On a Class of Zero-Preserving Bilateral Transforms and their Application to Precision Windowing

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Abstract

We introduce the Cosh Moment Sturm Transform (CMST) framework for constructing globally interlacing families of real-rooted functions. We prove that bilateral Laplace transforms $F(s) = \int_{-\infty}^{\infty} e^{-f(t^2)} e^{st} dt$ with convex $f(t)$ have all distinct zeros on the imaginary axis for a class of $f(t)$. The CMST construction produces infinite families maintaining global interlacing. Applications include a C_c^∞ window function for signal processing with super-algebraic sidelobe suppression.

1 Introduction

In this paper we show that a broad class of bilateral Laplace transforms arising from even, positive, exponentially decaying kernels have all their zeros located on the imaginary axis. Specifically, consider functions of the form

$$F(s) = \int_{-\infty}^{\infty} e^{-f(t^2)} e^{st} dt, \qquad f(t) = a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n \tag{1}$$

where $a_n \geq 0$ and $n > 1$ and

$$\sum_{i=1}^{n} a_i < \infty$$

Equivalently, this means $f(1) < \infty$.

Under these conditions $F(s)$ extends to an entire function that has distinct zeros that lie purely on the imaginary axis.

Thus the bilateral Laplace family

$$F(s) = \int_{-\infty}^{\infty} e^{-f(t^2)} e^{st} dt, \tag{2}$$

constitutes a real–symmetric, genus–zero or one subclass of entire functions whose zeros are confined to the imaginary axis, are distinct, and the zeros of $F(\sqrt{-s})$ interlace with the zeros of $F''(\sqrt{-s})$

We use a Cosh Transform defined as follows to produce families of functions with all real negative zeros.

$$\mathcal{C}_a[f](s) := \int_{-a}^{a} f(t) \cosh(\sqrt{s}\, t)\, dt \tag{3}$$

Using this, we define a "Cosh Moment Sturm Transform" (CMST) to produce families of functions with all real negative zeros which map to pairs of imaginary zeros in the bilateral Laplace Transform.

We finish by using these functions to produce a C_∞ window function with compact support and super-algebraic sidelobe suppression.

Definition 1.1 (Real-rooted functions)
Let $\mathcal{P}$ be the class of real entire functions with real, simple, zeros and genus at most 1. Thus

$$f(t) = c \prod_{k=1}^{N} (t + r_k) \tag{4}$$

where $N \in \mathbb{N} \cup \{\infty\}$ and $\sum_{k=1}^{N} \frac{1}{r_k} < \infty$ when $N = \infty$.

Definition 1.2 (P_{pos})
$\mathcal{P}_{\mathrm{pos}}$ is the set of $\mathcal{P}$ that has all real distinct roots ≤ 0 and all positive coefficients.
$\mathcal{P}_{\mathrm{pos}-}$ is the equivalent of $\mathcal{P}_{\mathrm{pos}}$ but with all the functions zeros > 0

Definition 1.3 (Rotational operator)
The *rotation* of two polynomials $f(t)$ and $g(t)$ by an angle $\theta \in \mathbb{R}$ is the linear combination

$$h_\theta(t) := \mathrm{Rot}_\theta(f; g)(t) := \cos\theta \cdot f(t) + \sin\theta \cdot g(t).$$

Definition 1.4 (Fisk interlacing notation)
Let $\mathrm{roots}(f) = (a_1 < \cdots < a_n)$ and $\mathrm{roots}(g) = (b_1 < \cdots < b_m)$

- Same degree, largest root in g: $g \ll f \iff a_1 < b_1 < \cdots < a_n < b_n$.

- g has one more root: $g \bowtie f \iff b_1 < a_1 < \cdots < b_n$. We say that g surrounds f

- Arrow shorthand: $f \longleftarrow g$ means $f \ll g$ or $f \bowtie g$; the arrow points to the polynomial with the largest root.

- Arrow shorthand: $f \leftrightarrow g$ means f and g strictly interlace without specifying which has the largest zero.

- $f \not\leftrightarrow g$ means f and g do not strictly interlace.

Lemma 1.5 (Interlacing can only break at a shared zero)
Let $f \in \mathbb{R}[x]$ have simple real zeros $a_1 < \cdots < a_n$, and let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator depending continuously on the coefficients. For $\lambda \in [0, 1]$, set $T_\lambda := (1 - \lambda)I + \lambda T$ and $g_\lambda := T_\lambda f$. Assume that each g_λ is real-rooted (of degree n or $n - 1$).

If f and $g_0 = f$ interlace, then the interlacing pattern can change at some $\lambda^\star$ only if

$$\exists i \in \{1, \ldots, n\} \text{ such that } f(a_i) = 0 = g_{\lambda^\star}(a_i).$$

In particular, if f and $T_\lambda f$ have no common zeros for all $\lambda \in [0, 1]$, then f and $T_\lambda f$ interlace for all λ (hence also f and Tf).

Definition 1.6 (Sturm–monotone transform (SMT_σ))
Let $\mathcal{F} \in \mathcal{P}_{\mathrm{pos}}$.

A linear operator $T : \mathcal{F} \to \mathcal{F}$ is called a *Sturm–monotone transform* (of degree–shift $\sigma \in \{-1, 0, +1\}$) if the following hold:

1. **Linearity:** $T[af + bg] = a\,T[f] + b\,T[g]$ for all $a, b \in \mathbb{R}$ and $f, g \in \mathcal{F}$.

2. **Degree/zero shift:** For all nonzero $f \in \mathcal{F}$,

$$\deg(Tf) = \deg(f) + \sigma$$

3. **Interlacing monotonicity:**

   - $f \leftrightarrow g \implies Tf \leftrightarrow Tg$.
   - $f \bowtie g \implies Tf \bowtie Tg$

4. **Positivity on $\mathcal{P}_{\mathrm{pos}}$:** T maps $\mathcal{P}_{\mathrm{pos}}$ into real–rooted functions.

We write $T \in \mathrm{SMT}_\sigma$ when T is Sturm–monotone with degree–shift σ.

2 Rotation $\iff$ Interlacing (Obreschkoff)

Proposition 2.1 (Obreschkoff criterion; sine–cosine form)
Let $f, g \in \mathcal{P}$. The following are equivalent:

1. $f \leftrightarrow g$

2. $Rot_\theta(f; g) \in \mathcal{P}$ for $\theta \in \mathbb{R}$

and

1. $f \not\leftrightarrow g$

2. $\exists \theta \in \mathbb{R}$ such that $Rot_\theta(f; g) \notin \mathcal{P}$

3 Sturm Summary

Let f be a real polynomial with real, simple zeros

$$r_1 < r_2 < \cdots < r_n.$$

The following classical facts describe the behaviour of zeros under differentiation, perturbation, reflection, and translation.

(S1) Sturm Separation. Between any two consecutive zeros of f there is exactly one zero of f':

$$r_1 < s_1 < r_2 < s_2 < \cdots < r_{n-1} < s_{n-1} < r_n, \qquad f'(s_k) = 0.$$

Thus f' has real, simple zeros strictly interlacing those of f. *(See Sturm (1836); also, e.g., Karlin–Studden,* Tchebycheff Systems, *Thm. 1.1.)*

(S2) Logarithmic derivative monotonicity.
The function

$$\phi(t) = \frac{f'(t)}{f(t)} = \sum_{i=1}^{n} \frac{1}{t - r_i}$$

is strictly decreasing on each interval (r_i, r_{i+1}), with $\phi(t) \to +\infty$ as $t \downarrow r_i$ and $\phi(t) \to -\infty$ as $t \uparrow r_{i+1}$. Thus $\phi(t) = c$ has exactly one solution in each such interval. *(Standard in Sturm theory; see Karlin–Studden, Ch. 1, or Marden,* Geometry of Polynomials, *Ch. 6.)*

(S3) First–order perturbations preserve real zeros.
For every real a,

$$f + af'$$

has real, simple zeros and shares a strict interlacing with f. (This follows from (S2) by solving $f + af' = 0$ via $\phi(t) = -1/a$.) *(See Marden, Ch. 6; also Obreschkoff, Verteilung und Berechnung der Nullstellen, §10.)*

(S4) Sturm Comparison. If f and g have strictly interlacing zeros, then f' and g' also have strictly interlacing zeros. More generally, any differential operator of the form $a_0(t)f + a_1(t)f'$ with $a_1(t)$ not changing sign preserves interlacing. *(See Sturm (1836); also, Gantmacher–Krein, Oscillation Matrices and Kernels, Ch. 1.)*

(S5) Translation preserves interlacing. For any real a, the shift operator

$$exp(aD)f(t) = f(t + a)$$

preserves real, simple zeros and preserves all interlacing relations. *(Immediate from the fact that translation is a homeomorphism of $\mathbb{R}$.)*

Together, **(S1)–(S5)** imply that the operators D, $1 + aD$, $e^a D$, and scalar multiples or compositions of these preserve real distinct zeros and preserve interlacing on P.

4 Exterior-root control and the sharp θ-range in $\mathcal{P}_{\mathrm{pos}}$

Lemma 4.1 (Exterior-root criterion in $\mathcal{P}_{\mathrm{pos}}$)
Let

$$f(t) = a_n t^n + \cdots + a_1 t + a_0, \qquad g(t) = b_n t^n + \cdots + b_1 t + b_0,$$

with $f, g \in \mathcal{P}_{\mathrm{pos}}$ with $f \to g$, and $h_\theta := \mathrm{Rot}_\theta(f; g)$. Then h_θ has at most one zero outside the zeros of g (the others lie in the gaps), and $h_\theta \in \mathcal{P}_{\mathrm{pos}}$ iff this unique exterior zero lies in $(-\infty, 0)$.

Then all zeros of h_θ are real; they are negative and interlace those of f and g *except possibly one* additional Exterior root in $[0, \infty]$. Precisely:

(i) (*Endpoint signs control the Exterior root*) Set

$$\theta_\infty := \arctan\left(-b_n/a_n\right)$$

and

$$\theta_0 := \arctan\left(-b_0/a_0\right)$$

Then h_θ has a (necessarily unique) zero in $(0, \infty)$ if and only if for $-\pi/2 > \theta > \pi/2$ then $\theta_\infty < \theta <= \theta_0$

(ii) (*Zero at infinity / degree drop*) If $\theta = \theta_\infty$, then the degree of $h_\theta \leq n - 1$ (the leading term cancels). In this case we say the exterior zero is at ∞; the polynomial has one fewer zero in $\mathbb{R}$.

(iii) (b_n/a_n and b_0/a_0)

Note that in $\theta_\infty = arctan(-b_n/a_n)$, b_n/a_n is the $\lim_{t\to\infty} g(t)/f(t)$ and in $\theta_0 := \arctan\left(-b_0/a_0\right)$, $b_0/a_0 = g(0)/f(0)$

Theta range	Interlacing	$\mathcal{P}_{\mathrm{pos}}$	Notes
$\frac{\pi}{2}$	$g = h \to f$	$h \in \mathcal{P}_{\mathrm{pos}}$	
$\frac{\pi}{2} \to 0$	$g \to h \to f$	$h \in \mathcal{P}_{\mathrm{pos}}$	
0	$g \to h = f$	$h \in \mathcal{P}_{\mathrm{pos}}$	
$0 \to \theta_0$	$g \to f \to h$	$h \in \mathcal{P}_{\mathrm{pos}}$	
θ_0	$g \to f \to h$	$h \in \mathcal{P}_{\mathrm{pos}}$	0 at zero
$\theta_0 \to \theta_\infty$	$g \to f \to h$	$h \notin \mathcal{P}_{\mathrm{pos}}$	
θ_∞	$g \to f \bowtie h$	$h \in \mathcal{P}_{\mathrm{pos}}$	0 at infinity
$\theta_\infty \to -\frac{\pi}{2}$	$h \to g \to f$	$-h \in \mathcal{P}_{\mathrm{pos}}$	
$-\frac{\pi}{2}$	$-h = g \to f$	$-h \in \mathcal{P}_{\mathrm{pos}}$	

Table 1: Interlacing and $\mathcal{P}_{\mathrm{pos}}$ Behaviour of $h_\theta = \cos\theta\, f + \sin\theta\, g$ as θ varies for $g \to f$, $(f, g \in \mathcal{P}_{\mathrm{pos}})$.

