1 Advection Equation

The goal of this homework is to try several different methods of solving partial differential equations on the specific case of the advection equation given by:

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0 \ . \tag{1}$$

I used the provided skeleton code and added the parts needed to implement each method for solving PDEs based on the expressions given in the lecture notes.

First I show the results for the upwind method (upper panels of Figure 1), for $\alpha = 0.6$. As expected, the solution in this case seems stable. The error grows as a function of time, but nearly linearly. The lower

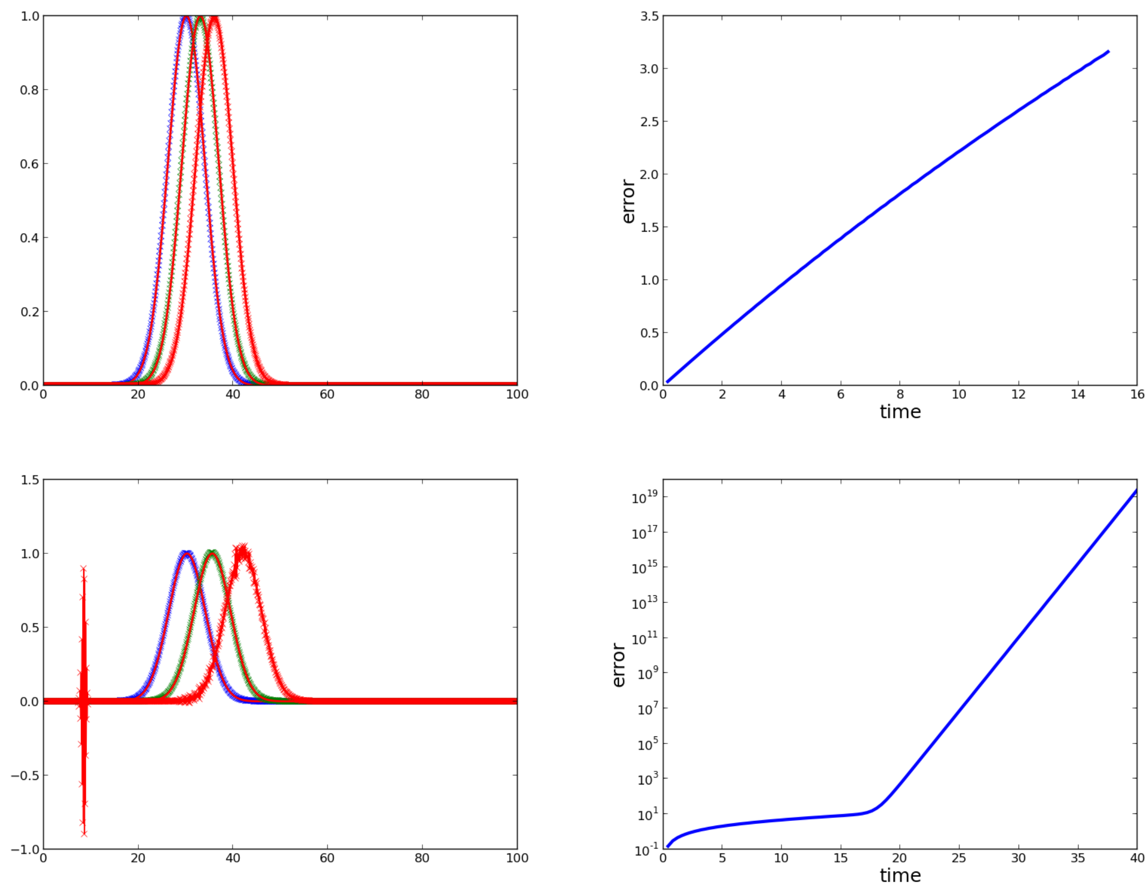

Figure 1: Results of the upwind method. Panels on the left show snapshots at three different times and error as a function of time is shown on the right. Top panels correspond to $\alpha = 0.6$, whereas lower panels show the results for $\alpha = 1.2$. Solid lines represent the analytic solution, and the numerical solution is denoted by crosses.

panels show the results for the upwind method, but with $\alpha = 1.2$. We can see that instabilities start to occur and that the error grows exponentially with time.

For a Gaussian solution with a 5 times smaller value of σ the results are shown in Figure 2. The Gaussian is narrower, but I didn't notice any change in the error compared to the larger value of σ, which makes sense because what determines the error is the value of α.

