Structural Dynamics Exam Preparation

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November 7, 2017

Contents

1 Discretization of continuous systems

1.1 Q1

Explain the elastodynamic equilibrium equations for continuous systems. Introduce the matrix notations to write the elastodynamic equations.

- The representation of a physical system by a discrete model is usually an idealized view.
- In reality: Main bodies composing a system are deformable, and the elastic elements connecting them do also have their own inertia.
- Each element has its own inertia, stiffness and damping $\Rightarrow$ *elastodynamics* time-dependent deformation relevant.
- Equations of motion are expressed in terms of the displacement field $u, v, w(x, y, z, t)$.
- Few closed form solutions $\Rightarrow$ we must discretize the system to approximate the real system.
- Definitions of an elastic body: Figure 1.1. Defining a continuous system.
- $S = S_u + S_\sigma$: surface of the body. S_u: The portion of surface on which displacements are imposed. S_σ: complementary part of the surface on which surface tractions are imposed.
- Assumptions:
  - Small deformations and displacements. Meaning we can write a linear expression for the strain tensor.
  - Material is linear and elastic. Can then relate stresses and strains in the body by $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$. C_{ijkl}: Elastic coefficients of the material (Young's modulus)
- Figure 1.2: Writing the equilibrium of a continuous system. Look at an infinitesimal volume inside the body.
- Watch the stress component (force per unit area), and write a moment equilibrium (stress times area).
- Solving the system gives you that $\sigma_{ij} = \sigma_{ji}$, proving the symmetry of the stress tensor.
- Finding translational eq.: $\sum_{x,y,z} F = 0$. Meaning add all stresses times area, and the applied body forces, minus density, acceleration: $(\bar{X}_1 - \rho\ddot{u}_1)dV$.
- Doing this in all directions yields

$$\frac{\partial}{\partial x_i}\sigma_{ji} + \bar{X}_j - \rho_0\ddot{u}_j = 0 \tag{1.1}$$

  in V_0.
  $\bar{X}_j$: applied body forces. ρ_0: density of the undeformed body. $\ddot{u}_j$: accelerations.
- Lastly, using the same relations as for the static case for friction: $t_j = n_i\sigma_{ij} = \bar{t}_j$ on S_σ.
- Eq. (1.1.11) summarizes the equations describing linear elastodynamic behaviour of a continuum body. These are the equations we are going to solve.

The matrix notations used for expressing equilibrium

- $\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ are used for collecting respectively displacements, stress and strain components.
- Spatial differential operator $\mathfrak{D}^T$: includes only partial differentiations so that on can write $\mathfrak{D}^T\boldsymbol{\sigma}$.

- $\boldsymbol{N}^T$ collects the direction cosines of the outward normal so that $\boldsymbol{N}^T\boldsymbol{\sigma} = \bar{\boldsymbol{t}}$
- $\boldsymbol{H}$ is the Hooke's matrix, collecting Young's modulus, Poisson's ratio ν and shear modulus $\boldsymbol{G}$. Then one can write $\boldsymbol{\sigma} = \boldsymbol{H}\boldsymbol{\varepsilon}$.
- All euations expressing linear dynamic eq. are then written in matrix form

$$\begin{aligned} \mathfrak{D}^T\boldsymbol{\sigma} + \bar{\boldsymbol{X}} - \rho\ddot{\boldsymbol{u}} &= \boldsymbol{0} \\ \boldsymbol{N}^T\boldsymbol{\sigma} &= \bar{\boldsymbol{t}} \\ \boldsymbol{\sigma} &= \boldsymbol{H}\boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} &= \mathfrak{D}\boldsymbol{u} \\ \boldsymbol{u} &= \bar{\boldsymbol{u}} \end{aligned}$$

1.2 Q2

Define the kinetic and potential energy of a continuous system. For a bar and a beam, find the equilibrium equations and the expressions of the energies

- Kinetic energy of cont. system in Fig. (1.1). Integration over the reference volume, similar to $1/2mv^2$:

$$\mathcal{T}(u) = \frac{1}{2}\int_{V_0} \rho_0 \dot{u}_i \dot{u}_i dV$$

  The 0 indicates it is taken for the non-deformed reference configuration.
- Total potential energy $\mathcal{V} = \mathcal{V}_{int} + \mathcal{V}_{ext}$
- $\mathcal{V}_{int}$ is the strain energy of the body.
- Strain energy is expressed as

$$\mathcal{V}_{int} = \int_{V_0} W(\varepsilon) dV$$

  where $W(\varepsilon_{ij}$ is the internal energy density. Similar to the area under the stress-strain curve. Can be approximated by the matrix notations as $W = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$ (similar to a simple area).
- The ext. pot. energy is computed by assuming existence of body forces $\bar{X}_i$ and surface tractions $\bar{t}_i$ on the portion S_σ of the surface:

$$V_{ext} = -\int_{V_0} \bar{X}_i(t) u_i dV - \int_{S_\sigma} \bar{t}_i(t) u_i dS$$

The bar

- Assuming disp. and stresses are constant over the cross section. Fig. 1.4 shows it.
- An axial load is at every end, $N(x) = A(x)\sigma_x(x) = AE\varepsilon_x$.
- The eq. for the bar element then writes

$$\frac{\partial}{\partial x}\Big(EA\frac{\partial u}{\partial x}\Big) - m\ddot{u} + \bar{X} = 0$$

- The system energies calc. same way by inserting into the eq. equations. See page 8.

The beam with no shear deformation

- Here, we are considering a beam of length l. Distributed to vertical load $p(x,t)$ and distributed moments $q(x,t)$ per unit length. Figure 1.5 shows the system. $\bar{\boldsymbol{T}}$: Shear loads at the end of the beam.

- Assumptions for analysis:
  - The beam cross section is not deformable
  - Transverse displacement is uniform, limited to the xz plane:
    $w = w(x)$
    $v = 0$.
  - Axial displacement results from rotation of the cross section. Rotation so that cross section remains orthogonal to the neutral axis.
    $\implies u(x,z) = -z\frac{\partial w}{\partial x}$
  - Can rewrite the strain expressions with this result (eq. page 9). We are then neglecting the shear deformation of the material.
- The rotational eq. of a beam element, width dx when $dx \to 0$:

$$T(x) = -\frac{\partial M}{\partial x} + mr^2\frac{\partial \ddot{w}}{\partial x} - q$$

- Transverse translational eq. gives (Fig. 1.7) when $dx \to 0$:

$$p + \frac{\partial T}{\partial x} = m\ddot{w}$$

- Substituting translational expression into transverse expression gives (1.1.42)
- From Figure 1.8, you can write moment $M(x)$ as: $M(x) = EI\frac{\partial^2 w}{\partial x^2}$
- Subst. in this gives the final equilibrium equation:

$$m\ddot{w} - \frac{\partial}{\partial x}\Big(mr^2\frac{\partial \ddot{w}}{\partial x}\Big) + \frac{\partial^2}{\partial x^2}\Big(EI\frac{\partial^2 w}{\partial x^2}\Big) = p - \frac{\partial q}{\partial x}$$

  Observations:
  - Fourth order differential equation in space
  - Assumption of no shear deformations is valid provided the ratio $\frac{I}{A} = r^2$ remains small.
  - If this is true, one can neglect the rotary inertia of the cross sections, and thereby neglect the term $mr^2\frac{\partial \ddot{w}}{\partial x}$
- The **energy expressions** are written by integrating over the whole volume, setting in correct expressions for strain.
- The kinetic energy has two terms. The first describes kinetic energy for vertical translation. The second is rotational kinetic energy of the cross section (often neglected, if you neglect the term mentioned above).

1.3 Q3

Explain how the Rayleigh-Ritz method discretizes linear dynamic equations. Give an interpretation of the discretized equations in terms of minimum weighted residual and show how they can be derived from the Lagrange equations.

- The Rayleigh-Ritz method results in a substitution problem which possesses a finite number n degrees of freedom and is described by ordinary differential equations.
- The solution improves with the number of degrees of freedom incorporated in its expression.