Remark 4.2 (Exceptional nullspace case: $Tf \equiv 0$)
Let T be a Sturm–monotone transform (linear, degree-shift $\sigma \in \{-1, 0, +1\}$, interlacing-monotone on its domain). If f lies in the nullspace (i.e. $Tf \equiv 0$), the usual interlacing statements must be read with care:

1. **Pairwise interlacing becomes vacuous.** If $Tf \equiv 0$ but $Tg \not\equiv 0$, then assertions like "Tf interlaces Tg" are void since Tf has no zeros to order.

2. **Finite–defect mechanism via annihilation of a reference.** Many operators annihilate a specific reference (e.g. $(1+D^2)\cos \equiv 0$). When one factor in a Wronskian identity is killed by T, the resulting interlacing with that reference fails only in a finite prefix (the "finite defect") but is intact thereafter.

Summary of Operator Properties

The Derivative Operator (D)
Effect: $D(t^n) = n t^{n-1}$
Stability: Preserves P (real roots) and $\mathcal{P}_{\mathrm{pos}}$ (non-positive roots).
Interlacing: Preserves interlacing for any real-rooted polynomial.
Sturm Class: $D \in \mathrm{SMT}_{-1}$

The Affine Derivative $(1 + aD)$
Effect: $t^n \mapsto t^n + a n t^{n-1}$
Stability: Preserves P for all $a \in \mathbb{R}$.
Note: Preserves $\mathcal{P}_{\mathrm{pos}}$ if $a \geq 0$. If $a < 0$, roots may shift out of the non-positive half-line.
Sturm Class: $1 + aD \in \mathrm{SMT}_0$

The Euler Operator (tD)
Effect: $t^n \mapsto n t^n$
Stability: Preserves P unconditionally (provided $f'(0) \neq 0$).
Interlacing: Preserves interlacing for $\mathcal{P}_{\mathrm{pos}}$ and $\mathcal{P}_{\mathrm{neg}}$.
Sturm Class: $tD \in \mathrm{SMT}_0$

The Generalized Euler Operator $(1 + atD)$
Effect: $t^n \mapsto (1 + an) t^n$
Stability: Preserves P for all $a > 0$.
Note: Preserves $\mathcal{P}_{\mathrm{pos}}$ if $a \geq 0$. If $a < 0$, stability cannot be guaranteed for $\mathcal{P}_{\mathrm{pos}}$.
Sturm Class: $1 + atD \in \mathrm{SMT}_0$

The Radial Operator $(D(4sD + 2))$, $\mathcal{T}_{\mathbf{rad}}$
  Effect: $t^n \mapsto 2n(2n - 1)t^{n-1}$
  Stability: Preserves $\mathcal{P}_{\text{pos}}$.
  Sturm Class: $\mathcal{T}_{\text{rad}} \in \text{SMT}_{-1}$

Theorem 4.3
Let $F_0, F_1, \ldots, F_m$ be real entire functions (or real polynomials) each having simple real zeros, all lying on the same side of the real line.

Assume:

  (i) (**Adjacent interlacing, same orientation**) For each $k = 0, 1, \ldots, m - 1$ we have $F_k \to F_{k+1}$

  (ii) (**Endpoint interlacing, same orientation**) We also have $F_0 \to F_m$

Then the entire family $\{F_0, \ldots, F_m\}$ is *pairwise interlacing*: for every $0 \leq i < k \leq m$

Proof. Since all the zeros of $\{F_1, \ldots, F_{m-1}\}$ are ordered and confined to be between the zeros of F_0 and F_m. Thus every pair of functions in the family interlaces, with the same orientation. $\square$

Theorem 4.4
If $\mathcal{C}_a[f](s) \in \mathcal{P}_{\text{pos}}$ and interlaces with $\mathcal{C}_a[g](s) \in \mathcal{P}_{\text{pos}}$ then $\mathcal{C}_a[t^2 f](s)$ interlaces with $\mathcal{C}_a[t^2 g](s)$ and both are in $\mathcal{P}_{\text{pos}}$

Proof. Consider $F(s^2) = \mathcal{C}_a[f](s^2)$ which will be a function with all imaginary roots. Take the second derivative of this.

$$\frac{\partial^2 F\left(s^2\right)}{\partial s^2} = 4s^2 F''\left(s^2\right) + 2F'\left(s^2\right) \tag{5}$$

and map s^2 back to s giving us the operator

$$4sF''(s) + 2F'(s) \text{ or } D(4sD + 2) := \mathcal{T}_{\text{rad}} \tag{6}$$

Both D and $(4sD + 2)$ preserve interlacing in $\mathcal{P}_{\text{pos}}$ and map $\mathcal{P}_{\text{pos}}$ to $\mathcal{P}_{\text{pos}}$

$\square$

5   Cosh–Moment Sturm Family (CMST) and Global Interlacing

Definition 5.1 (Cosh moment Sturm family)
For $n \in \mathbb{N}_0$ and $s \geq 0$ define

$$F_n(s) := (2n + 1) \int_{-1}^{1} t^{2n} \cosh(\sqrt{s}\, t)\, dt \tag{7}$$

Lemma 5.2 $(F_n(s) \in P_{\text{pos}})$

$$F_0(s) = 2\frac{\sinh\left(\sqrt{s}\right)}{\sqrt{s}} \tag{8}$$

which only has zeros on the negative real axis and noting that t^2 maps $P_{\text{pos}} \to P_{\text{pos}}$

Cosh Moment Sturm Tower (CMST)

Definition 5.3 (CMST family)
A *Cosine–Moment Sturm Tower* (CMST) is a collection

$$\mathcal{C}_a = \{F_\alpha : \alpha \in A\}$$

of real functions together with a distinguished total order $< A$ on the index set A, such that:

(i) (**Real–rootedness**) Each F_α has only real, simple zeros, all lying in a fixed interval. $(-\infty, 0)$.

(ii) (**Global interlacing**) For any $\alpha < \beta$ in A, the pair (F_α, F_β) has interlacing zeros in the direction
$$F_\alpha \;\to\; F_\beta.$$
In particular, every pair of distinct members of $\mathcal{C}_a$ interlaces with a consistent arrow direction.

(iii) (t^2 **interlacing**) $F_\alpha \to t^2 F_{\alpha+1}$

We say that $(\mathcal{C}_a)$ is a CMST structure if it satisfies (i)–(iii). We also say that $f(t) \in CMST$ if it preserves CMST. We will refer to $f(t)$ as even if its an even function and $f(t)$ not even if $f(t) \neq f(-t)$

Theorem 5.4 (Interlacing Direction by Reciprocal Roots)
Let $f(s)$ and $g(s)$ be two polynomials of degree n in $\mathcal{P}_{\text{pos}}$ that interlace. We normalize them such that $f(0) = g(0) = 1$, so they can be expressed in the product form:

$$f(s) = \prod_{k=1}^{n}\left(1 + \frac{s}{r_k}\right) \quad \text{and} \quad g(s) = \prod_{k=1}^{n}\left(1 + \frac{s}{s_k}\right)$$

where $r_k > 0$ and $s_k > 0$ are the magnitudes of the non-positive roots.

The coefficient of the linear term, s^1, is the sum of the reciprocal roots:

$$f(s) = 1 + a_1 s + \cdots \implies a_1 = \sum_{k=1}^{n}\frac{1}{r_k} \tag{9}$$

$$g(s) = 1 + b_1 s + \cdots \implies b_1 = \sum_{k=1}^{n}\frac{1}{s_k} \tag{10}$$

The direction of interlacing is determined by comparing these sums:

$$a_1 < b_1 \quad \Longleftrightarrow \quad f \to g \tag{11}$$

This means that if the sum of the reciprocal roots of f is smaller than that of g, then the roots of f are more negative than the roots of g.

Theorem 5.5 (Globally interlacing family)

$$F_n(s) := (2n+1)\int_{-1}^{1} t^{2n}\cosh(\sqrt{s}\,t)\,dt, \tag{12}$$

is a family of mutually interlacing functions with $F_n \to F_{n+a} \,\forall n \in \mathbb{N}_0, a \in \mathbb{N}$

Proof.

$$F_0(s) = \frac{\sinh\left(\sqrt{s}\right)}{\sqrt{s}} \tag{13}$$

$$F_1(s) = 3\left(\frac{(s+2)\sinh\left(\sqrt{s}\right)}{s^{3/2}} - \frac{2\cosh\left(\sqrt{s}\right)}{s}\right) \tag{14}$$

We know that $F_1(s) \in \mathcal{P}_{\text{pos}}$ If we look at the zeros of $F_0(s)$ we can see that $F_1(s)$ alternates in sign at each zero. Looking at the coefficients of s we get for $F_0(s)$ its 1/6 and for $F_1(s)$ its 3/10 therefore $F_0(s) \to F_1(s)$

Since t^2 preserves interlacing, $F_0(s) \to F_1(s) \implies F_1(s) \to F_2(s)$ and in general $F_n(s) \to F_{n+1}(s)$

$F_\infty(s) = \cosh\left(\sqrt{s}\right)$ so $F_0(s) \to F_\infty(s)$ and we have a complete family of mutually interlacing functions in $\mathcal{P}_{\text{pos}}$ with each having zeros strictly between $\frac{\sinh\left(\sqrt{s}\right)}{\sqrt{s}}$ and $\cosh\left(\sqrt{s}\right)$ except for the first and the last which are those functions.