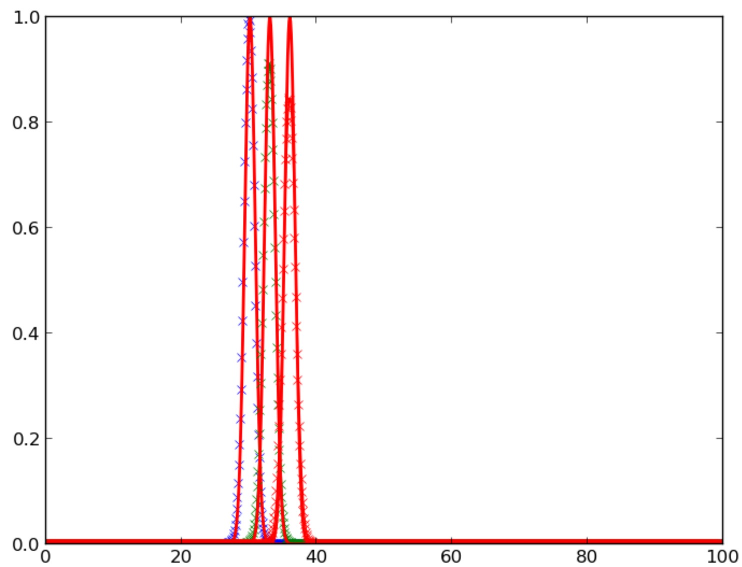

Figure 2: Results of a Gaussian with a 5 times smaller σ. The analytic solution is shown in a solid line and numerical is represented by crosses.

Implementing the downwind method does not seem like a very good way of solving PDEs because it results in instabilities even for $\alpha < 1$, as shown in Figure 3 which was obtained for $\alpha = 0.2$.

Other methods are shown in Figures 4, 5 and 6. The results of the FTCS method do not travel at the same velocity as the analytic solution. A similar problem occurs with the leapfrog method. The solution of Lax-Friedrich method moves at the right speed, but the Gaussian becomes wider and lower. Lax-Wendroff method seems to give the best result and it is convergent to second order, as shown in Figure 6.

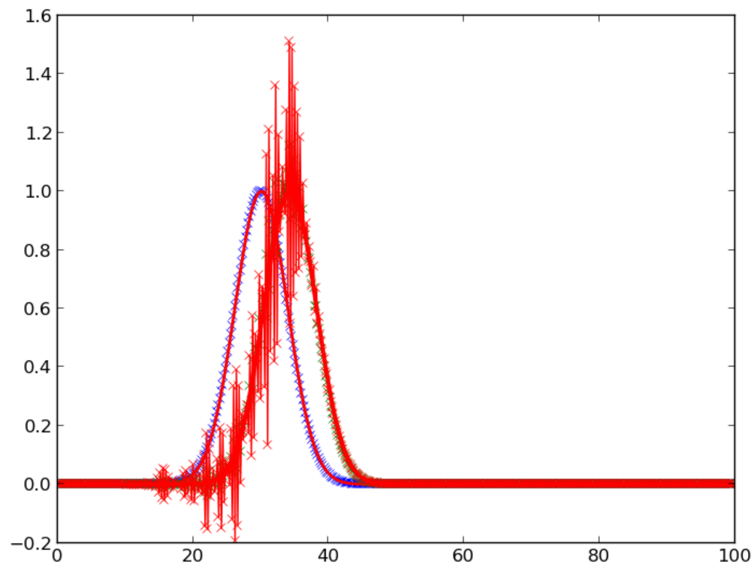

Figure 3: Results of the downwind method. Instabilities occur even for $\alpha < 1$.

