1 Root Finding: Eccentric Anomaly

Location of a planet moving around the Sun is given by

$$x = a\cos E \tag{1}$$

$$y = b\sin E \ , \tag{2}$$

where a and b are semi-major and semi-minor axes of the planet's orbit, respectively. E is the eccentric anomaly defined by

$$E = \omega t + e\sin E \ , \tag{3}$$

where $e = \sqrt{1 - b^2/a^2}$ is the eccentricity and ω the angular frequency of the planet. To find the eccentric anomaly E at a given time t, and consequently the position of the planet in terms of x and y, we need to find the roots of the following equation:

$$E - \omega t - e\sin E = 0 \ . \tag{4}$$

I have used the bisection method the find the roots of this equation for orbital parameters of the Earth, for three values of variable t. My results are given in Table 1. They were obtained in 34 iterations.

t (days)	E	x (a)	y (b)
91	1.58209228899	-0.0112957219748	0.999796755471
182	3.13096420068	-0.999943518526	0.0106267706405
273	4.67948910053	-0.0328939450291	-0.99931946851

If I change the eccentricity to $e = 0.99999$, the number of iterations needed to obtain the result doesn't change very much.

2 Root Finding: Polynomials with Multiple Roots

The aim of the second problem is to find all real roots of equation:

$$f(x) = 3x^5 + 5x^4 - x^3 \ . \tag{5}$$

We can easily solve this analytically. The roots are: $x_{1,2,3} = 0$, $x_4 = \frac{-5+\sqrt{37}}{6} \approx 0.1804$ and $x_5 = \frac{-5-\sqrt{37}}{6} \approx -1.847$. It is useful to have these values, so we can check how good our numerical results are.

I again used the bisection method for finding roots. I made use of a property of polynomials which I found on Wikipedia: for a polynomial $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ all (real) roots are located in the interval with endpoints

$$x_\pm = -\frac{a_{n-1}}{n a_n} \pm \frac{n-1}{n a_n}\sqrt{a_{n-1}^2 - \frac{2n}{n-1}a_n a_{n-2}} \ . \tag{6}$$

In our case $x_+ = 2.6$ and $x_- = -3.2$. I have divided this interval into ~ 60 subintervals. For each subinterval whose endpoints are of opposite sign, I applied the bisection method for finding roots. The algorithm found roots at these values: $x = -1.84712708838$, $x = 0.180460421741$ and a multiple root at $x = 0$ (the exact value was of the order of 10^{-15}, but that is less than the value of $\epsilon = 10^{-10}$ which I used in my bisection algorithm). Clearly, the calculated values match the true result very well.

I've tried my algorithm on a few other cases, to prove that it is working as it should:

- For
$$f(x) = x^2 + 3x - 1$$
the roots are $x_{1,2} = \frac{-3\pm\sqrt{13}}{2} \approx (-3.303, 0.303)$.
The results of my algorithm are $x_1 = -3.30277563773$ and $x_2 = 0.302775637732$.

- For
$$f(x) = x^3 - 5x^2 + 2x$$
the roots are $x_{1,2} = \frac{5\pm\sqrt{17}}{2} \approx (0.44, 4.56)$ and $x_3 = 0$.
The results of my algorithm are $x_1 = 0.438447187191$, $x_2 = 4.56155281281$ and $x_3 = 0$.