])
C = np.eye(2)
D = np.array([[0], [1]])
A, B = c2d(A_c, B_c, dt_sim)
eig_A = spla.eig(A_c)[0] # Eigenvalues of true system
etch(f"\lambda", eig_A)
etch(f"\omega_{{n}}", np.abs(eig_A))
etch(f"\zeta", -np.cos(np.angle(eig_A)))

# True simulation values
X_0_sim = np.zeros([n, 1]) # Zero initial condition
U_sim = np.zeros([cases, r, nt_sim]) # True input vectors
U_sim[0] = rng.normal(0, 0.1, [r, nt_sim]) # True input for case 1
U_sim[1] = spsg.square(2*np.pi*5*t_sim[:-1]) # True input for case 2
U_sim[2] = np.cos(2*np.pi*2*t_sim[:-1]) # True input for case 3
X_sim = np.zeros([cases, n, nt_sim + 1]) # True state vectors
Z_sim = np.zeros([cases, m, nt_sim]) # True observation vectors

# Sampled simulation values
U_sample = U_sim[:, :, ::interval] # Sampled input vectors
X_sample = np.zeros([cases, n, nt_sample + 1]) # Sampled state vectors
Z_sample = np.zeros([cases, n, nt_sample]) # Sampled observation vectors

# Separation into train and test data
U_train = U_sample[0, :r, :train_cutoff]
U_test = U_sample # Test input vectors
X_test = np.zeros([cases, n, nt_test + 1]) # Test state vectors
Z_test = np.zeros([cases, m, nt_test]) # Test observation vectors
V_train = np.zeros([r + m, nt_train]) # Train observation input vectors
V_test = np.zeros([cases, r + m, nt_test]) # Test observation input vectors
```

$$\lambda = \begin{bmatrix} 3.1225i \\ -3.1225i \end{bmatrix}$$

$$\omega_n = \begin{bmatrix} 3.1225 \\ 3.1225 \end{bmatrix}$$

$$\zeta = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$$

[2]:

```python
# OKID logistics
```

```
order = 10 # Order of OKID algorithm, number of Markov parameters to identify␣
 ↪after the zeroeth
alpha, beta = 3, 5 # Number of block rows and columns in Hankel matrices
n_era = 2 # Number of proposed states
X_0_okid = np.zeros([n_era, 1]) # Zero initial condition

print(f"Min. OKID Order: {max([alpha + beta, (n_era/m) + (n_era/r)]):n}")
print(f"Max. OKID Order: {(nt_train - r)/(r + m):n}")
print(f"Proposed OKID Order: {order:n}")
```

```
Min. OKID Order: 8
Max. OKID Order: 33.3333
Proposed OKID Order: 10
```

Note that we have set $l_0 = 10$, $\alpha = 3$, and $\beta = 5$ for this simulation. We choose the sampling frequency to be 5 Hz rather than 10 Hz to determine the effect of sampling frequency on the system identification.

[3]:
```
# OKID System Markov parameters
Y_okid = np.zeros([order + 1, m, r])
# OKID Observer Markov Gain parameters
Y_og_okid = np.zeros([order, m, m])
# OKID state vector, drawn from state space model derived from OKID/ERA
X_okid_train = np.zeros([n_era, nt_train + 1])
X_okid_test = np.zeros([cases, n_era, nt_test + 1])
X_okid_train_obs = np.zeros([n_era, nt_train + 1])
X_okid_test_obs = np.zeros([cases, n_era, nt_test + 1])
# OKID observations, drawn from state space model derived from OKID/ERA
Z_okid_train = np.zeros([n_era, nt_train])
Z_okid_test = np.zeros([cases, n_era, nt_test])
Z_okid_train_obs = np.zeros([n_era, nt_train])
Z_okid_test_obs = np.zeros([cases, n_era, nt_test])

# Singular values of the Hankel matrix constructed through OKID Markov␣
 ↪parameters
S_okid = np.zeros([min(alpha*m, beta*r)])
eig_A_okid = np.zeros([n_era], dtype = complex)

# OKID/ERA state space model
A_okid = np.zeros([n_era, n_era])
B_okid = np.zeros([n_era, r])
C_okid = np.zeros([m, n_era])
D_okid = np.zeros([m, r])
G_okid = np.zeros([m, m])
# OKID/ERA state space model augmented with observer
A_okid_obs = np.zeros([n_era, n_era])
B_okid_obs = np.zeros([n_era, r + m])
C_okid_obs = np.zeros([m, n_era])
```

```
D_okid_obs = np.zeros([m, r + m])
```

[4]:
```
# Simulation
for i in range(cases):
    X_sim[i], Z_sim[i] = sim_ss(A, B, C, D, X_0 = X_0_sim, U = U_sim[i], nt =␣
↪nt_sim)
    # Sample at lower frequency
    X_sample[i], Z_sample[i] = X_sim[i, :, ::interval], Z_sim[i, :, ::interval]
    if i == 0:
        # Split between train and test data for case 1
        X_train, Z_train = X_sample[i, :, :train_cutoff], Z_sample[i, :, :
↪train_cutoff]
        # Identify System Markov parameters and Observer Gain Markov parameters
        Y_okid, Y_og_okid = okid(Z_train, U_train,
                                 l_0 = order, alpha = alpha, beta = beta, n =␣
↪n_era)
        # Identify state space model using System Markov parameters for ERA
        A_okid, B_okid, C_okid, D_okid, S_okid = \
            era(Y_okid, alpha = alpha, beta = beta, n = n_era)
        # Construct observability matrix
        O_p_okid = np.array([C_okid @ np.linalg.matrix_power(A_okid, i)
                             for i in range(order)])
        G_okid = spla.pinv2(O_p_okid.reshape([order*m, n_era])) @ Y_og_okid.
↪reshape([order*m, m])
        # Augment state space model with observer
        A_okid_obs = A_okid + G_okid @ C_okid
        B_okid_obs = np.concatenate([B_okid + G_okid @ D_okid, -G_okid], 1)
        C_okid_obs = C_okid
        D_okid_obs = np.concatenate([D_okid, np.zeros([m, m])], 1)
        V_train = np.concatenate([U_train, Z_train], 0)
        # Simulate OKID realization with "raw" state and OKID realization with␣
↪estimated state
        X_okid_train, Z_okid_train = \
            sim_ss(A_okid, B_okid, C_okid, D_okid,
                   X_0 = X_0_okid, U = U_train, nt = nt_train)
        X_okid_train_obs, Z_okid_train_obs = \
            sim_ss(A_okid_obs, B_okid_obs, C_okid_obs, D_okid_obs,
                   X_0 = X_0_okid, U = V_train, nt = nt_train)
        # Display outputs
        etch(f"A_{{OKID}}(f_s = {1/dt_sample:0.2f})", A_okid)
        etch(f"B_{{OKID}}(f_s = {1/dt_sample:0.2f})", B_okid)
        etch(f"C_{{OKID}}(f_s = {1/dt_sample:0.2f})", C_okid)
        etch(f"D_{{OKID}}(f_s = {1/dt_sample:0.2f})", D_okid)
        etch(f"G_{{OKID}}(f_s = {1/dt_sample:0.2f})", G_okid)
        # Calculate and display eigenvalues
        eig_A_okid = spla.eig(d2c(A_okid, B_okid, dt_sample)[0])[0] #␣
↪Eigenvalues of identified system
```