$\square$

Remark 5.6
The s coefficient in $F_n(s)$ is $\frac{1}{2}\left(1 - \frac{2}{2n+3}\right)$ which can be seen to be increasing with n. As $n \to \infty$ the first term converges to 1/2 which is the sum of the reciprocals of the roots of $\cosh\left(\sqrt{s}\right)$

$$\sum_{n=1}^{\infty} \frac{4}{\pi^2(1-2n)^2} = 1/2 \tag{15}$$

Theorem 5.7 (An Application of Cauchy–Schwarz inequality)
Let $f(t)$ be a non-negative, continuous function on the interval $[-1, 1]$. The following inequality holds:

$$\frac{\int_{-1}^{1} t^4 f(t)\, dt}{\int_{-1}^{1} t^2 f(t)\, dt} > \frac{\int_{-1}^{1} t^2 f(t)\, dt}{\int_{-1}^{1} f(t)\, dt} \tag{16}$$

Proof. We define the $2k$-th moment of $f(t)$ over the interval $[-1, 1]$ as:

$$\mu_{2k} = \int_{-1}^{1} t^{2k} f(t)\, dt \tag{17}$$

Assuming $f(t) \geq 0$ and $f(t)$ is not identically zero, we have $\mu_0 > 0$, $\mu_2 > 0$, and $\mu_4 > 0$. The inequality is equivalent to proving the log-concavity of the moment sequence:

$$\mu_4 \mu_0 > (\mu_2)^2 \tag{18}$$

The Cauchy–Schwarz inequality for the weighted inner product $\langle g, h \rangle_f = \int_{-1}^{1} g(t)h(t)f(t)\, dt$ states that:

$$\left(\int_{-1}^{1} g(t)h(t)f(t)\, dt\right)^2 \leq \left(\int_{-1}^{1} g(t)^2 f(t)\, dt\right)\left(\int_{-1}^{1} h(t)^2 f(t)\, dt\right) \tag{19}$$

We choose the functions $g(t)$ and $h(t)$ as follows:

1. Let $\mathbf{g(t)} = \mathbf{1}$

2. Let $\mathbf{h(t)} = \mathbf{t^2}$

Substituting these into the Cauchy–Schwarz inequality:

- The Left-Hand Side (LHS) of the inequality becomes:

$$\left(\int_{-1}^{1} 1 \cdot t^2 f(t)\, dt\right)^2 = \left(\int_{-1}^{1} t^2 f(t)\, dt\right)^2 = (\mu_2)^2 \tag{20}$$

- The Right-Hand Side (RHS) of the inequality becomes:

$$\left(\int_{-1}^{1} 1^2 f(t)\, dt\right)\left(\int_{-1}^{1} (t^2)^2 f(t)\, dt\right) = \left(\int_{-1}^{1} f(t)\, dt\right)\left(\int_{-1}^{1} t^4 f(t)\, dt\right) = \mu_0 \mu_4 \tag{21}$$

Applying the Cauchy–Schwarz inequality yields:

$$(\mu_2)^2 \leq \mu_0 \mu_4 \tag{22}$$

Dividing both sides by the positive quantity $\mu_0 \mu_2$:

$$\frac{\mu_2^2}{\mu_0 \mu_2} \leq \frac{\mu_0 \mu_4}{\mu_0 \mu_2} \quad \Longrightarrow \quad \frac{\mu_2}{\mu_0} \leq \frac{\mu_4}{\mu_2} \tag{23}$$

The equality $(\mu_2)^2 = \mu_0 \mu_4$ holds if and only if the functions $g(t) = 1$ and $h(t) = t^2$ are linearly dependent, i.e., $t^2 = c \cdot 1$ for some constant c. This is impossible on the interval $[-1, 1]$ since t^2 is not a constant function.

Therefore, the inequality is strict:

$$\mu_4 \mu_0 > (\mu_2)^2 \tag{24}$$

which proves the original statement:

$$\frac{\int_{-1}^{1} t^4 f(t)\, dt}{\int_{-1}^{1} t^2 f(t)\, dt} > \frac{\int_{-1}^{1} t^2 f(t)\, dt}{\int_{-1}^{1} f(t)\, dt} \tag{25}$$

$$\square$$

Theorem 5.8
For $f(t) \in CMST_1$

$$\mathcal{C}_1(f)(s) \to \mathcal{C}_1(t^2 f)(s) \tag{26}$$

Proof. If we look at the normalised $\mathcal{C}_1(f)(s)$ and $\mathcal{C}_1(t^2 f)(s)$ and then look at the s term in the expansion, we see that using *theorem* 5.7 and *theorem* 5.4 we get this result. (Noting that $f(t)$ needs to be of constant sign to be in CMST.) $\square$

6 Assumption on Operator Convergence

Remark 6.1 (Convergence Assumption)
Unless explicitly stated otherwise, all differential and integral operators, particularly those defined on spaces of polynomials or analytic functions, are assumed to operate within domains where the resulting integrals and series expressions **converge**. Specifically, any results concerning zero-preserving properties or interlacing are valid only when the formal application of the operator yields a well-defined function. This includes, but is not limited to, the proper convergence of integrals used in generating function representations or the convergence of series expansions for associated polynomials.

7 CMST Zero-Preserving Operators

Definition 7.1 (CMSTZP)
An operator T acting on CMST members is a *CMST Zero-Preserving Operator* if: (i) $T(F) \in \mathcal{P}_{\text{pos}}$ whenever F is CMST; and (ii) $F \to t^2 F \implies T(F) \to T(t^2 F)$.

Proposition 7.2 (Simple even kernel)
Let
$$k(t^2) = (1 + at^2) \qquad a > 0. \tag{27}$$

Remark 7.3
This is $F_n(s) + (a(3 + 2n))/(1 + 2n)F_{n+1}(s)$ so all the zeros are between F_n and F_{n+1}

Proposition 7.4
If $f(t), g(t)$ both even maps CMST to CMST then $f(t)g(t)$ maps CMST to CMST.

Proposition 7.5 ($k(t) \in \mathcal{P}_{\text{pos}} \implies k(t^2) \in CMST$)
If $k(t) \in \mathcal{P}_{\text{pos}}$ then $k(t^2) \in CMST$ This is just *theorem* 7.2 repeated.

Remark 7.6
Note that this is a little too strict, since we can have repeated zeros and this still holds.

Proposition 7.7 (Positive even kernel)
Let
$$k(t^2) = \sum_{i \geq 1} a_i \, t^{2i}, \qquad a_i \geq 0. \tag{28}$$

We can see that at zeros of $F_0(s)$, $F_{n>}(s)$ are all the same sign alternating in signs at each zero of $F_0(s)$ Therefore $F_0(s) \to F_0 k(t^2)(s)$ and since t^2 preserves interlacing our whole family is preserved.

Proposition 7.8 (An even $T \in CMST \implies T^n \in CMST$)
This is just repeated application of the transform.

Proposition 7.9 ($k(t^2) \in CMST \implies e^{ak(t^2)} \in CMST, a > 0$)
If $k(t^2) \in CMST$ then so is $1 + k(t^2)/n \in CMST, n \in \mathbb{N}$ We then take the limit of $(1 + k(t^2)/n)^n$ as $n \to \infty$

8 Rotation and Interlacing under Operators

Corollary 8.1 (Local preservation on admissible arcs)
If $f \in CMST$ and $f(t) > 0, \forall, t \in [-1, 1]$ then $\text{Rot}_\theta(f, ft^2) \in \mathcal{P}_{\text{pos}}$ in the admissible range of *theorem* 4.1, Thus, along each admissible arc, the rotation orbit remains real-rooted after applying t^2.

In the case of $\mathcal{C}_1(\text{Rot}_\theta(f, ft^2))(s)$
θ_0 is when
$$\mathcal{C}_1(\text{Rot}_\theta(f(0), t^2 f(0)) = 0 \tag{29}$$

or more simply.
$$\int_{-1}^{1} \text{Rot}_\theta(f, ft^2) \, dt = 0 \tag{30}$$

In terms of the whole family, this doesnt help us much though, since

$$\int_{-1}^{1} \text{Rot}_\theta(f, ft^2) \, dt \to \int_{-1}^{1} t^2 \, \text{Rot}_\theta(f, ft^2) \, dt \tag{31}$$

so therefore, $\int_{-1}^{1} t^2 \operatorname{Rot}_\theta(f, ft^2)\, dt \notin \mathcal{P}_{\text{pos}}$ So to preserve the whole family in $\mathcal{P}_{\text{pos}}$ we need

$$\operatorname{Rot}_\theta\left(\cosh\left(\sqrt{s}\right), \cosh\left(\sqrt{s}\right)\right) \in \mathcal{P}_{\text{pos}} \tag{32}$$

This is true for $\theta \in [\pi/2, -\pi/4)$ and then, because $f \in CMST$ then

$$\mathcal{C}_1(\operatorname{Rot}_\theta(f, ft^2))(s) \in CMST \tag{33}$$

When $\theta = -\pi/4$ then
$$\operatorname{Rot}\left(\theta, \cosh\left(\sqrt{s}\right), \cosh\left(\sqrt{s}\right)\right) = 0 \tag{34}$$

This corresponds to $(1 - t^2) = 0$ at $t = 1$
We will call this function the Assassin **X**
θ_∞ is when
$$\mathcal{C}_1(\operatorname{Rot}_\theta(f, ft^2))(s) \tag{35}$$

has any positive zeros.
Looking at

$$\int_{-1}^{1} \left(1 - at^2\right) \cosh\left(\sqrt{st}\right)\, dt \tag{36}$$

we can see that for $a \le 1$ we will have no positive zeros since $\left(1 - at^2\right) \cosh\left(\sqrt{st}\right) \ge 0$ for $s > 0$ and $t \in [0, 1]$ and for $3 > a \ge 1$ we will have a positive zero since at $s = 0$, $\int_0^1 \left(1 - at^2\right) dt > 0$ and limit $\lim_{s \to \infty} \int_0^1 \left(1 - at^2\right) \cosh\left(\sqrt{st}\right) dt$ is negative. We don't have to worry about $a \ge 3$ since even though $(1 - at^2)$ will have no positive zeros, $t^2(1 - at^2)$ will or some power of t^2 will.
Thus $\theta_\infty = -\pi/4$
In summary
$$\mathcal{C}_1(\operatorname{Rot}_\theta(f, ft^2))(s) \tag{37}$$

preserves CMST in $\theta \in [\pi/2, -\pi/4)$
We will consider the case of $\theta = -\pi/4$ next

Remark 8.2 (Behaviour at $\theta = -\pi/4$)
At $\theta = -\pi/4$ we have

$$\int_{-1}^{1} \left(1 - t^2\right) \cosh\left(\sqrt{st}\right)\, dt \tag{38}$$

since $\theta = -\pi/4$ is $\theta = \theta_\infty$ we have

$$\int_{-1}^{1} \cosh\left(\sqrt{st}\right)\, dt \bowtie \int_{-1}^{1} \left(1 - t^2\right) \cosh\left(\sqrt{st}\right)\, dt \tag{39}$$

which preserves the family however applying it again gives

$$\int_{-1}^{1} \left(1 - t^2\right) \cosh\left(\sqrt{st}\right)\, dt \bowtie \int_{-1}^{1} \left(1 - t^2\right)^2 \cosh\left(\sqrt{st}\right)\, dt \tag{40}$$

which means that

$$\int_{-1}^{1} \left(1 - t^2\right)^2 \cosh\left(\sqrt{st}\right)\, dt \tag{41}$$

has two less zeros than

$$\int_{-1}^{1} \cosh\left(\sqrt{st}\right)\, dt \tag{42}$$

so interlacing is impossible. The zeros we have lost are the largest roots, so we have one defect in the interlacing.

Remark 8.3 (The Assassin $\mathbf{X}$)

In summary $\mathbf{X}$ removes a zero, kills our $\cosh\left(\sqrt{s}\right)$ function the tail of our interlacing and applied twice breaks our tower of mutually interlacing functions. We still have

$$\mathcal{C}_1(f))(s) \to \mathcal{C}_1(t^2 f))(s) \tag{43}$$

all the way up the tower, but we have lost the final interlacing which guaranteed that they were all mutually interlacing. Note that the Assassin appears in terms like $(1 - t^4) = (1 - t^2)(1 + t^2)$ and in general in terms like $(1 - t^{2n})$

Remark 8.4

CMST vs. General Laguerre-Pólya Operators: It is crucial to distinguish the stability of operators within the CMST framework from their behavior in the general Laguerre-Pólya class (P). In the broader space P, infinite-order differential operators such as e^{D^2} (the backward heat operator) are generally not zero-preserving; they can force real roots to migrate into the complex plane. However, the CMST class is a "protected subalgebra" defined by a strict constraint. Within this restricted domain, operators like e^{D^2} map CMST to CMST. Thus, the structural rigidity of the CMST allows for the use of powerful operators that are "forbidden" or unstable in the general theory of real-rooted functions.

Pólya–Levin Background (for reference)

Theorem 8.5 (Pólya; see Levin)

If $w : [0, 1] \to [0, \infty)$ is nondecreasing and not almost everywhere zero, then

$$G(a) := \int_0^1 w(t) \cos(at)\, dt \tag{44}$$

extends to an entire function of exponential type whose real zeros are simple and appear once in each interval $(k\pi, (k + 1)\pi)$; nonreal zeros lie on the imaginary axis.

