PMATH 464 - ASSIGNMENT 1

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1. Suppose that X is an algebraic set and L is a line in $A^n(k)$ such that $L \not\subset X$. Show that $L \cap X$ is either empty or a finite set of points.

Let S be the finite set of polynomials such that $V(S) = X$. Since $L \not\subset X$, we cannot have X be the whole space, so there exists a non-zero polynomial $f \in S$. We know that $V(S) \subseteq V(f)$, so if $L \cap X$ were an infinite set, then $L \cap V(f)$ would also be an infinite set. We denote $L = V(Y - (aX + b))$ and note that

2. Determine whether or not the following sets are algebraic.

(a) The set of points in $\mathbb{R}^2$ whose polar coordinates (r, θ) satisfy the equation $r = \theta$.

The equation $r = \theta + 2k\pi$ for $k \in \mathbb{N}$ defines an Archimedean spiral, so it will intersect *any* line in $\mathbb{R}^2$ infinitely many times. In polar coordinates, a line is specified by $\theta = a$, for some $a \in \mathbb{R}$. So take $a = 1$. Then $r = \theta + 2k\pi$ intersects with $\theta = 1$ at all the points $r = 1 + 2k\pi$ for $k \in \mathbb{N}$. That is, $r = 1, 1 + 2\pi, 1 + 4\pi + \dots$ are all points of intersection. Since there are infinitely many of them, the spiral cannot possibly be an algebraic set.

(b) The set of points in $\mathbb{R}^2$ whose polar coordinates (r, θ) satisfy $r = \cos(\theta)$.

(c) $\{(\cos t, 1, \sin t) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

Intuitively speaking, this set looks like a single circle centered around the point $(0, 1, 0)$ with radius 1, parallel to the xz-plane; thus intuitively we know it is the intersection of the vertical plane $y = 1$ and the cylinder of radius 1 centered around x axis, $x^2 + z^2 = 1$, both of which are polynomials in $\mathbb{A}^3(\mathbb{R})$.

The x and z components parametrize the circle $x^2 + z^2 = 1$ and the y component specifies that the circle is on the plane $y = 1$.

(d) $\{(\cos t, t, \sin t) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

3. Consider the real line $\mathbb{R}$ endowed with the Zariski topology. Verify that, given any two points $p, q \in \mathbb{R}$, any two open neighbourhoods of p and q, have a non-empty intersection, thus proving that the Zariski topology on $\mathbb{R}$ is not Hausdorff.

On $\mathbb{R}$, the only algebraic sets (and therefore the only closed sets) are empty, finite, or $\mathbb{R}$ itself. Let $X \subseteq \mathbb{R}$ be a closed set in $\mathbb{R}$. Therefore the only open sets are $\mathbb{R}$ with finitely many points missing. Since the neighbourhoods of p and q contain open sets, their intersection will also be an open set – namely $\mathbb{R}$ missing the points which both $N(p)$ and $N(q)$ were missing. This is nonempty, and therefore the Zariski topology on $\mathbb{R}$ is not Hausdorff.

If k is a finite field, show that every subset of $\mathbb{A}^n k$ is both open and closed in the Zariski topology. Is the Zariski topology Hausdorff in this case?

We know that if k is a finite field, any subset of $A^n(k)$ is an algebraic set. Indeed, if $\{v_0, v_1, \ldots, v_m\}$ is a subset of $A^n(k)$ then it is a finite union $\cup_{i=0}^m (v_i)$. Thus it suffices to prove that each v_i is an algebraic set. Well, if $v_i = (a_1, a_2, \ldots, a_n)$ then we have $V(X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n) = v_i$, so each v_i *is* an algebraic set.

Once we've established that all subsets are algebraic, it follows that all subsets are both closed (since they're algebraic) and open (since their complement is also a subset of $A^n(k)$ which must also be algebraic, hence closed). Therefore the Zariski topology over such fields is Hausdorff, since for any distinct points p, q, the sets $\{p\}$ and $\{q\}$ are open subsets with empty intersection.

5. Let k be any field and $M_{n\times n}(k)$ be the set of $n \times n$ matrices with entries in k. This set can naturally be identified with $\mathbb{A}^{n^2}(k)$, by considering the (ordered) n^2 entries of a matrix as a point in $\mathbb{A}^{n^2}(k)$, and is thus endowed with the Zariski topology. Show that the group $GL(n, k)$ of all invertible matrices in $M_{n\times n}(k)$ is open in the Zariski topology on $M_{n\times n}(k)$.

To show the open-ness of $GL(n, k)$, we must show that its complement (the non-invertible matrices) are an algebraic set. We know that the non-invertible matrices are precisely those with determinant 0. However, the determinant of a matrix is itself a polynomial $\det_n : \mathbb{A}^{n^2}(k) \to k$, and it will be 0 precisely when it is passed a (non-invertable) matrix of determinant 0. Thus the complement of $GL(n, k)$ is precisely $V(\det_n)$, a closed set.

6. Let $V \subset \mathbb{A}^n(k)$ and $W \subset \mathbb{A}^m(k)$ be algebraic sets. Show that

$$V \times W := \{(a_1, \ldots, a_n, b_1, \ldots, b_m) \mid (a_1, \ldots, a_n) \in V, (b_1, \ldots, b_m) \in W\}$$

is an algebraic set in $\mathbb{A}^{n+m}(k)$. It is called the *product* of V and W.

Since V, W are algebraic, there exist two finite sets of polynomials S_1, S_2 such that $V(S_1) = V$ and $V(S_2) = W$. Now, we note that for each polynomial $f(a_1, \ldots, a_n) \in \mathbb{A}^n(k)$ there is a corresponding polynomial $f' \in \mathbb{A}^{n+m}(k)$ such that $f'(a_1, a_2, \ldots, a_n, x_1, \ldots, x_m) = f(a_1, a_2, \ldots, a_n), \forall x_1, \ldots, x_m \in k$, and a corresponding polynomial $f''(x_1, \ldots, x_m, a_1, \ldots, a_n) = f(a_1, \ldots, a_n), \forall x_1, \ldots, x_m \in k$. Thus we can define the sets $S_1' = \{f' \mid f \in S_1\}$ and $S_2'' = \{g'' \mid g \in S_2\}$. These are now both sets of polynomials in $\mathbb{A}^{n+m}(k)$.

Let $I = \langle S_1' \rangle$ and $J = \langle S_2'' \rangle$. Then consider the ideal IJ. We claim that $V(IJ) = V \times W$.

To show that $V \times W \subseteq V(IJ)$, let $A \subseteq S_1$ be a subset such that $V(A) = (a_1, \ldots, a_n)$ and $B \subseteq S_2$ such that $V(B) = (b_1, \ldots, b_m)$. A and B must exist since V and W are algebraic.

Then

$$\prod_{f\in A, g\in B} f' + g''$$