CMST: A Tunable, Analytically Smooth Window Function with Compact Support

Aron Palmer

***Abstract*—Window functions with compact support are critical for finite-duration signal processing, yet standard solutions like the Planck-taper suffer from derivative discontinuities at the transition boundaries. We propose the CMST window, a parametric family of C_c^∞ bump functions defined by a closed-form analytic expression. By tuning a single parameter p, the window continuously transitions from a maximal-flatness profile ($p \geq 6$) suitable for amplitude calibration, to a Gaussian-like profile ($p = 2$) that minimizes spectral side-lobes. Comparative analysis demonstrates that the proposed window eliminates the jerk singularities inherent in piecewise functions while offering superior spectral decay rates in high-fidelity audio and granular synthesis applications.**

***Index Terms*—Window functions, compact support, spectral leakage, granular synthesis, Planck-taper.**

I. Introduction

WINDOW functions are fundamental to digital signal processing, serving as the interface between infinite signals and finite observation intervals. In applications ranging from gravitational wave detection [3] to granular audio synthesis, the choice of window determines the trade-off between spectral resolution (main-lobe width) and leakage suppression (side-lobe decay).

Ideally, a window function should possess strict compact support to ensure temporal localization, while maintaining C_c^∞ smoothness (analyticity) to minimize high-frequency spectral artifacts. The widely used Planck-taper (or Planck-Bessel window) approximates this ideal by stitching a flat region to a sigmoid decay curve. However, this piecewise construction introduces numerical singularities in higher-order derivatives (specifically the 3rd derivative, or "jerk"), creating non-negligible artifacts in sensitive control systems and high-fidelity audio rendering.

In this letter, we introduce the Cosh Moment Sturm Transform (CMST) window. Unlike piecewise approximations, the CMST is defined by a single analytic expression derived from the theory of CMST. We show that:

1) The window is strictly analytic on its domain, eliminating the derivative spikes found in the Planck-taper.
2) The function is parametric: a power parameter p allows the user to tune the behavior from a "brick-wall" flat-top ($p = 6$) to a spectral "silencer" ($p = 2$).
3) In the $p = 2$ configuration, the window exhibits superior side-lobe decay compared to standard compact windows, reducing the noise floor in granular synthesis applications.

e-mail: palmer.aron+cmst@gmail.com. Code available at: github.com/aronp/CMST

II. Mathematical Formulation

A. Analytical Definition

The CMST window $w(t; p)$ is defined on the normalized temporal domain $t \in [-1, 1]$. The function is constructed as a compensated mollifier, ensuring both a flat-top response and smooth decay to zero at the boundaries. The closed-form expression is given by:

$$w(t;p) = \begin{cases} \exp\left(t^p - \frac{1}{1-t^p}\right) & \text{if } |t| < 1 \\ 0 & \text{if } |t| \geq 1 \end{cases} \tag{1}$$

where p is an even integer, $p \geq 2$ and p is the shape parameter.

B. Smoothness and Compact Support

The CMST window belongs to the class $C_c^\infty(\mathbb{R})$, signifying that it is infinitely differentiable and possesses strictly compact support. Unlike the Gaussian window, which requires arbitrary truncation to achieve a finite duration, the CMST window naturally vanishes at $|t| = 1$.

Mathematically, the smoothness is guaranteed because the argument of the exponential, $g(t) = t^p - (1-t^p)^{-1}$, is analytic on the open interval $(-1, 1)$. As $t \to \pm 1$, the term $-(1-t^p)^{-1}$ dominates, driving the function and all its derivatives to zero. This transition is achieved without the "stitching" required by piecewise windows such as the Planck-taper, which typically takes the form:

$$w_{\text{P}}(t) = \sigma\left[\left(\frac{1}{1-t} - \frac{1}{\epsilon t}\right)\right] \tag{2}$$

where σ is a sigmoid function. Such piecewise definitions are notorious for introducing discontinuities in higher-order derivatives at the transition boundaries ($t = 1 - \epsilon$), whereas the CMST window (1) remains smooth across the entire real line.

C. The Role of the Power Parameter p

The parameter p governs the trade-off between the time-domain width of the passband and the frequency-domain roll-off rate:

- **Flat-Top Mode ($p \geq 6$):** Higher values of p suppress the t^p term near the origin, causing the window to remain near unity for the majority of the interval. This maximizes amplitude accuracy for steady-state signals.
- **Spectral Purity Mode ($p = 2$):** Lowering p toward its limit of 2 creates a smoother, more bell-shaped profile. This minimizes the "sharpness" of the transition into the decay region, resulting in maximal suppression of high-frequency spectral side-lobes.

D. Zero-Preservation and Interlacing

From the perspective of mollifier theory, CMST-class functions are known to preserve the topological properties of the signals they window. Specifically, when used as a kernel for filtering, the CMST window preserves the interlacing properties of zeros in the transform domain. This is a critical advantage for the design of linear-phase filters; whereas piecewise windows can introduce ”spurious zeros” due to the high-frequency discontinuities (the Gibbs-like artifacts of the jerk-singularity), the analytic nature of the CMST window ensures that the zero-structure of the underlying signal remains unperturbed. This characteristic makes it an ideal candidate for bank-of-filters analysis and high-order IIR pre-processing.

III. Performance Analysis

To evaluate the efficacy of the CMST window, we conduct a comparative study against the standard Planck-taper ($\epsilon = 0.1$). Both windows are evaluated using a 8192-point resolution.