Remark 8.6

References: G. Pólya [2, 4]; B. Ya. Levin [7, Chapter II, Thm. 7].

Theorem 8.7 (Obreschkoff, strict version)

Let p and q be real-rooted polynomials with simple zeros. Then the following are equivalent:

Every nontrivial linear combination $\alpha p + \beta q$ ($\alpha, \beta \in \mathbb{R}$) is real-rooted.

The zeros of p and q strictly interlace.

Proposition 8.8 (Contrapositive of Obreschkoff)

Let $p, q \in \mathbb{R}[x]$ be real-rooted polynomials. If the zeros of p and q do *not* interlace, then there exists $\lambda \in \mathbb{R}$ such that $p + \lambda q$ has a nonreal zero (equivalently, is not real-rooted).

Proof sketch. We first treat the generic case: p and q have simple zeros and no common zeros. Set the Wronskian

$$W(x) := p(x)q'(x) - p'(x)q(x). \tag{45}$$

A standard equivalent form of Obreschkoff's theorem states that p and q interlace if and only if W has constant sign on $\mathbb{R}$. Thus, if p and q do not interlace, W changes sign, so there exists $x_0 \in \mathbb{R}$ with $W(x_0) = 0$.

At such an x_0, the linear system

$$p(x_0) + \lambda q(x_0) = 0, \qquad p'(x_0) + \lambda q'(x_0) = 0 \tag{46}$$

has a real solution λ_0 (the determinant is $W(x_0) = 0$). Hence $r_{\lambda_0}(x) := p(x) + \lambda_0 q(x)$ has a multiple real root at x_0. As λ varies across λ_0, a standard root-continuity argument shows that

this double root splits into a pair of complex conjugate roots for nearby λ, so r_λ fails to be real-rooted for some real λ arbitrarily close to λ_0.

If p and q have a common factor, factor it out and apply the argument to the coprime quotients; if multiple roots occur, perturb coefficients slightly (or use a limiting argument) to reduce to the simple-root case and pass to the limit. $\qquad\square$

Proposition 8.9 (Positive combinations of iterates preserve interlacing)
Let $T \in \mathrm{SMT}_\sigma$ and suppose all iterates T^i act on $\mathcal{F}$ and satisfy the same interlacing clause as in the definition. For any finite family of nonnegative coefficients $a_i \geq 0$ (not all zero), define

$$S \;=\; \sum_{i=0}^{m} a_i\, T^i. \tag{47}$$

Then S preserves the corresponding interlacing relation:

- If $\sigma = 0$ and $f \to g$, then $Sf \to Sg$.

- If $\sigma = -1$ and $f \bowtie g$, then $Sf \bowtie Sg$ (one-sided removal persists).

- If $\sigma = +1$ and $f \to g$, then $Sf \to Sg$ (one-sided addition persists).

Theorem 8.10 (Pólya, 1918–1927)
For an even integer exponent $2q$ with $q \geq 1$,

$$U_q(s) \;=\; \int_0^\infty e^{-t^{2q}} \cos(st)\, dt \tag{48}$$

extends to an entire even function of s whose zeros are all real.

Proof sketch. Pólya's method places these cosine/Fourier transforms in the Laguerre–Pólya class by constructing them from kernels that are totally positive ("universal factors"). For $e^{-t^{2q}}$ (even, rapidly decaying, log-convex tails), one obtains an entire function of order $2q/(2q-1) \in (1,2)$ with only real zeros. See the survey of Dimitrov–Rusev for a modern consolidation [11], especially the discussion of Pólya's 1918 and 1927 papers and the explicit statement that

$$\int_0^\infty e^{-t^{2q}} \cos(zt)\, dt \quad \text{has only real zeros.} \tag{49}$$

$\qquad\square$

Remark 8.11
For non-even exponents $2 + \varepsilon$ (with $\varepsilon \notin 2\mathbb{N}$), the transform $\int_0^\infty e^{-t^{2+\varepsilon}} \cos(st)\, dt$ is still an entire even function of Genus 1 with an infinity of zeros

Definition 8.12 (Entire functions of genus one)
An entire function f is said to be of *genus 1* if it admits a canonical Weierstrass factorization

$$f(z) = e^{az+b}\, z^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) \exp\!\left(\frac{z}{z_k}\right), \tag{50}$$

and the zero sequence $\{z_k\}$ satisfies $\sum_k |z_k|^{-2} < \infty$ but $\sum_k |z_k|^{-1} = \infty$. Equivalently, f is of finite order $1 < \rho \leq 2$. Typical examples include $\sin(\pi z)$, $\cos z$, and $\int_0^\infty e^{-t^4} \cos(zt)\, dt$.

Definition 8.13 (Entire functions of genus zero)
An entire function f is of *genus 0* if it admits the Weierstrass product

$$f(z) = e^{az+b}\, z^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right), \qquad \sum_k \frac{1}{|z_k|} < \infty. \tag{51}$$

Equivalently, the canonical product converges without any exponential correction factors and f has finite order $\rho \le 1$. Typical examples include exponential functions and products with quadratically or faster growing zeros, such as $\prod_{n=1}^{\infty}(1 - \frac{z}{n^2})$.

Theorem 8.14 (Pólya)
Fix $\varepsilon > 0$ and set $m = 2 + \varepsilon$. Define

$$F_\varepsilon(s) \;=\; \int_0^{\infty} e^{-t^m} \cos(st)\, dt \qquad (s \in \mathbb{R}). \tag{52}$$

Then F_ε extends to an even entire function on $\mathbb{C}$ of order

$$\rho = \frac{m}{m-1} = \frac{2+\varepsilon}{1+\varepsilon} \in (1,2), \tag{53}$$

and hence of *genus 1* (since $1 < \rho < 2$).

Proof sketch. Entirety. For $s \in \mathbb{C}$ one has

$$\left|e^{-t^m} \cos(st)\right| \;\le\; e^{-t^m}\, e^{|\Im s|\, t}, \tag{54}$$

and $\int_0^{\infty} e^{-t^m + |\Im s|\, t}\, dt < \infty$ for every s because t^m ($m > 1$) dominates linearly as $t \to \infty$. Thus the integral converges absolutely and defines an entire function by dominated convergence (differentiate under the integral).

Order. By the Laplace/Legendre-transform asymptotics,

$$\log |F_\varepsilon(s)| \;=\; O\!\left(|s|^\rho\right) \quad \text{with} \quad \rho = \frac{m}{m-1}, \tag{55}$$

and this bound is sharp (saddle-point/steepest-descent at $t \sim c\, |s|^{1/(m-1)}$). Hence F_ε has order $\rho \in (1,2)$. For entire functions with noninteger order ρ, the *genus* equals $\lfloor \rho \rfloor$; therefore F_ε has genus 1. $\square$

Remark 8.15
The importance of this result for the paper is that functions of order $m = 2 + \varepsilon$ produce transforms with infinite zeros in most of the cases we will be looking at.

Remark 8.16 (Genus and reciprocal-sum criterion)
If an entire function admits the canonical product

$$f(s) = e^{as+b} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right), \tag{56}$$

then the product converges (and f is of genus 0) precisely when

$$\sum_k \frac{1}{|s_k|} < \infty. \tag{57}$$

If instead $\sum |s_k|^{-1}$ diverges but $\sum |s_k|^{-2} < \infty$, one must include the exponential correction e^{s/s_k}, yielding a genus-1 product. Thus, informally, *a finite sum of reciprocals of zeros corresponds to genus 0.*

Remark 8.17 (Genus of $\cos s$ versus $\cos\sqrt{s}$)

The zeros of $\cos s$ are $s_k = (k + \frac{1}{2})\pi$, which grow linearly. Hence the exponent of convergence of $\{s_k\}$ is $\lambda = 1$, and the minimal integer p with $p + 1 > \lambda$ is $p = 1$; thus $\cos s$ is an entire function of genus 1. Its canonical product can be paired symmetrically as

$$\cos s = \prod_{k=0}^{\infty} \left(1 - \frac{s^2}{\left((k + \frac{1}{2})\pi\right)^2}\right), \tag{58}$$

an even product that converges absolutely without exponential factors.

For $\cos\sqrt{s}$, the zeros are $s_k = ((k + \frac{1}{2})\pi)^2$, which grow quadratically. Then $\lambda = \frac{1}{2}$, so the genus is $p = 0$. Its Weierstrass product is

$$\cos\sqrt{s} = \prod_{k=0}^{\infty} \left(1 - \frac{s}{\left((k + \frac{1}{2})\pi\right)^2}\right), \tag{59}$$

showing explicitly that $\cos\sqrt{s}$ is entire of genus 0 while $\cos s$ is entire of genus 1.

Remark 8.18 (Known cases whose transform produces only real zeros)

(a) The Gaussian, e^{-t^2} is zero free (though I think we show here that on any finite domain it has an infinity of real zeros).

(b) For the monomial case $\phi(t) = e^{-t^{2m}}$ with $m \geq 2$, it is classical (Pólya) that $\widehat{\phi}$ has only real zeros

(c) Known "universal factor" results (Pólya–de Bruijn type) and total-positivity criteria cover several two– and three–term exponents (e.g. $e^{-(at^4 + bt^2)}$ and, more generally, $e^{-(at^{4q} + bt^{2q} + ct^2)}$ under constraints)

(d) $e^{-\cosh(t)}$ G. Pólya, "Über trigonometrische Integrale mit nur reellen Nullstellen" (1927)

9 Stability and Destruction in Mutually Interlacing Families

In this section, we examine the stability of mutually interlacing families in $\mathcal{P}_{\text{pos}}$ under linear combinations. Specifically, we analyze how the introduction of negative coefficients (the "Assassin" mechanism) affects the preservation of the family structure.

Definition 9.1 (Generated Family)

Let $\mathcal{F} = \{F_0, F_1, \ldots, F_n\}$ be a family of mutually interlacing functions in $\mathcal{P}_{\text{pos}}$, generated by an interlacing-preserving operator T (typically $Tf \approx t^2 f$), such that $F_k \to F_{k+1}$ for all k and T maps $\mathcal{P}_{\text{pos}}$ to $\mathcal{P}_{\text{pos}}$

9.1 Positive Coefficients: Absolute Stability

Proposition 9.2 (Convex Combinations Preserve Family)

Any linear combination of the family members with strictly positive coefficients:

$$H(s) = \sum_{k=0}^{n} c_k F_k(s), \quad c_k \geq 0 \tag{60}$$

remains in $\mathcal{P}_{\text{pos}}$ and preserves the global interlacing structure.

Proof. At the least negative root of the largest $F_k(s)$ all the terms are positive. At the next zero down they are all negative, so there is a zero between $F_0(s)$ and $F_k(s)$. This repeats all the way down the zeros. This means that $F_0(s) \to H(s) \to F_k(s)$. Since T maps $\mathcal{P}_{\text{pos}}$ and preserves interlacing this means that $TF_0(s) \to TH(s) \to TF_k(s)$ and a new family of mutually interlacing functions is produced. If we let the transform

$$T_+ = \sum_{k=0}^{n} c_k F_k(s) \quad c_k \geq 0 \tag{61}$$

We get $T_+ F_0(s) \to T_+ F_1(s) \cdots \to T_+ F_k(s)$ and $T_+ F_0(s) \to T_+ F_k(s)$

$\square$

Remark 9.3
Positive coefficients are very safe in interlacing terms and move zeros away from 0.