- The method starts with the choice of approximation functions for the displacements. Each component u_i is described by a series whose characteristic term is a function $f_{ij}(x_1, x_2, x_3)$ multiplied by a time-dependent amplitude, $q_j(t)$. The Rayleigh-Ritz approximation is then written in the form

$$u_i(x_1, x_2, x_3, t) = \sum_{j=1}^{n} f_{ij}(x_1, x_2, x_3) q_j(t)$$

- Important: Each set of functions must be admissible, meaning it must satisfy internal compatibility conditions (C0 continuity) and the set boundary conditions.

- The approximation can be written in matrix form as

$$\boldsymbol{u}(x_1, x_2, x_3, t) = \boldsymbol{F}(x_1, x_2, x_3)\boldsymbol{q}(t)$$

  where $\boldsymbol{q}(t)$ represents the vector of generalized coordinates, and matrix $\boldsymbol{F}$ is the displacement interpolation matrix.

- As strain is the derivate of displacement, one can then write

$$\boldsymbol{\varepsilon}(x, t) = \mathfrak{D}\boldsymbol{F}(x_1, x_2, x_3)\boldsymbol{q}(t) = \boldsymbol{B}(x_1, x_2, x_3)\boldsymbol{q}(t)$$

  where $\boldsymbol{B}$ is the derivatice of displacement interpolation matrix.

- The equil. equations (1.1.20), p. 6 can then be written, when knowing that $\boldsymbol{\sigma} = \boldsymbol{H\varepsilon} = \boldsymbol{HBq}$

$$\begin{aligned} \mathfrak{D}^T\boldsymbol{HBq} + \bar{\boldsymbol{X}} - \rho\boldsymbol{F}\ddot{\boldsymbol{q}} = \boldsymbol{0} \quad &\text{in V} \\ \boldsymbol{N}^T\boldsymbol{HBq} = \bar{\boldsymbol{t}} \quad &\text{on } S_\sigma \end{aligned}$$

- Because not all solutions are allowed by the Rayleigh-Ritz approximation, these equilibrium equations can not be satisfied exactly. Because the shape functions are probably not totally exact. Equilibrium is then written as

$$\begin{aligned} \mathfrak{D}^T\boldsymbol{HBq} + \bar{\boldsymbol{X}} - \rho\boldsymbol{F}\ddot{\boldsymbol{q}} = \boldsymbol{res}_V \quad &\text{in V} \\ \bar{\boldsymbol{t}} - \boldsymbol{N}^T\boldsymbol{HBq} = \boldsymbol{res}_S \quad &\text{on } S_\sigma \end{aligned}$$

- The resV and resS are force residuals in the volume and surface bound., corresponding to the equilibrium default.

- The problem must therefore be seen differently: We want to find the solution of the equilibrium eq. that satisfies the constraints inherent to $\boldsymbol{u} = \boldsymbol{Fq}$. The residual forces can then be seen as reaction forces associated to the constraints put on the solution field (boundary conditions, C0 cont.)

- We can then find the unknown amplitudes $\boldsymbol{q}$ of the shape functions by applying the virtual work principle: Require that the equilibrium eq. with residual forces must be satisfied for any compatible diplacement expressed by $\boldsymbol{u} = \boldsymbol{Fq}$.

- After an insane derivation using virtual work principle and integration by parts in space you get the expression in matrix form

$$\boldsymbol{Kq} + \boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{g}(t)$$

  where the discretized mass, stiffness and force matrices are

$$\begin{aligned} \boldsymbol{M} &= \int_V \rho\boldsymbol{F}^T\boldsymbol{F}dV \\ \boldsymbol{K} &= \int_V \boldsymbol{B}^T\boldsymbol{HB}dV \\ \boldsymbol{g} &= \int_{S_\sigma} \boldsymbol{F}^T\bar{\boldsymbol{t}}dS + \int_V \boldsymbol{F}^T\bar{\boldsymbol{X}}dV \end{aligned}$$

  Remember: $\boldsymbol{F}$ is the shape functions.

- The expression $\boldsymbol{Kq} + \boldsymbol{M\ddot{q}} = \boldsymbol{g}(t)$ expresses that

$$\int_V \boldsymbol{F}^T \boldsymbol{res}_V dV + \int_{S_\sigma} \boldsymbol{F}^T \boldsymbol{res}_S dS = \boldsymbol{0}$$

  which can be interpreted as:

  - The approximate solution $\boldsymbol{u} = \boldsymbol{Fq}$ is determined so as to minimize the error on the equilibrium equation weighted by the shape functions.
  - Therefore it is a best solution to the equilibrium problem in a minimum weighted residual sense.
  - When the shape functions in $\boldsymbol{F}$ are enriched, the solution will converge towards the exact solution.

- The expression $\boldsymbol{Kq} + \boldsymbol{M\ddot{q}} = \boldsymbol{g}(t)$ can also be derived from Lagrange's equation by substituting $\boldsymbol{u} = \boldsymbol{Fq}$ and $\boldsymbol{\varepsilon} = \boldsymbol{Bq}$ into expressions for kinetic and potential energy.

- Lagrange's equation:

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{T}}{\partial \boldsymbol{q}} + \frac{\partial \mathcal{V}}{\partial \boldsymbol{q}} = 0$$

1.4 Q4

Discuss how to find the free vibration modes and frequencies of a discretized problem. What can we say about the frequencies of a discretized system? Give an example.

- Free vibration: no external forces applied to a structure and all displacement boundary conditions are homogenous. The equilibrium equations then write

$$\begin{aligned} \mathfrak{D}^T \boldsymbol{\sigma} - \rho \ddot{\boldsymbol{u}} &= \boldsymbol{0} \quad \text{in V} \\ \boldsymbol{N}^T \boldsymbol{\sigma} &= \boldsymbol{0} \quad \text{on } S_\sigma \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } S_u \end{aligned}$$

- There exist free vibration modes described by a spatial displacement

$$\boldsymbol{u}_r(x_1, x_2, x_3, t) = \boldsymbol{u}_{(r)}(x_1, x_2, x_3) \cos \omega_r t$$

  inserted into the equilibrium equations, these become

$$\mathfrak{D}^T \boldsymbol{H} \mathfrak{D} \boldsymbol{u}_{(r)} + \omega_r^2 \rho \boldsymbol{u}_{(r)} = \boldsymbol{0} \quad \text{in } V$$

  ω_r are here the corresponding free vibration frequencies, the eigenfrequencies. Free vibration and harmonic motion assumptions yield the discretized eigenvalue problem

$$\boldsymbol{Kq} = \omega^2 \boldsymbol{Mq}$$

- The following can be said about the eigenfrequencies found. Notation: $\omega_{r,c}^2$: the approximate eigenfreq. computed by Rayeigh-Ritz method. $\omega_{r,e}^2$: exact eigenvalues of the original equilibrium eq. (in Q4).

- The eigenvalues $\omega_{r,c}^2$ are an upper bound to the exact eigenvalue $\omega_{r,c}^2$.
  $\implies \omega_{r,e}^2 \leq \omega_{r,c}^2$

- This means that the approximation $\boldsymbol{u} = \boldsymbol{Fq}$ stiffens the system by introducing artificial constraints, making it stiffer compared to reality.
  Example: the clamped-free uniform bar

- Read self, page 16-17.

- The essential: Using shape functions with a higher polynomial degree will make the eigenvalues converge to the exact solution.

1.5 Q5

Explain the Finite Element method for a simple bar system. How can the matrices be generalized for a bar in 2D such as in a truss frame?

- The Finite Element method can be seen as a particular application procedure of the Rayleigh-Ritz method.
- In general: Rayleigh-Ritz applied to a finite number of elements.
- The interpolation functions are chosen so they fullfill:
  - Interpolation is performed in terms of *piecewise continuous functions*. Meaning that inside each element, the disp. field is represented by a number of functions.
  - The intensity parameters, the generalized coordinates of the Rayleigh-Ritz method are local values of the displacement field (see Figure 1.9).
  - This means that if we set boundary conditions of one element to be equal to the neighbour's, the displacement continuity is kept.
  - A finite element that holds these criteria are said to be conforming.

