Code Guide

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October 26, 2016

1 Overview

The basis of the calibration is to iterate the operator:

$$\mathcal{T}\left[\hat{V}\right] = \max_{q} \left(q - \tau\right) p_{\hat{V}(\cdot),F(\cdot)}\left(q\right) + \left(1 - q\right)\left[\gamma + \delta \mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right) \mid \gamma\right]\right]$$

on candidate $\hat{V}(\cdot)$ functions until convergence, and then characterize the steady state of this. We will computationally do this by operating on quantile grids, with a total of $qres$ gridpoints. Any functions of γ, such as $V(\gamma)$, $WTP(\gamma)$, etc. will be represented as length $qres$ vectors, which for code clarity we'll keep together in a data.table. So, for example, we will have vectors:

- $Fgam = \left[\frac{1}{qres}, \frac{2}{qres}, \ldots \frac{qres-1}{qres}, 1\right]$
- $\gamma = \left[F^{-1}(0), F^{-1}\left(\frac{1}{qres}\right), \ldots F^{-1}\left(\frac{qres-1}{qres}\right)\right]$
- $V(\gamma) = \left[V\left(F^{-1}(0)\right), V\left(F^{-1}\left(\frac{1}{qres}\right)\right)\right]$

2 solve_value_function

2.1 Overview

solve_value_function takes as input a data table specifying the quantile vector γ, and an initial value function guess $V[\cdot]$, as well as parameters such as discount rate/decay rate, and iterates the $\mathcal{T}$ operator until convergence. It outputs the equilibrium value function, and some auxiliary information like equilibrium sale prices and probabilities, WTP's, etc.

2.2 Inputs

solve_value_function begins with a uniformly spaced quantile grid vector. Supposing we use $qres$, this will be:

The user specifies a value distribution $F(\cdot)$ over use values γ by inputting its quantile vector, as:

We will think of everything in terms of "quantile vectors." So, for example, $\gamma[q]$ refers to the value of γ at the q'th quantile. We will also need to specify a candidate value function $\hat{V}[q]$ to start the Bellman iteration. These vectors will be given to solve_value_function in a $qres$-row data table, with the following columns:

- $Fgam$: Quantile vector $\left[\frac{1}{qres}, \frac{2}{qres}, \ldots \frac{qres-1}{qres}, 1\right]$
- $fgam$: Density at q, by construction equal to $\frac{1}{qres}$

- γ: Use value at qth quantile of F
- V: Candidate value function at qth quantile

In addition, we specify the following parameters:

- δ: discount rate
- $decay\_rate$: beta decay parameter
- $beta\_shape$: beta shape parameter
- max_runs: max iterations of Bellman operator (never reached in practice)
- $Vtol$: sup norm tol of Bellman operator
- $quiet$: whether to print output
- tau_try: value of tax rate τ

2.3 Outputs

The output of solve_value_function appends the following vectors to the data table:

- EV: Period $t+1$ expected value
- WTP: Willingness to pay, where we have $WTP[q] = \gamma[q] + \delta EV[q]$
- V: Value function of the qth quantile asset owner
- $best\_saleprob$: optimal sale probability
- $best\_p$: optimal price, equal to $WTP[1 - best\_saleprob[q]]$ (i.e. willingness to pay of the marginal buyer)

2.4 Details

2.4.1 Continuation value calculation

We can implement the decay operator $\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}(\gamma') \mid \gamma\right]$ as a matrix; since G is defined in quantile space, this interfaces well with the quantile grid. In particular, let $P_\beta(x)$ be the Beta CDF respectively, shape parameters $C\beta,\ C(1-\beta)$. For given quantile $q \in \{1, \ldots, qres\}$, we derive the decay vector:

$$y_{\beta,q} = \left[P_\beta\left(\frac{1}{q}\right),\ P_\beta\left(\frac{2}{q}\right) - P_\beta\left(\frac{1}{q}\right), \ldots 1 - P_\beta\left(\frac{q-1}{q}\right), 0, 0, \ldots 0\right]$$

i.e. a grid approximation to the decay distribution. Then, conditional on q, we can implement the expectation operator numerically as:

$$\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(F^{-1}(q')\right) \mid q\right] = y_{\beta,q} \cdot \hat{V}$$

In particular we can stack these decay vectors into a matrix:

$$decay\_matrix = \begin{bmatrix} y_{\beta,1} \\ y_{\beta,2} \\ \vdots \\ y_{\beta,1000} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ P_\beta\left(\frac{1}{2}\right) & 1 - P_\beta\left(\frac{1}{2}\right) & 0 & \cdots \\ P_\beta\left(\frac{1}{3}\right) & P_\beta\left(\frac{2}{3}\right) - P_\beta\left(\frac{1}{3}\right) & 1 - P_\beta\left(\frac{2}{3}\right) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

And this lets us express the "continuation utility vector" as a matrix operator:

$$E\hat{V} = decay\_matrix \cdot \hat{V}$$

2.4.2 Maximization

Note that we can also derive a WTP vector, which we'll call $WTP_{\hat{V}}$, naturally, and an associated inverse demand distribution...