```
        etch(f"\hat{{\lambda}}(f_s = {1/dt_sample:0.2f})", eig_A_okid)
        etch(f"\hat{{\omega}}_{{n}}(f_s = {1/dt_sample:0.2f})", np.
↪abs(eig_A_okid))
        etch(f"\hat{{\zeta}}(f_s = {1/dt_sample:0.2f})", -np.cos(np.
↪angle(eig_A_okid)))
    X_test[i], Z_test[i] = X_sample[i], Z_sample[i]
    X_okid_test[i], Z_okid_test[i] = \
        sim_ss(A_okid, B_okid, C_okid, D_okid,
               X_0 = X_0_okid, U = U_test[i], nt = nt_test)
    V_test[i] = np.concatenate([U_test[i], Z_test[i]], 0)
    X_okid_test_obs[i], Z_okid_test_obs[i] = \
        sim_ss(A_okid_obs, B_okid_obs, C_okid_obs, D_okid_obs,
               X_0 = X_0_okid, U = V_test[i], nt = nt_test)
```

```
Rank of H(0): 5
Rank of H(1): 5
```

$$A_{OKID}(f_s = 5.00) = \begin{bmatrix} 0.83094 & 0.74587 \\ -0.46364 & 0.81261 \end{bmatrix}$$

$$B_{OKID}(f_s = 5.00) = \begin{bmatrix} -0.1028 \\ 0.22645 \end{bmatrix}$$

$$C_{OKID}(f_s = 5.00) = \begin{bmatrix} 0.04672 & 0.0903 \\ -0.25341 & 0.23411 \end{bmatrix}$$

$$D_{OKID}(f_s = 5.00) = \begin{bmatrix} -0.00029 \\ 0.99433 \end{bmatrix}$$

$$G_{OKID}(f_s = 5.00) = \begin{bmatrix} -0.66531 & -0.46935 \\ -0.88417 & -1.05222 \end{bmatrix}$$

$$\hat{\lambda}(f_s = 5.00) = \begin{bmatrix} 0.05207 + 3.1053i \\ 0.05207 - 3.1053i \end{bmatrix}$$

$$\hat{\omega}_n(f_s = 5.00) = \begin{bmatrix} 3.10574 \\ 3.10574 \end{bmatrix}$$

$$\hat{\zeta}(f_s = 5.00) = \begin{bmatrix} -0.01677 \\ -0.01677 \end{bmatrix}$$

The sampling frequency being halved leads to the detection of slight damping in the system where there is in fact none. The identified natural frequency is close to the true value. However, the fact that the detected eigenvalues have positive (unstable) real parts means that the identification as a whole is quite poor, even with the observer.

[5]:
```
RMS_train = np.sqrt(np.mean((Z_okid_train - Z_train)**2, axis = 1))
print(f"RMS Error of sim. for system found via OKID for train data, sampling␣
↪frequency = {1/dt_sample:0.2f}: {RMS_train}")
RMS_test = np.zeros([cases, m])
for i in range(cases):
```