Generation of a bar element

- The bar is divided into N elements of length l.
- The displacement field is then linearly interpolated by the formula

$$u(x,t) = u_1(t)\phi_1(x) + u_2(t)\phi_2(x)$$

- The shape functions ϕ must be chosen so that $u(0,t) = u_1(t)$ and $u(l,t) = u_2(t)$
- The shape functions can result from a linear interpolation

$$\phi_1(x) = 1 - \frac{x}{l}$$
$$\phi_2(x) = \frac{x}{l}$$

- The displacement field u can be expressed in matrix form

$$u(x,t) = \boldsymbol{F}_e(x)\boldsymbol{q}_e(t) \quad x \in \text{element } e$$

- The element's kinetic and strain energies can be expressed in terms of the mass and stiffness *elementary* matrices:

  $$\mathcal{T}_e = \frac{1}{2}\dot{\boldsymbol{q}}_e^T \boldsymbol{M}_e \dot{\boldsymbol{q}}_e$$

  for the kinetic energy for every element, and

  $$\mathcal{V}_{int,e} = \frac{1}{2}\boldsymbol{q}_e^T \boldsymbol{K}_e \boldsymbol{q}_e$$

  for the potential strain energy.

- For the bar element, the mass and stiffness matrices are given by

$$\boldsymbol{M}_e = \int_0^l m\boldsymbol{F}_e^T\boldsymbol{F}_e dx$$
$$\boldsymbol{K}_e = \int_0^l EA\frac{d\boldsymbol{F}_e^T}{dx}\frac{d\boldsymbol{F}_e}{dx}dx$$

- The virtual work of external forces in the element:

$$\delta \mathcal{V}_{ext,e} = -\delta \boldsymbol{q}_e^T \boldsymbol{g}_e(t)$$

- The generalized load $\boldsymbol{g}(t)$:

$$\boldsymbol{g}_e(t) = \int_0^l \boldsymbol{F}_e^T p(x,t) dx + \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}$$

  P_1, P_2 are the end loads of the element.

- Doing the integration of mass and stiffness matrices with the linear interpolation functions, we obtain

$$\boldsymbol{K}_e = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{M}_e = \frac{ml}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Note that summing up all elements of the mass matrix, $\sum_{k,s} \boldsymbol{M}_{ks} = ml$

  Assembly process

- Connecting local displacements with global displacements through the localization operator $\boldsymbol{L}_e$:

$$\boldsymbol{q}_e = \boldsymbol{L}_e \boldsymbol{q}$$

- Look at Figure 1.11, and corresponding localization operator.
- Total kinetic and pot. energy of the system is then found by summing all elements.
- Similarly for the mass matrix:

$$\boldsymbol{M} = \sum_{e=1}^{N} \boldsymbol{L}_e^T \boldsymbol{M}_e \boldsymbol{L}_e$$

  and the structural stiffness matrix:

$$\boldsymbol{K} = \sum_{e=1}^{N} \boldsymbol{L}_e^T \boldsymbol{M}_e \boldsymbol{L}_e$$

- Look at figure 1.12 to see how the assembly of the matrices are. **The truss frame**
- Transforming coordinates: $\boldsymbol{q}_{eL} = \boldsymbol{R}\boldsymbol{q}_{eS}$ where
  $\boldsymbol{q}_{eL}$: element degrees of freedom in local axes.
  $\boldsymbol{q}_{eS}$: element degrees of freedom in structural (global) axes.
- The transformation of the stiffness matrix is then $\boldsymbol{K}_{eS} = \boldsymbol{R}^T \boldsymbol{K}_{eL} \boldsymbol{R}$
- See page 24 + 25 for the assembly of the whole structure.

1.6 Q6

Explain how to derive the matrices for a beam finite element. Give as application example the analysis of a clamped-clamped beam modeled with 2 and then 3 finite elements.

- We see that when integrating the strain energy over the beam, the function $w(x,t)$ must be continuous (C1).
- We use one-dimensional variable $\xi = \frac{x}{l}$ to write the cubic approximation, which can be written in the form

$$\begin{aligned} w(\xi) &= w_1 F_1(\xi) + \psi F_2(\xi) + w_2 F_3(\xi) + \psi F_4(\xi) \\ &= \boldsymbol{F}_e(\xi)\boldsymbol{q}_e(t) \end{aligned}$$

- The shape functions must be third-order Herimitan polynomials
- The derivative of the shape function and the shape function itself must be 0 at $\xi = 0$.
- The elementary stiffness and mass matrices are then computed with an integral of $d\xi$, from 0 to 1
- With $\boldsymbol{M}$ and $\boldsymbol{K}$ matrices built, one can see that the **translation inertia** of the system is equal to $\boldsymbol{u}_{trans}^T \boldsymbol{M} \boldsymbol{u}_{trans} = ml$. And the **rotary inertia** is equal to $\boldsymbol{u}_{rot}^T \boldsymbol{M} \boldsymbol{u}_{rot} = \frac{ml^3}{12}$

The beam clamped at both ends

- Results: Beam clamped at both ends can only give the first eigenfrequency with acceptable accuracy. The second eigenmode will not be precise.
- For the beam clamped at both ends, but with two nodes, and four degrees of freedom, one could compute its **symmetric** eigensolutions by taking advantage of symmetry in the system: $w_3 = w_2, \quad \psi_3 = -\psi_2$
- Then, by setting

$$\boldsymbol{y}^T = \begin{bmatrix} w_2 \\ \psi_2 \end{bmatrix}$$

  we can write the displacement as $\boldsymbol{q} = \boldsymbol{C}\boldsymbol{y}$, with

$$\boldsymbol{C}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

  and then computing reduced matrices $\boldsymbol{M}$ and $\boldsymbol{K}$: $\boldsymbol{K}^* = \boldsymbol{C}^T\boldsymbol{K}\boldsymbol{C}$ and $\boldsymbol{M}^* = \boldsymbol{C}^T\boldsymbol{M}\boldsymbol{C}$.
- For the **antisymmetric** eigenmodes, the same is done, but with $w_3 = -w_2$ and $\psi_3 = \psi_2$.
- The symmetric eigenmodes correspond to eigenmode 1 and 3 of the whole system, while the anitsymmetric eigenmodes correspond to eigenmodes 2 and 4.
- One could also see that the nearly converged eigenvalues are equal to half of the number of degrees of freedom in the model.
- All approximations are upper bounds to the exact ones, resulting from the conformity of the displacement field.

1.7 Q7

How can one build a finite element model of a full 3 dimensional beam?

- A full 3-dimensional beam needs to have a strain energy expression consisting of four terms:
  1. Bending energy in plane $O'xy$
  2. Bending energy in plane $O'xz$
  3. Extension energy of the beam, computed in terms of axial displacement $u(x)$ and axial stiffness EA.
  4. Torsional deformation energy of the beam. Computed in terms of local rotation ψ_x and torsional stiffness GJ_x.
- This will again be expressed in matrix notation:

$$\mathcal{V}_{int,e} = \frac{1}{2}\boldsymbol{q}_{eL}^T \boldsymbol{K}_{eL} \boldsymbol{q}_{eL}$$

- Similarly, the kinetic energy could be discretized as

$$\mathcal{T}_e = \frac{1}{2}\dot{\boldsymbol{q}}_{eL}^T \boldsymbol{M}_{eL} \dot{\boldsymbol{q}}_{eL}$$

- To express the elementary stiffness and mass matrices in the structural axes, we most use a transformation (p. 37) so that $\boldsymbol{q}_{eL} = \boldsymbol{T}\boldsymbol{q}_{eS}$
- The resulting stiffness and mass matrices of the element expressed in structural axes are then

$$\boldsymbol{K} = \boldsymbol{T}^T \boldsymbol{K}_{eL} \boldsymbol{T}, \quad \boldsymbol{M} = \boldsymbol{T}^T \boldsymbol{M}_{eL} \boldsymbol{T}$$

2 Linear Solvers in Finite Element

2.1 Q8

What are the general problem characteristics that need to be looked at when choosing a linear solver? Explain the basic Gauss elimination procedure for non-symmetric and regular systems.

