PMATH 464/664: Assignment 2

Due: Friday, 30 January, 2009.

Note: Only solutions to the problems preceded by $*$ have to be handed in.

$*$ 1. Let k be an infinite field. Prove that any non-empty Zariski open subset of $\mathbb{A}^n(k)$ is dense.

$*$ 2. In this exercise, we prove that the spiral $r = \theta$ is dense in $\mathbb{R}^2$.

(a) Let V and W be two algebraic sets in $\mathbb{A}^n(k)$. Show that $V = W$ if and only if $I(V) = I(W)$.

(b) Let k be an infinite field and $f \in k[x, y]$ be an irreducible polynomial such that $V(f)$ is infinite set of points in $\mathbb{A}^2(k)$. Prove that if E is any infinite subset of $V(f)$, then $\overline{E} = V(f)$ in the Zariski topology.

(c) Let E be the curve in $\mathbb{R}^2$ given in polar coordinates by $r = \theta$. Show that E is dense in $\mathbb{R}^2$ in the Zariski topology.

3. *Irreducible topological spaces.* The notion of irreducibility extends to any topological space. More precisely, a non-empty subset Y of a topological space X is *irreducible* if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper non-empty subsets Y_1 and Y_2 of Y that are both closed in the induced topology on Y. Note that this definition also applies to subsets of X that are not closed.

   (a) Show that any non-empty open subset of an irreducible topological space is irreducible and dense. *(Note: This generalises question 1.)*

   (b) Show that if Y is an irreducible subset of a topological space X, then the closure $\bar{Y}$ is also irreducible.

   (c) Show that any two non-empty open subsets of an irreducible topological space have a non-empty intersection. Describe all irreducible Hausdorff topological spaces.

$*$ 4. Let V and W be algebraic sets in $\mathbb{A}^n(k)$ such that $V \subset W$. Show that each irreducible component of V is contained in some irreducible component of W.

$*$ 5. (a) Show that although the algebraic curve $X = V(y^2 + x^2(x-1)^2(x^2+1))$ is irreducible in $\mathbb{C}^2$, it is reducible in $\mathbb{R}^2$. What does this imply about the ideal of X? Verify your answer by giving an explicit description of $I(X)$ in both $\mathbb{R}[x, y]$ and $\mathbb{C}[x, y]$.

(b) Sketch the algebraic curve $X = V(y^2 - x^2(x - 1))$ in $\mathbb{R}^2$. Is X irreducible in the Zariski topology on $\mathbb{R}^2$? Is your answer consistent with the sketch you drew? Explain. What happens if you consider the the curve in $\mathbb{C}^2$ instead?

6. Let k be an algebraically closed field, which may have characteristic 2. Find the irreducible components of the following algebraic sets.

   Note: Don't forget to consider the case where the characteristic is 2.

   $*$ (a) $V(x^3 + x^2y - 2xy - 2y^2)$ in $\mathbb{A}^2(k)$.

   (b) $V(xy, xy^2 - x^2)$ in $\mathbb{A}^2(k)$.

   (c) $V(x^2 + y^2 + z^2, x^2 - y^2 - z^2 + 1)$ in $\mathbb{A}^3(k)$.

   $*$ (d) $V(x^2 - yz, xz - x)$ in $\mathbb{A}^3(k)$.

7. Let k be any field. Are the following ideals prime, radical, or closed in $k[x, y]$?

   (a) $< x, y^2 - 1 >$.

   $*$ (b) $< x + y, xy >$.

   $*$ (c) $< x^2 - y^2 >$.