```
    RMS_test[i] = np.sqrt(np.mean((Z_okid_test[i] - Z_test[i])**2, axis = 1))
    print(f"RMS Error of sim. for system found via OKID for test data, case␣
 ↪{i}, sampling frequency = {1/dt_sample:0.2f}: {RMS_test[i]}")
```

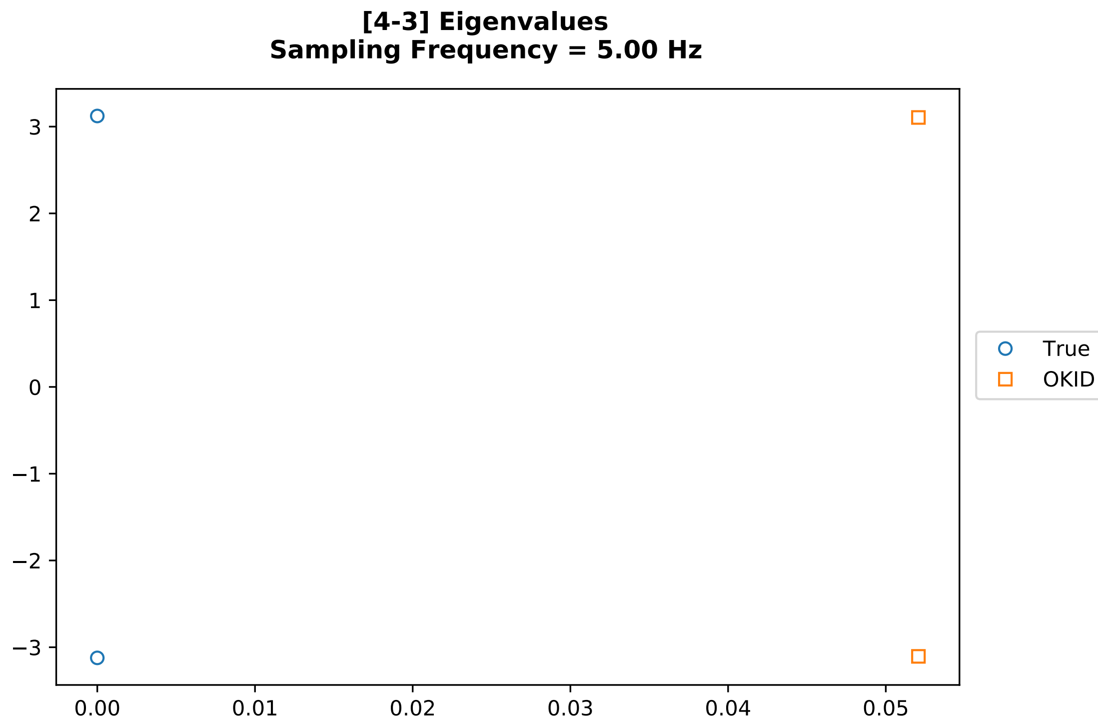
```
RMS Error of sim. for system found via OKID for train data, sampling frequency =
5.00: [0.01258072 0.02951865]
RMS Error of sim. for system found via OKID for test data, case 0, sampling
frequency = 5.00: [0.06219972 0.24769425]
RMS Error of sim. for system found via OKID for test data, case 1, sampling
frequency = 5.00: [0.09802495 0.37483967]
RMS Error of sim. for system found via OKID for test data, case 2, sampling
frequency = 5.00: [0.0483794  0.19170376]
```

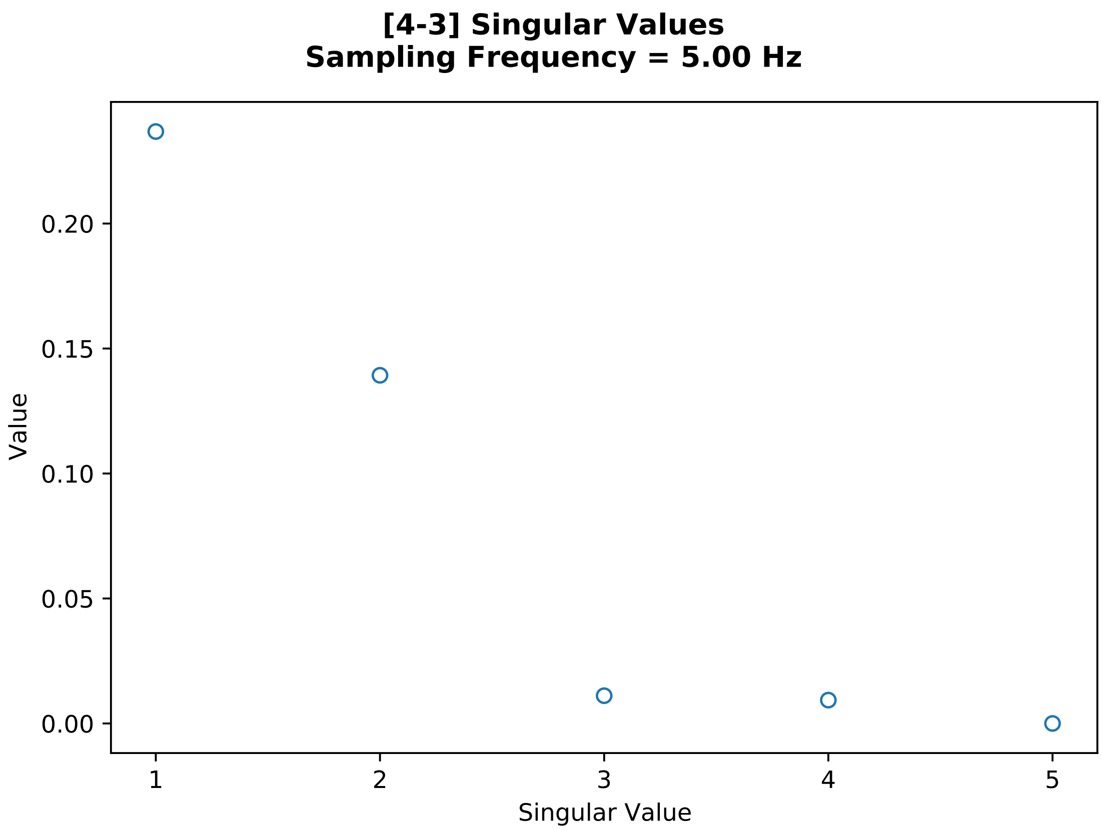
The RMS error in the estimation is quite high for both the training and testing data. The training case has an input frequency of double the sampling frequency, again contributing to the poor accuracy of the identified system as it cannot cope with this.

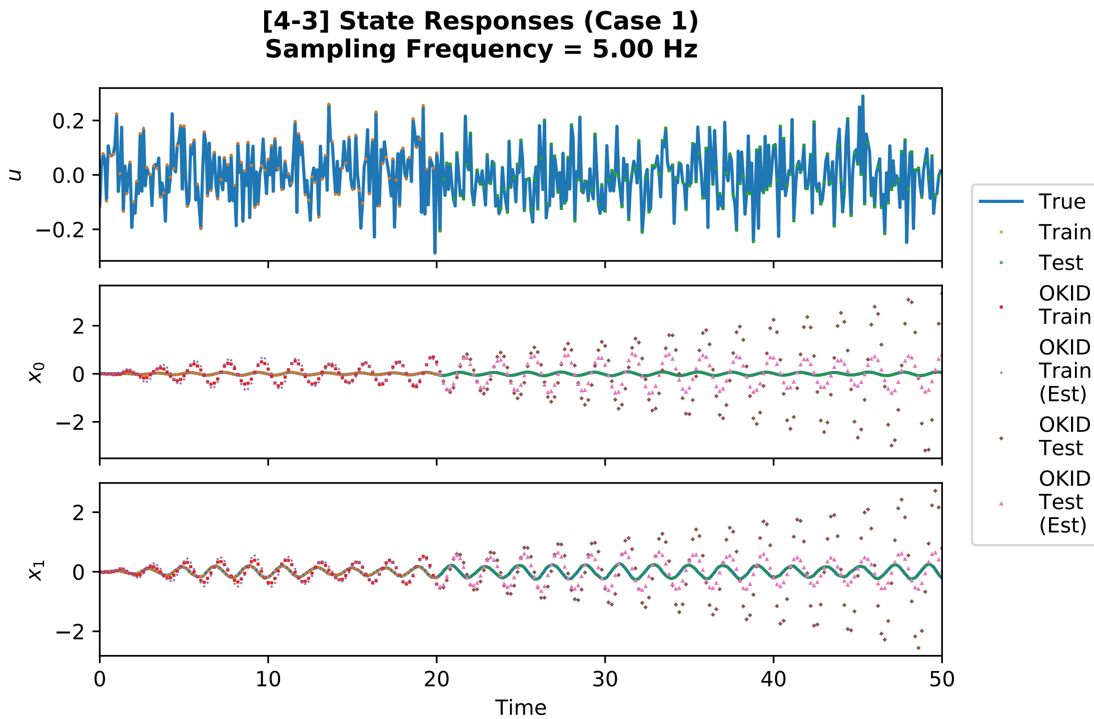
[6]:
```
# Eigenvalue plots
fig, ax = plt.subplots(constrained_layout = True) # type:figure.Figure
fig.suptitle(f"[{prob}] Eigenvalues\nSampling Frequency = {1/dt_sample:0.2f}␣
 ↪Hz", fontweight = "bold")