- Things to consider when choosing a method for solving the system of linear equations:
  - Properties: symmetry, definite positiveness, singularity, sparsity
  - The size of the system (how big is n)
  - Numerical conditioning of the system of equations (many large numbers? Combination of large and small numbers?)
  - Capability to efficiently handle multiple right-hand sides:

$$Ax_1 = F_1$$
$$Ax_2 = F_2$$
$$Ax_3 = F_3$$

  - The burden of its implementation
  - Its ability to run on vector or parallel computers
- Here, we will solve non-singular linear systems on the form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A}$ is square and non-singular.
- The Gauss elimination turns $\boldsymbol{A}$ into an upper and lower triangular matrix: $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$

- Making $\boldsymbol{U}$: Row reducing $\boldsymbol{A}$ to upper-triangular form.
- Making $\boldsymbol{L}$: A lower triangular matrix with unit diagonal $l_{ii} = 1$ where $l_{ik} = \frac{u_{ik}}{u_{kk}}$
- The matrix equations then writes

$$\begin{aligned} \boldsymbol{Ax} &= \boldsymbol{b} \\ \boldsymbol{LUx} &= \boldsymbol{b} \\ \boldsymbol{L\tilde{b}} &= \boldsymbol{b} \end{aligned}$$

  where $\tilde{\boldsymbol{b}} = \boldsymbol{Ux}$
- To solve the system, we then solve succesively the triangular systems

$$\begin{aligned} \boldsymbol{L\tilde{b}} &= \boldsymbol{b} \quad \text{(forward substitution)} \\ \boldsymbol{Ux} &= \tilde{\boldsymbol{b}} \quad \text{(backward substitution)} \end{aligned}$$

- Additional remarks:
  - The diagonal element u_{kk} is called a pivot and must be non-zero. To have this, you must apply permutations on the rows and coloumns.
  - To minimize memory requirements, the $\boldsymbol{A}$ matrix is usually overwritten by the coefficients of $\boldsymbol{U}$ and $\boldsymbol{L}$.

2.2 Q9

For symmetric and regular systems describe the LDLT and the Cholesky decomposition procedures. Discuss specific storage techniques for sparse matrices.

- If the matrix is symmetric, the LDL^T factorization requires only half the number of operations than the LU factorization would.
- We introduce a diagonal matrix $\boldsymbol{D}$ that contains the diagonal values of $\boldsymbol{U}$ such that $\boldsymbol{A} = \boldsymbol{LD\tilde{U}}$. Since $\boldsymbol{A}$ is symmetric, we must have that $\tilde{\boldsymbol{U}} = \boldsymbol{L}^T$, meaning that $\boldsymbol{A}$ can be factorized into the symmetric form

$$\boldsymbol{A} = \boldsymbol{LDL}^T$$

- Comparing to the LU factorization, for the symmetric case, $\boldsymbol{U} = \boldsymbol{DL}^T$
- This system is also solved in the same manner, by forwards- and backwards-substitution.
- Remarks to the method:
  - We also here need non-zero pivots. To remain symmetry, the same permutation must be applied to the rows and coloumns of $\boldsymbol{A}$.
  - If all elements of $\boldsymbol{A}$ are greater than zero, the factorization can be given as

$$\boldsymbol{A} = \boldsymbol{LDL}^T = \boldsymbol{LD}^{\frac{1}{2}}\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{L}^T = \boldsymbol{CC}^T$$

    which is the Cholesky factorization.
- Doing this, you often get sparce matrices (only entries along rows/coloumns close to the diagonal, zeros rest), which you use special storage techniques to store.
- To avoid storing zeros of a sparse matrix, you could store the index $[i, j]$ of its non-zero entries, together with their values.
- Unfortunately, when factorizing a sparse matrix, many zero entries become non-zero, destroying the sparsity.
- For storing a symmetric matrix, one could introduce a "skyline storage" where one stores all entries between the skyline and the diagonal.

2.3 Q10

Explain the concept of static condensation. Discuss how it relates to factorization and to the frontal method.

- The concept of static condensation, you rewrite the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ to

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}$$

  $\boldsymbol{x}_1$ can then be expressed as

$$\boldsymbol{x}_1 = \boldsymbol{A}_{11}^{-1}(\boldsymbol{b}_1 - \boldsymbol{A}_{12}\boldsymbol{x}_{12})$$

  Substituting this in gives the block triangular form

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 - \boldsymbol{A}_{21}\boldsymbol{A}_{12}^{-1}\boldsymbol{b}_1 \end{bmatrix}$$

  or

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_0 & \boldsymbol{A}_{22}^* \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2^* \end{bmatrix}$$

- This implies that the entry $\boldsymbol{A}_{22}^*$ represents the stiffness operator coupling the variables $\boldsymbol{x}_2$ when $\boldsymbol{x}_1$ are left free.
- Advantages of this block factorization approach:
  - Having several blocks, one can perform LU factorization on smaller matrices at a time.
  - Factorization must only be applied to the subpart that is being modified.
  - The procedure is exact for static analysis.

  The frontal method

- Used for saving storing space. Only the "front" of the matrix needs to be known to reduce the matrix.
- The front of the matrix is stored using static condensation.

2.4 Q11

Recall the concept of singular matrices and nullspace. Discuss the existence of solutions for a singular problem. Introduce the concept of generalized inverse.

- If a structure has rigid body modes, the stiffness matrix will be singular. This means that there exists displacement modes that create no deformation energy.
- We will here consider the case of a square, symmetric matrix $\boldsymbol{A}$ of dimension $n \times n$.
- The nullspace of $\boldsymbol{A}$ is the set of m independent solutions $\boldsymbol{r}_s$ of $\boldsymbol{A}\boldsymbol{r}_s = \boldsymbol{0}$.
- For a stiffness matrix, $\boldsymbol{r}_s$ represents the rigid body modes if not enough boundary conditions exist to restrain the structure.
- Assume that the system can be partitioned as

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}$$

  where $\boldsymbol{A}_{11}$ is the largest non-singular submatrix of $\boldsymbol{A}$.

- Since the m coloumns labeled 2 are linearly dependent on the $n-m$ first ones, there exists a matrix $\boldsymbol{W}$ of dimension $(n-1)\times m$ such that

$$\begin{bmatrix}\boldsymbol{A}_{12}\\ \boldsymbol{A}_{22}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}_{11}\\ \boldsymbol{A}_{21}\end{bmatrix}\boldsymbol{W}$$

  This can again be written as

$$\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{bmatrix}\begin{bmatrix}-\boldsymbol{W}\\ \boldsymbol{I}_{m\times m}\end{bmatrix} = \boldsymbol{0}$$

  Meaning that the nullspace of $\boldsymbol{A}$ can be represented by

$$\boldsymbol{R} = \begin{bmatrix}\boldsymbol{r}_1 & \dots & \boldsymbol{r}_m\end{bmatrix} = \begin{bmatrix}-\boldsymbol{W}\\ \boldsymbol{I}_{m\times m}\end{bmatrix} = \begin{bmatrix}-\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\\ \boldsymbol{I}_{m\times m}\end{bmatrix}$$

  Solution of singular problems

- For singular systems, a solution to the system $\boldsymbol{Ax}=\boldsymbol{b}$ exists if and only if $\boldsymbol{b}$ is orthogonal to the nullspace of $\boldsymbol{A}$, $\boldsymbol{A}$ being symmetric.
- If $\boldsymbol{A}$ is a stiffness matrix, this shows that a static solution exists if and only if the applied forces are self-equilibrated with respect to the rigid body modes. The rigid body modes are then arbitrary.
- Examples p. 51 and 52 are useful!

  Generalized inverses

- Since solutions $\boldsymbol{x}$ exist for singular systems, we can define the operator $\boldsymbol{A}^+$ as a generalized inverse, such that the solution $\boldsymbol{x}$ is expressed as

$$\boldsymbol{x} = \boldsymbol{A}^+\boldsymbol{b}$$

  which gives

$$\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{b} = \boldsymbol{b}$$

- Since $\boldsymbol{b}$ is in the image of $\boldsymbol{A}$ if a solution exists (the image of $\boldsymbol{A}$: the set of $\boldsymbol{Ay}$ for any vector $\boldsymbol{y}$). That implies

$$\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{y} = \boldsymbol{y}$$

- If $\boldsymbol{A}$ is non-singular, $\boldsymbol{A}^+ = \boldsymbol{A}^{-1}$
- The solution could become

$$\boldsymbol{x} = \boldsymbol{A}^+\boldsymbol{b} + \boldsymbol{B}\boldsymbol{R}^T(\boldsymbol{A}\boldsymbol{y}) + \boldsymbol{R}\boldsymbol{C}\boldsymbol{b} = \boldsymbol{A}^+\boldsymbol{b} + \boldsymbol{R}\boldsymbol{C}\boldsymbol{b}$$

  differing from the solution $\boldsymbol{x} = \boldsymbol{A}^+\boldsymbol{b}$ only by its rigid body mode content.

