COE 321K: Arbitrary N-Dimensional Truss Solver

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1 Introduction

This assignment computes the reaction of an arbitrary N-dimensional truss system to applied forces and constrained displacements with a Python script and auxiliary graphical user interface module. It assumes the joints are pins, that bars are light, and that strains are small and proportional to stresses. It accepts input through files in the form provided in class, and saves results to new files. The product is applied to a two-dimensional truss in Section 4.

1.1 New Features

This program has a few expansions in functionality. It now optionally identifies the critical elements most inclined to yield, crush, or buckle. Note that when this option is enabled, the positively-signed yield stress σ_y, negatively-signed crushing stress σ_c, and cross-sectional second moment of area I must all be provided, space delimited, on a per-element basis at the end of each line in the elements file.

It also now offers a user interface for ease-of-use. Simply running the main script opens a new window, seen in Figure 1. Input files may be selected individually, or automatically with the "Select Directory" button. This presumes that all of the input files are in the directory, named `nodes`, `elements`, `displacements`, and `forces`, and outputs results to the same location. While the program is running, a progress bar will mark the solution's progress, seen in Figure 2.

While this feature has not yet been rigorously tested, it is also now designed to solve systems defined in any number of Euclidian spatial dimensions. This is achieved simply by generalizing the local stiffness matrix, since all other parts of the code were already generalized. See Section 2.1 for the generalized form of the local stiffness matrix.

Figure 1: Startup window before and after selecting options

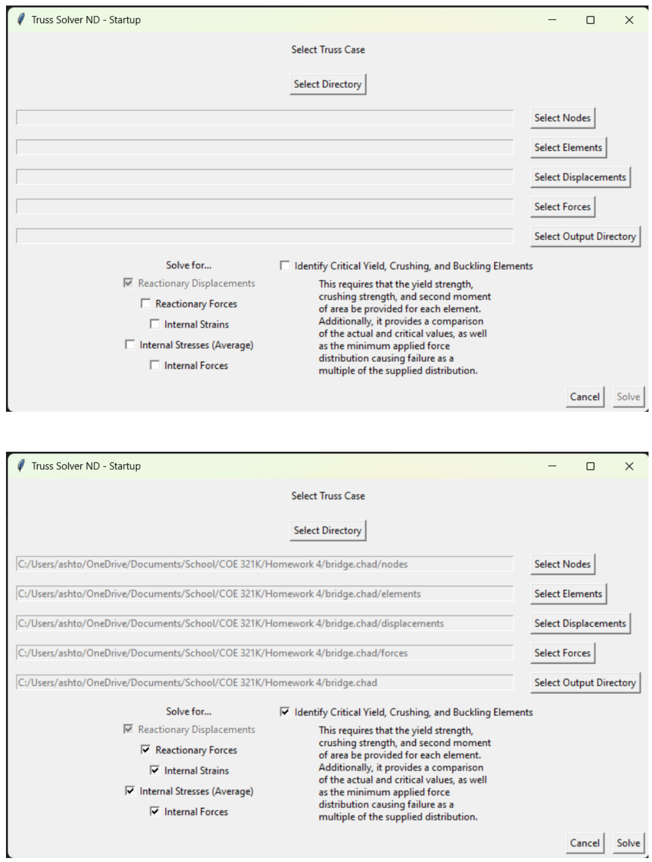
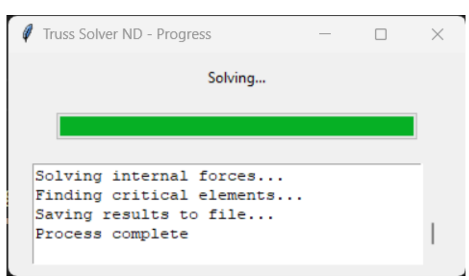


Figure 2: Progress window showing a successful execution



2 Stiffness of Arbitrary 3D Truss Element

We want to find the stiffness matrix for an arbitrary, weightless, pin-ended truss element. In an arbitrary coordinate system, one end is located at (x_1, y_1, z_1), and the other is at (x_2, y_2, z_2). It has a Young's modulus E, and a cross-sectional area A. Its length is simply the Euclidian distance formula.