ax.plot(np.real(eig_A), np.imag(eig_A),
         "o", mfc = "None")
ax.plot(np.real(eig_A_okid), np.imag(eig_A_okid),
         "s", mfc = "None")

fig.legend(labels = ("True", "OKID"),
           bbox_to_anchor = (1, 0.5), loc = 6)
fig.savefig(figs_dir / f"midterm_{prob}_eigval.pdf",
            dpi = 80, bbox_inches = "tight")

# Singular Value plots
fig, ax = plt.subplots(constrained_layout = True) # type:figure.Figure
fig.suptitle(f"[{prob}] Singular Values\nSampling Frequency = {1/dt_sample:0.
 ↪2f} Hz", fontweight = "bold")

ax.plot(np.linspace(1, len(S_okid), len(S_okid)), S_okid,
         "o", mfc = "None")
plt.setp(ax, xlabel = f"Singular Value", ylabel = f"Value",
         xticks = np.arange(1, len(S_okid) + 1))

fig.savefig(figs_dir / f"midterm_{prob}_singval.pdf",
            dpi = 80, bbox_inches = "tight")

# Response plots
ms = 0.5 # Marker size
for i in range(cases):
```

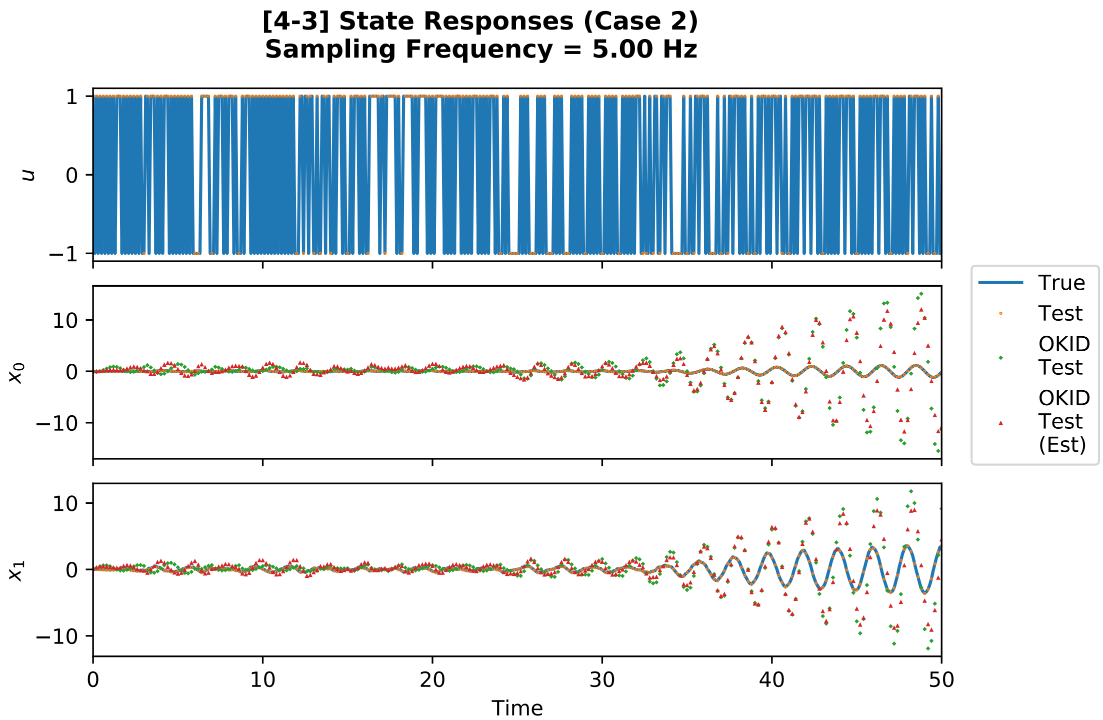
```
    fig, axs = plt.subplots(1 + n, 1,
                            sharex = "col",
                            constrained_layout = True) # type:figure.Figure
    fig.suptitle(f"[{prob}] State Responses (Case {i + 1})\nSampling Frequency␣
↪= {1/dt_sample:0.2f} Hz",
                 fontweight = "bold")

    if i == 0:
        axs[i].plot(t_sim[:-1], U_sim[i, 0])
        axs[i].plot(t_train, U_train[0],
                    "o", ms = ms, mfc = "None")
        axs[i].plot(t_test[train_cutoff:-1], U_test[i, 0, train_cutoff:],
                    "s", ms = ms, mfc = "None")
        plt.setp(axs[i], ylabel = f"$u$", xlim = [0, t_max])

        for j in range(n):
            axs[j + 1].plot(t_sim, X_sim[i, j])
            axs[j + 1].plot(t_train, X_train[j],
                            "o", ms = ms, mfc = "None")
            axs[j + 1].plot(t_test[train_cutoff:], X_test[i, j, train_cutoff:],
                            "o", ms = ms, mfc = "None")
            axs[j + 1].plot(t_train, X_okid_train[j, :-1],
                            "s", ms = ms, mfc = "None")
            axs[j + 1].plot(t_train, X_okid_train_obs[j, :-1],
                            "*", ms = ms, mfc = "None")
            axs[j + 1].plot(t_test[train_cutoff:], X_okid_test[i, j,␣
↪train_cutoff:],
                            "D", ms = ms, mfc = "None")
            axs[j + 1].plot(t_test[train_cutoff:], X_okid_test_obs[i, j,␣
↪train_cutoff:],
                            "^", ms = ms, mfc = "None")
            plt.setp(axs[j + 1], ylabel = f"$x_{j}$", xlim = [0, t_max])
            if j == 1:
                plt.setp(axs[j + 1], xlabel = f"Time")
        fig.legend(labels = ["_", "_", "_", "True", "Train", "Test",
                             "OKID\nTrain", "OKID\nTrain\n(Est)",
                             "OKID\nTest", "OKID\nTest\n(Est)"],
                   bbox_to_anchor = (1, 0.5), loc = 6)
    else:
        axs[0].plot(t_sim[:-1], U_sim[i, 0])
        axs[0].plot(t_test[:-1], U_test[i, 0],
                    "o", ms = ms, mfc = "None")
        plt.setp(axs[0], ylabel = f"$u$", xlim = [0, t_max])