2.5 Q12

Explain how factorization techniques can be used to compute a generalized inverse and the corresponding nullspace. Using as an example a free-free beam element, show how the factorization can still be applied to find the solution of a singular problem, when a solution exist.

- If a solution exists, it can be calculated by the general inverse $\boldsymbol{A}^{[11]+}$ so that

$$\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{11}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}$$

- This means we are solving the system $\boldsymbol{Ax} = \boldsymbol{b}$ for a maximum set of independent variables and equations, $\boldsymbol{x}_1 = \boldsymbol{A}_{11}^{-1}\boldsymbol{b}_1$ and setting to zero the remaining unknowns $\boldsymbol{x}_2$
- The physical interpretation of this is that we are adding fictious links to the system to render it statically determined. Since the forces applied must be self-equilibrated for a static solution to exist if the structure is floating, applying temporary constraints for the system to be exactly statically determined does not affect the solution as far as deformations are concerned, but simply sets the rigid body displacement of the solution.
- The example on p. 55 shows how the rigid body modes are found as a bi-product of the factorzation, by using the concept of generalized inverses.

3 Free vibration and eigenvalue solvers

3.1 Q13

Summarize the concept of free vibration modes and explain the property of orthogonality between modes (including modes of identical eigenfrequency). Introduce Rayleigh's quotient.

- Mostly covered by the report!
- **Rigid body modes**: When the system exhibits rigid body modes, $\boldsymbol{u}$, they satisfy the fundamental relation

$$\boldsymbol{Ku} = \boldsymbol{0}$$

- Rigid-body modes can be interpreted as eigenmodes of zero eigenfrequencies.
- Orthogonality of eigenmodes covered by the report.
- When a structure exhibits some form of symmetry, different modes can exist for the same eigenfrequency.

3.2 Q14

Show how a dynamic response can be obtained by modal superposition and how system matrices can be expressed as spectral expansion of eigenmodes

- Knowing the eigenmodes, eigenfrequencies and modal masses of a structure hold the same information as knowing its complete mass and stiffness matrices.
- The transient time response of a system subjected to external forces are governed by the equation

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{Kq} = \boldsymbol{p}(t)$$

- The system response may then be expresed through modal expansion as

$$\boldsymbol{q}(t) = \sum_{s=1}^{n} \eta_s(t)\boldsymbol{x}_{(s)}$$

  η_s is the time-dependent amplitudes of the modal components.

- This can be inserted into the above equation, and we then see that
  In the absence of damping, calculating the response of an n-degree-of-freedom-system reduces to the solution of n uncoupled single-degree-of-freedom systems excited by the external forces.

  Spectral expansion

- The important: By using spectral expansion, you can build $\boldsymbol{A}$, $\boldsymbol{K}$, $\boldsymbol{M}$ and $\boldsymbol{x}$ by knowing the eigenmodes $\boldsymbol{x}_{(r)}$

3.3 Q15

What criteria have to be considered when choosing an eigensolver? Discuss the basic principle of the power iteration to compute the lowest eigenfrequency

- Criterias to consider when choosing an eigensolver:
  - Number of degrees of freedom
  - Number of eigensolutions that must be computed
  - Bandwith of the frequencies looked after (how large area are they spanning over?)
  - Preservation of sparsity of matrices. For memory purposes, and to not store unnecessary values close to zero ($10^{-8} \approx 0$)
  - Ability of the algorithm to extract rigid-body modes
  - Ability to compute nearly identical eigenvalues
  - Convergence rate of the algorithm
  - Computational cost
- For the power iteration, we write the eigenvalue problem in the form $\boldsymbol{X}\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}$ where $\boldsymbol{D} = \boldsymbol{K}^{-1}\boldsymbol{M}$.
- The eigenvalues are then $\lambda_i = \frac{1}{\omega_i^2}$ which mean that we will first find the *largest* eigenvalue, because $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i$.
- Hence, we will find the *smallest* eigenfrequency since $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_i$
- Advantages of the power iteration:
  - Can limit the solving of the eigenvalue problem to the number of required solutions.
  - Convergence rate is independent of the matrix size. Makes it ideal for large systems.
- The power iteration is obtained by forming successive iterates $\boldsymbol{z}_p$ of an arbitrary starting vector $\boldsymbol{z}_0$ by

$$\boldsymbol{z}_{p+1} = \boldsymbol{D}\boldsymbol{z}_p$$

- Using the above mentioned principle of modal superposition, the starting vector can be written as a series of normal modes to the structure:

$$\boldsymbol{z}_0 = \sum_{i=1}^{n} \alpha_i \boldsymbol{x}_{(i)}$$

- By doing an expansion of the iterate, eq. 3.4.12, the iterates go to

$$\boldsymbol{z}_p \to \lambda_1^p \alpha_1 \boldsymbol{x}_{(1)}$$

- More on convergence rate in SD report

3.4 Q16

Describe the orthogonal deflation technique for computing higher modes. Explain how the power iteration can be organized as an inverse iteration procedure. Indicate how spectral shifting works and why it is useful.

- Orthogonal deflation and inverse iteration: SD report
- Spectral shifting is useful if you want to compute the eigenvalues in a given part of the spectrum: If you have an engine vibrating at 100Hz, you are only interested in eigenfrequencies around this.
- To do this, we change the expression of the eigenvalue problem by a shift μ towards a given band of the eigenvalue spectrum:

$$(\boldsymbol{K} - \mu \boldsymbol{M})\boldsymbol{x} = (\omega^2 - \mu)\boldsymbol{M}\boldsymbol{x}$$

3.5 Q17

Discuss the subspace iteration method. In particular, explain why it is more efficient than the power iteration.

- Subspace iteration method makes us of what the inverse iteration and power iteration had as drawbacks:
  - For every iteration, the iteration starts from scratch for every eigensolution. All information produced while finding the previous solution is thrown away.
  - Every eigensolution converges without knowledge of higher eigensolutions and therefore the convergence rate is poor for close eigenvalues.
- The idea of the subspace iteration method is to perform inverse iterations *simultaneously* on several vectors.
- At every iteration step, a vector remains orthogonal to its previous.
- We now construct a sequence of matrices $\boldsymbol{Z}_0, \boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_k$. k : iteration number.
- $\boldsymbol{Z}$ is of size $(n \times p)$ where p is the number of eigenmodes you want to find.
- The algorithm is then the same as for power iteration, but applied to several vectors at once:

$$\begin{aligned} \boldsymbol{V}_k &= \boldsymbol{M}\boldsymbol{Z}_k \\ \boldsymbol{K}\boldsymbol{Z}_{k+1} &= \boldsymbol{V}_k \end{aligned}$$

- A Gram-Schmidt process ensures orthogonality (not $\boldsymbol{M}$ orthogonality) between the modes:
  - Construct
  $$\boldsymbol{S}_k = \tilde{\boldsymbol{Z}}_k^T \tilde{\boldsymbol{Z}}_k$$
  - Perform Choleski decomposition
  $$\boldsymbol{S}_k = \boldsymbol{C}\boldsymbol{C}^T$$
  - Construct the new orthogonal directions $\boldsymbol{Z}_k$ by solving the systems
  $$\boldsymbol{C}\boldsymbol{Z}_k^T = \tilde{\boldsymbol{Z}}_k^T$$

3.6 Q18

Explain the Arnoldi method for finding the eigensolutions. Show how it simplifies to the Lanczos method in case of symmetric systems (using the simplified proof of footnote 6 p.79, not the full derivation of p.78-79). Shortly discuss the difficulties of the method.