9.2 Negative Coefficients: The Rightward Shift

Remark 9.4 (The Assassin's Shift)
Introducing a negative coefficient acts as a perturbation that shifts roots to the right. Consider $H_\epsilon = F_0 - \epsilon F_k$ with $\epsilon > 0$. As ϵ increases, the largest root of H_ϵ moves towards the origin. If ϵ exceeds a critical threshold, the root crosses zero onto the positive real axis, causing the function to exit $\mathcal{P}_{\text{pos}}$.

9.3 The Critical Value: Surrounding

Proposition 9.5 (Creation of Surrounding Relation)
There exists a critical value ϵ^* for the negative coefficient such that the largest zero is removed (pushed to infinity or the boundary). At this value, the new function H_{ϵ^*} has one fewer root than the base family. Consequently, the original family members now *surround* the new function:

$$F_k \bowtie H_{\epsilon^*} \tag{62}$$

This state represents a "Finite Defect."

9.4 High-End Negative Terms: The Assassin

Proposition 9.6 (Lowest Negative Term Principle)
If the negative coefficients are confined to high-order terms (e.g., terms corresponding to F_k for large k), there exists a value that removes a zero and produces the surrounding relation. Specifically, if

$$H(s) = F_0(s) - \lambda F_m(s) \tag{63}$$

the negative tail acts as a single block. The dominant negative term at infinity cancels the positive behavior of lower terms, executing the "Assassin" mechanism and reducing the effective degree of the function.

9.5 One Sign Change: Variation Diminishing

Theorem 9.7 (Admissibility of One Sign Change)
Let the sequence of coefficients $\{c_k\}$ in the combination $H(s) = \sum c_k F_k(s)$ have exactly one sign change (from positive to negative). Then $H(s)$ retains real-rootedness (with at most one positive root, which can be pushed to infinity).

Proof. This follows from the variation diminishing property of totally positive kernels (as established by Schoenberg/Fisk). One sign change in the coefficient sequence implies at most one sign change in the function values (roots) on the positive half-line. By adjusting the magnitude of the negative tail, this single "bad" root is pushed to the boundary, leaving the remaining roots in $\mathcal{P}_{\text{pos}}$. $\square$

9.6 Family Size Limitation

Conclusion 9.8 (Truncation of Mutual Interlacing)
The term $(1 - F_m)$ is only guaranteed to preserve mutual interlacing for the first m terms of the family.

While the lower-order members $F_0, \ldots, F_{m-1}$ interlace with the modified function (due to the surrounding property $F \bowtie G$), terms higher than F_m rely on geometric root positioning that is destroyed by the defect introduced at index m. Thus, the "Assassin" creates a ceiling: the global mutual interlacing of the tower is truncated, and the sustainable family size is determined by the index m of the lowest negative term.

Theorem 9.9 (Infinite series exponent (part 1))
Let $f(t) = a_1 t + a_2 t^2 + a_3 t^3 + \cdots$ be a power series with $a_i > 0$ and $\sum_{i=1}^{\infty} a_i < \infty$. Then the bilateral Laplace transform

$$F(s) = \int_{-1}^{1} e^{-f(t^2)} e^{st} \, dt \tag{64}$$

has all its zeros on the imaginary axis.

Proof. Let $a = \sum_{i=1}^{\infty} a_i$ which is finite by assumption.
Then

$$(1 - (a_1 t^2 + a_2 t^4 + a_3 t^6 \cdots)/a) \geq 0 \in [-1, 1] \tag{65}$$

Therefore

$$\mathcal{C}_1(1 - (a_1 t^2 + a_2 t^4 + a_3 t^6 \cdots)/n)(s) \in \mathcal{P}_{\text{pos}} \text{ if } n \geq a \tag{66}$$

and this is true for any n larger than a.
So taking the limit

$$(1 - (a_1 t^2 + a_2 t^4 + a_3 t^6 \cdots)/n)^n = e^{-f(t^2)} \text{ as } n \to \infty \tag{67}$$

we get

$$F(s) = \mathcal{C}_1 e^{-f(t^2)}(s) \in \mathcal{P}_{\text{pos}} \tag{68}$$

and in fact

$$\mathcal{C}_1 e^{-f(t^2)}(s) \to \mathcal{C}_1 t^2 e^{-f(t^2)}(s) \tag{69}$$

So provided that any terms of the series expansion of $e^{-f(t^2)}$ of $1 - t^{2n}$ don't fall in $[-1, 1]$ the full family is preserved.

$\square$

10 To Infinity and Beyond

In this section we examine what happens to the interlacing structure of our families as we extend the domain of integration from $[-1, 1]$ to $[-\infty, \infty]$. We discover that *global mutual interlacing* is destroyed, though *sequential interlacing* survives and some families do survive.

10.1 On the Infinite Domain: The Damage

Theorem 10.1

All is not lost though. Using the above techniques we arrive at another proof of Polya's result that $\mathcal{C}_\infty e^{-t^{2k}}(s)$ has all real distinct zeros in $\mathcal{P}_{\text{pos}}$ and that they interlace with $\mathcal{C}_\infty t^2 e^{-t^{2k}}(s)$. We now show that $\mathcal{C}_\infty e^{-t^{2k}}(s)$ has a family size of $k+1$.

Proof. Moving back to the bilateral Laplace, lets consider

$$\frac{\partial}{\partial s} \int_{-\infty}^{\infty} \exp(-st) \frac{\partial}{\partial t} \exp\left(-t^{2k}\right) \, dt \tag{70}$$

which we can express as

$$-2k \int_{-\infty}^{\infty} e^{-t^{2k}} t^{2k} e^{-st} \, dt \tag{71}$$

The operator we have applied is Ds so this clearly preserves interlacing so back in CMST we have

$$\mathcal{C}_\infty[e^{-t^{2k}}](s) \to \mathcal{C}_\infty[t^{2k} e^{-t^{2k}}](s) \tag{72}$$

and since we know that

$$\mathcal{C}_\infty[e^{-t^{2k}}](s) \to \mathcal{C}_\infty[t^2 e^{-t^{2k}}](s) \tag{73}$$

and that t^2 preserves interlacing we have a family back of mutually interlacing functions.

$$\mathcal{C}_\infty[e^{-t^{2k}}](s) \to \mathcal{C}_\infty[t^2 e^{-t^{2k}}](s) \to \cdots \to \mathcal{C}_\infty[t^{2k} e^{-t^{2k}}](s) \tag{74}$$

which is of size $k+1$ $\qquad\qquad\square$

Remark 10.2

The kernel $e^{-t^{2k}}$ serves as a critical benchmark for the stability of the CMST on infinite domains. It demonstrates that the 'Assassin' mechanism does not induce a catastrophic collapse of the zero set. Instead, it imposes a finite defect, truncating the mutually interlacing family to a precise size of k+1.

Remark 10.3

$$\lim_{n \to \infty} e^{-t^{2n}} \tag{75}$$

is basically compact support so we can see some of the differences between compact support and infinite support. Compact support means that we can have infinite size families and often do. Infinite support means finite size families, and indeed in order to converge we need an "Assassin" term that limits the size of the family.

11 The Gaussian Limit

11.1 The Vanishing Act: Limit of the Finite Domain

The Gaussian kernel arises as the singular limit of the polynomial family defined on compact domains. Consider the standard generator sequence from the "Assassin" class:

$$f_N(t) = \left(1 - \frac{t^2}{N}\right)^N, \quad t \in [-\sqrt{N}, \sqrt{N}]. \tag{76}$$

The zeros of $f_N(t)$ are located at $t = \pm\sqrt{N}$. As $N \to \infty$, these zeros migrate to infinity, leaving the limit function devoid of finite roots:

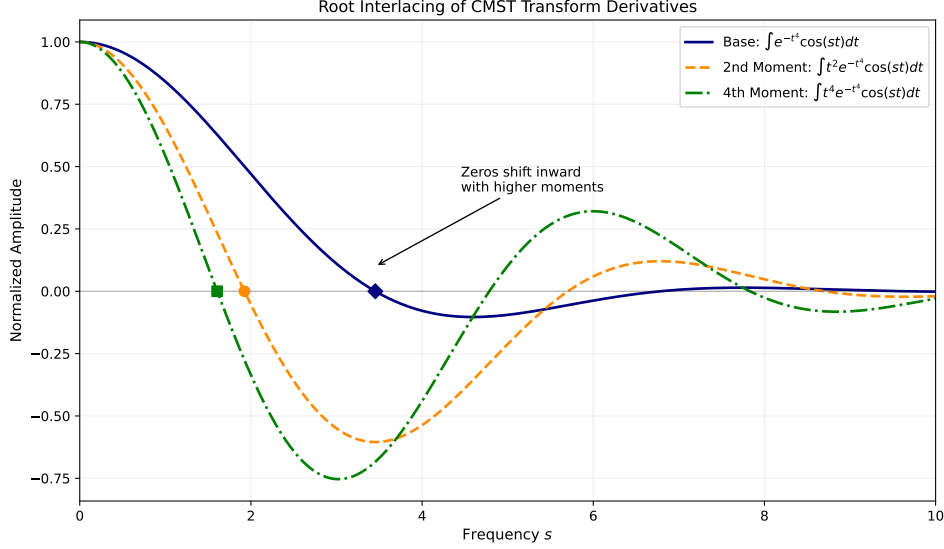$$\lim_{N \to \infty} f_N(t) = e^{-t^2}. \tag{77}$$

Figure 1: **Visualisation of the family root-interlacing property.** The zeros of the Fourier transform of the moments $t^k e^{-4}$ (Order 2 and 4) strictly interlace with the zeros of the base function (Order 0).

The Gaussian is on the limit of zero destruction. As we increase the range of the integral it manages to destroy zeros at the same rate so that by ∞ they are all gone. Maybe of interest, but on any finite range it has an infinite number of real zeros.

The Gaussian is an entire function of order 2 that has no zeros. It can pull out a set of functions for which $f(n+1) \bowtie f(n)$. This is also the only place in this paper where you can actually see the $\bowtie$ working.

$$f(n) = \mathcal{C}_\infty t^{2n} e^{-t^2}(s) \in \mathcal{P}_{\text{pos}} \tag{78}$$

which is a modified form of the even Hermite polynomials.

Remark 11.1

Note we can do this with other exponential terms and get families of mutually interlacing polynomials.

$$\mathcal{C}_\infty t^{2n} e^{-t^{2k}}(s) \in \mathcal{P}_{\text{pos}} \tag{79}$$

resulting in functions with infinite numbers of zeros with $f(n) \to f(n+1)$ and a family of rolling mutually interlacing functions of size $k+1$

12 Asymptotic Distribution of Zeros

This section is all known theory based on the works of *E. C. Titchmarsh*, *G. Pólya* and *F. W. J. Olver*

We wont be using these results further on, but if you are here, looking at zeros, their distribution is probably of interest.

If we think of

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{st} \, dt \tag{80}$$

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where $f(t)$ is very log-concave then if we normalise the integral we get a sharper and shaper peak, approaching a Dirac delta function at the point where $\log f'(t) = -s$. Lets call point that $z(s)$ This gives us that

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{st}\,dt \sim \lambda \int_{-\infty}^{\infty} \delta(t - z(s))f(z)e^{st} = \lambda e^{sz(s)} \tag{81}$$

Where λ is just a normalisation constant.