        for j in range(n):
            axs[j + 1].plot(t_sim, X_sim[i, j])
            axs[j + 1].plot(t_test, X_test[i, j],
```

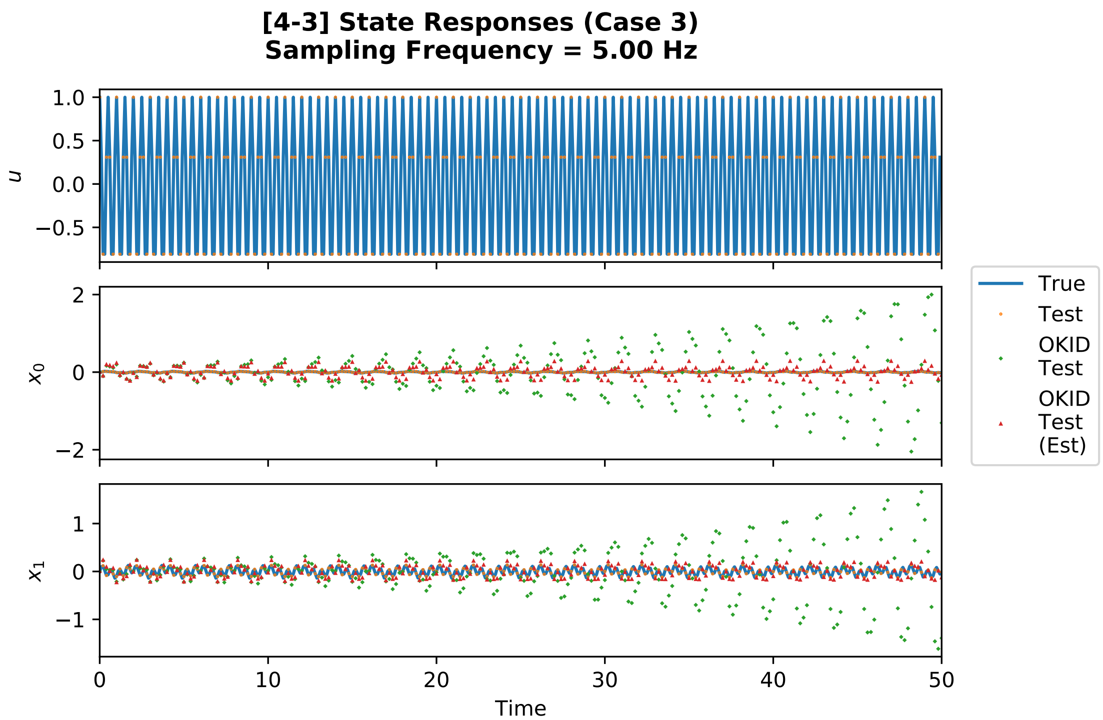
```
                            "o", ms = ms, mfc = "None")
            axs[j + 1].plot(t_test, X_okid_test[i, j],
                            "D", ms = ms, mfc = "None")
            axs[j + 1].plot(t_test, X_okid_test_obs[i, j],
                            "^", ms = ms, mfc = "None")
            plt.setp(axs[j + 1], ylabel = f"$x_{j}$", xlim = [0, t_max])
            if j == 1:
                plt.setp(axs[j + 1], xlabel = f"Time")
        fig.legend(labels = ["_", "_", "True", "Test",
                             "OKID\nTest", "OKID\nTest\n(Est)"],
                   bbox_to_anchor = (1, 0.5), loc = 6)
    fig.savefig(figs_dir / f"midterm_{prob}_states_case{i + 1}.pdf",
                bbox_inches = "tight")
```




[4-3] Singular Values
Sampling Frequency = 5.00 Hz
Value
Singular Value
0.00
0.05
0.10
0.15
0.20
1
2
3
4
5


[4-3] State Responses (Case 1)
Sampling Frequency = 5.00 Hz
u
x0
x1
Time
True
Train
Test
OKID Train
OKID Train (Est)
OKID Test
OKID Test (Est)

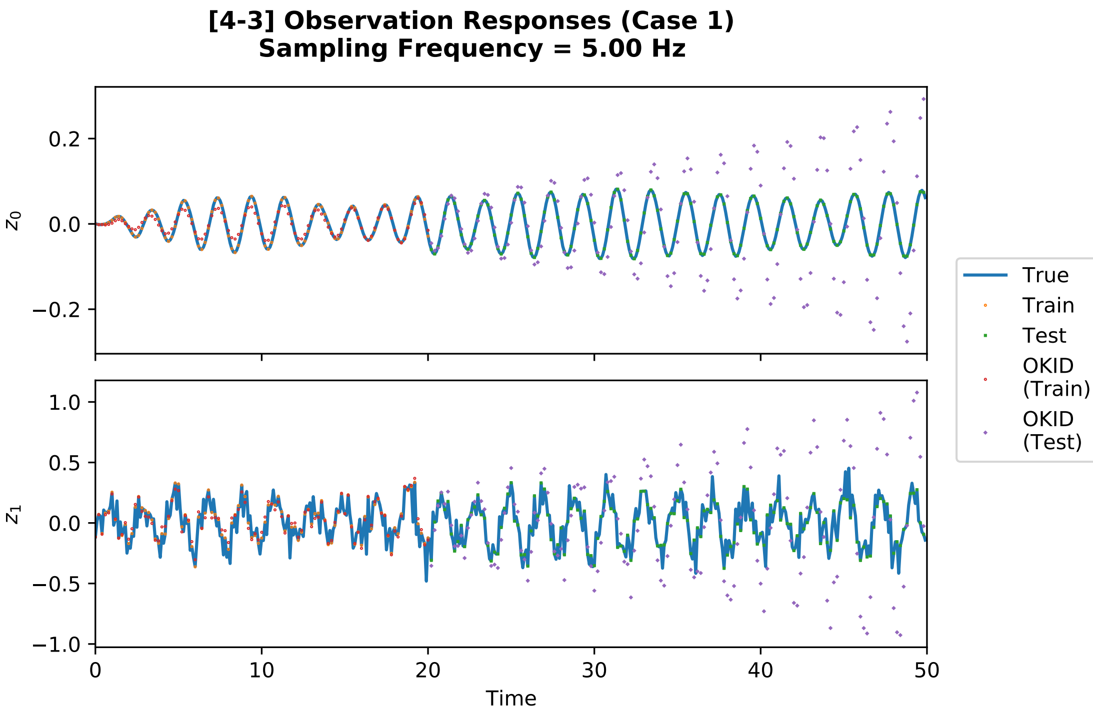

[4-3] State Responses (Case 2)
Sampling Frequency = 5.00 Hz
u
x0
x1
Time
True
Test
OKID Test
OKID Test (Est)

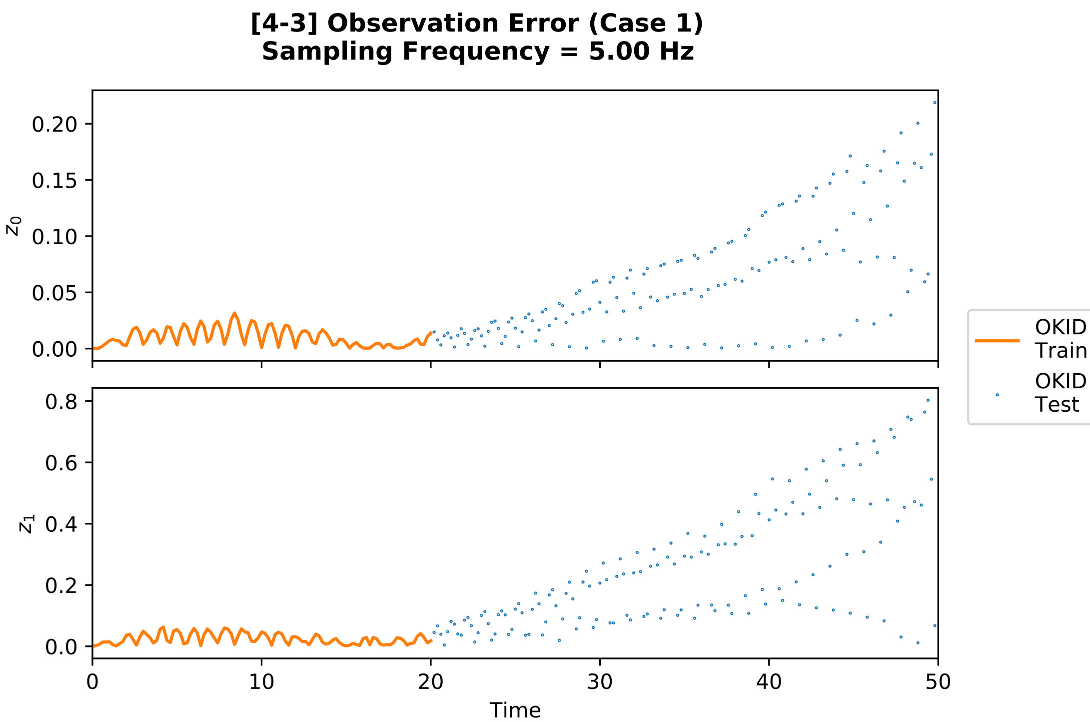

[4-3] State Responses (Case 3)
Sampling Frequency = 5.00 Hz
u
x0
x1
Time
True
Test
OKID Test
OKID Test (Est)

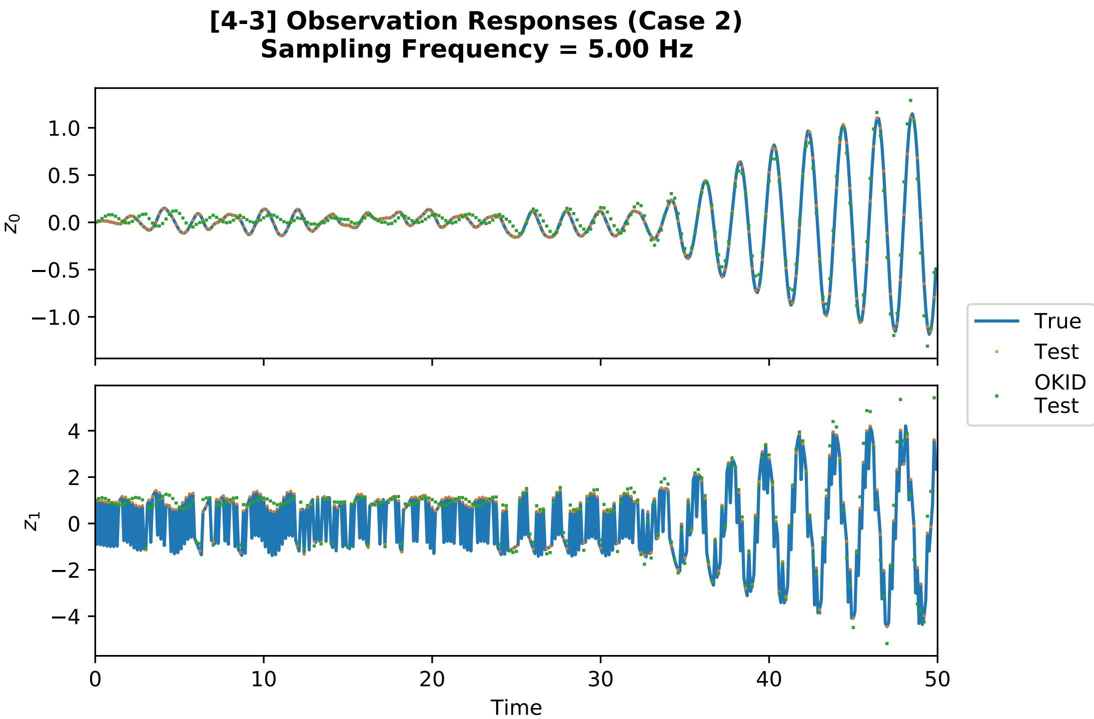