- Lanczos compared to power iteration: like in the inverse iteration one applies the operator $\boldsymbol{K}^{-1}\boldsymbol{M}$ recursively on a direction. But now one keeps all directions computed during the iterations and one looks for estimate of the eigenmodes in the so-generated subspace.
- Compared to the subspace algorithm, the Lanczos method is similar in the way it looks for eigensolutions in a subspace, but it is different in that the subspace is generated and growing recursively.
- In the Lanczos method, no information is discarded since all iterates are used to find the best approximation in a growing subspace.

  Building iteratively a Krylov space for the Arnoldi method

- We mass-normalize all vectors generated in the Krylov sequence:

$$\boldsymbol{z}_0 \leftarrow \frac{\boldsymbol{z}_0}{\sqrt{\boldsymbol{z}_0^T \boldsymbol{M} \boldsymbol{z}_0}}$$

- Next vector of the Krylov basis is computed as

$$\tilde{\boldsymbol{z}}_1 = \boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{z}_0$$

  Note that this is not yet orthogonal, hence the tilde.

- We then $\boldsymbol{M}$ orthogonalize $\tilde{\boldsymbol{z}}_1$ with respect to $\boldsymbol{z}_0$:

$$\boldsymbol{z}_1 = \tilde{\boldsymbol{z}}_1 - a_{00}\boldsymbol{z}_0 \quad \text{with} \quad a_{00} = \boldsymbol{z}_0^T \boldsymbol{M} \tilde{\boldsymbol{z}}_1$$

- We then ensure the new vector is mass-normalized by

$$\boldsymbol{z}_1 \leftarrow \frac{\boldsymbol{z}_1}{\sqrt{\boldsymbol{z}_1^T \boldsymbol{M} \boldsymbol{z}_1}}$$

- This whole process can be written in matrix form (p.77):

$$\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{Z} = \boldsymbol{Z}\boldsymbol{T} + \boldsymbol{E}$$

- We use $\boldsymbol{Z}$ as a subspace in which we want to find an approximation of the eigenmodes, namely

$$\boldsymbol{x}_{(s)} \approx \boldsymbol{Z}\boldsymbol{y}_{(s)}$$

  where $\boldsymbol{y}_{(s)}$ is found from

$$(\frac{1}{\omega_s^2}\mathbf{I} - \boldsymbol{T})\boldsymbol{y}_{(s)} = \mathbf{0}$$

- This is called the *Arnoldi iteration*. Problem: Much required memory for storing all Krylov vectors.
- The Lanczos method is a special case of the Arnoldi when the system matrices are symmetric.
- Proof that $\boldsymbol{T}$ is tridiagonal when premultiplying by $\boldsymbol{Z}^T\boldsymbol{M}$:

$$\boldsymbol{Z}^T\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{Z} = \boldsymbol{Z}^T\boldsymbol{M}\boldsymbol{Z}\boldsymbol{T} + \boldsymbol{Z}^T\boldsymbol{M}\boldsymbol{E} = \mathbf{I}\boldsymbol{T} + \mathbf{0} = \boldsymbol{T}$$

- Notes to the method:
  - Two to four inverse iterations per converged eigenvalue. Independent of problem size.
  - Not strongly affected by the closeness of eigenvalues.

4 Mode superposition in practice

4.1 Q19

Describe the mode displacement method and discuss why some modes are important and others not in the superposition.

- The mode displacement method is used when you ony know $k < n$ eigenmodes. This is often the case in practice. As every iteration of the inverse iteration technique requires solving a static problem, for systems with high number of degrees of freedom, computing all the eigenmodes is too expensive.
- In order to get an approximation of the dynamic response when only knowing k eigenmodes is the neglect the higher modes. Inserted in the modal displacement method, you then get

  $$\boldsymbol{q}(t) = \sum_{s=1}^{k} \eta_s(t)\boldsymbol{x}_{(s)}$$

  Inserting this into the dynamic equation, you get that $\eta_s(t)$ is the solution of the equation

  $$\ddot{\eta}_r(t) + \omega_r^2\eta_r(t) = \phi_r(t) \qquad r = 1, \ldots, k$$

  where

  $$\phi_r(t) = \frac{\boldsymbol{x}_{(r)}^T\boldsymbol{p}(t)}{\mu_r}$$

- For analyzing the approximation by such a truncated modal series, we consider the **forced** harmonic response of the form

  $$\boldsymbol{q}(t) = \boldsymbol{z}\sin\omega t$$

  to a harmonic excitation

  $$\boldsymbol{p}(t) = \boldsymbol{s}\sin\omega t$$

  ω and $\boldsymbol{s}$ being respectively a given excitation frequency and amplitude, and $\boldsymbol{z}$ being the forced response amplitude.
- The dynamic equation then writes

  $$(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{z} = \boldsymbol{s}$$

- As the **complete** mode superposition expression for the forced response amplitude is

  $$\boldsymbol{z} = \sum_{s=1}^{n} \eta_s\boldsymbol{x}_{(s)}$$

  where the modal contributions are determined by

  $$-\omega^2\eta_r + \omega_r^2\eta_r = \frac{\boldsymbol{x}_{(t)}^T\boldsymbol{s}}{\mu_r}$$

  Solved for η_r:

  $$\eta_r = \frac{\boldsymbol{x}_{(r)}^T\boldsymbol{s}}{\mu_r} \cdot \frac{1}{(\omega_r^2 - \omega^2)}$$

  Hence the **forced** harmonic response is given by

  $$\boldsymbol{q}(t) = \boldsymbol{z}\sin\omega t = \sin\omega t \sum_{s=1}^{n} \frac{\boldsymbol{x}_{(s)}\boldsymbol{x}_{(s)}^T\boldsymbol{s}}{\mu_s(\omega_s^2 - \omega^2)}$$

- This expansion is governed by two types of convergence:
  - **Convergence of quasi-static type** occurring if the applied load distribution $\boldsymbol{s}$ admits a sufficiently accurate spatial representation in the basis of the k retained eigenmodes. This is equivalent to assuming that $\boldsymbol{p}$ must be nearly orthogonal to the $(n-k)$ omitted eigenmodes which therefore will not be excited or will only be slightly excited.
  - A **convergence of spectral type** conditioned by the convergence to zero of the dynamic amplification factor $\frac{1}{\omega_s^2-\omega^2}$ when progressing in the eigenspectrum of the system. It thus depends both on the frequency content of the excitation and on the system eigenspectrum.

4.2 Q20

Explain the correction term introduced in the mode acceleration method.

- In the mode acceleration method we include in the truncated solution the quasi-static response due to higher modes without having to actually compute these higher modes. This can be seen as an "error term" added to the mode displacement method.
- We then assume that the eigenfrequencies associated to modes $k+1,\ldots,n$ are much **higher** than the frequency obtained in the excitation. In that case these higher modes respond as if the excitation was static, similar to a mass-spring system with small mass and high stiffness.
- We can then write the solution as

$$\boldsymbol{q}(t) \approx \sum_{r=1}^{k} \boldsymbol{x}_{(r)}\eta_r(t) + \sum_{s=k+1}^{n} \frac{\boldsymbol{x}_{(s)}\boldsymbol{x}_{(s)}^T\boldsymbol{p}(t)}{\mu_s\omega_s^2}$$

- We then see this as a purely static problem, and the quasi-static solution can then be expressed as

$$\boldsymbol{x}_{quasi-static}(t) = \boldsymbol{K}^{-1}\boldsymbol{p}(t) = \sum_{s=1}^{n} \frac{\boldsymbol{x}_{(s)}\boldsymbol{x}_{(s)}^T}{\omega_s^2\mu_s}\boldsymbol{p}(t)$$

  where $\boldsymbol{K}^{-1}$ is found from spectral expansion.
- From this we see that the last term in $\boldsymbol{q}(t)$ corresponds to the contribution of the higher modes to the quasi-static response. This term can then be expanded as

$$\sum_{s=k+1}^{n} \frac{\boldsymbol{x}_{(s)}\boldsymbol{x}_{(s)}^T\boldsymbol{p}(t)}{\mu_s\omega_s^2} = \boldsymbol{K}^{-1}\boldsymbol{p}(t) - \sum_{s=1}^{k} \frac{\boldsymbol{x}_{(s)}\boldsymbol{x}_{(s)}^T\boldsymbol{p}(t)}{\mu_s\omega_s^2}$$

  which gives the solution

$$\boldsymbol{q}(t) \approx \sum_{r=1}^{k} \boldsymbol{x}_{(r)}\eta_r(t) + \boldsymbol{K}^{-1}\boldsymbol{p}(t) - \sum_{s=1}^{k} \frac{\boldsymbol{x}_{(s)}\boldsymbol{x}_{(s)}^T\boldsymbol{p}(t)}{\mu_s\omega_s^2}$$

- We here see that the mode acceleration method consists in complementing the mode displacement solution with the missing terms from the modal expansion of the static response.