For each quantile q we can thus evaluate the objective function, as:

$$\mathcal{T}\left[\hat{V}\right][q] = \max_{q'} \left(q' - \tau\right) p_{\hat{V}(\cdot),F(\cdot)}\left(q'\right) + \left(1 - q'\right)\left[\gamma + \delta \mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(F^{-1}\left(q'\right)\right) \mid q\right]\right]$$

$$= \max_{q'} \left(q' - \tau\right) WTP_{\hat{V}}\left[q'\right] + \left(1 - q'\right) F^{-1}\left(q\right) + \left(1 - q'\right) \delta \left(M_{\beta}\hat{V}\right)[q]$$

Since we can calculate $M_{\beta}\hat{V}$, and we know $WTP_{\hat{V}}$, this is now a fairly straightforwards maximization problem, so to evaluate $\mathcal{T}\left[\hat{V}\right]$ we can just loop over all 1000 q values in each iteration. Somewhat counterintuitively, it turned out, in my numerical experiments, that rather than looping, it is actually faster to generate a 1000 x 1000 grid and use a group-by operation with data.table to do all the maximizations together, so this is the approach I adopt in the code, although it is somewhat awkward.

3 solve_steadystate

For a given τ, the stationary equilibrium defines a Markov process over quantiles q. By solving for the stationary equilibrium of the Markov process, we can characterize steady-state values, etc.

3.1 Overview

The Markov transition process can be decomposed into two steps:

1. Trade: Quantile q sets a saleprob q^*, and sells to all agents with higher values
2. Decay: Owner's value decays by the Beta decay process

3.2 Inputs

Takes as input a data_table output from solve_value_function, that is, with the columns:

- $Fgam$
- $fgam$
- γ
- EV
- WTP
- V
- $best\_saleprob$
- $best\_p$

In addition we specify the incidental parameters:

- $decay\_rate$: Beta decay parameter
- $beta\_shape$: Beta shape parameter
- $quiet$: Whether to print output
- $efficient$: Whether we want equilibrium behavior, or socially efficient behavior (hacky, always set to 0 for equilibrium)

3.3 Outputs

Appends the following rows to the data table:

- ss: Stationary density over γ values of owners
- val_ss: Stationary density over γ values of users – slightly different from owners, because if the owner sells in period t, we count the γ value of the buyer here.
- $buyer\_dist$: In steady state, distribution over buyers' quantiles
- $seller\_dist$: In steady state, distribution over sellers' quantiles

3.4 Details

3.4.1 Trade

Given a quantile q and her optimal choice q_q^*, transition is uniform from q_q^* upwards, so,

$$T_q = \left[0, \dots 0, \frac{1}{1-q_q^*}, \frac{1}{1-q_q^*}, \dots \frac{1}{1-q_q^*}\right]$$

This can be stacked into a matrix:

$$tradeprob\_matrix = \begin{bmatrix} 0 & \frac{1}{1-q_1^*} & \frac{1}{1-q_1^*} & \frac{1}{1-q_1^*} & \cdots \\ 0 & 0 & \frac{1}{1-q_2^*} & \frac{1}{1-q_2^*} & \cdots \\ 0 & 0 & \frac{1}{1-q_3^*} & \frac{1}{1-q_3^*} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

3.4.2 Decay

Decay matrix is just M_β from above:

$$decay\_matrix = \begin{bmatrix} y_{\beta,1} \\ y_{\beta,2} \\ \vdots \\ y_{\beta,1000} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ P_\beta\left(\frac{1}{2}\right) & 1 - P_\beta\left(\frac{1}{2}\right) & 0 & \cdots \\ P_\beta\left(\frac{1}{3}\right) & P_\beta\left(\frac{2}{3}\right) - P_\beta\left(\frac{1}{3}\right) & 1 - P_\beta\left(\frac{2}{3}\right) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

3.4.3 Stationary distribution

Now we can get the full transition matrix as:

$$M = decay\_matrix \cdot tradeprob\_matrix$$

This is always ergodic. So we can use a formula from Resnick (Adventures in Stochastic Processes, pg 138) to get the unique stationary distribution:

$$\pi' = (1, \dots, 1)\,(I - M + ONE)^{-1}$$

The stationary distribution over q values then allows us to compute all the values of interest.