The observer appears to cause the state estimate to grow in time for all three cases, which is testament to the fact that the identification of the system is very poor.

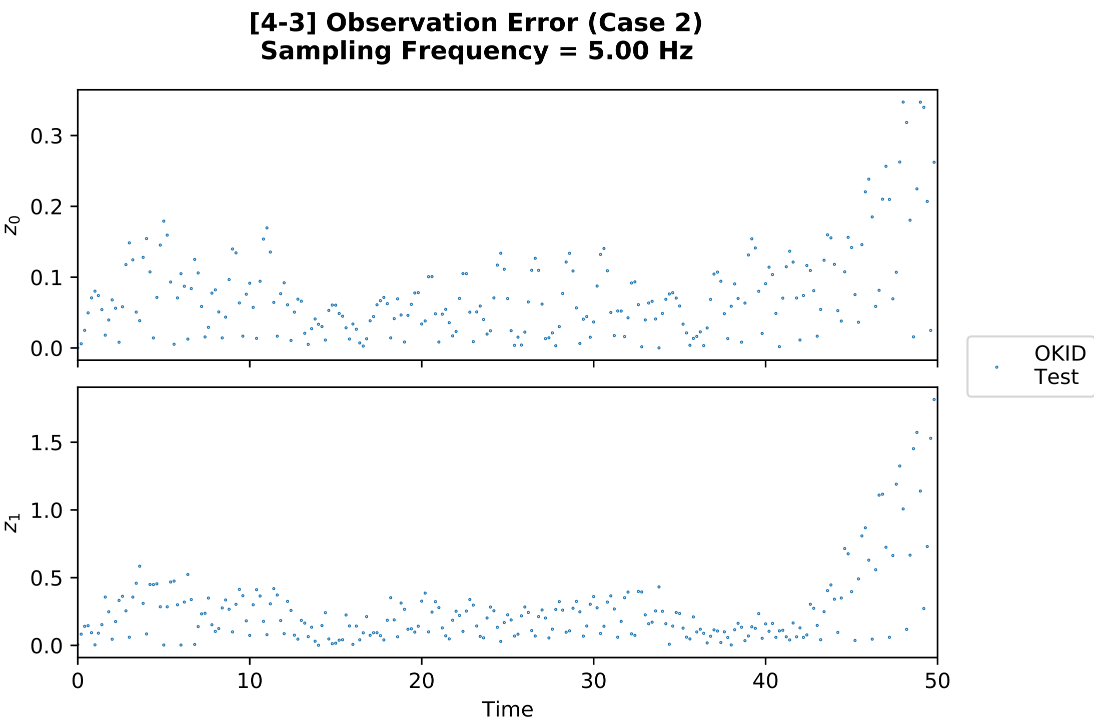
[7]:
```
# Observation plots
for i in range(cases):
    fig, axs = plt.subplots(m, 1,
                            sharex = "col",
                            constrained_layout = True) # type:figure.Figure
    fig.suptitle(f"[{prob}] Observation Responses (Case {i + 1})\nSampling␣
↪Frequency = {1/dt_sample:0.2f} Hz",
                 fontweight = "bold")
    if i == 0:
        for j in range(m):
            axs[j].plot(t_sim[:-1], Z_sim[i, j])
            axs[j].plot(t_train, Z_train[j],
                        "o", ms = ms, mfc = "None")
            axs[j].plot(t_test[train_cutoff:-1], Z_test[i, j, train_cutoff:],
                        "s", ms = ms, mfc = "None")
            axs[j].plot(t_train, Z_okid_train[j],
                        "o", ms = ms, mfc = "None")
            axs[j].plot(t_test[train_cutoff:-1], Z_okid_test[i, j, train_cutoff:
↪],
                        "D", ms = ms, mfc = "None")
            plt.setp(axs[j], ylabel = f"$z_{j}$",
                     xlim = [0, t_max])
            if j == (m - 1):
                plt.setp(axs[j], xlabel = f"Time")
        fig.legend(labels = ["True", "Train", "Test",
                             "OKID\n(Train)", "OKID\n(Test)"],
                   bbox_to_anchor = (1, 0.5), loc = 6)
    else:
        for j in range(m):
            axs[j].plot(t_sim[:-1], Z_sim[i, j])
            axs[j].plot(t_test[:-1], Z_test[i, j],
                        "o", ms = ms, mfc = "None")