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The bar can also be considered in another locally-based right-handed coordinate system with one axis a along the axis of the bar, from point 1 to point 2, and two arbitrary transverse axes t and s. In these coordinates, the strain of the bar is approximately linear.

$$\varepsilon_{12} \approx \frac{u_a^{(2)} - u_a^{(1)}}{L}$$

This is because displacements transverse to the bar on the same scale have a far smaller impact on the axial strain of the bar than the axial displacement. From this, we can find the axial force F_{12} in the bar, and infer the axial components of the forces $F_a^{(1)}$ and $F_a^{(2)}$ applied at the nodes.

$$\begin{aligned}
\sigma_{12} &= \frac{F_{12}}{A} = E\varepsilon_{12} \\
F_{12} &= \frac{EA}{L}(u_a^{(2)} - u_a^{(1)}) \\
F_a^{(1)} &= -\frac{EA}{L}(u_a^{(2)} - u_a^{(1)}) \\
F_a^{(2)} &= \frac{EA}{L}(u_a^{(2)} - u_a^{(1)})
\end{aligned}$$

The transverse components of those forces must be zero in order to maintain equilibrium, so, in local coordinates, the stiffness equation is as follows.

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_a^{(1)} \\ u_t^{(1)} \\ u_s^{(1)} \\ u_a^{(2)} \\ u_t^{(2)} \\ u_s^{(2)} \end{bmatrix} = \begin{bmatrix} F_a^{(1)} \\ F_t^{(1)} \\ F_s^{(1)} \\ F_a^{(2)} \\ F_t^{(2)} \\ F_s^{(2)} \end{bmatrix}$$

Now we will try to relate displacements our two coordinate systems. The direction cosines of the bar can be defined as follows.

$$
\begin{aligned}
c_1 &= \frac{x_2 - x_1}{L} \\
c_2 &= \frac{y_2 - y_1}{L} \\
c_3 &= \frac{z_2 - z_1}{L}
\end{aligned}
$$

The axial displacements u_a at each node, being parallel to the bar, can be shown, with simple geometry, to relate to the xyz displacements u, v, and w as follows.

$$u_a = c_1 u + c_2 v + c_3 w$$

Thus, the rotation matrix from xyz to ats is as follows.

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u_a \\ u_t \\ u_s \end{bmatrix}$$

The second and third lines can be left unknown. This is not because u_t and u_s are zero, but because the t and s axes are arbitrary, and their displacements are not needed. They do not contribute to the axial force, and the transverse forces are zero. Similar logic applies when rotating the force coordinates. With all of this, we can find the stiffness equation in global coordninates. We can also leverage the fact that the inverse of a rotation matrix is its transpose.

$$
\frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 & c_2 & c_3 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \end{bmatrix}\begin{bmatrix} u^{(1)} \\ v^{(1)} \\ w^{(1)} \\ u^{(2)} \\ v^{(2)} \\ w^{(2)} \end{bmatrix} =
$$

$$
\begin{bmatrix} c_1 & c_2 & c_3 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \end{bmatrix}\begin{bmatrix} F_x^{(1)} \\ F_y^{(1)} \\ F_z^{(1)} \\ F_x^{(2)} \\ F_y^{(2)} \\ F_z^{(2)} \end{bmatrix}
$$

$$
\frac{EA}{L}\begin{bmatrix} c_1 & & & 0 & 0 & 0 \\ c_2 & & & 0 & 0 & 0 \\ c_3 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & & \\ 0 & 0 & 0 & c_2 & & \\ 0 & 0 & 0 & c_3 & & \end{bmatrix}\begin{bmatrix} c_1 & c_2 & c_3 & -c_1 & -c_2 & -c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & -c_2 & -c_3 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u^{(1)} \\ v^{(1)} \\ w^{(1)} \\ u^{(2)} \\ v^{(2)} \\ w^{(2)} \end{bmatrix} =
$$

