UiO : Department of Physics
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Project 5: The diffusion equation

FYS3150 - Computational physics

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Contents

Abstract

Hei.

1 Introduction

2 Mathematical theory

We will now take a look into how we can numerically approximate the given problem. As there exists many different approaches, we have chosen to look at three different methods, namely forward Euler, backward Euler and Crank-Nicolson.

2.1 Forward Euler

2.1.1 Derivation and error analysis

The Forward Euler scheme is an explicit scheme based on Taylor polynomials. To find an approximation of the time derivative of u at point (x_i, t_j), a first order Taylor polynomial around x_i, t_j is used to calculate $u(x_i, t_{j+1})$ is used:

$$u_{i,j+1} = u_{i,j} + \Delta t \frac{\partial u_{i,j}}{\partial t} + \frac{1}{2}(\Delta t)^2 \frac{\partial^2 u(x_i, \tilde{t})}{\partial t^2} \implies \frac{\partial u_{i,j}}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} + \frac{1}{2}\Delta t \frac{\partial^2 u(x_i, \tilde{t})}{\partial t^2}$$

where $\tilde{t} \in (t_j, t_{j+1})$ and the last term is the truncation error, which is proportional to Δt.

Similarly, the three point approximation to the second derivative (derived in [1])) with its error term is used to approximate the second derivative of u at point (x_i, t_j) with respect to position:

$$\frac{\partial^2 u_{i,j}}{\partial x^2} = \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} +$$

2.1.2 Stability analysis

2.2 Backward Euler

The idea of backward Euler is similar to the forward Euler. Actually, the only difference lies in how we approximate the time derivative of u.

2.2.1 Derivation and error analysis

We can also use Taylor polynomials of $u(x_i, t_{j-1})$to write an approximation of the time derivative of u around x_i, t_j:

$$u_{i,j-1} = u_{i,j} - \Delta t \frac{\partial u_{i,j}}{\partial t} + \frac{1}{2}(\Delta t)^2 \frac{\partial^2 u(x_i, \tilde{t})}{\partial t^2} \implies \frac{\partial u_{i,j}}{\partial t} = \frac{u_{i,j} - u_{i,j-1}}{\Delta t} + \frac{1}{2}\Delta t \frac{\partial^2 u(x_i, \tilde{t})}{\partial t^2}$$

where $\tilde{t}$ and the second term is the same as defined in section 2.1.1 on the preceding page. We have also the very same approximation of the second derivative of u with respect to x as in [1].

truncation error

If we set up the approximations of the derivatives as in the given problem, we get

$$
\begin{aligned}
\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} &= \frac{u_{i,j} - u_{i,j-1}}{\Delta t} \\
r(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) &= u_{i,j} - u_{i,j-1}, \qquad r = \frac{\Delta t}{h^2} \\
u_{i+1,j} &= \frac{1}{r}\big(u_{i,j}(1+2r) - u_{i,j-1} - r u_{i-1,j}\big) \\
\frac{1}{r}u_{i,j-1} &= \frac{1+2r}{r}u_{i,j} - u_{i+1,j} - u_{i-1,j}
\end{aligned}
$$

2.2.2 Stability analysis

2.3 Crank-Nicolson

2.3.1 Derivation and error analysis

2.3.2 Stability analysis

References

[1] Anders Johansson. "Project 1". In: *FYS3150, Computational Physics* (Sept. 2016), pp. 4–6. URL: https://github.com/anjohan/Offentlig/blob/master/FYS3150/Oblig1/Johansson_Anders_FYS3150_Oblig1.pdf.