            axs[j].plot(t_test[:-1], Z_okid_test[i, j],
                        "s", ms = ms, mfc = "None")
            plt.setp(axs[j], ylabel = f"$z_{j}$",
                     xlim = [0, t_max])
            if j == (m - 1):
                plt.setp(axs[j], xlabel = f"Time")
        fig.legend(labels = ["True", "Test", "OKID\nTest"],
                   bbox_to_anchor = (1, 0.5), loc = 6)
    fig.savefig(figs_dir / f"midterm_{prob}_obs_case{i + 1}.pdf",
                dpi = 80, bbox_inches = "tight")
```

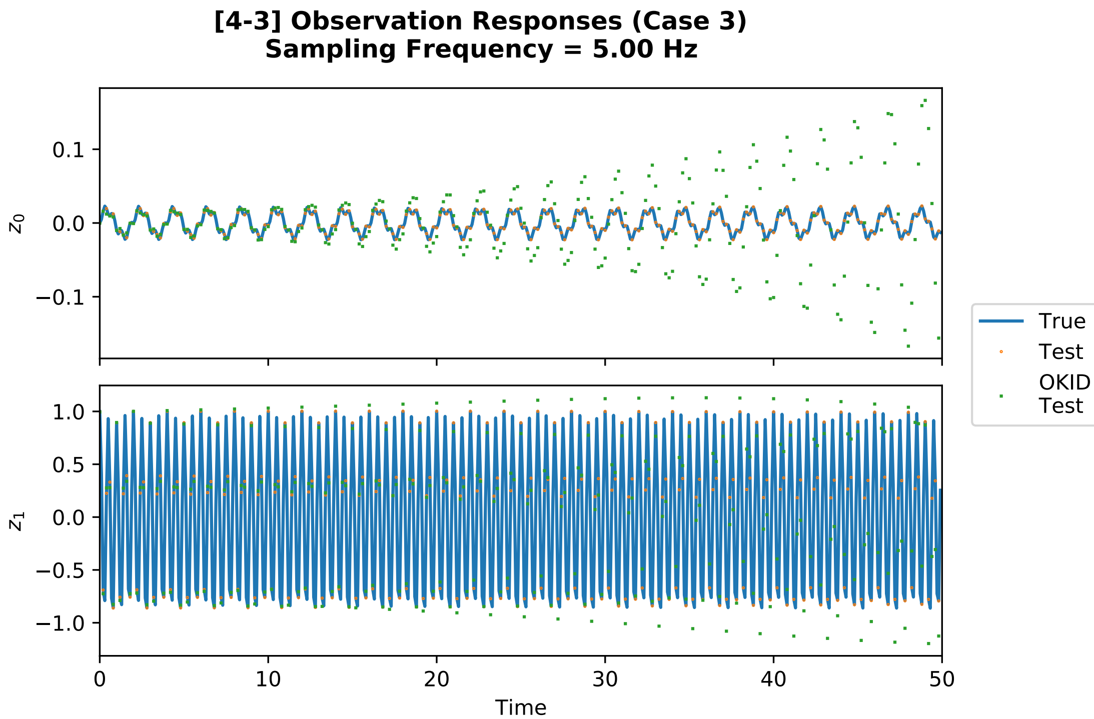
```python
    fig, axs = plt.subplots(m, 1,
                            sharex = "col",
                            constrained_layout = True) # type:figure.Figure
    fig.suptitle(f"[{prob}] Observation Error (Case {i + 1})\nSampling Frequency = {1/dt_sample:0.2f} Hz",
                 fontweight = "bold")
    if i == 0:
        for j in range(m):
            axs[j].plot(t_train, np.abs(Z_okid_train[j] - Z_train[j]),
                        c = "C1")
            axs[j].plot(t_test[train_cutoff:-1], np.abs(Z_okid_test[i, j, train_cutoff:] - Z_test[i, j, train_cutoff:]),
                        "o", ms = ms, mfc = "None", c = "C0")
            plt.setp(axs[j], ylabel = f"$z_{j}$",
                     xlim = [0, t_max])
            if j == (m - 1):
                plt.setp(axs[j], xlabel = f"Time")
        fig.legend(labels = ["OKID\nTrain", "OKID\nTest"],
                   bbox_to_anchor = (1, 0.5), loc = 6)
    else:
        for j in range(m):