So

$$F(s) + F(-s) \sim \lambda \cosh(sz) \tag{82}$$

By performing the substitution $s \to is$, we move to the cosine series:

$$F(is) + F(-is) = \int_{-\infty}^{\infty} f(t)\cos st\,dt \sim \lambda \cos(sz(s)) \tag{83}$$

Remark 12.1
Asymptotically we should expect the function to behave like $\cos(sz(s))$ This approximation should get better and better for higher values of s as the function becomes more Dirac delta like

Remark 12.2
We wont be proving this, but the second derivative looks very like a term that would interlace with the original function. We almost get the Wronskian and I think the bilateral Laplace transform preserves the Wronskian, but thats for another day.

$$\left(z'(s) + z(s)\right)^2 \cos(sz(s)) + \sin(sz(s))\left(sz''(s) + 2z'(s)\right) \tag{84}$$

Remark 12.3
It becomes very clear if you try to numerically integrate functions like

$$\int_0^{\infty} e^{st - \cosh(t)}\,dt \tag{85}$$

near say $s = 100$ where the normalisation term is of the order of $\sim 10^{274}$ and the function becomes really sharp. And since its Dirac delta like, we get that the terms in s have the ratio of roughly the saddle point and since the shape is skewed to the high side each term will increase in ratio over the previous terms. This results in the terms having a strict Log Concavity. This is not meant to be rigourous, just an insight into whats going on with large exponential type functions.

13 Beyond

1. The Role of Parity in the Transform

So far, our analysis has primarily focused on even functions, $f(t) = f(-t)$, due to their algebraic simplicity. For the linear action of the operator $\mathcal{C}_a$, non-even functions can often be simplified because the operator annihilates the odd component. Specifically, if we decompose any function into even and odd parts, the odd component vanishes under the transform:

$$\mathcal{C}_a\left[f(t) - f(-t)\right](s) \equiv 0. \tag{86}$$

However, linearity is not the only concern. To build this theory, we must consider the *multiplicative structure* of the family. When multiplying two non-even functions $f(t)$ and $g(t)$, the odd components interact to produce a new even component that survives the transform:

$$f(t)g(t) = \underbrace{(f_{\text{even}}g_{\text{even}} + f_{\text{odd}}g_{\text{odd}})}_{\text{Surviving Even Part}} + \underbrace{(f_{\text{even}}g_{\text{odd}} + f_{\text{odd}}g_{\text{even}})}_{\text{Vanishing Odd Part}}. \tag{87}$$

Therefore, to ensure that the product $f(t)g(t)$ preserves interlacing, we must impose conditions on the coefficients of the full expansion, not just the even ones.

2. The Positive Coefficient Condition

We established previously that even functions of the form:

$$f(t) = \sum_{n=0}^{\infty} a_n t^{2n}, \quad a_n \geq 0 \tag{88}$$

preserve the interlacing property of the kernel, mapping to $\mathcal{C}_a f(t)(s)$. We can extend this to general power series. Consider:

$$f(t) = \sum_{n=0}^{\infty} a_n t^n, \quad a_n \geq 0. \tag{89}$$

Under the action of $\mathcal{C}_a$, all odd powers vanish, leaving only the sum over even powers with positive coefficients. Thus, strict positivity of the Maclaurin coefficients is a sufficient condition for preserving interlacing. Consequently, if $f(t)$ and $g(t)$ possess non-negative coefficients, their product $f(t)g(t)$ will also possess non-negative coefficients (by Cauchy convolution), preserving the "safe" status of the function.

Remark 13.1
Note that this works for negative odd coefficients as well, we just have to make sure that we don't multiply a function with negative odd coefficents with a function with positive odd coefficients.

3. Connection to Stability (Hermite-Biehler)

This condition links directly to the **Hermite-Biehler Theorem**. If a polynomial (or entire function) $f(t)$ has all its zeros in the left half-plane (including the imaginary axis, $\text{Re}(t) \leq 0$), it implies a specific structure in its coefficients. For real polynomials, strict Hurwitz stability implies that the coefficients are all of the same sign (typically positive). Thus, functions in the Hurwitz class serve as valid generators for our sequence.

Remark 13.2
This can be seen in the expansion of

$$\left(1 + \frac{s}{a - ib}\right)\left(1 + \frac{s}{a + ib}\right) \tag{90}$$

which is

$$\frac{a^2 + 2as + b^2 + s^2}{a^2 + b^2} \tag{91}$$

with the only odd term being $2as$. So any product of these terms will have all positive coefficients.

4. Building the Tower: Exponentials and Double Exponentials

We begin with the elementary exponential functions:

1. $f(t) = e^t$: All coefficients are strictly positive. Stable.

2. $f(t) = e^{-t}$: Even function $\cosh(t)$, odd function $\sinh(t)$ valid class.all zeros on the imaginary axis and interlacing.

Remark 13.3
e^{-t} has negative odd coefficients so we should really only combine it with operators with negative odd coefficients

Remark 13.4
Note: While e^t looks like a shift operator, in the context of CMST it behaves differently.

By taking the composition limit, we can construct the double exponential:

$$\lim_{n \to \infty} \left(1 - \frac{e^{-t}}{n}\right)^n = e^{-e^{-t}}.$$

(92)

Since this is derived from stability-preserving operations, $e^{-e^{-t}}$ is an interlacing-preserving function.

Remark 13.5

$e^{-e^{-t}}$ doesnt really have any zeros and if it does they are at $-\infty$, so we are in the stable region.

By extension, the following bounded forms are also interlacing preserving:

$$1 - e^{-e^{-t}} > 0 \quad \text{and} \quad 1 + e^{-e^{-t}} > 0$$

(93)

Lemma 13.6 (CMST Property of the Operator)

If $f(t) \in CMST$ then so is $\lambda^2 f''(t) - f(t)$ provided its ≥ 0 We note that

$$\frac{\int_{-\infty}^{\infty} f(t)(\lambda s + 1) \cosh\left(\sqrt{s}t\right) dt}{\lambda s + 1} = \int_{-\infty}^{\infty} f(t) \cosh\left(\sqrt{s}t\right) dt$$

(94)

so we havent really done anything. But another expression of this via the bilateral Laplace transform is

$$\frac{\int_{-\infty}^{\infty} \cosh\left(\sqrt{s}t\right) \left(\lambda^2 f''(t) - f(t)\right) dt}{\lambda s + 1}$$

(95)

provided that $\left(\lambda^2 f''(t) - f(t)\right) > 0$ if needed we can multiply by $(1 + \lambda s)$ to get a function in $\mathcal{P}_{\text{pos}}$

14   Interlacing of Zeros for Convex Kernels

Theorem 14.1 (Interlacing Property)

Let $g(t)$ be a real-valued function on $[0, 1]$ satisfying the following conditions:

1. Boundary Conditions: $g(0) = 0$ and $g'(0) = 0$.

2. Growth: $g(1) > 0$ and $g'(1) > 0$.

3. Convexity: $g(t)$ is strictly convex, i.e., $g''(t) > 0$ for $t \in (0, 1]$.

Then, the zeros ρ_n of the finite Fourier transform

$$F(s) = \int_0^1 g(t) \cos(st) dt$$

are all real and strictly interlace with the zeros of the Sinc function, $z_n = n\pi$. Specifically, for large n, each zero lies in the interval:

$$n\pi - \frac{\pi}{2} < \rho_n < n\pi$$

Proof. We analyze the asymptotic behavior of $F(s)$ by performing integration by parts twice to isolate the boundary contributions.

1. Expansion via Integration by Parts

Applying integration by parts to $F(s)$:

$$F(s) = \left[g(t) \frac{\sin(st)}{s} \right]_0^1 - \frac{1}{s} \int_0^1 g'(t) \sin(st)\, dt \tag{96}$$

Using the condition $g(0) = 0$:

$$F(s) = g(1) \frac{\sin(s)}{s} - \frac{1}{s} \int_0^1 g'(t) \sin(st)\, dt$$

Applying integration by parts a second time to the integral term:

$$\int_0^1 g'(t) \sin(st)\, dt = \left[-g'(t) \frac{\cos(st)}{s} \right]_0^1 + \frac{1}{s} \int_0^1 g''(t) \cos(st)\, dt$$

Using the condition $g'(0) = 0$, the lower boundary term vanishes. Substituting back into the expression for $F(s)$, we obtain the master equation:

$$F(s) = g(1) \frac{\sin(s)}{s} + g'(1) \frac{\cos(s)}{s^2} - \frac{1}{s^2} \underbrace{\int_0^1 g''(t) \cos(st)\, dt}_{R(s)} \tag{97}$$

2. Sign Analysis at the Grid Points $s = n\pi$

We evaluate $F(s)$ at the zeros of the Sinc function, $s_n = n\pi$ for integers $n \geq 1$. Note that $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$. Equation (97) becomes:

$$F(n\pi) = \frac{(-1)^n}{(n\pi)^2} \left[g'(1) - (-1)^n \int_0^1 g''(t) \cos(n\pi t)\, dt \right]$$

By the Fundamental Theorem of Calculus, $g'(1) = \int_0^1 g''(t)\, dt$. Since $g''(t) > 0$, we have the strict inequality:

$$\left| \int_0^1 g''(t) \cos(n\pi t)\, dt \right| < \int_0^1 g''(t)\, dt = g'(1)$$

Consequently, the term in the square brackets is strictly positive. The sign of $F(n\pi)$ is determined solely by the pre-factor:

$$\mathrm{sgn}(F(n\pi)) = (-1)^n \tag{98}$$

3. Sign Analysis at the Midpoints $s = n\pi - \pi/2$

Let $s^* = n\pi - \frac{\pi}{2}$. At this point, $\cos(s^*) = 0$ and $\sin(s^*) = (-1)^{n+1}$. Using the first-order expansion of $F(s)$:

$$F(s^*) = \frac{(-1)^{n+1}}{s^*} \left[g(1) - (-1)^{n+1} \int_0^1 g'(t) \sin(s^* t)\, dt \right]$$

By a similar domination argument, $\left| \int g'(t) \sin(s^* t) \right| < g(1)$, making the bracket term strictly positive. Thus:

$$\mathrm{sgn}\left(F\left(n\pi - \frac{\pi}{2} \right) \right) = (-1)^{n+1} = -(-1)^n \tag{99}$$

4. Conclusion

Comparing the signs at the endpoints of the interval $I_n = (n\pi - \frac{\pi}{2}, n\pi)$:

- At the left endpoint, the sign is $-(-1)^n$.

- At the right endpoint, the sign is $(-1)^n$.

Since $F(s)$ is continuous, by the Intermediate Value Theorem, there must exist at least one real zero $\rho_n \in I_n$. This establishes that the zeros of $F(s)$ are real and strictly interlace with the zeros of the canonical Sinc function. $\qquad \square$

15 More generally Log concavity produces real distinct zeros

Theorem 15.1
For functions of the form $e^{-f(t^2)}$ where $f(t)$ is convex

$$\mathcal{C}_a e^{-f(t^2)} \in \mathcal{P}_{\text{pos}} \tag{100}$$

Proof. We have that

$$\mathcal{C}_1(1 - f(t^2)) \in \mathcal{P}_{\text{pos}}, \quad \text{provided } 1 - f(t^2) \geq 0 \in [-1, 1] \tag{101}$$

We then take the limit of $(1 - f(t^2)/n)^n$ as $n \to \infty$ and extend the range.