$$
\begin{bmatrix} F_x^{(1)} \\ F_y^{(1)} \\ F_z^{(1)} \\ F_x^{(2)} \\ F_y^{(2)} \\ F_z^{(2)} \end{bmatrix}
$$

$$
\frac{EA}{L}\begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 & -c_1^2 & -c_1c_2 & -c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 & -c_1c_2 & -c_2^2 & -c_2c_3 \\ c_1c_3 & c_2c_3 & c_3^2 & -c_1c_3 & -c_2c_3 & -c_3^2 \\ -c_1^2 & -c_1c_2 & -c_1c_3 & c_1^2 & c_1c_2 & c_1c_3 \\ -c_1c_2 & -c_2^2 & -c_2c_3 & c_1c_2 & c_2^2 & c_2c_3 \\ -c_1c_3 & -c_2c_3 & -c_3^2 & c_1c_3 & c_2c_3 & c_3^2 \end{bmatrix}\begin{bmatrix} u^{(1)} \\ v^{(1)} \\ w^{(1)} \\ u^{(2)} \\ v^{(2)} \\ w^{(2)} \end{bmatrix} = \begin{bmatrix} F_x^{(1)} \\ F_y^{(1)} \\ F_z^{(1)} \\ F_x^{(2)} \\ F_y^{(2)} \\ F_z^{(2)} \end{bmatrix}
$$

As expexted, the blank entries had no bearing on the final result. This is the element's stiffness contribution.

2.1 Extension into Arbitrary Dimensions

Similar logic can be applied to direction cosines in any N dimensions to create a $2N \times 2N$ stiffness matrix. This formula is now implemented in the truss solver.

$$\mathbf{S} = \begin{bmatrix} c_1^2 & c_1c_2 & \cdots & c_1c_N \\ c_2c_1 & c_2^2 & \cdots & c_2c_N \\ \vdots & \vdots & \ddots & \vdots \\ c_Nc_1 & c_Nc_2 & \cdots & c_N^2 \end{bmatrix}$$

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} \mathbf{S} & -\mathbf{S} \\ -\mathbf{S} & \mathbf{S} \end{bmatrix}$$

3 Setup and Execution

The files `trussolverND.py` and `mygui.py` are needed to run the solver. Additionally, the modules `numpy` and `tkinter` must be installed for the matrix operations and user interface, respectively. The module `tkinter` is a part of the Python standard library, but if a user interface is not desired, the method `solve_case()` may be called with manually input file paths and a boolean 6-tuple of whether the user wants to solve for reactionary displacements (required), reactionary forces, internal strains, internal stresses, internal forces, and critical failure elements, respectively. By default, only reactionary displacements are solved.

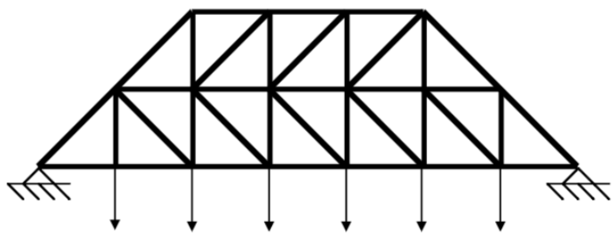
After this, the main script `trussolverND.py` can simply be interpreted in a Python 3 environment. Reactionary forces are output to `reactionary_forces`, reactionary displacements to `reactionary_displacements`, internal strains to `internal_strains`, internal stresses to `internal_stresses`, internal forces to `internal_forces`, and critical elements to `critical`, respectively.

4 Results

For this assignment, we consider a two-dimensional truss, shown in Figure 3. The horizontal and vertical bars are 5 m long, while the diagonal ones are $5\sqrt{2}$ m long. All have square cross-sections of side length 0.1. For each bar, the Young's modulus E is 210×10^9 Pa, the cross-sectional area A is 0.01, the yield strength σ_y is 250×10^6 Pa, the crushing strength σ_c is -250×10^6 Pa, and the second moment of the cross-sectional area I is $\frac{bh^3}{12} = \frac{s^4}{12} = 8.\bar{3} \times 10^{-6}$. The applied forces are all set to 1 N. Since our assumptions make the system linear, a scaling of these forces scales all resultant displacements, strains, stresses, and forces by the same amount.