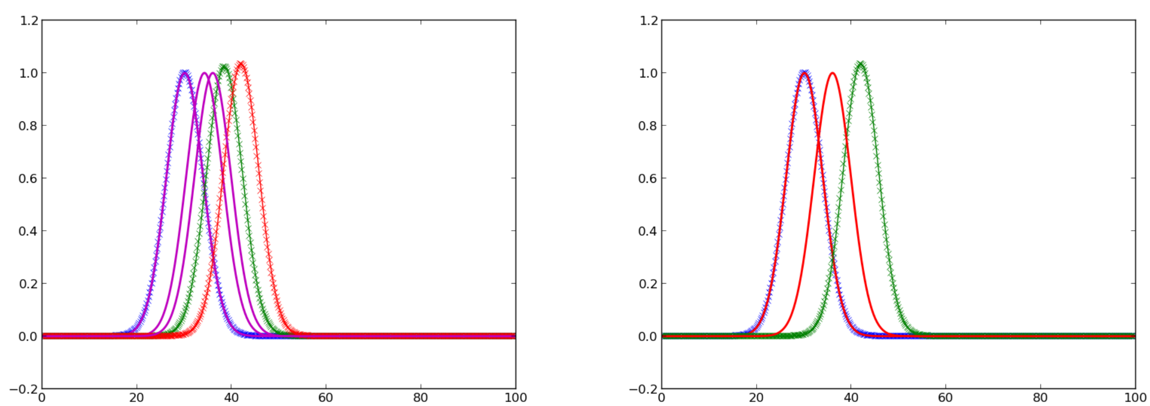

Figure 4: Results of the FTCS (left) and leapfrog (right) methods. The numerical solutions move at a different velocity from the analytic one, given in solid lines.

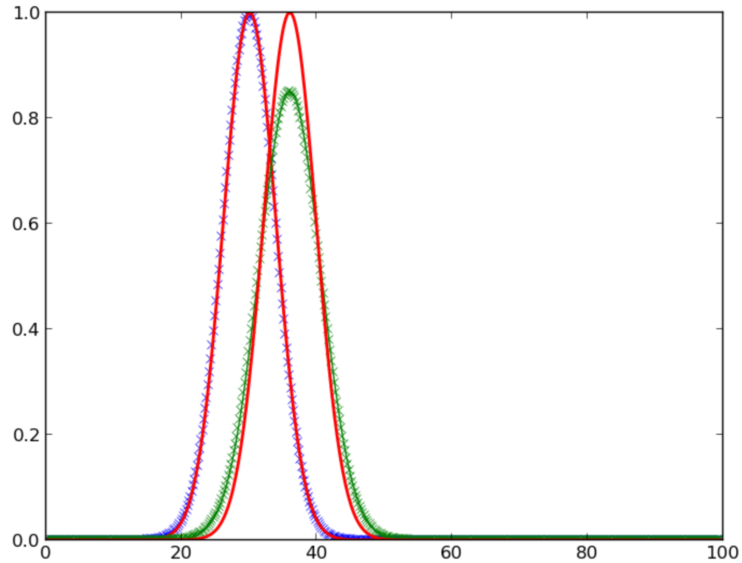

Figure 5: Results of the Lax-Friedrich method. The Gaussian becomes lower and wider, but travels at the same speed as the analytic solution (solid lines).

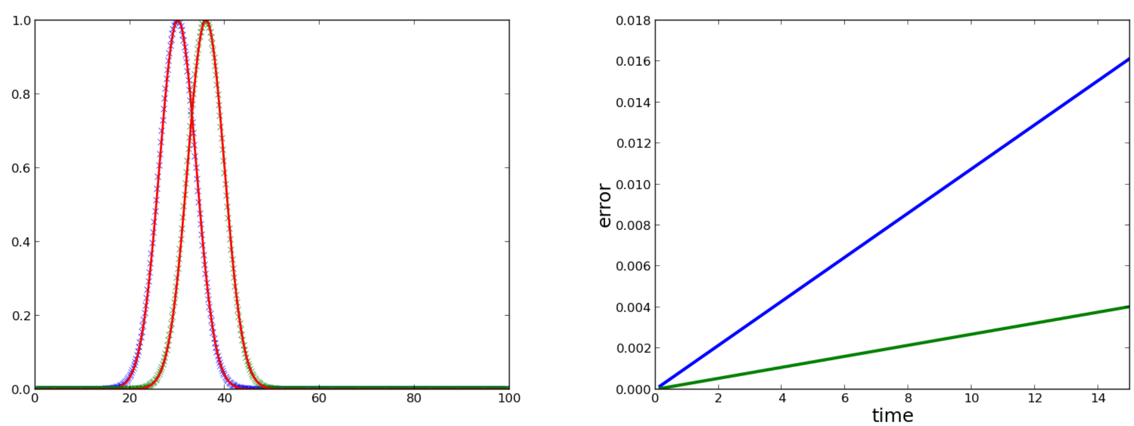

Figure 6: *Left:* Results of the Lax-Wendroff method. Numerically calculated solution matches the analytic one fairly well. *Right:* Errors induced by the Lax-Wendroff method for two different resolutions. The resolution for the result shown in green is twice that of the result in blue. Errors differ by a factor of 4, hence we can conclude that the Lax-Wendroff method is second order convergent.