While Log-Concavity is preserved in the limit, the existence of zeros requires the kernel to be "stiffer" than a Gaussian. The strict convexity condition $f''(t) \geq 0$ ensures the kernel decays like $e^{-t^{(2+\epsilon)}}$ (or faster), which satisfies the Newman condition for infinite real zeros [12]

$\square$

Theorem 15.2
For functions of the form $e^{-f(t^2)}$ where $f(t)$ is convex

$$\mathcal{C}_a e^{-f(t^2)} \leftrightarrow \mathcal{T}_{\text{rad}} f'(\mathcal{T}_{\text{rad}}) \mathcal{C}_a e^{-f(t^2)} \tag{102}$$

Proof. That is simply the result of applying sD , the Euler operator, to the transform, and moving the t^2 terms outside to the integral to their $\mathcal{T}_{\text{rad}}$ equivalents. $\square$

Remark 15.3
Note that the the terms of $f'(\mathcal{T}_{\text{rad}})$ are positive. This shifts the zeros to the left which means that these functions may not be $\in CMST$

16 The CMST Window

We consider the specific kernel defined by the infinite product of exponential monomials, which corresponds to the following compact-support function on $[-1, 1]$:

$$\Phi(t) = \prod_{n=0}^{\infty} e^{-t^{2nk}} = \exp\left(\frac{1}{t^{2k} - 1}\right), \quad t \in (-1, 1) \tag{103}$$

and $\Phi(t) = 0$ for $|t| \geq 1$. This function is C^{∞} and vanishes with all its derivatives at the boundaries $t = \pm 1$.

Theorem 16.1 (CMST Membership of the Geometric Kernel)
The function $\Phi(t) = \exp\left(\frac{1}{t^{2k} - 1}\right)$ belongs to the CMST class on the interval $[-1, 1]$. Consequently, its CMST $F_n(s) = \mathcal{C}_1[t^{2n}\Phi(t)](s)$ is a interlacing family in $\mathcal{P}_{\text{pos}}$

Proof. First, we observe the power series expansion of the exponent for $t \in (-1, 1)$:

$$\frac{1}{(at)^{2k} - 1} = -\sum_{n=0}^{\infty}(at)^{2nk} = -(1 + (at)^{2k} + (at)^{4k} + \dots) \tag{104}$$

Which converges on $(-1, 1)$ for $0 < a < 1$ The kernel can be expressed as $e^{-f(t^2)}$ where $f(t) = \sum_{n=0}^{\infty} a_n t^n$ with $a_n < 1$ for all n. We can then take the limit as $a \to 1$ which for $e^{-f(t^2)}$ is finite. $\square$

Remark 16.2 (The Infinite Product of Hyper-Gaussians)
The factorization

$$\Phi(t) = \prod_{n=0}^{\infty} e^{-t^{2nk}} = \exp\left(\frac{1}{t^{2k}-1}\right) = e^{\sum_{n=0}^{\infty} t^{2nk}} \tag{105}$$

reveals a fundamental structural property of the CMST window. Each factor $e^{-t^{2nk}}$ represents a "Hyper-Gaussian" of order $2nk$. This implies that the window $\Phi(t)$ can be interpreted as the infinite intersection of all Hyper-Gaussian stability classes.

This product structure is analogous to the Euler product of the Zeta function, where the additive structure of the singularity (the geometric series sum in the exponent) generates the multiplicative structure of the function.

$$\zeta(s) = \prod_{p}\left(1-p^{-s}\right)^{-1} = \exp\left(\sum_{k=1}^{\infty}\frac{P(ks)}{k}\right) = \sum_{i=1}^{\infty} i^{-s} \tag{106}$$

Theorem 16.3 (Analytic form for $\int_{-a}^{a}\frac{\cos(st)}{1-t^2}\,dt$)
We show that

$$\int_{-a}^{a}\frac{\cos(st)}{1-t^2}\,dt = \cos(s)(\mathrm{Ci}((a+1)s)-\mathrm{Ci}((1-a)s))+\sin(s)(\mathrm{Si}((a-1)s)+\mathrm{Si}((a+1))s) \tag{107}$$

Where $\mathrm{Si}(x)$ is the Sine Integral

$$\mathrm{Si}(x) = \int_{0}^{x}\frac{\sin(t)}{t}\,dt \tag{108}$$

and $\mathrm{Ci}(x)$ is the Cosine Integral

$$\mathrm{Ci}(x) = \gamma + \ln(x) + \int_{0}^{x}\frac{\cos(t)-1}{t}\,dt \tag{109}$$

Proof. We note that

$$\frac{1}{1-t^2} = \frac{t^2}{1-t^2} + 1 \tag{110}$$

giving us

$$\int_{-a}^{a}\frac{\cos(st)}{1-t^2}\,dt = \int_{-a}^{a}\left(\frac{t^2}{1-t^2}+1\right)\cos(st)\,dt \tag{111}$$

If we let

$$\mathcal{F}_a(s) = \int_{-a}^{a}\frac{\cos(st)}{1-t^2}\,dt \tag{112}$$

we get

$$\mathcal{F}_a(s) = -\mathcal{F}''(s) + \frac{2\sin(as)}{s} \tag{113}$$

Which we can solve giving us the general solution

$$\mathcal{F}_a(s) = \cos(s)(-\mathrm{Ci}((1-a)s)+\mathrm{Ci}((a+1)s)+c_1)+\sin(s)(\mathrm{Si}((a-1)s)+\mathrm{Si}((a+1)s)+c_2) \tag{114}$$

Noting that the function is even and equating coefficients gives us $c_1 = 0$, $c_2 = 0$ which gives us our final result

$$\mathcal{F}_a(s) = \cos(s)(-\mathrm{Ci}((1-a)s)+\mathrm{Ci}((a+1)s))+\sin(s)(\mathrm{Si}((a-1)s)+\mathrm{Si}((a+1)s)) \tag{115}$$

$\square$

Theorem 16.4 ($\lim_{a\to 1}\mathcal{F}_a(s)$)

Proof.

$$\int_{-a}^{a} \frac{\cos(st)}{1 - t^2} \, dt \tag{116}$$

does not coverge as $a \to 1$ however if we use the substitution $u = it$ the integral does converge giving us the analytic continuation.

$$I(s) = \int_{-1}^{1} \frac{\cosh(st)}{1 + t^2} \, dt \tag{117}$$

$$I(s) = i \cosh(s) \left[\text{Shi}\left(s(1 - i)\right) - \text{Shi}\left(s(1 + i)\right) \right] - \sinh(s) \left[\text{Chi}\left(s(1 - i)\right) - \text{Chi}\left(s(1 + i)\right) \right] \tag{118}$$

Which with some rearranging gives

$$I(b) = \frac{\pi}{2} \cosh(b) - 2b \int_{0}^{1} \sinh(bt) \arctan(t) \, dt \tag{119}$$

$\square$

Theorem 16.5 ($\int_{-1}^{1} \exp\left(\frac{1}{t^{2k}-1}\right) cos(st), dt = I(\imath s) * \int_{-1}^{1} \exp\left(\frac{1}{t^{2k}-1}\right) cos(st) \, dt$)

Proof. Let

$$w(t) = e^{-f(t)} \tag{120}$$

then

$$w'(t) = -f'(t)e^{-f(t)} \tag{121}$$

Taking the Fourier transform of both sides give

$$\imath s W(s) = -F(f'(t)) * W(s) \tag{122}$$

where $*$ is convolution If we now use $f(t) = \frac{1}{t^{2k}-1}$ we get

$$\imath s W(s) = -\imath \, s \, I(\imath s) * W(s) \tag{123}$$

or

$$W(s) = -\, I(\imath s) * W(s) \tag{124}$$

$\square$

Remark 16.6
This result reveals that the CMST window is a 'spectral fixed point.' It demonstrates that the function's C_∞ smoothness is not an arbitrary construction, but a solution to a convolution equation generated by the geometry of its own boundary. Boundaries usually create noise (ringing/Gibbs phenomenon) because the function hits a wall, this window effectively eliminates that.

Remark 16.7
The $e^{t^{2k}}$ operators are benign on families of size $k + 1$ so we can flatten this function with

$$e^{1+t^{2k}+\frac{1}{t^{2k}-1}} \tag{125}$$

and produce a tunable window suitable for signal processing with compact support and good zero preserving properties.

The kernel CMST window exhibits a distinct advantage over classical window functions (e.g., Kaiser-Bessel, Blackman-Harris). Standard windows typically possess only finite regularity (C^n

continuity) at the boundaries, leading to spectral sidelobes that decay polynomially ($\mathcal{O}(\omega^{-(n+1)})$). In contrast, the CMST kernel is C_c^∞ (smooth with compact support). This smoothness guarantees super-algebraic decay of the Fourier transform side-lobes. As a result, while the main-lobe width (resolution) is comparable to a Kaiser window, the dynamic range (sidelobe suppression) is vastly superior, avoiding the "spectral floor" caused by boundary discontinuities. Windows like the Planck window while smooth at $t = 1$ introduce discontinuities elsewhere.

17 Differential Structure and Efficient Computation of the CMST Window

While the integral definition of the CMST window $\Phi(t) = \exp\left(\frac{1}{t^2-1}\right)$ ensures its theoretical properties in C_c^∞, its practical application in digital signal processing (such as FIR filter design) benefits from a differential formulation. This section derives the governing differential equations in both the time and spectral domains, yielding a stable recurrence relation for high-precision generation of the window's transform.

17.1 The Time-Domain Governing Equation

Differentiating the kernel $\Phi(t)$ with respect to t reveals a distinct non-linear relationship.

$$\Phi'(t) = \frac{d}{dt}\exp\left(\frac{1}{t^2-1}\right) = \frac{-2t}{(t^2-1)^2}\Phi(t) \tag{126}$$

Rearranging terms eliminates the singularity in the denominator and linearizes the expression, yielding the defining Ordinary Differential Equation (ODE) for the window:

$$(1-t^2)^2\Phi'(t) + 2t\Phi(t) = 0 \tag{127}$$

This equation encodes the boundary behavior: the term $(1-t^2)^2$ enforces a "soft landing" at $t = \pm 1$, ensuring that the derivative vanishes faster than the function itself, a necessary condition for the preservation of smoothness.

17.2 The Spectral Differential Equation

To determine the behavior of the Fourier transform $F(s) = \mathcal{C}[\Phi](s)$, we map the time-domain operators of eq. (127) to the frequency domain using the standard correspondences $\Phi'(t) \leftrightarrow sF(s)$ and $t^n(\cdot) \leftrightarrow i^n D^n(\cdot)$. Expanding the polynomial term $(1-t^2)^2 = 1 - 2t^2 + t^4$, the transformed operator acts on the product $sF(s)$:

$$\left(1 + \frac{d^2}{ds^2}\right)^2[sF(s)] + 2F'(s) = 0 \tag{128}$$

Expanding the differential operator yields a fourth-order linear differential equation with polynomial coefficients:

$$sF''''(s) + 4F'''(s) + 2sF''(s) + 6F'(s) + sF(s) = 0 \tag{129}$$

17.3 Power Series Solution and Recurrence Relation

Since $\Phi(t)$ is an even function, $F(s)$ admits a power series expansion of the form:

$$F(s) = \sum_{k=0}^{\infty} a_k s^{2k} \tag{130}$$

Substituting this series into eq. (129) and matching coefficients of powers of s yields a recurrence relation that allows for the precise computation of the transform without numerical integration. For $k \geq 1$:

$$a_{k+1} = -\frac{8k(k+1)a_k + a_{k-1}}{4k(k+1)(2k+1)(2k+3)} \tag{131}$$

Since the ratio $a_1/a_0 := \psi$ define the whole series, it would be nice to get an expression for this ratio.

