UiO : Department of Physics
University of Oslo

Project 5: The diffusion equation

FYS3150 - Computational physics

Kristine Baluka Hein
Anders Johansson

Contents

Abstract

Hei.

1 Mathematical theory

1.1 Forward Euler

1.1.1 Derivation and error analysis

The Forward Euler scheme is an explicit scheme based on Taylor polynomials. To find an approximation of the time derivative of u at point (x_i, t_j), a first order Taylor polynomial around x_i, t_j is used to calculate $u(x_i, t_{j+1})$ is used:

$$u_{i,j+1} = u_{i,j} + \Delta t \frac{\partial u_{i,j}}{\partial t} + \frac{1}{2}(\Delta t)^2 \frac{\partial^2 u(x_i, \tilde{t})}{\partial t^2} \implies \frac{\partial u_{i,j}}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} + \frac{1}{2}\Delta t \frac{\partial^2 u(x_i, \tilde{t})}{\partial t^2}$$

where $\tilde{t} \in (t_j, t_{j+1})$ and the last term is the truncation error, which is proportional to Δt.

Similarly, the three point approximation to the second derivative (derived in [1]) with its error is used to approximate the second derivative of u at point (x_i, t_j) with respect to position:

$$\frac{\partial^2 u_{i,j}}{\partial x^2} = \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} - \frac{1}{12}h^2 \frac{\partial^4 u(\tilde{x}, t_j)}{\partial x^4}$$

where the last term is the truncation error, which for this approximation is proportional to h^2. The observation $\tilde{x} \in (x_{i-1,j} + x_{i+1,j})$ is shown in the referenced report.

Inserting these two expressions into the diffusion equation gives

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} + \frac{1}{2}\Delta t \frac{\partial^2 u(x_i, \tilde{t})}{\partial t^2} = \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} - \frac{1}{12}h^2 \frac{\partial^4 u(\tilde{x}, t_j)}{\partial x^4}$$

The goal is to find the value of u at the next time step, i.e. $u_{i,j+1}$. Multiplying by Δt on both sides of the equation and moving one term to the right, we get

$$u_{i,j+1} = u_{i,j} + \frac{\Delta t}{h^2}\left(u_{i+1,j} + u_{i-1,j} - 2u_{i,j}\right) + \frac{1}{2}(\Delta t)^2 \frac{\partial^2 u(x_i, \tilde{t})}{\partial t^2} - \Delta t \cdot \frac{1}{12}h^2 \frac{\partial^4 u(\tilde{x}, t_j)}{\partial x^4}$$

The quantity $\Delta t/h^2$ can be defined as α, which, with a slight reorganisation yields the final expression

$$u_{i,j+1} = (1-2\alpha)u_{i,j} + \alpha\left(u_{i+1,j} + u_{i-1,j}\right) + \frac{1}{2}(\Delta t)^2 \frac{\partial^2 u(x_i, \tilde{t})}{\partial t^2} - \Delta t \cdot \frac{1}{12}h^2 \frac{\partial^4 u(\tilde{x}, t_j)}{\partial x^4}$$

The two error terms are proportional to $(\Delta t)^2$ and $\Delta t \cdot h^2$. As per usual, the global error is one order lower, as the error is accumulated. This gives one error term proportional to Δt and one propotional to h^2. The order of h is not reduced, as this error is not accumulated.

1.1.2 Stability analysis

1.2 Backward Euler

hei

1.2.1 Derivation and error analysis

1.2.2 Stability analysis

1.3 Crank-Nicolson

1.3.1 Derivation and error analysis

1.3.2 Stability analysis

References

[1] Anders Johansson. “Project 1”. In: *FYS3150, Computational Physics* (Sept. 2016), pp. 4–6. URL: https://github.com/anjohan/Offentlig/blob/master/FYS3150/Oblig1/Johansson_Anders_FYS3150_Oblig1.pdf.