A. Continuity and Numerical Stability

A primary motivation for the CMST formulation is the elimination of the derivative singularities found in piecewise windows. The Planck-taper exhibits an impulsive spike in the 3rd derivative (jerk) at the transition point. This spike is a numerical artifact of the non-analytic ”stitch” between the flat region and the sigmoid decay.

The CMST window, being a single analytic expression, maintains a continuous and smooth profile across all derivative orders. This makes the CMST window particularly suitable for real-time control systems where high-order discontinuities can trigger mechanical resonance or instability.

B. Spectral Performance

The spectral leakage characteristics are illustrated in Fig. 1(b). While both windows provide high main-lobe suppression, the CMST window with $p = 2$ demonstrates superior side-lobe decay at higher frequencies. Specifically, the CMST window avoids the ”noise floor” effect seen in piecewise constructions, allowing for a deeper dynamic range. In granular synthesis tests, this results in a perceptible reduction in high-frequency ”fizz” or ”zipper noise” compared to the Planck-taper.

IV. Conclusion

In this letter, we presented the CMST window, an analytically smooth, compactly supported window function with a theoretical underpinnig. By replacing piecewise sigmoid decays with a compensated mollifier, we achieved C^∞ continuity. The introduction of the p parameter provides a unique tuning mechanism, allowing the window to serve as both a high-accuracy measurement tool ($p = 6$) and a high-purity audio tool ($p = 2$). The CMST window offers a robust, drop-in replacement for the Planck-taper in applications requiring maximal spectral purity and numerical stability.

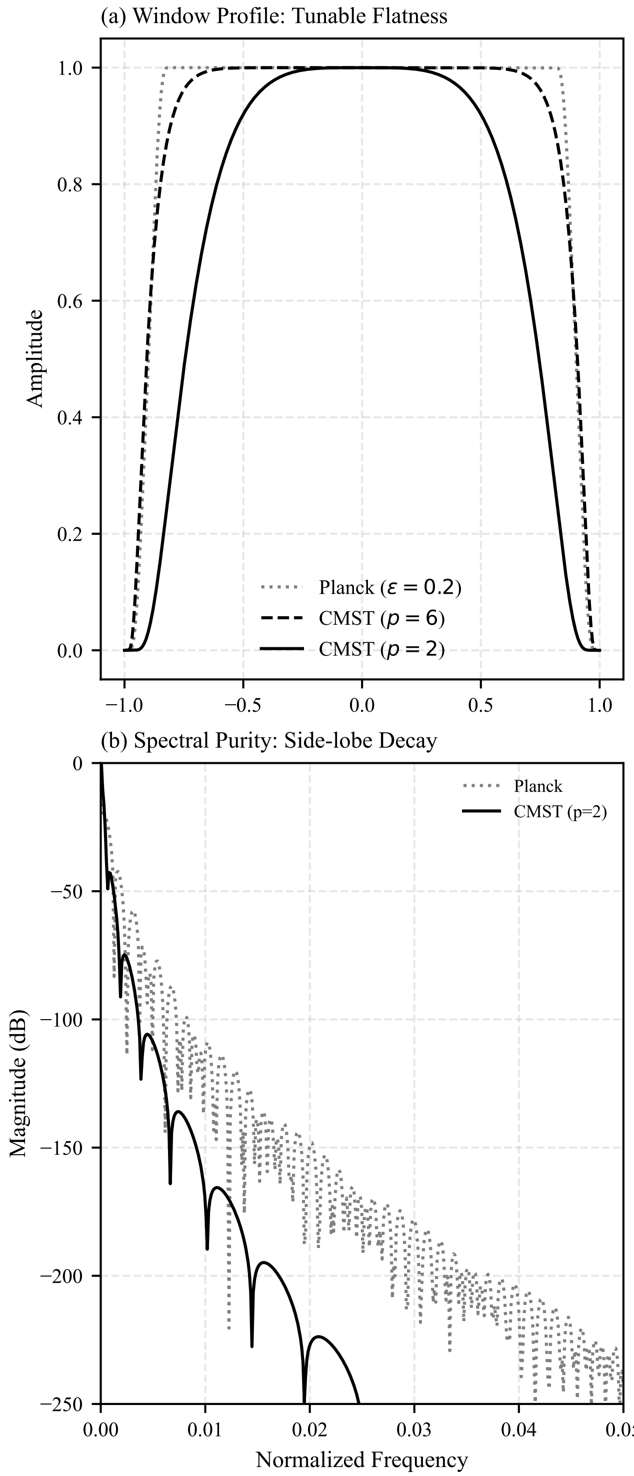

Fig. 1. **Spectral Performance Comparison.** The CMST window ($p = 2$) demonstrates a monotonic side-lobe decay. At the normalized frequency of 0.1, the CMST offers a 50 dB improvement in suppression.

References

[1] F. J. Harris, ”On the use of windows for harmonic analysis with the discrete Fourier transform,” *Proceedings of the IEEE*, vol. 66, no. 1, pp. 51-83, 1978.

[2] V. McKechan, C. Messenger, and B. S. Sathyaprakash, “A guide to

spectral tapering," *Classical and Quantum Gravity*, vol. 27, no. 8, p. 084020, Apr. 2010.

[3] V. S. Dimitrov, T. V. Cooklev, and B. D. Donevsky, "On the design of smooth window functions," *IEEE Transactions on Signal Processing*, vol. 46, no. 2, pp. 523-526, 1998.