            axs[j].plot(t_test[:-1], np.abs(Z_okid_test[i, j] - Z_test[i, j]),
                        "o", ms = ms, mfc = "None")
            plt.setp(axs[j], ylabel = f"$z_{j}$",
                     xlim = [0, t_max])
            if j == (m - 1):
                plt.setp(axs[j], xlabel = f"Time")
        fig.legend(labels = ["OKID\nTest"],
                   bbox_to_anchor = (1, 0.5), loc = 6)
    fig.savefig(figs_dir / f"midterm_{prob}_obs-error_case{i + 1}.pdf",
                dpi = 80, bbox_inches = "tight")
```

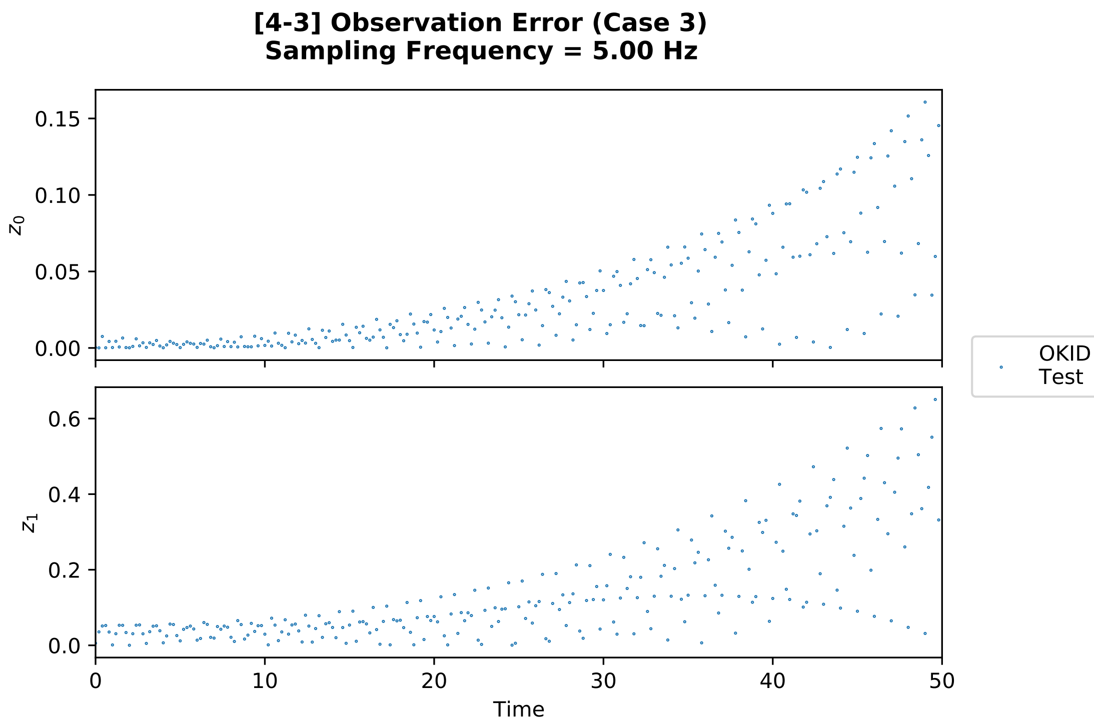

[4-3] Observation Responses (Case 1)
Sampling Frequency = 5.00 Hz
$z_0$
$z_1$
Time
True
Train
Test
OKID (Train)
OKID (Test)


[4-3] Observation Error (Case 1)
Sampling Frequency = 5.00 Hz
$z_0$
$z_1$
Time
OKID Train
OKID Test


[4-3] Observation Responses (Case 2)
Sampling Frequency = 5.00 Hz
$z_0$
$z_1$
Time
True
Test
OKID Test


[4-3] Observation Error (Case 2)
Sampling Frequency = 5.00 Hz
$z_0$
$z_1$
Time
OKID Test


[4-3] Observation Responses (Case 3)
Sampling Frequency = 5.00 Hz
$z_0$
$z_1$
Time
True
Test
OKID Test


[4-3] Observation Error (Case 3)
Sampling Frequency = 5.00 Hz
$z_0$
$z_1$
Time
OKID Test

On the whole, the estimation is quite poor in this case due to the loss of half of all the random input and the corresponding output data. Due to this reason, decreasing the sample frequency significantly worsened the accuracy of the estimation. For a simpler training dataset where the input was more uniform and less random, the estimation accuracy for lower sampling frequencies may have been slightly improved.