4.3 Q21

Discuss the way a structure excited through its supports can be analysed. Explain also the additional mass method to tackle this problem.

- This is the case of imposed ground motion (earthquakes, vibration of equipment carried by a vehicle, etc.). Corresponding to Figure 4.1.

- We can then partition its degrees of freedom in two: n_1 displacements $\boldsymbol{q}_1$ remaining completely free.
- n_2 displacements imposed at the support level $\boldsymbol{q}_2$.
- By doing this, the equations of motion take the form

$$\begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_1 \\ \ddot{\boldsymbol{q}}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{r}_2(t) \end{bmatrix}$$

- When following the idea of the mode acceleration method, we can first compute the quasi-static response of the unrestrained part of the structure, neglecting the inertia force terms (with $\boldsymbol{M}$):

$$\boldsymbol{q}_1^{qs}(t) = -\boldsymbol{K}_{11}^{-1}\boldsymbol{K}_{12}\boldsymbol{q}_2 = \boldsymbol{S}\boldsymbol{q}_2$$

- From this, we can write the complete response in the form

$$\boldsymbol{q}(t) = \begin{bmatrix} \mathbf{I} & \boldsymbol{S} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{q}_1 \end{bmatrix}$$

  where $\boldsymbol{S} = -\boldsymbol{K}_{11}^{-1}\boldsymbol{K}_{12}$, the **static condensation matrix**. $\boldsymbol{y}_1(t)$ represents the dynamic part of the response additional to the static response, i.e. the vibration of the structure on its support relative to the static deformation.

- Substituting this into the the equations of motion, you get

$$\boldsymbol{M}_{11}\ddot{\boldsymbol{y}}_1 + \boldsymbol{K}_{11}\boldsymbol{y}_1 = \boldsymbol{g}_1(t)$$

  which is the equation governing the motion of the unrestrained degrees of freedom. The load $\boldsymbol{g}_1(t)$ comprises the virtual forces related to the reference quasi-static solution and the forces transmitted from the support through mass coupling:

$$\begin{aligned} \boldsymbol{g}_1(t) &= -\boldsymbol{M}_{11}\ddot{\boldsymbol{q}}_1^{qs}(t) - \boldsymbol{M}_{12}\ddot{\boldsymbol{q}}_2(t) \\ &= -[\boldsymbol{M}_{11}\boldsymbol{S} + \boldsymbol{M}_{12}]\ddot{\boldsymbol{q}}_2(t) \end{aligned}$$

- This force is in effect a ficticious force due to the fact that the reference frame follows the quasi-static deformation, in a way similar to the forces experienced by a person in an elevator).
- In the **additional mass method**, one tries to get away from partitioning the structural matrices as seen before. Here, we add a large mass to the "$\boldsymbol{q}_2$ part" of the structure. And we think the structure no longer as being subjected to acceleration on the $\boldsymbol{q}_2$, but to external forces $\boldsymbol{f}_2(t)$.
- The masses associated to subsystem $\boldsymbol{q}_2$ are augmented so that $\boldsymbol{M}_{22}$ becomes $\boldsymbol{M}_{22} + \boldsymbol{M}_{22}^0$
- The force $\boldsymbol{f}_2(t)$ is taken as

$$\boldsymbol{f}_2(t) = (\boldsymbol{M}_{22} + \boldsymbol{M}_{22}^0)\ddot{\boldsymbol{q}}_2(t)$$

- The additional mass must be large enough to keep the modelling error small, between 10^2 and 10^4 times the magnitude of the coefficients of $\boldsymbol{M}_{11}$.

5 Reduction methods and dynamic substructuring

5.1 Q22

How can one build a reduced system based on a reduced basis to represent the dynamic response? Discuss Guyan's reduction method.

- The finite element model mesh is normally very refined, giving a high number of degrees of freedom. This gives very high computation time when solving free vibration modes, harmonic and transient responses. As we do not want to redo the mesh, *reduction methods* are used to reduce the existing mesh.
- The idea by this technique is to replace the degrees of freedom by a set of global variables representing the amplitudes of possible displacement modes:

  $$\boldsymbol{q} \approx \boldsymbol{R}\tilde{\boldsymbol{q}}$$

  where $\boldsymbol{R}$ is a reduction matrix of dimension $(n \times r), \quad r < n$. Applying this to the equations of motion, you get

  $$\tilde{\boldsymbol{M}}\ddot{\tilde{\boldsymbol{q}}} + \tilde{\boldsymbol{C}}\dot{\tilde{\boldsymbol{q}}} + \tilde{\boldsymbol{K}}\tilde{\boldsymbol{q}} = \tilde{\boldsymbol{p}}(t)$$

  where the structural matrices and load are calculated from

  $$\begin{aligned}\tilde{\boldsymbol{M}} &= \boldsymbol{R}^T\boldsymbol{M}\boldsymbol{R}\\ \tilde{\boldsymbol{C}} &= \boldsymbol{C}^T\boldsymbol{M}\boldsymbol{C}\\ \tilde{\boldsymbol{K}} &= \boldsymbol{K}^T\boldsymbol{M}\boldsymbol{K}\\ \tilde{\boldsymbol{p}}(t) &= \boldsymbol{R}^T\boldsymbol{p}(t)\end{aligned}$$

- Doing this reduction implies that you are stiffening the system.
- The **Guyan's reduction** is a method for reducing the system.
- Here, we partition the system into degrees of freedom $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, where $\boldsymbol{q}_1$ is considered the "slave", and $\boldsymbol{q}_2$ the "master", implying that no forces are applied to $\boldsymbol{q}_1$.
- One sees the $\boldsymbol{q}_1$ as being represented by two contributions:

  $$\boldsymbol{q}_1 = \boldsymbol{q}_{1,stat.} + \boldsymbol{q}_{1,dyn.}$$

  with the static part deduced from

  $$\boldsymbol{q}_{1,stat.} = -\boldsymbol{K}_{11}^{-1}\boldsymbol{K}_{12}\boldsymbol{q}_2$$

- From this, the static condensation algorithm consists of neglecting $\boldsymbol{q}_{1,dyn.}$ and then building the reduction

  $$\begin{bmatrix}\boldsymbol{q}_2\\ \boldsymbol{q}_1\end{bmatrix} = \boldsymbol{R}\boldsymbol{q}_2 = \begin{bmatrix}\mathbf{I}\\ \boldsymbol{S}\end{bmatrix}\boldsymbol{q}_2$$

- From this you can build the statically condensed stiffness and mass matrices.
- If this algorithm is applied to static problems, the exact solution is found.
- When applied to dynamic problems, an approximation is introduced by neglecting the dynamic response of the interior of the substructure and thereby assuming that all internal nodes respond quasi-statically to the interface displacements.

- **Dynamic substructuring** is about defining subparts of the structure, corresponding for example in an aircraft to wings, fuselage, stabilizers and tail. Every substructure is denoted by $\Omega^{(s)}$
- For every subpart, define a reduction matrix $\boldsymbol{R}^{(s)}$ such that

$$\boldsymbol{q}^{(s)} = \boldsymbol{R}^{(s)} = \begin{bmatrix} \boldsymbol{q}_2^{(s)} \\ \boldsymbol{\xi}^{(s)} \end{bmatrix}$$

  where $\boldsymbol{\xi}^{(s)}$ are generalized degrees of freedom.
- The reduced matrices $\tilde{\boldsymbol{K}}^{(s)} = \boldsymbol{R}^{(s)^T}\boldsymbol{K}^{(s)}\boldsymbol{R}^{(s)}$ and $\tilde{\boldsymbol{M}}^{(s)} = \boldsymbol{R}^{(s)^T}\boldsymbol{M}^{(s)}\boldsymbol{R}^{(s)}$, which easily can be shared between different design teams, without revealing potentially sensitive information about the finite element model or geometry for the structure.