Nodes are given five-digit ID's starting with a 1. The next number, ranging from 0 to 2, refers to their vertical level in the truss, while the last digit counts from left to right. Elements are given five-digit ID's starting with a 2. The next number, ranging from 0 to 4, refers to their vertical level in the truss, while the last digit counts from left to right. In short, elements and nodes are numbered from left to right, and bottom to top, starting at the left pin joint.

Figure 3: Truss before displacement



All output files are attached in Appendix A. Stresses and forces will be left there for the reader to inspect. Upon inspection, all results seem reasonable. For example, all of the vertical displacements are negative. The horizontal reactionary forces are opposing each other directed inwards, while the vertical reactions are equal and directed upwards. This is consistent with a truss being pulled downwards.

From Listing 6, we find that the critical tensile element for yielding is element 23002. This is a vertical bar, the first vertical bar and second bar overall from left, in the fourth row from the bottom. It would yield with loads scaled to approximately 1.618×10^6 N each. The critical compression element for both crushing and buckling is element number 21001. This is a diagonal bar at the far left in the second row of bars from the bottom. It would crush with loads scaled to approximately 5.893×10^5 N each, and buckle with loads scaled to approximately 8.142×10^4 N each. Note that this element shares the same strain, stress, and force as its right-end counterpart element 21013, so both of these would fail at the same point. This is sensible, since these bars transmit most of the loads in the truss to the two pin attachments. From this, we can conclude that the truss would fail when those nodes buckeled, with each applied force at 8.142×10^4 N.

5 Conclusions

The script proves to be a quick and effective way to solve a light, pin-ended, arbitrary truss. It takes less than a second to execute a system of equations with dozens of degrees of freedom. Once again, it was tested on example problems to verify its results.

Nevertheless, this script could still be improved. It still could still use better error-checking to inform the user. Higher dimensions still need to be thoroughly tested. Finally, a visualization tool could be developed for one-, two-, and three-dimensional structures to more easily visually verify results.

A Full Results

Listing 1: Reactionary Displacements

```
32
10002 1 -7.28878651101755e-10
10002 2 -3.220092353409013e-08
10003 1 -1.4577573022035102e-09
10003 2 -4.3957776956546476e-08
10004 1 -1.1036565785453938e-09
10004 2 -5.066047928629766e-08
10005 1 -1.0361040704738303e-10
10005 2 -5.002524165122809e-08
10006 1 6.786466653301036e-10
10006 2 -4.3267261934338e-08
10007 1 7.288786511017632e-10
10007 2 -3.187246254193129e-08
11002 1 9.616920262093718e-09
11002 2 -2.981997115313775e-08
11003 1 5.070009131621606e-09
11003 2 -4.265980395035398e-08
11004 1 1.612159486422051e-09
11004 2 -4.8925472353185185e-08
11005 1 -1.4101119605364798e-09
11005 2 -4.742650017115521e-08
11006 1 -5.349285615235679e-09
11006 2 -4.015428446677979e-08
11007 1 -8.5093486330614580e-09
11007 2 -2.8712399524105488e-08
12003 1 8.372255317247241e-09
12003 2 -3.898087856320913e-08
12004 1 4.693329930102398e-09
12004 2 -4.719046542007273e-08
12005 1 -7.206023901548934e-10
12005 2 -4.7208711072034706e-08
12006 1 -6.35232380953269e-09
12006 2 -3.782041763609499e-08
```

Listing 2: Reactionary Forces

```
4
10001 1 3.3061290334627103
10001 2 2.9999999999999734
10008 1 -3.3061290334627125
10008 2 2.999999999999972
```