Derivation of ψ

We define the two components of the recurrence denominator to distinguish the scaling logic from the geometric kernel:

- **Scaling Factor:** $\Lambda_k = 4k(k+1)$

- **Geometric Factor:** $\Omega_k = (2k+1)(2k+3)$

1. The Recurrence Relation

The recurrence relation for the coefficients a_k of the Cosine transform is given by:

$$a_{k+1} = -\frac{2\Lambda_k a_k + a_{k-1}}{\Lambda_k \Omega_k} \tag{132}$$

2. Rearranging for the Ratio

To find the ratio of successive moments $r_k = \frac{a_k}{a_{k-1}}$, we multiply by the denominator $\Lambda_k \Omega_k$ and isolate the terms:

$$\Lambda_k \Omega_k a_{k+1} = -2\Lambda_k a_k - a_{k-1} \tag{133}$$

Dividing the entire equation by a_k:

$$\Lambda_k \Omega_k \left(\frac{a_{k+1}}{a_k} \right) = -2\Lambda_k - \frac{a_{k-1}}{a_k} \tag{134}$$

Substituting $r_{k+1} = \frac{a_{k+1}}{a_k}$ and noting that $\frac{a_{k-1}}{a_k} = \frac{1}{r_k}$:

$$\Lambda_k \Omega_k r_{k+1} = -2\Lambda_k - \frac{1}{r_k} \tag{135}$$

3. The Recursive Step

Isolating the reciprocal ratio $\frac{1}{r_k}$ and factoring out the common scaling term Λ_k:

$$\frac{1}{r_k} = -\Lambda_k \left[2 + \Omega_k r_{k+1} \right] \tag{136}$$

Taking the reciprocal yields the fundamental step for the continued fraction:

$$r_k = \frac{-1}{\Lambda_k \left[2 + \Omega_k r_{k+1} \right]} \tag{137}$$

4. Continued Fraction Expansion

By iterating this step for $k = 1, 2, 3, \ldots$, we obtain the continued fraction for the first moment ratio $\frac{a_1}{a_0}$:

$$\psi = \frac{a_1}{a_0} = \cfrac{-1}{\Lambda_1 \left[2 + \Omega_1 \cfrac{-1}{\Lambda_2 \left[2 + \Omega_2 \cfrac{-1}{\Lambda_3 [2 + \ldots]} \right]} \right]} \tag{138}$$

For a continued fraction, this converges really slowly to a value $\psi \approx -0.0790568$. I cant find any other relationships with this number, but it generates our whole FIR and so it is important here.

5. First few terms

So if we normalise the first two terms to be 1 and ϕ we get the following.

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= \phi \\
a_2 &= \frac{-1 - 16\phi}{120} \\
a_3 &= \frac{2 + 27\phi}{4200} \\
a_4 &= \frac{-157 - 2032\phi}{12,700,800} \\
a_5 &= \frac{10,360 + 133,735\phi}{44,008,272,000}
\end{aligned}
\tag{139}
$$

Remark 17.1 (From the moments of the transform to moments of the function)

$$a_n = \frac{(-1)^n}{(2n)!} \frac{\mu_{2n}}{\mu_0} \tag{140}$$

Rearranging for the moment μ_{2n}

$$\mu_{2n} = (-1)^n (2n)! \, a_n \, \mu_0 \tag{141}$$

Noting that μ_0 has been normalised

Theorem 17.2 (Analytic Representation of the Normalization Constant)
The area M_0 of the standard bump function $\psi(t) = \exp\left(\frac{-1}{1-t^2}\right)$ is determined analytically by its peak value $1/e$ and the spectral shape factors:

$$M_0 = \frac{1}{e} \cdot \left[\sum_{k=0}^{\infty} (-1)^k \frac{4k+1}{2} P_{2k}(0) \sum_{j=0}^{k} d_{k,j} \frac{M_{2j}}{M_0} \right]^{-1} \tag{142}$$

where

$$d_{k,j} = \frac{(-1)^{k-j}(2k+2j)!}{2^{2k}(k-j)!(k+j)!(2j)!} \tag{143}$$

and $P_{2k}(0)$ are the Legendre polynomials at the origin, and the moment ratios $\mu_{2j} = M_{2j}/M_0$ are rational numbers derived from the recurrence relation induced by the differential equation $(1 - t^2)^2 \psi' + 2t\psi = 0$:

$$(n + 4)M_{n+3} - (2n + 6)M_{n+1} + nM_{n-1} = 0 \tag{144}$$

When we plug in

$$\mu_{2n} = (-1)^n (2n)! \, a_n \, \mu_0 \tag{145}$$

we get

$$M_0 = \frac{1}{e} \cdot \left[\sum_{k=0}^{\infty} \frac{4k + 1}{2^{4k+1}} \binom{2k}{k} \sum_{j=0}^{k} \frac{(2k + 2j)!}{(k - j)!(k + j)!} \, a_j \right]^{-1} \tag{146}$$

or

$$\int_{-1}^{1} \exp\left(\frac{1}{t^2 - 1}\right) dt = \frac{1}{e} \cdot \left[\sum_{k=0}^{\infty} \frac{4k + 1}{2} \sum_{j=0}^{k} \left[\frac{\binom{2k}{k}}{2^{2k}} \cdot \frac{\binom{2k}{k-j}}{2^{2k}} \right] \binom{2k + 2j}{2k} (2j)! \, a_j \right]^{-1} \tag{147}$$

which is the intergral of the Schwartz function. [15]

Remark 17.3

Note that numeric convergence of the above series is hugely dependant on the accuracy of ϕ. A 7 decimal place accurate value for ϕ blows up around the 5th term for instance.

18 Spectral Decay Analysis

As described by Boyd [14], the rate of convergence of the Fourier coefficients is governed by the smoothness of the function at its boundaries...

Theorem 18.1 (Super-Algebraic Decay)
Let $\Phi(t) = \exp\left(\frac{1}{t^2 - 1}\right)$ for $|t| < 1$ and 0 otherwise. The Fourier transform $\hat{\Phi}(\omega)$ satisfies:

$$\lim_{\omega \to \infty} |\omega|^N \hat{\Phi}(\omega) = 0 \quad \forall N \in \mathbb{N} \tag{148}$$

implying decay faster than any polynomial order (super-algebraic).

Proof. Consider the Fourier integral definition:

$$\hat{\Phi}(\omega) = \int_{-1}^{1} \Phi(t) e^{-i\omega t} dt \tag{149}$$

Integrating by parts once, let $u = \Phi(t)$ and $dv = e^{-i\omega t} dt$:

$$\hat{\Phi}(\omega) = \left[\frac{\Phi(t) e^{-i\omega t}}{-i\omega} \right]_{-1}^{1} + \frac{1}{i\omega} \int_{-1}^{1} \Phi'(t) e^{-i\omega t} dt \tag{150}$$

The boundary term vanishes because $\Phi(\pm 1) = 0$. Repeating this process k times yields:

$$\hat{\Phi}(\omega) = \frac{1}{(i\omega)^k} \int_{-1}^{1} \Phi^{(k)}(t) e^{-i\omega t} dt \tag{151}$$

Since $\Phi(t) \in C_c^{\infty}(-1, 1)$, all derivatives $\Phi^{(k)}(t)$ vanish at the boundaries $t = \pm 1$. The remaining integral is bounded by the maximum of the k-th derivative $||\Phi^{(k)}||_{\infty}$. Thus:

$$|\hat{\Phi}(\omega)| \leq \frac{2||\Phi^{(k)}||_{\infty}}{|\omega|^k} \tag{152}$$

Since this holds for any arbitrarily large integer k, the decay is super-algebraic. $\qquad \square$

18.1 Asymptotic Decay Analysis

While the super-algebraic property guarantees decay faster than any polynomial $O(\omega^{-N})$, it is useful to derive the precise asymptotic envelope. The window function $\Phi(t) = \exp((t^2 - 1)^{-1})$ belongs to the Gevrey class $\mathcal{G}^s$ with index $s = 2$. The bounds on its derivatives satisfy the Gevrey condition:

$$\max_{t\in[-1,1]} |\Phi^{(k)}(t)| \leq C^{k+1}(2k)! \tag{153}$$

for some constant $C > 0$. Using the standard bound for the Fourier transform derived via repeated integration by parts:

$$|\hat{\Phi}(\omega)| \leq \frac{2\max|\Phi^{(k)}(t)|}{|\omega|^k} \approx \frac{2C^{k+1}(2k)!}{|\omega|^k} \tag{154}$$

To find the tightest bound, we minimize the right-hand side with respect to k. Using Stirling's approximation $(2k)! \approx \sqrt{4\pi k}(2k/e)^{2k}$, the term behaves approximately as $(4k^2/\omega)^k$. The minimum occurs when $k \sim \sqrt{\omega}$. Substituting this optimal order back into the bound yields a root-exponential decay envelope:

$$|\hat{\Phi}(\omega)| \leq A\exp\left(-B\sqrt{|\omega|}\right) \tag{155}$$

where A and B are positive constants. This confirms that the spectral leakage decays significantly faster than the polynomial rates of standard cosine-sum windows (e.g., Blackman-Harris), approaching the ideal band-limited behavior.

Theorem 18.2 (Asymptotic Decay Rate)
For the window function $\Phi(t) = \exp((t^2 - 1)^{-1})$, the Fourier transform magnitude behaves asymptotically as:
$$|\hat{\Phi}(\omega)| \sim \mathcal{O}\left(|\omega|^{-3/4}\exp\left(-\sqrt{|\omega|}\right)\right) \quad \text{as } |\omega| \to \infty \tag{156}$$

This establishes the Gevrey index $s = 2$, corresponding to a root-exponential decay rate with coefficient $B = 1$.

Acknowledgments

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Figure 2: **Visualisation spectral leakage vs other filters.**
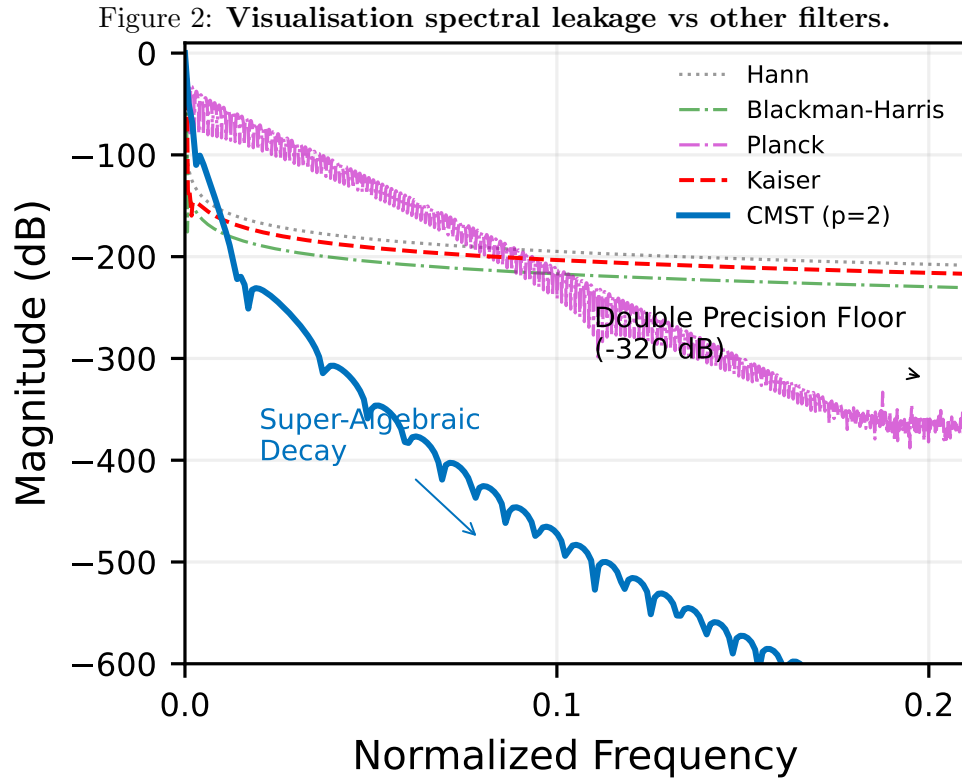
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