5.2 Q23

Explain the Craig-Bampton method and discuss the reduction basis that is used in this method

- The Craig-Bampton method can be seen as a substructuring method that uses substructure vibration modes in order to compute the response of the internal nodes to the interface displacements $\boldsymbol{q}_2$.
- The dynamic behaviour of a mechanical subsystem can be fully described in terms of two types of information: static modes resulting from unit forces on the boundary degrees of freedom, and the internal vibration modes put forward by fixing the subsystem on its boundary with the neighboring subsystems. Corresponding to Figure 5.1.
- The reduction basis of the Craig-Bampton model is based on these two types.

$$\begin{bmatrix} \boldsymbol{M}_{22}^{(s)} & \boldsymbol{M}_{21}^{(s)} \\ \boldsymbol{M}_{12}^{(s)} & \boldsymbol{M}_{11}^{(s)} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_2^{(s)} \\ \ddot{\boldsymbol{q}}_1^{(s)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{22}^{(s)} & \boldsymbol{K}_{21}^{(s)} \\ \boldsymbol{K}_{12}^{(s)} & \boldsymbol{K}_{11}^{(s)} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_2^{(s)} \\ \boldsymbol{q}_1^{(s)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}_2^{(s)} \\ \boldsymbol{0} \end{bmatrix}$$

- The static modes, when we do not look at the inertia forces, would give a Guyan-Iron's reduction of the substructure similar to

$$\begin{bmatrix} \boldsymbol{q}_2^{(s)} \\ \boldsymbol{q}_1^{(s)} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \boldsymbol{S}^{(s)} \end{bmatrix} \boldsymbol{q}_2^{(s)}$$

- The vibration solution space can be found by solving the vibration problem of the substructure fixed on its boundary:

$$\boldsymbol{K}_{11}^{(s)}\boldsymbol{q}_1^{(s)} = \omega^2 \boldsymbol{M}_{11}^{(s)}\boldsymbol{q}_1^{(s)}$$

- The substructure degrees of freedom are then represented by

$$\boldsymbol{q}^{(s)} = \begin{bmatrix} \boldsymbol{q}_2^{(s)} \\ \boldsymbol{q}_1^{(s)} \end{bmatrix} = \boldsymbol{R}^{(s)} \begin{bmatrix} \boldsymbol{q}_2^{(s)} \\ \boldsymbol{\xi}^{(s)} \end{bmatrix}$$

  where $\boldsymbol{\xi}^{(s)}$ are the intensity parameters of the substructure's internal vibration modes.

6 Time-integration

6.1 Q24

Discuss how one can integrate the scalar equations of the modal amplitudes. Give an interpretation of Duhamel's integral in terms of impulse response.

- When solving dynamic equations of motion under arbitrary excitation, two approaches can be considered: modal superposition techniques and direct time integration.
- This section is about how to compute a solution to the normal equations

$$\ddot{\eta}_r(t) + \omega_r^2 \eta_r(t) = \phi_r(t) \qquad r = 1, \ldots, n$$

- The normal equations may be integrated in the form of a convolution product by the Laplace transforms:

$$\bar{\eta}_r(s) = \mathcal{L}\{\eta_r(t)\} = \int_0^{+\infty} e^{-st} \eta_r(t) dt$$
$$\bar{\phi}_r(s) = \mathcal{L}\{\phi_r(t)\} = \int_0^{+\infty} e^{-st} \phi_r(t) dt$$

- Applying this, and then the inverse Laplace transform, you obtain the solution $\eta_r(t)$:

$$\eta_r(t) = \eta_r(0)\cos(\omega_r t) + \dot{\eta}_r(0) h_r(t) + \int_0^t \phi_r(\tau) h_r(t-\tau) d\tau$$

- The first two terms of this solution represents the response to the initial conditions. The last term is the **convolution product** know as *Duhamel's integral*. This represents the forced response to the external loads $\boldsymbol{p}(t)$.
- Physically, this equation can be interpreted with the concept of the impulse response. When an impulse $\phi(\tau)d\tau$ is applied at time τ to a one-degree-of-freedom system, the response is a jump in velocity so that

$$\Delta\dot{\eta}(\tau) = \phi(\tau)d\tau$$

- Assuming that the force can be decomposed in a series of impulses, seen in Figure 7.1, the convolution product is understood as the sum of the responses to impulses at times τ between $t = 0$ and t.

6.2 Q25

Compare mode superposition and direct time-integration. Discuss the basics of direct time-integration, in particular the issues of initial conditions, integration schemes and consistency, accuracy, stability.

- Modal expansion techniques are effective when fundamental modes predominate in the response. In the opposite case, where the frequency spectrum requires the inclusion of a high number of modes so as to ensure good quasi-static and spectral convergence, modal expansion techniques should be replaced by direct integration methods.
- Direct time-integration should be used in
  - Non-linear systems. Contrary to the modal superposition, time integration can be extended to solve non-linear systems.
  - Response to shocks. When the spectral content of the excitation has a large frequency band compared to the frequency content of the structure.

  - Highly-damped structures. When the damping in a structure is non-proportional (high damping or non-viscous damping).

- We want to solve the space-discretized dynamic equations

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{p}(t)$$

- To solve this, we must solve a set of second order differential equations from an initial time t_0 to a final time t_f. Therefore, two boundary conditions in time are needed to determine the solution.

- Knowing that accelaration, velocity and displacement are related by

$$\ddot{\boldsymbol{q}}_n = \lim_{h\to 0} \frac{\dot{\boldsymbol{q}}_n - \dot{\boldsymbol{q}}(t_n - h)}{h}$$
$$\dot{\boldsymbol{q}}_n = \lim_{h\to 0} \frac{\boldsymbol{q}_n - \boldsymbol{q}(t_n - h)}{h}$$

- A general, direct multistep integration method for first-order systems can be stated in the form

$$\boldsymbol{u}_{n+1} = \sum_{j=1}^{m} \alpha_j \boldsymbol{u}_{n+1-j} - h \sum_{j=0}^{m} \beta_j \dot{\boldsymbol{u}}_{n+1-j}$$

- A time integration scheme consists in approximating these limits by finite differences.

- Two major classes of integration schemes are found:
  - Implicit schemes. Used when $\beta_0 \neq 0$, meaning that the state vector at time t_{n+1} is a function of its own derivative and the scheme is called *implicit*.
  - Explicit schemes. For $\beta_0 = 0$. Then $\boldsymbol{u}_{n+1}$ can be deduced directly from the results at the previous time-steps.
  - When $\alpha_j = \beta_j = 0$ for $j > 1$, we have a single-step method.

- From the multistep integration method, we kan deduce three basic integration formulas:
  - Trapezoidal rule (implicit):

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \frac{h}{2}(\dot{\boldsymbol{u}}_n + \dot{\boldsymbol{u}}_{n+1})$$

  - Euler backward formula (implicit):

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + h\dot{\boldsymbol{u}}_{n+1}$$

  - Euler forward formula (explicit):

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + h\dot{\boldsymbol{u}}_n$$

- These are said to be consistent becuase as the time step $h \to 0$, the integration schemes are equivalent to the true definition of the derivatives.

6.3 Q26

Explain how to derive Newmark's family of integration schemes and indicate how they can be used in a time-stepping algorithm.

- Covered by SD report

6.4 Q27

Shortly discuss the consistency, stability and accuracy of Newmark's integration formula. Explain the HHT scheme. and discuss the so-called explicit Newmark algorithm.

- Newmark integration covered by SD report.
- The HHT method is a way to introduce damping in the Newmark method without degrading the order of accuracy.
- If the constant $\alpha = 0$, the HHT method reduces to Newmark's scheme. If $\alpha \in [0, \frac{1}{3}], \gamma = \frac{1}{2} + \alpha, \beta = \frac{1}{4}(1+\alpha)^2$ the result is an unconditionally stable second-order scheme.
- The HHT scheme is excellent when requiring an unconditionally stable implicit algorithm which introduces high numerical damping in the high frequency range while nearly preserving the maximum accuracy attribute of the average constant acceleration method for the low frequency band. In particular, the HHT method is recommended for systems having constrained degrees of freedom.