Listing 3: Internal Strains

```
37
20001 −1.45775730220351e−10
20002 −1.4577573022035106e−10
20003 7.082014473162331e−11
20004 2.0000923429960216e−10
20005 1.5645141447549735e−10
20006 1.0046397154331909e−11
20007 −1.4577573022035263e−10
21001 −2.0203050891044036e−09
21002 4.761904761904757e−10
21003 3.063128239111491e−10
21004 2.5959460123849917e−10
21005 1.8270096257766791e−10
21006 3.470013866224956e−10
21007 −6.160005954265324e−11
21008 5.19748296014577e−10
21009 −2.0704796109506262e−10
21010 6.225954935116433e−10
21011 −2.2036576585110556e−10
21012 6.320126035651597e−10
21013 −2.020305089104403e−09
22001 −9.093822260944225e−10
22002 −6.915699290399109e−10
22003 −6.044542893917063e−10
22004 −7.878347309398398e−10
22005 −6.320126035651561e−10
23001 −1.0405572354917853e−09
23002 7.357850774289692e−10
23003 −4.907340671237954e−10
23004 3.470013866224912e−10
23005 −6.160005954264608e−11
23006 4.355781982409964e−11
23007 4.663870686064011e−10
23008 4.667733661369602e−10
23009 −1.1265042935518268e−09
24001 −7.357850774289686e−10
24002 −1.0827864640514584e−09
24003 −1.1263442838755594e−09
```

Listing 4: Internal Stresses

```
37
20001 −30.61290334627371
20002 −30.612903346273722
20003 14.872230393640896
```

```
20004  42.001939202916454
20005  32.85479703985444
20006  2.109743402409701
20007  -30.612903346274052
21001  -424.2640687119247
21002  99.9999999999999
21003  64.32569302134131
21004  54.51486626008482
21005  38.36720214131026
21006  72.87029119072407
21007  -12.936012503957182
21008  109.14714216306116
21009  -43.48007182996315
21010  130.7450536374451
21011  -46.27681082873217
21012  132.72264674868353
21013  -424.26406871192466
22001  -190.97026747982872
22002  -145.22968509838128
22003  -126.93540077225832
22004  -165.44529349736635
22005  -132.72264674868276
23001  -218.5170194532749
23002  154.51486626008352
23003  -103.05415409599702
23004  72.87029119072315
23005  -12.936012503955677
23006  9.147142163060924
23007  97.94128440734423
23008  98.02240688876164
23009  -236.56590164588363
24001  -154.5148662600834
24002  -227.38515745080625
24003  -236.53229961386748
```

Listing 5: Internal Forces

```
37
20001  -1.45775730220351e-10
20002  -1.4577573022035106e-10
20003  7.082014473162331e-11
20004  2.0000923429960216e-10
20005  1.5645141447549735e-10
20006  1.0046397154331909e-11
20007  -1.4577573022035263e-10
21001  -2.0203050891044036e-09
21002  4.761904761904757e-10
```

```
21003 3.063128239111491e-10
21004 2.5959460123849917e-10
21005 1.8270096257766791e-10
21006 3.470013866224956e-10
21007 -6.160005954265324e-11
21008 5.19748296014577e-10
21009 -2.0704796109506262e-10
21010 6.225954935116433e-10
21011 -2.2036576585110556e-10
21012 6.320126035651597e-10
21013 -2.020305089104403e-09
22001 -9.093822260944225e-10
22002 -6.915699290399109e-10
22003 -6.044542893917063e-10
22004 -7.878347309398398e-10
22005 -6.320126035651561e-10
23001 -1.0405572354917853e-09
23002 7.357850774289692e-10
23003 -4.907340671237954e-10
23004 3.470013866224912e-10
23005 -6.160005954264608e-11
23006 4.355781982409964e-11
23007 4.663870686064011e-10
23008 4.667733661369602e-10
23009 -1.1265042935518268e-09
24001 -7.357850774289686e-10
24002 -1.0827864640514584e-09
24003 -1.1263442838755594e-09
```

Listing 6: Critical Nodes

```
Failure Type - Element ID - Critical Value -
    ObservedValue - Factor of Load Causing Failure
YieldStress 23002 250000000.0 154.51486626008352
    1617967.2937048876
CrushingStress 21001 -250000000.0 -424.2640687119247
    589255.6509887949
BucklingForce 21001 -345436.0158636659
    -4.2426406871192475 81420.04976108804
```